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INFINITESIMAL DEFORMATIONS AND STABILITY OF SINGULAR LEGENDRE SUBMANIFOLDS*

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Abstract. We give the characterization of Arnol'd-Mather type for stable singular Legendre immersions in connection with the classification problem. The most important building block of the theory is providing a module structure on the space of infinitesimal integral deformations by means of the notion of natural liftings of differential systems and of contact Hamiltonian vector fields.

Key words. contact stability, Legendre stability, open Whitney umbrella

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1. Introduction.

First let us make clear the fundamental problems for the classification of singular Legendre submanifolds and thus make clear the significance of the present paper.

When we study singularities in contact geometry, we consider the local classification problem of parametrized singular Legendre submanifolds $f : N^n \to W^{2n+1}$ in two cases:

(C) The space W^{2n+1} is endowed with a contact structure only ("contact singularity theory").

(L) The space W^{2n+1} is endowed with a contact structure and a Legendre fibration ("Legendre singularity theory").

In case (C), it is natural to require that a diffeomorphism of W^{2n+1} preserves the contact structure, a *contactomorphism*, and, in case (L), it is natural to require that it is also respects the Legendre fibration, a *Legendre diffeomorphism*.

As for the case (C), we note that non-singular Legendre submanifolds are locally transformed by contactomorphisms to each other (see for instance [4]). However, the classification of singular Legendre submanifolds by contactomorpshims turns out to be very fruitful. In particular in the case of curves ($n = 1$) we have much progress of the classification theory recently ([3][39][40][41][20]). In this paper we give the exact classification, in the case of any dimension, of stable singularities of corank one (Theorem 3.1).

As for the case (L), the framework of Legendre singularity theory for Legendre immersions is established [5]: The singularity of a Legendre immersion via a Legendre fibration is embodied in a family of hypersurfaces, namely, the generating family of the Legendre immersion, and the stability of such singularity is expressed by mean of a notion, $\mathcal{K}$-versality, for its generating family. However, since a singular Legendre immersion has no generating family in general, the direct characterization should be worthwhile for the understanding of the stability of singular Legendre immersions, which we are going to provide in this paper (Theorem 4.1).

Take a contact manifold W^{2n+1} with the contact structure D and a Legendre fibration $\pi : W^{2n+1} \to Z^{n+1}$. Let $f, f' : (N^n, x_0) \to W^{2n+1}$ be germs of singular Legendre submanifolds. Then we consider the following conditions:

(d): f and f' are diffeomorphic;

(c): f and f' are contactomorphic with respect to D;

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(l): f and f' are Legendre equivalent with respect to D and π.

Clearly we have the implications (l) $\Rightarrow$ (c) $\Rightarrow$ (d).

In the case $n = 1$, it is proved by M. Zhitomirskii([40][41]) that (d) implies (c) in the real analytic category (resp. in C^∞ category except for an infinite codimensional set of germs). See also [20]. The general criterion on the classification (c) is given in [39], for general subset-germs, not necessarily Legendre, in a contact manifolds. Moreover we even conjecture that, for arbitrary n, (d) implies (c) except for an infinite codimensional set of germs. Actually in this paper we show that, at least for contact stable germs, (d) is equivalent to (c) (Theorem 3.1, Corollary 3.2).

On the other hand, we see that (c) does not implies (l), even for Legendre stable germs. For example two germs $x = t, y = 0, p = 0$ and $x = t^2, y = t^3, p = \dfrac{3}{2}t$ are contactomorphic but not Legendre equivalent. (See also the discussion below.) Also we have such examples of singular Legendre manifolds for instance, the $A_{2,1}$ singularity and the $A_{3,1}$ singularity in the sense of [16] for map-germs $(\mathbf{R}^3, 0) \to (\mathbf{R}^7, 0) \to (\mathbf{R}^4, 0)$ (Example 4.2). They are Legendre stable projections of open Whitney umbrellas of type 1 (§2).

In this paper we discuss the stable Legendre projections of open Whitney umbrellas in detail (Theorems 4.1 and 4.3).

The significance of Legendre singularity theory, as well as contact singularity theory, has increased recently by the trend of differential geometry treating (wave) fronts as generalised objects of hypersurfaces. Moreover the point of view in the micro-local analysis provides the motivation for the study of Legendre submanifolds as the description of singularities of solutions to partial differential equations.

The most simple singularity of front is given by

$$x = t^2, \quad y = t^3,$$

near $(x, y) = (0, 0)$, the $(2, 3)$-cusp on the (x, y)-plane. The front lifts to the Legendre curve

$$x = t^2, \quad y = t^3, \quad p = \frac{3}{2}t,$$

which is an immersion. Then the stability of the front is well described by the lifted non-singular Legendre submanifold via the Legendre equivalences induced by diffeomorphisms on the (x, y)-plane.

Consider then the similarly simple plane curve

$$x = t^2, \quad y = t^5,$$

the $(2, 5)$-cusp near $(x, y) = (0, 0)$. Then the natural lifting has the form:

$$x = t^2, \quad y = t^5, \quad p = \frac{5}{2}t^3,$$

which is an integral curve to the contact distribution $dy - pdx = 0$ and not an immersion at $t = 0$. Therefore, restricted ourselves to Legendre immersion without singularities we can not treat this very simple curve in the framework of Legendre singularity theory. Thus we are going to study, in this paper, singular Legendre immersions, in particular, the nature of their deformations in canonical way.

For example, consider the "stable"deformation of the $(2, 5)$-cusp

$$x = t^2, \quad y = t^5 + \lambda t^3$$

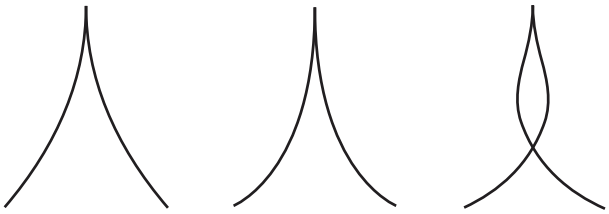

FIG. 1. *An integral deformation of* $(2,5)$*-cusp.*

inducing smooth deformation of tangent lines. See figure 1.

Then we understand, via our general theory, the stable deformation forms the stable projection (front) of the open Whitney umbrella of type 1, introduced in this paper, which is contactomorphic to

$$x = t^2, \quad y = t^5 + \lambda t^3, \quad p = \frac{5}{2}t^3 + \frac{3}{2}\lambda t, \quad \mu = t^3,$$

in the (x, y, λ, p, μ)-space with the contact structure $dy - pdx - \mu d\lambda = 0$.

We have given in [18] the characterizations for symplectic stability of Lagrange varieties and Lagrange stability in symplectic geometry. Therefore the present paper can be regarded as a contact or Legendre counterpart to [18]: We observe surely the parallelism between Lagrange and Legendre singularity theories, as well as symplectic and contact geometries. In fact we use several results in Lagrange singularities proved in [18] to deduce several results in Legendre singularities. Nevertheless we need to break through several difficulties for obtaining the characterizations (Theorem 4.1).

In particular, we realize that the direct characterization needs the deep understanding of *the space of Legendre submanifolds*. Since the space of submanifolds can be treated as the space of immersions, we consider, in a contact manifold, the space of integral mappings, parametrizations of integral submainfolds of the contact distribution. The space of integral mappings turns out to be our central object. Its tangent space at an integral mapping is naturally regarded as the space of infinitesimal deformations of the integral mapping among integral mappings. The fact, then, we observe in this paper is that the tangent space to the space of integral mappings has the structure of not merely a vector space, but the very natural module structure. It reminds us the "modularity"in the sense of Mather [29]. However, in this paper we introduce the module structure for functions not on the source manifold but for functions on the target manifold.

We understand the modularity of tangent spaces to the space of integral mappings in a contact manifold without difficulty as follows: An infinitesimal deformations on a contact manifold, namely, a contact vector field is locally given by a *contact Hamilton vector field* X_K with a Hamiltonian function K on the contact manifold, if we fix a local contact form α. Then we see, for functions H, K,

$$X_{HK} = H \cdot X_K + K \cdot X_H - (HK) \cdot X_1.$$

Thus, we can give the module structure $*$ of functions on the space of contact Hamilton vector fields by identifying it with the space of functions: The formula reads as

$$H * X_K := H \cdot X_K + K(X_H - H \cdot X_1).$$

Note that the interior product $i_{X_K}\alpha$ is equal to K. Let f be an integral mapping. The vector field $X_K \circ f$ along f is a kind of integral infinitesimal deformations of f. Then we set

$$H * (X_K \circ f) := f^*H \cdot (X_K \circ f) + f^*K(X_H - H \cdot X_1) \circ f.$$

Note that $i_{X_K \circ f}\alpha$ is equal to f^*K. We proceed even further. We define the multiplication $*$ by a function H of any infinitesimal integral deformation v of f by the formula:

$$H * v := f^*H \cdot v + (i_v\alpha)(X_H - H \cdot X_1) \circ f.$$

Moreover we observe that the multiplication is *intrinsic*: The definition of multiplications looks like depending on the choice of a local contact form α, but in fact it is independent of it and is determined only by the contact structure.

Note that the module structure is effectively used in [20] for the classifying of singular Legendre curves in the contact three space.

In §2, we introduce the class of singular Legendre submanifolds, *open Whitney umbrellas*, in contact manifolds by explicit forms. In §3, we give the characterization of open Whitney umbrellas as contact stable integral map-germs of corank at most one (Theorem 3.1). Also we characterize contact versal deformations (Theorem 3.3). In §4, we formulate the characterizations of Legendre stability and Legendre versality; Theorems 4.1 and 4.3. To prove Theorems 3.1 and 4.1, we need to clarify the infinitesimal condition on Legendre stability. For this, we introduce the notion of natural liftings ([34][35]) of differential forms and differential systems in §5. After reviewing the notion of contact Hamilton vector fields in §6, we formulate exactly infinitesimal conditions in §7. In §8, we study the relation of integral mappings and isotropic mappings, and, in §9, we study on the integral jet spaces. In §10, we give results on finite determinacy of integral map-germs. We give, using all results given in the previous sections, the proofs of Theorems 3.1 and 4.1 in §11. In §12, we mention on the proofs of contact and Legendre versality theorems (Theorems 3.3 and 4.3).

In this paper, all manifolds and mappings we treat are assumed to be of class C^∞ (in case $\mathbf{K} = \mathbf{R}$) or complex analytic (in case $\mathbf{K} = \mathbf{C}$).

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2. Integral mappings. Now we are going to describe in detail the objects to which we apply our theory, before formulating the main theorems.

Let (W, D) be a real or complex contact manifold of dimension $2n+1$ ([1][2][5]). Here $D \subset TW$ stands for the contact structure on W, namely, a completely non-integrable distribution of codimension one. A typical example is $W = \mathbf{K}^{2n+1}$, $\mathbf{K} = \mathbf{R}$

or $\mathbf{C}$, with coordinates (p, q, r), and

$$D = \{dr - \sum_{i=1}^{n} p_i dq_i = 0\} \subset T\mathbf{K}^{2n+1}.$$

By Darboux's theorem, any contact manifold is locally contactomorphic to this standard model.

A mapping $f : N \to W$ from an n-dimensional manifold N is called an *integral mapping* if, for any $x \in N$, $f_*(T_xN) \subset D_{f(x)}$, where $f_* : T_xN \to T_{f(x)}W$ is the differential mapping (the linearization) of f at x. If D is given by a contact form α, then f is an integral mapping if and only if $f^*\alpha \equiv 0$. Thus the notion of integral mappings generalizes that of (immersed) integral manifolds in the contact manifold W.

Two map-germs $f : (N, x_0) \to (W, D)$ and $f' : (N', x_0') \to (W', D')$ to contact manifolds (W, D) and (W', D') respectively, are called *contactomorphic* (resp. *diffeomorphic*) if there exist a diffeomorphism $\sigma : (N, x_0) \to (N', x_0')$ and a contactomorphism (resp. a diffeomorphism) $\tau : (W, f(x_0)) \to (W', f'(x_0'))$, $\tau_* D = D'$, such that $f' \circ \sigma = \tau \circ f$. In this case we call also the pair (σ, τ) a *contactomorphism* (resp. a *diffeomorphism*) of f and f'.

Let $f : (N^n, x_0) \to W^{2n+1}$ be an integral map-germ. Suppose that f is of corank ≤ 1, namely that the kernel of the differential map $f_* : T_{x_0}N \to T_{f(x_0)}W$ is zero or one dimensional. Then there exists a contactomorphism (σ, τ) from f to $f' = \tau \circ f \circ \sigma : (\mathbf{K}^n, 0) \to (\mathbf{K}^{2n+1}, 0)$ such that

$$(q_1, \ldots, q_{n-1}, q_n) \circ f' = (x_1, \ldots, x_{n-1}, u(x_1, \ldots, x_{n-1}, x_n),$$

for some function u, where (x_1, x_{n-1}, x_n) is the standard coordinate of $\mathbf{K}^n$. Then, setting $v := p_n \circ f'$, we easily see that the components $p_1, \ldots, p_{n-1}$ and r of f' are uniquely determined by the condition

$$d(r \circ f) = \sum_{i=1}^{n} (p_i \circ f) d(q_i \circ f).$$

Actually we have:

PROPOSITION 2.1. (Pre-normal form of integral map-germ of corank at most one.) *Let $f : (N^n, x_0) \to W^{2n+1}$ be an integral map-germ of corank ≤ 1. Then there exist functions-germs $u, v : (\mathbf{K}^n, 0) \to (\mathbf{K}, 0)$ such that f is contactomorphic to the integral map-germ $f' : (\mathbf{K}^n, 0) \to (\mathbf{K}^{2n+1}, 0)$ defined by*

$$(q_1, \ldots, q_{n-1}, q_n, p_n) \circ f' := (x_1, \ldots, x_{n-1}, u, v),$$

$$p_i \circ f' := \int_0^{x_n} \left(\frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_n} - \frac{\partial v}{\partial x_n} \frac{\partial u}{\partial x_i} \right) dx_n, (1 \leq i \leq n-1),$$

and

$$r \circ f' := \int_0^{x_n} \left(v \frac{\partial u}{\partial x_n} \right) dx_n.$$

In particular, our main objects of the study are introduced as follows: For an integer k with $0 \leq k \leq \dfrac{n}{2}$, we define a map-germ $f = f_{n,k} : (\mathbf{K}^n, 0) \to (\mathbf{K}^{2n+1}, 0)$ by $q_1 \circ f = x_1, \ldots, q_{n-1} \circ f = x_{n-1}$ and

$$u = q_n \circ f = \frac{x_n^{k+1}}{(k+1)!} + x_1 \frac{x_n^{k-1}}{(k-1)!} + \cdots + x_{k-1}x_n,$$

$$v = p_n \circ f = x_k \frac{x_n^k}{k!} + \cdots + x_{2k-1}x_n,$$

and the property $f^*\alpha = 0$. The components $p_1, \ldots, p_{n-1}$ and r of f are defined as in Proposition 2.1 so that

$$d(r \circ f) = \sum_{i=1}^{n} (p_i \circ f) d(q_i \circ f).$$

Then we call a map-germ $f : (N, x_0) \to W$ an *open Whitney umbrella* (or an *unfurled Whitney umbrella*) *of type* k $(0 \leq k \leq \dfrac{n}{2})$, if it is contactomorphic to the normal form $f_{n,k}$.

An open Whitney umbrella is an integral map-germ of corank at most one. It is an immersion, namely, *Legendre immersion*, exactly when $k = 0$: A map-germ $f : (N^n, x_0) \to (W^{2n+1}, D)$ is a Legendre immersion if and only if f is an open Whitney umbrella of type 0. If $k > 0$, then the singular locus of an open Whitney umbrella of type k is non-singular and of codimension 2 in N.

If $k \neq k'$, then an open Whitney umbrella of type k and that of type k' are not diffeomorphic, therefore they are not contactomorphic.

Open Whitney umbrellas are intrinsically characterized via the notion of "contact stability" in §3.

3. Contact stability and versality. In this section, we treat the case (C), the contact singulatity theory. First we define the notion of contact stability of map-germs: Roughly speaking, an integral map-germ $f : (X, x_0) \to W$ is *contact stable* if, by any sufficiently small integral perturbations, the contact equivalence class of f_{x_0} is not removed but remains nearby x_0.

More exactly, an integral map-germ $f : (N, x_0) \to W$ is *contact stable* if, for any integral representative $f : U \to W$ of f, there exists a neighborhood Ω in $C_I^\infty(N, W)$ such that, for any $f' \in \Omega$, the original germ f is contact equivalent to $f'_{x'_0}$ for some $x'_0 \in U$ (cf. [5] page 325).

An integral map-germ $f : (N, x_0) \to W$ is called *homotopically contact stable* if any one-parameter integral deformation $F = (f_t)$ of f is trivialized by contactomorphisms:

$$\tau_t \circ f_t \circ \sigma_t^{-1} = f,$$

(σ_t, τ_t) being contactomorphism of f_t and f. Here σ_t may move base points of germs. (See the definition of contactomorphism in §1).

To characterize the Legendre stability by means of transversality, we introduce the notion of *integral jet spaces*. Denote by $J_I^r(N, W)$ the set of r-jets of integral map-germs $f : (N, x_0) \to (W, w_0)$ *of corank at most one*:

$$J_I^r(N, W) = \{j^r f(x_0) \mid f : (N, x_0) \to (W, w_0) \text{ integral, } \operatorname{corank}_{x_0} f \leq 1\}.$$

Then $J^r_I(N,W)$ is a submanifold of the ordinary jet space $J^r(N,W)$ (§9). Moreover, for $j^rf(x_0) \in J^r_I(N,W)$, the *contactomorphism class* of $j^rf(x_0)$, namely, the set of r-jets of map-germs which are contactomorphic to $f : (N,x_0) \to (W,w_0)$ form a submanifold of $J^r_I(N,W)$.

If $f : N \to W$ is an integral mapping of corank at most one, then the image of the r-jet extension $j^rf : N \to J^r(N,W)$ is contained in $J^r_I(N,W)$. Then we regard j^rf as a mapping to $J^r_I(N,W)$. Based on a Legendre version of transversality theorem (§9), contact stability is characterized by the transversality.

In the case $\mathbf{K} = \mathbf{C}$, we understand the definition of stability by the transversality.

For a manifold-germ (N,x_0), we denote by $\mathcal{E}_{N,x_0}$ the $\mathbf{K}$-algebra consisting of C^∞ function-germs $(N,x_0) \to \mathbf{K}$, and by m_{N,x_0} the unique maximal ideal of $\mathcal{E}_{N,x_0}$. If the base point x_0 is clear in the context, we abbreviate $\mathcal{E}_{N,x_0}$ and m_{N,x_0} to $\mathcal{E}_N$ and m_N respectively.

For a map-germ $f : (N,x_0) \to (W,w_0)$, we denote by $f^* : \mathcal{E}_W \to \mathcal{E}_N$ the homomorphism defined by the pull-back by f. Moreover we set

$$R_f := \{h \in \mathcal{E}_W \mid dh \in \mathcal{E}_N d(f^*\mathcal{E}_W)\},$$

for the exterior differential d ([18]).

Then we have:

THEOREM 3.1. (*Classification of contact stable germs.*) *Let $f : (N,x_0) \to W^{2n+1}$ be an integral map-germs of corank at most one. Then the following conditions are equivalent:*

(cs) *f is contact stable.*

(hcs) *f is homotopically contact stable.*

(ics) *f is infinitesimally contact stable.*

(fics) *f is infinitesimally contact stable on the level of formal series.*

(owu) *f is an open Whitney umbrella.*

(ca) *$R_f = f^*\mathcal{E}_W$ and f is diffeomorphic (i.e. $\mathcal{A}$-equivalent) to an analytic map-germ $f' : (\mathbf{K}^n,0) \to (\mathbf{K}^{2n+1},0)$ (not necessarily integral) such that the codimension of the singular locus of the complexification $f'_{\mathbf{C}}$ of f' is greater than or equal to 2.*

(ct) *The jet extension $j^rf : (N,x_0) \to J^r_I(N,W)$ is transversal to the contactomorphism class of $j^rf(x_0)$, for an integer $r \geq \dfrac{n}{2} + 1$.*

We must explain the notion of infinitesimal contact stablity (ics): Of course, it is the infinitesimal counterpart of contact stability.

Now recall the notion of infinitesimal stability due to Mather [28] for a general C^∞ map-germ $f : (N,x_0) \to (W,w_0)$. A map-germ f is called infinitesimally stable if $V_f = tf(V_N) + wf(V_W)$, where V_N (resp. V_W, V_f) is the module consisting of all germs of vector fields over (N,x_0) (resp. over (W,w_0), along f), and $tf : V_N \to V_f$ (resp. $wf : V_W \to V_f$) is defined by $tf(\xi)(x) = f_*(\xi(x)), (\xi \in V_N, x \in (N,x_0))$ (resp. $wf(\eta)(x) = \eta(f(x)), (\eta \in V_W, x \in (N,x_0))$).

Similarly we call an integral map-germ $f : (N,x_0) \to (W,w_0)$ *infinitesimally contact stable* if

$$VI_f = tf(V_N) + wf(VH_W).$$

where $VH_W(\subset V_W)$ (resp. $VI_f(\subset V_f)$) is the module of all germs of Hamiltonian vector fields over (W,w_0) (resp. all germs of infinitesimal integral deformations of f over (N,x_0)). See §7 for details.

Also we can consider the notion of infinitesimal contact stability on the level of formal series: f is called *infinitesimally contact stable on the level of formal series* or *formally infinitesimally contact stable* if any formal integral deformation along f is written as the sum of the push-forward of a formal vector field by f and the pull-back of a formal Hamilton vector filed over (W, w_0) by f.

Let $f, f' : (N^n, x_0) \to W^{2n+1}$ be contact stable integral map-germs of corank ≤ 1. Then f and f' are open Whitney umbrellas by Theorem 3.1. Therefore f (resp. f') is an open Whitney umbrella of type k (resp. of type k'), for some k (resp. k'). Suppose f and f' are diffeomorphic. Then we easily see that $k = k'$. Therefore f and f' are contactomorphic to the same normal form $f_{n,k}$, hence, they are contactomorphic. Thus we have:

COROLLARY 3.2. *Contact stable integral map-germs of corank at most one are contactomorphic if and only if they are diffeomorphic.*

The notions in Theorem 3.1 are discussed in detail along the following sections, in particular in §7 and in §9. The proof of Theorem 3.1 will be given in §11.

As in the ordinary singularity theory, we can show also *contact versality theorem*, which gives an alternative proof of the equivalence of (hcs) and (ics).

An integral deformation $F = (f_\lambda) : (N \times \mathbf{K}^r, (x_0, 0)) \to W$ of an integral map-germ $f : (N, x_0) \to W$ is called *contact versal* if any other integral deformation $G = (g_\mu) : (N \times \mathbf{K}^s, (x_0, 0)) \to W$ of f is induced from F up to contactomorphisms, namely if there exist a map-germ $\varphi : (\mathbf{K}^s, 0) \to (\mathbf{K}^r, 0)$ and a family of contactomorphisms $(\sigma_\mu, \tau_\mu), (\mu \in (\mathbf{K}^s, 0))$ such that $g_\mu = \tau_\mu \circ f_{\varphi(\mu)} \circ \sigma_\mu^{-1}$ for any $(\mu \in (\mathbf{K}^s, 0))$, where $g_\mu(x) = G(x, \mu)$.

F is called *infinitesimally contact versal* if

$$VI_f = \left\langle \left.\frac{\partial F}{\partial \lambda_1}\right|_{\lambda=0}, \ldots, \left.\frac{\partial F}{\partial \lambda_r}\right|_{\lambda=0} \right\rangle_{\mathbf{R}} + tf(V_N) + wf(VH_W).$$

THEOREM 3.3. (*Contact versality theorem.*) *An integral deformation $F : (N \times \mathbf{K}^r, (x_0, 0)) \to W$ of an integral map-germ $f : (N, x_0) \to W$ of corank at most one, is contact versal if and only if F is infinitesimally contact versal. Any contact versal deformations of f with the same number of parameters are contactomorphic to each other. An integral map-germ $f : (N, x_0) \to W$ of corank at most one has a contact versal deformation if and only if $tf(V_N) + wf(VH_W)$ is of finite codimension over $\mathbf{K}$ in VI_f.*

We give an example of contact versal deformations:

EXAMPLE 3.4. Let $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^5, 0)$ be defined by

$$p_1 = v^3 + u^2 v,\ q_1 = v^2,\ p_2 = \frac{4}{3}uv^3,\ q_2 = u,\ r = \frac{2}{5}v^5 + \frac{2}{3}u^2v^3.$$

Then f is not contact stable and of contact codimension 1. The germ f has the one-parameter contact versal deformation $f_\lambda : (\mathbf{R}^2, 0) \to (\mathbf{R}^5, 0)$ defined by

$$p_1 = v^3 + u^2 v - \lambda v,\ q_1 = v^2,\ p_2 = \frac{4}{3}uv^3,\ q_2 = u,\ r = \frac{2}{5}v^5 + \frac{2}{3}(u^2 - \lambda)v^3.$$

This deformation is also Legendre versal (§4). The deformation describes the "creation-annihilation" of two folded umbrellas.

4. Legendre stability and versality. Now we turn to the case (L), Legendre singularity theory.

A fibration $\pi : W^{2n+1} \to Z^{n+1}$ is called a *Legendre fibration* if the fibers of π are Legendre submanifolds of W. Note that any Legendre fibrations are locally isomorphic (See for instance [4]). Then we concern with the relative position of the image of an integral mapping with respect to a Legendre fibration: We consider an integral map-germ $f : (N, x_0) \to (W, w_0)$ together with a germ of Legendre fibration $\pi : (W, w_0) \to (Z, z_0)$, where $w_0 = f(x_0), z_0 = \pi(w_0) = (\pi \circ f)(x_0)$. Then a contactomorphism-germ $\tau : (W, w_0) \to (W, w_0)$ is called a *Legendre diffeomorphism-germ* if τ maps π-fibers to π-fibers, or more exactly, if there exists a diffeomorphism-germ $\bar{\tau} : (Z, z_0) \to (Z, z_0)$ such that $\bar{\tau} \circ \pi = \pi \circ \tau$.

A pair (f, π) is *Legendre equivalent* to another pair (f', π) if there exists a contact equivalence (σ, τ) of f and f' such that τ is a Legendre diffeomorphism. In this case, we call (σ, τ) a *Legendre equivalence* of (f, π) and (f', π).

An integral map-germ $f : (N, x_0) \to (W, w_0)$ is called *homotopically Legendre stable* if any integral deformation (f_t) of f is trivialized under Legendre equivalence:

$$\tau_t \circ f_t \circ \sigma_t^{-1} = f,$$

(σ_t, τ_t) being Legendre equivalences of f_t and f. Here σ_t may move base points of germs.

Moreover we can define, over the $\mathbf{R}$, the notion of Legendre stability of map-germs similarly to the contact stability: An integral map-germ $f : (N, x_0) \to W$ is Legendre stable with respect to an Legendre fibration $\pi : W \to Z$ if, by any sufficiently small integral perturbations, the Legendre equivalence class of (f, π) is not removed. To formulate accurately, denote by $C_I^\infty(N, W)$ the space of C^∞ integral mappings from N to W, endowed with the Whitney C^∞ topology. Then an integral map-germ $f : (N, x_0) \to W$ is *Legendre stable* if, for any integral representative $f : U \to W$ of f, there exists a neighborhood Ω in $C_I^\infty(U, W)$ of f such that, for any $f' \in \Omega$, the original pair (f_{x_0}, π) of germs is Legendre equivalent to $(f'_{x'_0}, \pi)$ for some $x'_0 \in U$ (cf. [5]).

We can define the notion of infinitesimal Legendre stablity as the infinitesimal counterpart of Legendre stability: We call an integral map-germ $f : (N, x_0) \to W$ *infinitesimally Legendre stable* if $VI_f = tf(V_N) + wf(VL_W)$, where $VL_W (\subset V_W)$ is the module of all germs of infinitesimal Legendre transformations over (W, w_0) (resp. infinitesimal integral deformations of f over (N, x_0)). See §7 for details.

Moreover, for algebraic characterizations of Legendre stability, we set

$$r_0 = \inf\{r \in \mathbf{N} \mid f^*\mathcal{E}_W \cap m_N^{r+1} \subset f^* m_W^{n+2}\}.$$

If $f : (N, x_0) \to W$ is an open Whitney umbrella, then f is, in particular, finite, namely, $\mathcal{E}_N$ is a finite $\mathcal{E}_W$-module via $f^* : \mathcal{E}_W \to \mathcal{E}_N$. Therefore r_0 is a finite positive integer, determined by n and k, the type of the open Whitney umbrella. Actually r_0 depends only on the right-left equivalent class (diffeomorphism class) of f.

The main purpose of the present paper is to show the following:

THEOREM 4.1. (*Arnol'd-Mather type characterization of Legendre stability.*). *For an integral map-germ $f : (N^n, x_0) \to (W^{2n+1}, w_0)$ of corank at most one, the following conditions are equivalent to each other:*

(ls) *f is Legendre stable.*

(hls) *f is homotopically Legendre stable.*

(ils) *f is infinitesimally Legendre stable.*

(fils) *f is infinitesimally Legendre stable on the level of formal series.*

(la) *f is an open Whitney umbrella and $f^*\mathcal{E}_W$ is generated by $1, p_1 \circ f, \ldots, p_n \circ f$ as $\mathcal{E}_Z$-module via $(\pi \circ f)^*$.*

(la$'$) *f is an open Whitney umbrella and $Q(f) := f^*\mathcal{E}_W/(\pi \circ f)^* m_Z \mathcal{E}_W$ is generated over $\mathbf{K}$ by $1, p_1 \circ f, \ldots, p_n \circ f$.*

(la$''_r$) $(r \geq r_0)$. *f is an open Whitney umbrella and $Q_{r+1}(f) := f^*\mathcal{E}_W/\{(\pi \circ f)^* m_Z f^*\mathcal{E}_W + f^*\mathcal{E}_W \cap m_N^{r+2}\}$ is generated by $1, p_1 \circ f, \ldots, p_n \circ f$ over $\mathbf{K}$.*

(lt$_r$) $(r \geq r_0)$. *The jet extension $j^r f : (N, x_0) \to J_I^r(N, W)$ is transversal to the Lagrange equivalence class of $j^r f(x_0)$.*

We give examples of Legendre stable germs in connection with the classification problem:

EXAMPLE 4.2. Consider two integral map-germs $(\mathbf{R}^3, 0) \to (\mathbf{R}^7, 0)$ considered with the Lagrange projections $\pi : (\mathbf{R}^7, 0) \to (\mathbf{R}^4, 0), \pi(p, q, r) = (q, r)$:

$$\begin{aligned}
A_{2,1} \quad : \quad & q_1 = x_1,\ q_2 = x_2,\ q_3 = x_3^2,\ r = \frac{1}{3}x_1x_3^3 \\
& p_1 = \frac{1}{3}x_3^3,\ p_2 = 0,\ p_3 = x_1x_3^2 \\
A_{3,1} \quad : \quad & q_1 = x_1,\ q_2 = x_2,\ q_3 = x_3^3 + x_1x_3, \\
& r = \frac{3}{5}x_3^5 + \frac{3}{4}x_2x_3^4 + \frac{1}{3}x_1x_3^3 + \frac{1}{2}x_1x_2x_3^2 \\
& p_1 = -\frac{2}{3}x_3^3 - \frac{1}{2}x_2x_3^2,\ p_2 = \frac{3}{4}x_3^4 + \frac{1}{2}x_1x_3^2,\ p_3 = x_3^2 + x_2x_3.
\end{aligned}$$

Then they are Legendre stable. They are contactomorphic to open Whitney umbrella of type 1 and they are not Legendre equivalent to each other.

The equivalence of (hls) and (ils) is one of consequences of *Legendre versality theorem*: We introduce the notion of Legendre versality of integral deformations of integral map-germs.

A deformation $F : (N \times \mathbf{K}^r, (x_0, 0)) \to W$ of an integral map-germ $f : (N, x_0) \to W$ is called *integral* if each $f_\lambda = F|_{N \times \{\lambda\}}, (\lambda \in (\mathbf{K}^r, 0))$ is integral, for a representative of F. We write $F = (f_\lambda)$ in short. An integral deformation F of f is called *Legendre versal* if any other integral deformation $G : (N \times \mathbf{K}^s, (x_0, 0)) \to W$ of f is induced from F up to Legendre equivalence, namely if there exist a map-germ $\varphi : (\mathbf{K}^s, 0) \to (\mathbf{K}^r, 0)$ and a Legendre deformation $(\sigma_\mu, \tau_\mu), (\mu \in (\mathbf{K}^s, 0))$ such that $g_\mu = \tau_\mu \circ f_{\varphi(\mu)} \circ \sigma_\mu^{-1}$ for any $(\mu \in (\mathbf{K}^s, 0))$, where $g_\mu(x) = G(x, \mu)$. F is called *infinitesimally Legendre versal* if

$$VI_f = \left\langle \left.\frac{\partial F}{\partial \lambda_1}\right|_{\lambda=0}, \ldots, \left.\frac{\partial F}{\partial \lambda_r}\right|_{\lambda=0} \right\rangle_{\mathbf{K}} + tf(V_N) + wf(VL_W).$$

Then we also mention in this paper on a proof of the following:

THEOREM 4.3. (*Legendre versality theorem.*) *An integral deformation $F : (N \times \mathbf{K}^r, (x_0, 0)) \to W$ of an integral map-germ $f : (N, x_0) \to W$ of corank at most one, is Legendre versal if and only if F is infinitesimally Legendre versal. Any Legendre versal deformations of f with the same number of parameters are Legendre equivalent to each other. An integral map-germ $f : (N, x_0) \to W$ of corank at most one has a Legendre versal deformation if and only if $tf(V_N) + wf(VL_W)$ is of finite codimension over $\mathbf{K}$ in VI_f.*

Setting $r = 0$ we have again that f is homotopically Legendre stable if and only if f is infinitesimally Legendre stable.

We give an example of Legendre versal deformations:

EXAMPLE 4.4. Let $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^5, 0)$ be defined by

$$p_1 = v^2,\ q_1 = v^3 + uv,\ p_2 = -\frac{2}{3}v^3,\ q_2 = u,\ r = \frac{3}{5}v^5 + \frac{1}{3}uv^3.$$

Then f is contactomorphic to open Whitney umbrella of type 1 but f is not Legendre stable and of Legendre codimension 1. The Legendre versal deformation $f_\lambda : (\mathbf{R}^2, 0) \to (\mathbf{R}^5, 0)$ of f is given by

$$p_1 = v^2, q_1 = v^3 + uv + \lambda v^2, p_2 = -\frac{2}{3}v^3, q_2 = u, r = \frac{3}{5}v^5 + \frac{1}{3}uv^3 + \frac{1}{2}\lambda v^4.$$

The deformation describes the "collision-penetration" of one swallowtail with one folded umbrella together with one A_1A_2-multi-singularity.

5. Lie derivative. Let N, W be manifolds, and $f : N \to W$ a mapping. A mapping $v : N \to TW$ is called a *vector field along* f or an *infinitesimal deformation* of f, if $\pi \circ v = f$, for the projection $\pi : TW \to W$. We denote by V_f the module of all vector field along f. By the fiberwise addition and scalar multiplication on TW, V_f turns out to be a module over the function-algebra $\mathcal{E}_N$ on N.

It is easy to see that there exists a one-parameter deformation $F : U \to W$ of f defined on an open neighborhood U in $N \times \mathbf{K}$ of $N \times \{0\} \cong N$ such that $F|_{N\times\{0\}} = f$. We write as $F = (F_t)$ so that $F_0 = f$. Then we define, for a differential p-form α on W, a differential p-form $L_v\alpha$ on N by

$$L_v\alpha = \left.\frac{d}{dt}\right|_{t=0} F_t^*\alpha.$$

For this, see also [18] p.225. Then $L_v\alpha$ does not depend on the choice of F but depends only on v. We call $L_v\alpha$ the *Lie derivative* of α by v. Moreover we define the interior product $i_v\alpha$, that is a differential $(p-1)$-form on N by

$$i_v\alpha(Z_1, \ldots, Z_{p-1})(x) = \alpha(v(x), f_*Z_1(x), \ldots, f_*Z_{p-1}(x)),$$

for vector fields $Z_1, \ldots, Z_{p-1}$ over N.

EXAMPLE 5.1. Let $N = TW$ and $f = \pi : TW \to W$. We regard the identity map $1 : TW \to TW$ as a vector field along π. Then, for a p-form α on W, we have defined the p-form $L_1\alpha$ and $(p-1)$-form $i_1\alpha$ on TW. □

LEMMA 5.2. *We have the following fundamental formulae:*

$$\begin{array}{lrcl}
(1) & i_v(\lambda\alpha + \mu\beta) & = & \lambda(i_v\alpha) + \mu(i_v\beta), \\
(2) & i_{\lambda u+\mu v}\alpha & = & (f^*\lambda)(i_u\alpha) + (f^*\mu)(i_v\alpha), \\
(3) & L_v\alpha & = & i_v(d\alpha) + d(i_v\alpha), \\
(4) & L_v(\alpha \wedge \beta) & = & (L_v\alpha) \wedge f^*\beta + f^*\alpha \wedge (L_v\beta), \\
(5) & i_v(\alpha \wedge \beta) & = & (i_v\alpha) \wedge f^*\beta + (-1)^r f^*\alpha \wedge (i_v\beta).
\end{array}$$

Here u, v are vector fields along a mapping $f : N \to W$, λ, μ are functions on W, α, β are differential forms on W, and α is an r-form.

In particular, we refer (3) as the Cartan type formula: $L_v = di_v + i_v d$.

Proof. (1) and (2) are straightforward from the definition. The proof of (3) is given in [18] Lemma 3.3. (4), (5) are easily proved similarly to the ordinary case $W = N$ and f is the identity mapping. □

The following formulae are proved from the definitions in the straightforward way.

LEMMA 5.3. *Let* $f : N \to W$ *be a mapping,* $v : N' \to TN$ *a vector field along a mapping* $N' \to N$, $w : W \to TW'$ *a vector field along a mapping* $W \to W'$, α *a differential form on* W *and* α' *a differential form on* W'. *Then we have*

$$\begin{array}{lll} \text{(i)} & L_{w\circ f}\alpha' = f^*(L_w\alpha'), & L_{f_*v}\alpha = L_v(f^*\alpha), \\ \text{(ii)} & i_{w\circ f}\alpha' = f^*(i_Y\alpha'), & i_{f_*v}\alpha = i_v(f^*\alpha). \end{array}$$

Here $w\circ f$ *is the pull-back of* w *by* f, *and* f_*v *is the push-forward of* v *by* f: $(w\circ f)(x) = w(f(x)), (x \in N)$, $(f_*v)(x') = f_*(v(x')), (x' \in N')$.

In particular we have

$$\begin{array}{lll} \text{(i')} & L_{Y\circ f}\alpha = f^*(L_Y\alpha), & L_{f_*X}\alpha = L_X(f^*\alpha), \\ \text{(ii')} & i_{Y\circ f}\alpha = f^*(i_Y\alpha), & i_{f_*X}\alpha = i_X(f^*\alpha), \end{array}$$

for a vector field X *over* N, *a vector field* Y *over* W, *and for a differential form* α *on* W. *Here* $Y \circ f$ *is the pull-back of* Y *by* f, *and* f_*X *is the push-forward of* X *by* f.

Then the fundamental concept of this paper is introduced as follows:

PROPOSITION 5.4. *Let* W *be a manifold and* α *a differential form on* W.

(1) *There exists a unique differential form* $\widetilde{\alpha}$ *on* TW *such that, for any vector field* $X : W \to TW$ *over* W, $X^*\widetilde{\alpha} = L_X\alpha$ *holds.*

(2) *Moreover,* $\widetilde{\alpha}$ *of* (1) *satisfies* $v^*\widetilde{\alpha} = L_v\alpha$, *for any vector field* $v : N \to TW$ *along a mapping* $f : N \to W$.

(3) $d\widetilde{\alpha} = \widetilde{d\alpha}$ *and* $\widetilde{f^*\alpha} = (f_*)^*(\widetilde{\alpha})$, *where* $f_* : TN \to TM$ *is the bundle homomorphism defined by differential of* f.

In fact, we have $\widetilde{\alpha} = L_1\alpha$. We call $\widetilde{\alpha}$ the *natural lifting* of α. The notion of natural liftings is first defined, even for general tensors, in [34][35] in a different manner: This fact is pointed out to the author by H. Sato. Though our construction is limited to differential forms, it seems more direct and useful for the infinitesimal study of differential systems. We are going to apply, in this paper, the notion of natural liftings for the infinitesimal study of stability of integral mappings in contact geometry.

Proof of Proposition 5.4: (1) We set $\widetilde{\alpha} = L_1\alpha$, for the identity mapping $1 : TW \to TW$. Then $X^*\widetilde{\alpha} = X^*L_1\alpha = L_{1\circ X}\alpha = L_X\alpha$. Similarly we have (2). Let, for another β, $X^*\beta = X^*\widetilde{\alpha}$, for any vector field over W. Then, for any $z \in TW$ and any $v \in T_z(TW) \setminus K$, there exists a vector field X over W and $u \in T_{\pi(z)}W$ such that $X_*(u) = v$. Here $\pi : TW \to W$ the canonical projection and K is the kernel of $\pi_* : T_z(TW) \to T_{\pi(z)}W$. Then $\langle\beta, v\rangle = \langle X^*\beta, u\rangle = \langle X^*\widetilde{\alpha}, u\rangle = \langle\widetilde{\alpha}, v\rangle$. Thus β and $\widetilde{\alpha}$ coincide on $T_z(TW) \setminus K$ thus on $T_z(TW)$, the linear-hull of $T_z(TW) \setminus K$, for any $z \in TW$. Therefore $\beta = \widetilde{\alpha}$. (3) follows from the uniqueness of the natural lifting of $d\alpha$ and $f^*\alpha$: For example, $X^*(f_*)^*(\widetilde{\alpha}) = (f_*X)^*(\widetilde{\alpha}) = L_{f_*X}\alpha = L_X(f^*\alpha)$, for any vector field X over N. □

EXAMPLE 5.5. Let M a symplectic manifold, and ω the symplectic form on M. Since ω is non-degenerate, ω induces an isomorphism $TM \cong T^*M$. On the other hand, T^*M is endowed with the canonical symplectic form $d\theta_M$, which is independent of the symplectic structure of M. Therefore $d\theta_M$ is regarded as a symplectic form on TM. This coincides with the natural lifting $\widetilde{\omega}$.

EXAMPLE 5.6. Let (p, q, r) be a Darboux coordinates of (W, D) at a point $w_0 \in W$. Then the standard contact form $\alpha = dr - pdq$ gives the contact distribution $D \subset TW$. Let $(p, q, r; \phi, \xi, s)$ be the induced local coordinates of the tangent bundle TW; (ϕ, ξ, s) being fiber coordinates. Then we have

$$\widetilde{\alpha} = d(s - p\xi) + \xi dp - \phi dq.$$

Remark that $\widetilde{\alpha}$ is linear in the fiber coordinates (ϕ, ξ, s).

In general we have

LEMMA 5.7. *Let $f : N \to W$ be a mapping, and α a differential form on W. Then, for $v_1, v_2 \in V_f$, we have $i_{v_1+v_2}\alpha = i_{v_1}\alpha + i_{v_2}\alpha$, and $(v_1 + v_2)^*\widetilde{\alpha} = v_1^*\widetilde{\alpha} + v_2^*\widetilde{\alpha}$.*

Proof. The first equality follows from the definition of interior product. The second equality follows from Proposition 5.4 (2) and the Cartan's formula $L_v = di_v + i_v d$. □

The notion of natural liftings is defined also for differential systems. Let W be a manifold and Ω the sheaf of differential forms on W. A subsheaf $I \subset \Omega$ is called a *differential system* on W if it is a d-closed ideal of the differential algebra Ω, namely, if, for any section α of I and for any section β of Ω (defined on the same open subset of W), $\alpha \wedge \beta$ and $d\alpha$ are sections of I. Let S be a set of differential forms on open subsets of W. Then the differential system $\langle S \rangle$ generated by S has the stalk $\langle S \rangle_x$, for each $x \in W$, consisting of the functional linear combination of elements $\alpha_x \wedge \beta_x$ and $d\alpha_x \wedge \beta_x$, for those $\alpha \in S$ and differential forms β defined over x.

For example, a contact structure $D \subset TW$ on W may be defined also as the differential system generated by local sections of $D^\perp \subset T^*W$, local contact forms compatible with D.

Let I be a differential system on W. Then the *natural lifting* $\widetilde{I}$ of I is defined as the differential system on TW generated by the natural liftings $\widetilde{\alpha}$ of all sections α of I. If $f : N \to W$ is a mapping, then f^*I denotes the differential system generated by $f^*\alpha$ for all sections of I. Then we have by Proposition 5.4 (3):

LEMMA 5.8. *Let I be a differential system on W. Then $\widetilde{f^*I} = (f_*)^*(\widetilde{I})$, where $f_* : TN \to TW$ is the differential mapping of f.*

6. Contact Hamilton vector fields. Let (W, D) be a contact manifold, and α a local contact form representing D. There does not necessarily exist α globally; α can be taken over an open subset of W where the contact distribution D is co-oriented. A vector field X over W is called a *contact vector field* if the Lie derivative $L_X\alpha = \mu\alpha$ for a function μ, namely if X preserves the contact distribution D.

Deleting W if necessary, we assume a contact form α is taken over W. Let $H : W \to \mathbf{K}$ be a function. Then there exists a unique contact vector field $X = X_H$ over W with the condition $i_X\alpha = H$. The contact vector field X_H is called the *contact Hamilton vector field with Hamilton function H*.

If $\alpha = dr - \sum_{i=1}^{n} p_i dq_i$, then X_H is explicitly given by

$$X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q_i} + p_i \frac{\partial H}{\partial r} \right) \frac{\partial}{\partial p_i} - \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} + \left(H - \sum_{i=1}^{n} p_i \frac{\partial H}{\partial p_i} \right) \frac{\partial}{\partial r}.$$

Conversely, any contact vector field is locally a contact Hamilton vector field with some Hamiltonian function.

Associated to a contact form α, we define the *Reeb vector field* R by $i_R\alpha = 1, i_R d\alpha = 0$. Note that, since α is a contact form, R is characterized uniquely. If $\alpha = dr - pdq$, then $R = \dfrac{\partial}{\partial r}$. Then we have:

LEMMA 6.1. *Let α be a contact form on W, and $H : W \to \mathbf{K}$ a function. Then we have*

(1) *$L_{X_H}\alpha = R(H)\alpha$ and $i_{X_H} d\alpha = R(H)\alpha - dH$.*
(2) *Let η be a vector field on W. If $i_\eta d\alpha = 0$, then $\eta = (i_\eta \alpha)R$.*
(3) *$X_1 = R$.*

Proof. (1) The first equality holds for a system of coordinates (p, q, r) with $\alpha = dr - pdq$. Remark that X_H and R are defined intrinsically from the contact form α. The latter equality follows from $L_{X_H}\alpha = i_{X_H} d\alpha + d i_{X_H}\alpha = i_{X_H} d\alpha + dH$. (2) Set $\eta' = \eta - (i_\eta\alpha)R$. Then $i_{\eta'} d\alpha = 0$ and $i_{\eta'}\alpha = 0$. Therefore $\eta' = 0$, and we have $\eta = (i_\eta\alpha)R$. (3) By (1), we have $i_{X_1} d\alpha = 0$. Since $i_{X_1}\alpha = 1$, we see $X_1 = R$. □

We have the following formula for the contact Hamilton vector field with the sum (resp. product) of two contact Hamilton functions:

LEMMA 6.2. *For functions K, H on W, we have*

$$X_{K+H} = X_K + X_H,$$

$$X_{KH} = K \cdot X_H + H \cdot X_K - (KH) \cdot R = K \cdot X_H + H \cdot X_K - (KH) \cdot X_1.$$

In particular, $X_{aH} = aX_H, (a \in \mathbf{K})$.

Proof. The first one is clear. To show the second equality, we set $\eta = K \cdot X_H + H \cdot X_K - X_{KH}$. Then

$$\begin{aligned} i_\eta d\alpha &= K(R(H)\alpha - dH) + H(R(K)\alpha - dK) - (R(KH)\alpha - d(KH)) \\ &= (KR(H) + HR(K) - R(KH))\alpha = 0. \end{aligned}$$

Moreover, $i_\eta\alpha = KH + HK - KH = KH$. Therefore, by Lemma 6.1, $\eta = (KH) \cdot R = (KH) \cdot X_1$. □

We denote by VH_W the vector space of contact Hamilton vector fields over W and by $\mathcal{E}_W$ the $\mathbf{K}$-algebra of functions on W. Define a linear map $\Phi : \mathcal{E}_W \to VH_W$ by $\Phi(H) = X_H$. Then Φ is an isomorphism of vector spaces. Therefore VH_W is endowed with $\mathcal{E}_W$-module structure induced from Φ, namely, $K * X_H = X_{KH}$. Here, we distinguish this new functional multiplication, using $*$, with the ordinary functional multiplication in V_W, the $\mathcal{E}_W$-module consisting of all vector fields over W.

In term of the local coordinates p, q, r of (W, w_0) with $\alpha = dr - \sum_{i=1}^{n} p_i dq_i$, we define the order of function-germs $h = h(p, q, r) \in \mathcal{E}_W$ by setting

$$\text{weight}(p_i) = \text{weight}(q_j) = 1, (1 \le i, j \le n), \quad \text{and weight}(r) = 2;$$

namely, $\mathrm{ord}(h) \geq r$ if the Taylor expansion of h has no monomials of weight $< r$. We set $m_W^{(r)} := \{h \in \mathcal{E}_W \mid \mathrm{ord}(h) \geq r\}$.

If $\tau : (W, w_0) \to (W, w_0)$ is a contactomorphism, then $\mathrm{ord}(h \circ \tau) = \mathrm{ord}(h)$. Then we can define, on the local ring $\mathcal{E}_W$, the filtration

$$\mathcal{E}_W \supset m_W^{(1)} \supset m_W^{(2)} \supset \cdots \supset m_W^{(r)} \supset \cdots .$$

Note that

$$m_W^{(2r)} \subset m_W^r \subseteq m_W^{(r)}, (r = 0, 1, 2, \dots).$$

In particular $m_W^2 \subset m_W^{(2)} \subset m_W$.

In the $\mathcal{E}_W$-module VH_W introduced above, we have

$$m_W^2 * VH_W \subseteq VH_W \cap m_W V_W = m_W^{(2)} * VH_W.$$

Let $\pi : W \to Z$ be a Legendre fibration. Then a contact vector field X over W is called a *Legendre vector field* if, X is lowerable, namely, if there exists a vector field Y over Z such that $t\pi(X) = w\pi(Y)$ as vector fields along π. Then easily we have:

PROPOSITION 6.3. *Let $(p_1, \dots, p_n, q_1, \dots, q_n, r)$ be a Darboux coordinate, so that $\alpha = dr - pdq$. Then a contact Hamilton vector field X_H with Hamilton $H = H(p, q, r)$ is a Legendre vector field if and only if H is an affine function, namely, H is of form*

$$H(p, q, r) = a_0(q, r) + a_1(q, r)p_1 + \cdots + a_n(q, r)p_n$$

We denote by $VL_W = VL_{(W,\pi)}$, the totality of Legendre vector fields over W with respect to π.

7. Infinitesimal deformations. Let $f : (N, x_0) \to W$ be an integral map-germ. The space of infinitesimal integral deformations of f is, at least formally, given by

$$VI_f = \{v : (N, x_0) \to TW \mid v^*\widetilde{\alpha} = 0,\ \pi \circ v = f\},$$

where $\pi : TW \to W$ is the natural projection, and $\widetilde{\alpha}$ is the natural lifting to TW of a contact 1-form α locally defining D near $w_0 = f(x_0) \in W$.

Recall that VH_W denotes the $\mathcal{E}_W$-module of contact Hamilton vector fields over W. Define a linear mapping $wf : VH_W \to VI_f$ by $wf(H) = X_H \circ f, (H \in \mathcal{E}_W)$.

For $v \in VI_f$, we call $i_v\alpha \in \mathcal{E}_N$ the *generating function* of v. The linear mapping $e : VI_f \to R_f$ is defined by taking generating function. Here

$$R_f := \{h \in \mathcal{E}_N \mid dh \in \mathcal{E}_N d(f^*\mathcal{E}_W)\}.$$

In local coordinates, we have $e(v) = s \circ v - \sum(p \circ f)(\xi \circ v)$ and

$$0 = v^*\widetilde{\alpha} = d(e(v)) + \sum(\xi \circ v)d(p \circ f) - \sum(\phi \circ v)d(q \circ f).$$

Therefore $e(v) \in R_f$. Note that $i_v(\lambda\alpha) = (\lambda \circ f)i_v\alpha$.

We see the mapping e is surjective. In fact, for any $h \in R_f$, dh is a functional linear combination of the exterior derivatives of components of f. Since f is integral, $r \circ f$ is a functional linear combination of $d(p \circ f), d(q \circ f)$, and so is dh. Therefore, choosing $\xi \circ v, \varphi \circ v$ and $s \circ v$ properly, we get $v \in VI_f$ with $e(v) = h$.

Note that

LEMMA 7.1. *We have* $i_{X_H \circ f}\alpha = f^*(i_{X_H}\alpha) = f^*H$. *Therefore the generating function of* $X_H \circ f$ *is equal to the pull-back* f^*H *of the Hamiltonian function* H.

We need a result proved in page 222 of [18]:

LEMMA 7.2. *Let* $f : (N, x_0) \to W$ *be of corank* ≤ 1. *If*

$$R_f := \{e \in \mathcal{E}_N \mid de \in \mathcal{E}_N d(f^*\mathcal{E}_W)\}$$

is a finite $\mathcal{E}_W$*-module if and only if* f *is a finite map-germ, namely,* $\mathcal{E}_N$ *is a finite* $\mathcal{E}_W$*-module via* $f^* : \mathcal{E}_W \to \mathcal{E}_N$.

Now set $VI'_f = \mathrm{Ker}(e : VI_f \to R_f)$. Then we have the exact sequence of vector spaces:

$$0 \longrightarrow VI'_f \longrightarrow VI_f \xrightarrow{e} R_f \longrightarrow 0.$$

Remark that $R_f \subset \mathcal{E}_N$ is an $\mathcal{E}_W$-submodule via $f^* : \mathcal{E}_W \to \mathcal{E}_N$.

Now, in V_f, the $\mathcal{E}_N$-module consisting of vector fields along f, we have

$$VI'_f = \{v \in V_f \mid i_v\alpha = 0, i_v d\alpha = 0\},$$

and $VI'_f \subset V_f$ is an $\mathcal{E}_N$-submodule, therefore, an $\mathcal{E}_W$-submodule via f^*.

To proceed algebraic calculation, we are going to provide also VI_f a module structure.

As in the previous section, we denote by X_H the contact Hamilton vector field with Hamilton function H.

PROPOSITION 7.3. VI_f *is an* $\mathcal{E}_W$*-module by the multiplication*

$$H * v = f^*H \cdot v + (i_v\alpha)(X_H - H \cdot R) \circ f,$$

for $H \in \mathcal{E}_W, v \in VI_f$. *The multiplication is independent on the choice of contact form* α, *but it depends only on the contact structure (and on* H, v). *Moreover the sequence*

$$0 \longrightarrow VI'_f \longrightarrow VI_f \xrightarrow{e} R_f \longrightarrow 0$$

is $\mathcal{E}_W$*-exact.*

REMARK 7.4. For a constant function c, we have $X_c = cR$ and $c * v = cv$.

To verify Proposition 7.3, we need several lemmas:

LEMMA 7.5. $i_{H*v}\alpha = f^*H \cdot i_v\alpha$.

Proof. Since $i_{(HR - X_H)\circ f}\alpha = f^*(i_{HR - X_H}\alpha) = f^*(Hi_R\alpha - i_{X_H}\alpha) = f^*(H - H) = 0$, we see $i_{H*v}\alpha = i_{f^*H \cdot v}\alpha = f^*H \cdot i_v\alpha$. □

LEMMA 7.6. *Set* $\alpha' = \lambda\alpha$, *for a non-vanishing function* λ. *Then* $i_v\alpha' = f^*\lambda i_v\alpha$ *for any vector field along a mapping* $f : N \to W$. *If we denote by* R', X'_H *the Reeb vector field and the contact Hamilton vector field of* H *with respect to* α', *respectively, and if* $f : N \to W$ *is integral, then*

$$(X'_H - HR') \circ f = \left\{\frac{1}{\lambda}(X_H - HR)\right\} \circ f.$$

Therefore we have

$$(i_v\alpha')(X'_H - HR') \circ f = (i_v\alpha)(X_H - HR) \circ f.$$

Proof. That $i_v\alpha' = f^*\lambda i_v\alpha$ follows by Lemma 5.2 (1).

Set $u = (X'_H - HR') \circ f$ and $v = \left\{\dfrac{1}{\lambda}(X_H - HR)\right\} \circ f$. Then, by Lemma 5.3, $i_u\alpha' = i_{X'_H \circ f}\alpha' - i_{(HR')\circ f}\alpha' = f^*H - f^*H = 0$. Similarly we have $i_v\alpha = 0$. So we have $i_v\alpha' = (f^*\lambda)(i_v\alpha)$.

We will show $i_u d\alpha' = i_v d\alpha' = f^*(-dH)$. Then, since α' is a contact form, we have $u = v$.

Now in fact, since f is integral, we have $f^*\alpha' = f^*\alpha = 0$, and therefore we have, by Lemma 6.1,

$$\begin{aligned}
i_u d\alpha' &= f^*(i_{X'_H - HR'} d\alpha') \\
&= f^*(R'(H)\alpha' - dH - Hi_{R'}d\alpha') \\
&= f^*(-dH). \\
i_v d\alpha' &= (f^*\lambda)(i_v d\alpha) + i_v(d\lambda \wedge \alpha) \\
&= f^*(i_{X_H - HR} d\alpha) + (i_v d\lambda) f^*\alpha - (i_v\alpha) f^*(d\lambda) \\
&= f^*(R(H)\alpha - dH) \\
&= f^*(-dH). \quad \square
\end{aligned}$$

Remark 7.7. The terms $(i_v\alpha)X_H \circ f$ and $(i_v\alpha)(H \cdot R) \circ f$ do depend on the choice of α. Just the difference is intrinsically defined as seen in Lemma 7.6.

Proof of Proposition 7.3: We compare

$$(KH) * v = f^*(KH) \cdot v - (i_v\alpha)(KH \cdot R - X_{KH}) \circ f$$

with

$$K * (H * v) = f^*K(f^*H \cdot v - (i_v\alpha)(H \cdot R - X_H) \circ f) - (i_{H*v}\alpha)(K \cdot R - X_H) \circ f.$$

By Lemma 7.5, the right hand side of the latter equals to

$$f^*(KH) \cdot v - (i_v\alpha)(2KH \cdot R - KX_H - HX_K) \circ f,$$

which is equal to the right hand side of the former, by Lemma 6.2. By Lemma 7.5, e is an $\mathcal{E}_W$-epimorphism. By Lemma 7.6, we see the multiplication depends only on the contact structure. The remaining parts are clear. $\square$

The following is a consequence of Proposition 7.3, Lemma 7.1 and Proposition 6.3:

Lemma 7.8. *If we set*

$$VH'_{W,f} = \{X_H \in VH_W \mid H \circ f = 0\},$$

then we have an $\mathcal{E}_W$-exact sequence,

$$0 \longrightarrow VI'_f/wf(VH'_{W,f}) \longrightarrow VI_f/wf(VH_W) \longrightarrow R_f/\mathcal{E}_W \longrightarrow 0.$$

If we set

$$VL'_{W,f} = \{X_H \in VL_W \mid H \circ f = 0\},$$

then we have an $\mathcal{E}_Z$-exact sequence,

$$\begin{aligned} 0 \to VI'_f/wf(VH'_{W,f}) &\to VI_f/wf(VL_W) \\ &\to R_f/(E_Z + \textstyle\sum_{i=1}^n E_Z(p_i \circ f)) \to 0. \end{aligned}$$

Let $f : (N, x_0) \to (W, w_0)$ be an integral mapping. We define an $\mathcal{E}_W$-homomorphism $tf : V_N \to VI_f$ by $tf(\xi) := f_*(\xi), \xi \in V_N$.

LEMMA 7.9. *Let $f : (N, x_0) \to (W, w_0)$ be an integral map-germ. Then $tf(V_N) \subseteq VI'_f$.*

Proof. Take $f_*(\xi) \in tf(V_N)$. Then we have $e(f_*(\xi)) = i_{f_*(\xi)}\alpha = i_\xi f^*\alpha = 0$. □

Under a condition, the converse inclusion holds:

PROPOSITION 7.10. *Let $f : (N, x_0) \to (W, w_0)$ be an integral map-germ. Suppose that f is diffeomorphic to an analytic map-germ $f' : (\mathbf{K}^n, 0) \to (\mathbf{K}^{2n+1}, 0)$ (not necessarily integral) such that the codimension of the singular locus of the complexification $f'_{\mathbf{C}}$ of f' is greater than or equal to 2. Then we have $VI'_f \subseteq tf(V_N)$. Therefore we have an isomorphism of $\mathcal{E}_W$-modules*

$$VI_f/\{tf(V_N) + wf(VH_W)\} \cong R_f/\mathcal{E}_W,$$

and an isomorphism of $\mathcal{E}_Z$-modules

$$VI_f/\{tf(V_N) + wf(VL_W)\} \cong R_f/(E_Z + \sum_{i=1}^n E_Z(p_i \circ f)).$$

Proof. Let $v \in VI'_f$. Set

$$v = (p \circ f, q \circ f, r \circ f; \phi \circ v, \xi \circ v, s \circ v).$$

Then, since $e(v) = s \circ v - \sum(p \circ f)(\xi \circ v) = 0$, we have

$$\sum_{i=1}^n (\xi_i \circ v) d(p_i \circ f) - \sum_{i=1}^n (\phi_i \circ v) d(q_i \circ f) = 0.$$

This means, for any regular point $x \in (\mathbf{K}^n, 0)$, that $v(x) \in D_{f(x)}$ and $v(x)$ belongs to the skew orthogonal complement to $f_*(T_x\mathbf{K}^n)$ with respect to the symplectic structure $\sum_{i=1}^n dp_i \wedge dq_i$ on D. Therefore we have $v(x) \in f_*(T_x\mathbf{K}^n)$. Since f and f' are diffeomorphic, any vector field in VI'_f is transformed to a vector field along f' which is tangent to the image of f' off the singular locus of f'.

Let $v \in V_{f'_{\mathbf{C}}}$. This means that $v : (\mathbf{C}^n, 0) \to T\mathbf{C}^{2n+1}$ is a holomorphic vector field along $f'_{\mathbf{C}} : (\mathbf{C}^n, 0) \to (\mathbf{C}^{2n+1}, 0)$. Suppose, for each regular point $x \in (\mathbf{C}^n, 0)$ of $f'_{\mathbf{C}}$ that $v(x) \in f'_{\mathbf{C}*}(T_x\mathbf{C}^n)$. Then we can find a vector field w over $\mathbf{C}^n \setminus \mathrm{Sing}(f'_{\mathbf{C}})$ satisfying $v = (f'_{\mathbf{C}})_*(w)$ on $\mathbf{C}^n \setminus \mathrm{Sing}(f'_{\mathbf{C}})$, where $\mathrm{Sing}(f'_{\mathbf{C}})$ is the locus of singular points of $f'_{\mathbf{C}}$. Since $\mathrm{Sing}(f'_{\mathbf{C}})$ is of codimension ≥ 2 in $\mathbf{C}^n$, w extends to a holomorphic vector field

on $(\mathbf{C}^n, 0)$ still called w, by Hartogs theorem. Then we have $v = f'_{\mathbf{C}*}(w)$. This proves that $VI'_f \subseteq tf(V_N)$ in the case $\mathbf{K} = \mathbf{C}$.

In the case $\mathbf{K} = \mathbf{R}$, we set $T \subset V_{f'}$ as the set of vector fields along f' such that, for each regular point $x \in (\mathbf{R}^n, 0)$ of f', $v(x) \in f'_*(T_x\mathbf{R}^n)$.

Take $v \in T$. Suppose v is real analytic. Then considering the complexification of v, we see that there exists a real analytic $w \in V_n$ such that $v = f'_*(w)$ over $(\mathbf{R}^n, 0)$. This means that T is generated formally by $tf'(V_n)$ in the sense of [25], and, by Whitney's spectral theorem, we have that T is contained in the closure of $tf'(V_n)$ for a representative of f'. Since f' is analytic, we see $tf'(V_n)$ itself is closed, and we have $T \subseteq tf'(V_n)$. See also [32]. This shows that $VI_f \subseteq tf(V_N)$.

The remaining parts are clear. □

We call f *infinitesimally contact stable* if

$$VI_f = tf(V_N) + wf(VH_W).$$

Then we have:

COROLLARY 7.11. *Let $f : (N, x_0) \to (W, w_0)$ be an integral mapping. Then the condition (ca) of Theorem 3.1 implies that f is infinitesimally contact stable, namely the condition (ics).*

Proof. Since $R_f = f^*\mathrm{E}_W$, we see $0 = R_f/\mathcal{E}_W \cong VI_f/\{tf(V_N) + wf(VH_W)\}$. Therefore we have $VI_f = tf(V_N) + wf(VH_W)$. □

LEMMA 7.12. *If an integral map-germ of corank at most one $f : (N, x_0) \to W$ is infinitesimally contact stable then f is a finite map-germ.*

Proof. By taking generating functions of both sides of the equality $VI_f = tf(V_N) + wf(VH_W)$, we have $R_f = f^*\mathcal{E}_W$. Therefore R_f is a finite $\mathcal{E}_W$-module. Therefore, by Lemma 7.2, we see f is finite. □

Let (f_t) be an integral deformation of f. To show f is homotopically contact (resp. Legendre) stable, we need to find a deformation (σ_t) of id_N and an integral deformation (τ_t) of id_W (resp. an integral deformation τ_t of id_W covering a deformation $(\bar{\tau}_t)$ of id_Z via $\pi : W \to Z$) satisfying $\tau_t^{-1} \circ f_t \circ \sigma_t = f$. For this, it is sufficient to solve $df_t/dt = \eta_t \circ f_t - Tf_t \circ \xi_t (= wf_t(\eta_t) - tf_t(\xi_t)) : N \times \mathbf{K} \to TW$ with $\xi_t \in V_N$ and $\eta_t \in VH_W$ (resp. $\eta_t \in VL_W$), (cf. [28]).

For an unfolding $F = (f_t, t) : N \times J \to W \times J$, $t \in J = (\mathbf{K}, 0)$, we set

$$VI_{F/J} = \{v : N \times J \to TW \mid v_t \in VI_{f_t},\, t \in J\}.$$

If (f_t) is an integral deformation of f, then we have $(df_t/dt)_{t\in J} \in VI_{F/J}$. We define an $\mathcal{E}_{W\times J}$-module structure on $VI_{F/J}$ by

$$a_t * v_t = (f_t^* a_t) \cdot v_t + (i_{v_t}\alpha)(X_{a_t} - a_t \cdot R) \circ f_t,$$

for $v_t \in VI_{F/J}, a_t \in \mathcal{E}_{W\times J}$. Compare with Proposition 7.3. Then we have

COROLLARY 7.13. *If f is finite and of corank at most one, then the quotient $VI_{F/J}$ is a finite $\mathcal{E}_{W\times J}$-module.*

Now assume f is integral and f_t is an integral deformation of f. We define $tF/J : V_N \to VI_{F/J}$ by $v \mapsto (tf_t(v))_{t\in J}$. We set

$$S_{F/J} = VI_{F/J}/((wF/J)(VH_W) + (tF/J)(V_N)),$$

which is an $\mathcal{E}_{W\times J}$-module, and set

$$S_f = VI_f/(wf(VH_W) + tf(V_N)),$$

which is an $\mathcal{E}_W$-module. Then we have:

LEMMA 7.14. *The quotient $S_{F/J}/m_J S_{F/J}$ is isomorphic to S_f as an $\mathcal{E}_W$-modules.*

Proof. Consider the morphism $\Phi : S_{F/J} \to S_f$ defined by $\Phi([v_t]) = [v_t|_{t=0}]$. We will show that the kernel of Φ is equal to $m_J S_{F/J}$. Let $v_t \in VI_{F/J}$. Assume $v_t|_{t=0} = wf(\eta)+tf(\xi)$, for some $\xi \in V_N$, $\eta \in VI_W$. Set $w_t = v_t - wf_t(\eta) - tf_t(\xi)$. Then $w_t|_{t=0} = 0$. Therefore $w_t = tw'_t$, for some $w'_t \in V_{f_t}$. We see $\Pi_\sharp(w'_t) \in VI_{g_t}$. Here $g_t = \Pi\circ f_t$ is the family of isotropic map-germs induced from f_t. In fact $\Pi_\sharp(w_t)^\flat = t\Pi_\sharp(w'_t)^\flat$ and so $0 = (\Pi_\sharp(w_t)^\flat)^* d\theta_{T^*Q} = t(\Pi_\sharp(w'_t)^\flat)^* d\theta_{T^*Q}$. Thus $(\Pi_\sharp(w'_t)^\flat)^* d\theta_M = 0$. This means $w'_t \in VI_{f_t}$. Since x-derivative of t is equal to zero, we have $w_t = tw_t = t * w'_t$ and $[v_t] = [w_t] = t[w'_t] \in m_J S_{F/J}$. □

8. Relation to Isotropic Mappings. Let Q be a manifold of dimension n. Then $T^*Q \times \mathbf{K} \cong J^1(Q, \mathbf{K}) \subset PT^*(Q \times \mathbf{K})$ has the canonical contact structure, whereas T^*Q has the canonical symplectic structure $\omega = d\theta_Q$, θ_Q being Liouville form on Q, $\theta_Q = \sum_{i=1}^n p_i dq_i$, for a system of local symplectic coordinates. A contact form on $T^*Q \times \mathbf{K}$ is given by $dr - \theta_Q$, for the coordinate r on $\mathbf{K}$.

Let $g : N \to T^*Q$ be a mapping from a manifold N of dimension n. Then g is called *isotropic* if $g^*\omega = 0$. The singularities of isotropic mappings of corank at most one is studied in [18] in detail. In particular, we have a series of singularities, "open Whitney umbrellas", which are symplectic counterparts of objects we have introduced in this paper.

Two isotropic map-germs $g : (N, x_0) \to T^*Q$ and $g' : (N, x'_0) \to T^*Q$ are called *symplectomorphic* (or *symplectically equivalent*) if there exist a symplectomorphism

$$\tau : (T^*Q, g(x_0)) \to (T^*Q, g'(x'_0))$$

and a diffeomorphism $\sigma : (N, x_0) \to (N, x'_0)$ satisfying $\tau \circ f = f' \circ \sigma$. Then we call also the pair (σ, τ) a symplectomorphism between g and g'.

Let $f : (N, x_0) \to T^*Q \times \mathbf{K}$ be a map-germ. Set $g : (N, x_0) \to T^*Q$ to be $g = \Pi \circ f$, where $\Pi : T^*Q \times \mathbf{K} \to T^*Q$ is the natural projection along the flow of Reeb vector field $\dfrac{\partial}{\partial r}$.

Then we have by [18]:

LEMMA 8.1.

(1) *f is an integral map-germ if and only if g is an isotropic map-germ.*

(2) *If $g = \Pi \circ f$ and $g' = \Pi \circ f'$ are symplectomorphic, then f and f' are contactomorphic.*

(3) *$R_f = R_g$.*

(4) *f is an open Whitney umbrella of type k (as an integral map-germ) if and only if g is an open Whitney umbrella of type k (as an isotropic map-germ). In particular, f is a Legendre immersion if and only if g is a Lagrange immersion.*

REMARK 8.2. The converse of (2) of Lemma 8.1 does not hold in general. For example, consider integral map-germs $f_\lambda : (\mathbf{K}, 0) \to (\mathbf{K}^3, 0)$, $\lambda > 0$ defined by $g(t) = (t^3, t^7 + \lambda t^8, \dfrac{3}{10}t^{10} + \dfrac{3}{11}\lambda t^{11})$. Then $g_\lambda = \Pi \circ f_\lambda : (\mathbf{K}, 0) \to (\mathbf{K}^2, 0)$, $g_\lambda(t) = (t^3, t^7 + \lambda t^8)$

is not symplectomorphic to $g_{\lambda'}$ if $\lambda' \neq \lambda$, while all f_λ are contactomorphic to each other ([21][20]).

Set $W = T^*Q \times \mathbf{K}$. The projection $\Pi : W \to T^*Q$ induces the projection $\Pi_* : TW \to T(T^*Q)$; by using local coordinates, it is given by

$$\Pi_*(p, q, r; \phi, \xi, s) = (p, q; \phi, \xi).$$

Then Π_* induces $\mathbf{K}$-linear mapping $\Pi_\sharp : V_f \to V_g$ by $\Pi_\sharp(v) = \Pi_* \circ v, (v \in V_f)$.

Now we observe the following:

LEMMA 8.3. *$\Pi_\sharp$ restricts to a $\mathbf{K}$-linear epimorphism $\Pi_\sharp : VI_f \to VI_g$, to an $\mathcal{E}_N$-isomorphism $\Pi_\sharp : VI'_f \to VI'_g$ and $\mathcal{E}_{T^*Q}$-epimorphism $\Pi_\sharp : VI_f \to VI_g/wg(VH_{T^*Q})$ over the ring morphism $\Pi^* : \mathcal{E}_{T^*Q} \to \mathcal{E}_W$. Furthermore we have the following commutative diagram which consists of exact sequences:*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & wf(VH'_{W,g}) & \to & wf(VH'_W) & \to & f^*\mathcal{E}_W & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & VI'_f & \to & VI_f & \stackrel{\widetilde{e}}{\to} & R_f & \to & 0 \\
 & & \downarrow & & \overline{\Pi}_\sharp\downarrow & & \downarrow & & \\
0 & \to & VI'_g/wg(VH'_{T^*Q,g}) & \to & VI_g/wg(VH_{T^*Q}) & \stackrel{\overline{e}}{\to} & R_g/g^*\mathcal{E}_{T^*Q} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The kernel of $\Pi_\sharp$ is generated by $R \circ f = \dfrac{\partial}{\partial r} \circ f$ over $\mathbf{R}$.

Proof. We show that $\mathrm{Ker}(\Pi_\sharp) = \left\langle \dfrac{\partial}{\partial r} \circ f \right\rangle_{\mathbf{R}}$. Let $v = (p \circ f, q \circ f, r \circ f; \phi, \xi, s) \in VI_f$. Recall that $d(s - (p \circ f)\xi) + \xi d(p \circ f) - \phi d(q \circ f) = 0$. Suppose that $\Pi_* \circ v = 0$. Then $\xi = 0, \phi = 0$. Then we have $ds = 0$. Thus s is constant. The remaining parts are clear. □

We have also

LEMMA 8.4. *For any $\eta \in VH_{T^*Q}$ (resp. $\eta \in VL_{T^*Q}$), there exists an $\widetilde{\eta} \in VH_W$ (resp. $\widetilde{\eta} \in VL_W$), such that $\Pi_\sharp wf(\widetilde{\eta}) = wg(\eta)$. Here $wf(\widetilde{\eta}) = \widetilde{\eta} \circ f$ and $wg(\eta) = \eta \circ g$. Here VL_{T^*Q} means the set of Lagrange vector fields of the Lagrange fibration $T^*Q \to Q$ ([18]).*

If $\eta \in m_{T^*Q} * VH_{T^*Q}$ (resp. $\eta \in m_Q * VL_{T^*Q}$), then we can take $\widetilde{\eta}$ from $m_W^2 * VH_W$ (resp. from $m_Z^2 * VL_W$), where $Z = Q \times \mathbf{R}$.

Proof. If η has a symplectic Hamiltonian function H on T^*Q, $H(0) = 0$, then we may set $\widetilde{\eta} = X_{\Pi^* H}$, the contact Hamiltonian vector field for the pull-back $\Pi^* H$ of H by Π. □

LEMMA 8.5. *Let $f : (N, x_0) \to W = T^*Q \times \mathbf{K}$ be an integral mapping. If $g = \Pi \circ f : (N, x_0) \to T^*Q$ is infinitesimally symplectically (resp. Lagrange) stable, then f is infinitesimally contact (resp. Legendre) stable.*

Proof. Suppose g is infinitesimally symplectic (resp. Lagrange) stable. Then, for any $v \in VI_f$, there exist $\xi \in V_N$ and $\eta \in VH_{T^*Q}$ (resp. $\eta \in VL_{T^*Q}$) satisfying $\Pi_\sharp v = tg(\xi) + wg(\eta)$. Then we see, using Lemma 8.4

$$\Pi_\sharp(v - tf(\xi) - wf(\widetilde{\eta})) = 0.$$

Thefore, by Lemma 8.3, there exists $s_0 \in \mathbf{R}$ such that

$$v - tf(\xi) - wf(\widetilde{\eta}) = s_0 \frac{\partial}{\partial r} \circ f.$$

Thus we have $v = tf(\xi) + wf(\widetilde{\eta} + s_0 \dfrac{\partial}{\partial r})$. □

PROPOSITION 8.6. *If f is an open Whitney umbrella, then we have $R_f = f^*\mathcal{E}_W$. Therefore f satisfies the condition (ca) of Theorem 3.1.*

Proof. $R_f = R_g = g^*\mathcal{E}_Z \subseteq f^*\mathcal{E}_W \subseteq R_f$. □

COROLLARY 8.7. *If f is an open Whitney umbrella, then f is infinitesimally contact stable. Moreover we have the isomorphism*

$$VI_f/(tf(V_N) + wf(VL_W)) \cong R_f/(E_Z + \sum_{i=1}^{n} E_Z(p_i \circ f))$$

of $\mathcal{E}_Z$-modules via $\pi^* : \mathcal{E}_Z \to \mathcal{E}_W$.

9. Integral Jets. We consider the *integral jet space* $J^r_I(n, 2n+1)$ consisting of $j^r f(0) \in J^r(n, 2n+1)$ for integral map-germs $f : (\mathbf{K}^n, 0) \to (\mathbf{K}^{2n+1}, 0)$ of corank at most one. Then we see $J^r_I(n, 2n+1)$ is a submanifold of $J^r(n, 2n+1)$.

REMARK 9.1. The projection $\Pi^r : J^r(n, 2n+1) \to J^r(n, 2n)$ defined by $\Phi^r(j^r f(0)) := j^r(\Pi \circ f)(0)$ induces a diffeomorphism of $J^r_I(n, 2n+1)$ and the isotropic jet space $J^r_I(n, 2n) \subset J^r(n, 2n)$ ([17]). In fact, for any $j^r g(0) \in J^r_I(n, 2n)$, we set $j^r f(0) = j^r(g, e)(0)$, where e is the generating function of g, $de = g^*\theta_Q, e(0) = 0$. Then $j^r f(0) \in J^r_I(n, 2n+1)$ and $\Pi^r(j^r f(0)) = j^r g(0)$.

Let $f : (N, x_0) \to (W, w_0)$ be an integral map-germ of corank at most one. Then we set

$$VI^s_f = \{v \in VI_f \mid j^s v(x_0) = 0\} = VI_f \cap m_N^{s+1} V_f, \ (s = 0, 1, 2, \dots).$$

Let $z = j^r f(x_0) \in J^r_I(n, 2n+1)$. Define $\pi_r : VI^0_f \to T_z J^r(n, 2n+1)$ as follows: For each $v \in VI^0_f$, take an integral deformation (f_t) of f with $v = \left.\dfrac{df_t}{dt}\right|_{t=0}$, and set $\pi_r(v) = \left.\dfrac{d(j^r f_t(x_0))}{dt}\right|_{t=0}$. Then the image of the linear map π_r coincides with $T_z J^r_I(n, 2n+1)$.

Let $z \in J^r_I(n, 2n+1)$ and $z = j^r f(x_0)$ for a $f : (N, x_0) \to (W, w_0)$. Then under the identification $T_z J^r(n, 2n+1) \cong m_N V_f / m_N^{r+1} V_f$ we have

$$T_z J^r_I(n, 2n+1) \cong VI^0_f / VI^r_f.$$

If we denote by $\mathcal{C}^r z$ (resp, $\mathcal{L}^r z$) the orbit of z under the contactomorphisms (resp. Legendre diffeomorphisms), we have

$$T_z \mathcal{C}^r z \cong \{(tf(m_N V_N) + wf(m_W^{(2)} * VH_W)) + VI^r_f\}/VI^r_f,$$

$$T_z \mathcal{L}^r z \cong \{(tf(m_N V_N) + wf(m_Z^{(2)} * VL_W)) + VI^r_f\}/VI^r_f.$$

Set $z = j^r f(x_0)$. For $(w, v) \in T_{x_0}N \oplus VI_f$, take a curve x_t in N with the velocity vector w at $t = 0$ and take an integral deformation f_t of f with $v = \left.\dfrac{df_t}{dt}\right|_{t=0}$ (cf. [18], Lemma 3.4), and define a linear map

$$\Pi_r : T_0N \oplus VI_f \to T_z J^r(N, W),$$

by

$$\Pi_r(w, v) = \frac{j^r df_t(x_t)}{dt}|_{t=0}.$$

Then $\Pi_r(T_0N \oplus VI_f) = T_z J_I^r(N, W)$ and $\mathrm{Ker}\Pi_r = \{0\} \oplus VI_f^r$. Moreover we have, for the Legendre equivalence class,

$$[z] = \{j^r f'(x) \mid x \in N,\ f' \text{ is Legendre equivalent to } f\}$$

in $J_I^r(N, W)$,

$$T_z[z] = \Pi_r(T_{x_0}N \oplus (tf(m_N V_N) + wf(VL_W))).$$

For the jet extension $j^r f : (N, x_0) \to J_I^r(N, W)$, we have

$$(j^r f)_*(\frac{\partial}{\partial x_i}) = \Pi_r(\frac{\partial}{\partial x_i}, f_*(\frac{\partial}{\partial x_i})).$$

LEMMA 9.2. *The transversality condition* (t_r) *is equivalent to the condition*

$$VI_f = tf(V_N) + wf(VL_W) + VI_f^r.$$

Proof. The condition (lt_r) that $j^r f$ is transverse to $[z] = [j^r f(x_0)]$ at x_0 is equivalent to the condition

$$(j^r f)_*(T_{x_0}N) + T_z[z] = T_z J_I^r(N, W),$$

and to the condition that

$$(\Pi_r)^{-1}((j^r f)_*(T_{x_0}N)) + T_{x_0}N \oplus (tf(m_N V_N) + wf(VL_W)) + \{0\} \oplus VI_f^r$$

coincides with $T_{x_0}N \oplus VI_f$. This condition is equivalent to that

$$VI_f = \langle f_*(\frac{\partial}{\partial x_1}), \ldots, f_*(\frac{\partial}{\partial x_n})\rangle_{\mathbf{K}} + tf(m_N V_N) + wf(VL_W) + VI_f^r,$$

namely that

$$VI_f = tf(V_N) + wf(VL_W) + VI_f^r. \quad \square$$

Similarly we have the following:

LEMMA 9.3. *The condition that $j^r f$ is transverse to the orbit for contactomorphisms through $j^r f(x_0)$ at x_0 is equivalent to the condition*

$$VI_f = tf(V_N) + wf(VH_W) + VI_f^r.$$

Moreover the transversality condition on $j^r f$ implies that f is an open Whitney umbrella:

PROPOSITION 9.4. *Let $f : (N, x_0) \to (W, w_0)$ be an integral map-germ of corank ≤ 1, and k a non-negative integer. If the $k+1$ extension $j^{k+1}f : (N, x_0) \to J_I^{k+1}(N, W)$ is transverse to the contactomorphism-orbit through $j^{k+1}f(x_0)$, then f is an open Whitney umbrella of type $\leq k$.*

Proof. Since f is an integral map-germ of corank ≤ 1, f is contactomorphic to $f' : (\mathbf{K}^n, 0) \to (\mathbf{K}^{2n+1}, 0)$ with

$$\varphi := (q_1, \ldots, q_{n-1}, q_n, p_n) \circ f' = (x_1, \ldots, x_{n-1}, u(x), v(x)).$$

Since f' is contactomorphic to f, we see that also

$$j^{k+1}f' : (\mathbf{K}^n, 0) \to J_I^{k+1}(\mathbf{K}^n, \mathbf{K}^{2n+1})$$

is transverse to the contactomorphism-orbit through $j^{k+1}f'(0)$, therefore to $\mathcal{K}$-orbit through $j^{k+1}f'(0)$. Then we see $j^{k+1}\varphi : (\mathbf{K}^n, 0) \to J^{k+1}(\mathbf{K}^n, \mathbf{K}^{n+1})$ is transverse to $\mathcal{K}$-orbit through $j^{k+1}\varphi(0)$. Then f' is an open Whitney umbrella of type $\leq k$. Therefore f is an open Whitney umbrella of type $\leq k$. □

For an n-dimensional manifold N and a contact manifold W of dimension $2n+1$, we set

$$C_I^\infty(N, W)^1 := \{f : N \to W \mid f \text{ is integral of corank} \leq 1\}.$$

We endow $C_I^\infty(N, W)^1$ with the relative topology of the Whitney C^∞ topology of $C^\infty(N, W)$. Then we have the following Legendre transversality theorem:

PROPOSITION 9.5. *Let r be a non-negative integer and U a locally finite family of submanifolds of $J^r(N, W)$. Then*

$$T_U := \{f \in C_I^\infty(N, W)^1 \mid j^r f \text{ is transverse to all of } U\}$$

is dense in $C_I^\infty(N, W)^1$.

10. Finite determinacy.

LEMMA 10.1. *Let $f, f' : (N, x_0) \to W$ be integral map-germs. If f is an open Whitney umbrella of type k and $j^{k+1}f'(x_0) = j^{k+1}f(x_0)$, then f' is an open Whitney umbrella of type k.*

Proof. By definition there exist a contactomorphism (σ, τ) such that $\tau \circ f \circ \sigma^{-1} = f_{n,k}$, the normal form. Set $f'' = \tau \circ f' \circ \sigma^{-1}$. Then $j^{k+1}f''(x_0) = j^{k+1}f_{n,k}(x_0)$. Set $\varphi = (q_1, \ldots, q_{n-1}, q_n, p_n) \circ f'' : (N, x_0) \to \mathbf{K}^{n+1}$ and $\varphi_{n,k} = (q_1, \ldots, q_{n-1}, q_n, p_n) \circ f_{n,k}$. Then $j^{k+1}\varphi(x_0) = j^{k+1}\varphi_{n,k}(x_0)$. Now $j^k\varphi : (N, x_0) \to J^k(N, \mathbf{K}^{n+1})$ is transverse at x_0 to Thom-Boardman strata as well as $j^k\varphi_{n,k}$ is. In [15], we have shown that $g'' = (q, p) \circ f'' : (N, x_0) \to \mathbf{K}^{2n}$ is symplectomorphic to $g_{n,k} = (q, p) \circ f_{n,k}$. Then f'' and $f_{n,k}$ are contactomorphic. Since f and f'' are contactomorphic, we see f and $f_{n,k}$ are contactomorphic, therefore f is an open Whitney umbrella of type k. □

An integral map-germ $f : (N, x_0) \to (W, w_0)$ is called *r-determined by contactomorphisms* if, for any integral map-germ $f' : (N, x_0) \to (W, w_0)$ with $j^r f'(x_0) = j^r f(x_0)$, f and f' are contactomorphic.

Let $\pi : (W, w_0) \to (Z, z_0)$ be a fixed Legendre fibration. An integral map-germ $f : (N, x_0) \to (W, w_0)$ is called *Legendre r-determined* if, for any integral map-germ $f' : (N, x_0) \to (W, y_0)$ with $j^r f'(x_0) = j^r f(x_0)$, then (f', π) and (f, π) are Legendre equivalent.

Then we have:

LEMMA 10.2. *An open Whitney umbrella of type k is $(k+1)$-determined by contactomorphisms.*

Proof. Suppose f is an open Whitney umbrella of type k. Let $f' : (N, x_0) \to W$ be an integral map-germ with $j^{k+1} f'(x_0) = j^{k+1} f(x_0)$. Then f' is also an open Whitney umbrella of type k. Therefore both f and f' are contactomorphic to the normal form $f_{n,k}$. Thus f and f' are contactomorphic. □

LEMMA 10.3. *Let $f : (N, x_0) \to (W, y_0)$ be an open Whitney umbrella. Suppose that f is infinitesimally Legendre stable, namely that*

$$VI_f = tf(V_N) + wf(VL_W).$$

Take a positive integer r satisfying

$$f^*\mathcal{E}_W \cap m_N^{r+1} \subseteq f^* m_W^{n+2}.$$

Then we have

(1) *$R_f = f^*\mathcal{E}_W$ is generated as $\mathcal{E}_Z$-module by $1, p_1 \circ f, \ldots, p_n \circ f$.*
(2) *$m_W^{n+1} R_f \subseteq m_Z R_f$.*
(3) *$VI_f^r \subseteq tf(m_N V_N) + wf(VL_W \cap m_W^{(2)} * VH_W)$.*
(4) *f is Legendre r-determined.*

Proof. (1) : Taking generating functions both sides of $VI_f = tf(V_N) + wf(VL_W)$, we have

$$R_f = \langle 1, p_1 \circ f, \ldots, p_n \circ f \rangle_{\mathcal{E}_Z}.$$

Moreover, since f is an open Whitney umbrella, we have $R_f = f^*\mathcal{E}_W$ (Lemma 8.6).

(2) : Set $Q_f := R_f / m_Z R_f$. Then Q_f is generated by $1, p_1 \circ f, \ldots, p_n \circ f$ over $\mathbf{K}$. Therefore $\dim_{\mathbf{K}} Q_f \leq n+1$. Considering the sequence

$$Q_f \supseteq m_W Q_f \supseteq \cdots \supseteq m_W^{n+1} Q_f,$$

and using Nakayama's lemma, we have $m_W^{n+1} Q_f = 0$. Therefore we have $m_W^{n+1} R_f \subseteq m_Z R_f$.

(3) : Let $v \in VI_f^r$. Then the generating function $e(v) = i_v \alpha$ of v belongs to $f^*\mathcal{E}_W \cap m_N^{r+1}$. Now

$$f^*\mathcal{E}_W \cap m_N^{r+1} \subseteq f^*(m_W^{n+2}) \subseteq m_Z f^* m_W.$$

Therefore there exist functions $a_1, \ldots, a_s \in m_Z$ and $b_1, \ldots, b_s \in m_W$ such that

$$e(v) = (a_1 b_1 + \cdots + a_s b_s) \circ f.$$

For each $b_j \circ f$, there exist $c_{j0}, c_{j1}, \ldots, c_{jn} \in \mathcal{E}_Z$ satisfying

$$b_j \circ f = c_{j0} \cdot 1 + c_{j1}(p_1 \circ f) + \cdots + c_{jn}(p_n \circ f).$$

Note that, since $b_j(x_0) = 0$, we see $c_{j0}(x_0) = 0$, therefore $c_{j0} m_Z$. Set

$$h = \sum_{j=1}^{s} a_j(c_{j0} + c_{j1}p_1 + \cdots + c_{jn}p_n).$$

Then h is an affine function with respect to $p_1, \ldots, p_n$ and $h \in m_W^2$. So the Hamilton vector field X_h belongs to $VL_W \cap m_W^2 * VH_W \subseteq VL_W \cap m_W^{(2)} * VH_W$. Then the generating function of $u := v - X_h \circ f$ is equal to zero. Then the vector field u is tangent to f along the regular locus of f. Since f is an open Whitney umbrella, f is analytic and the singular locus of the complexification of f is at least 2. Therefore there exists a vector field $\xi \in V_N$ satisfying $u = tf(\xi)$ (Proposition 7.10). So we have $v = tf(\xi) + wf(X_h)$. Remark that, since f is an open Whitney umbrella, the kernel of the differential mapping $f_* : T_{x_0}N \to T_{w_0}W$ is not tangent to the Boardman strata containing x_0. The vector $\xi(x_0)$ belongs to the kernel. On the other hand $\xi(x_0)$ must be tangent to the Boardman stratum. Therefore we have $\xi(x_0) = 0$ namely $\xi \in m_N V_N$. Thus we have

$$v \in tf(m_N V_N) + wf(VL_W \cap m_W^{(2)} * VH_W).$$

(4) : Let $f' : (N, x_0) \to W$ be an integral map-germ with $j^r f'(x_0) = j^r f(x_0)$. Note that $r \geq n+1$. Therefore f' is also an open Whitney umbrella of the same type as f. By the argument of Proposition 3.5 in [19], we can connect f and f' by a family of integral map-germs f_t satisfying

$$VI_{f_t}^r \subseteq tf_t(m_N V_N) + wf_t(VL_Z \cap m_W^{(2)} * VH_W).$$

Thus by the homotopy method we see f and f' are Legendre equivalent.

11. Stability. First we show the following:

LEMMA 11.1. *Let $f : (N^n, x_0) \to W^{2n+1}$ be an integral map-germ of corank at most one. If f is contact stable then f is an open Whitney umbrella.*

Proof. Because all notions involved are local and invariant under the contactomorphisms, we may assume, by the Darboux theorem, $f : (\mathbf{K}^n, 0) \to (\mathbf{K}^{2n+1}, 0)$, $f^*\alpha = 0$, $\alpha = dr - pdq$ and f is of corank ≤ 1. Take a representative $f : U \to \mathbf{K}^{2n+1}$ of the germ f. We may assume the representative is also integral and of corank ≤ 1.

Set $g = (p \circ f, q \circ f) : U \to T^*\mathbf{K}^n$. Then g is isotropic and of corank ≤ 1. Here g is called *isotropic* if $g^*\omega = 0$, for the symplectic form $\omega = d(pdq)$ on $T^*\mathbf{K}^n$. In fact, since $g^*(pdq) = d(r \circ f)$ we have $g^*\omega = d(g^*(pdq)) = 0$. Furthermore, if there is a plane in $T_x\mathbf{K}^n$, for some $x \in U$, included in the kernel of $g_* : T_x\mathbf{K}^n \to T_g(x)T^*\mathbf{K}^n$, then $d(p_i \circ f), d(q_i \circ f), (1 \leq i \leq n)$ vanish on the plane, and then also $d(r \circ f) = d(p \circ fd(q \circ f))$ vanishes on the plane. This means that the plane is included in the kernel of $f_* : T_x\mathbf{K}^n \to T_{f(x)}\mathbf{K}^{2n+1}$. Therefore if f is of corank ≤ 1, then g is necessarily of corank ≤ 1.

Now, by [15] Theorem 2, g is approximated by an isotropic mapping $\widetilde{g} : U \to T^*\mathbf{K}^n$ of corank ≤ 1 such that, at any point $x \in U$, the germ of $\widetilde{g}$ at x is an (symplectic) open Whitney umbrella. Then there exist a symplectomorphism $\kappa : (T^*\mathbf{K}^n, \widetilde{g}(x)) \to (T^*\mathbf{K}^n, 0)$ and a diffeomorphism $\sigma : (\mathbf{K}^n, x) \to (\mathbf{K}^n, 0)$ such that $\kappa \circ \widetilde{g} \circ \sigma^{-1} : (\mathbf{K}^n, 0) \to (T^*\mathbf{K}^n, 0)$ coincides with $(p \circ f_{n,k}, q \circ f_{n,k})$ in §2.

Let $e : (\mathbf{K}^n, x) \to \mathbf{K}$ be a generating function of $\widetilde{g}_x$, $\widetilde{g}_x^*(pdq) = de$. Remark that two generating functions e, e' differ by just the addition of a constant function. Then

$(\widetilde{g}, e) : (\mathbf{K}^n, x) \to \mathbf{K}^{2n+1}$ is an integral map-germ and it is contact equivalent to $f_{n,k}$ by the contactomorphism (σ, τ), $\tau(p, q, r) = (\kappa(p, q), r + c)$ for some constant c.

Since g is of corank ≤ 1, if the perturbation $\widetilde{g}$ of g is sufficiently small, then we can take e on U, deleting U smaller if necessary. Then $(\widetilde{g}, e)$ is an integral perturbation of f on U, which we can take near f arbitrarily. Since the original germ f is contact stable, it is contact equivalent to some $(\widetilde{g}, e) : (\mathbf{K}^n, x) \to \mathbf{K}^{2n+1}$. Thus f is an open Whitney umbrella in the sense of §2. □

Proof of Theorem 3.1.

(cs) ⇒ (owu) is already proved in Lemma 11.1.

(owu) ⇒ (ics): Note that the infinitesimal contact stability is invariant under contactomorphisms. Let $f_{n,k}$ be the normal form of an open Whitney umbrella. Then the corresponding isotropic map-germ $\Pi \circ f_{n,k}$ is an open Whitney umbrella as an isotropic map-germs. Then it is proved in [18] that $\Pi \circ f_{n,k}$ is symplectically stable. Then by Lemma 8.5, we see $f_{n,k}$ is infinitesimally contact stable.

(owu) ⇒ (ca) : It follows from Lemma 8.6.

(ca) ⇒ (ics) : It follows from Corollary 7.11.

(ics) ⇒ (hcs) : The condition (ics) is equivalent to that $S_f = 0$, which is equivalent to that $S_{F/J} = m_J S_{F/J} = m_{W\times J} S_{F/J}$, by Lemma 7.14. By Corollary 7.13, $S_{F/J}$ is a finite $\mathcal{E}_{W\times J}$-module. So by Nakayama's lemma, we see $S_{F/J} = 0$. Therefore any integral deformation of f is trivialised with respect to contactomorphisms. Thus we have (hcs).

(hcs) ⇒ (ics) : Let $v \in VI_f$. Then there exists an integral deformation (f_t) of f with $(df_t/dt)|_{t=0} = v$. Since f is homotopically contact stable, f_t is trivialised by contactomorphisms: $f_t = \tau_t^{-1} \circ f \circ \sigma_t$. Differentiating both sides by t and setting $t = 0$, we have $v = (df_t/dt)|_{t=0} = tf(\xi) + wf(\eta)$, for some $\xi \in V_N, \eta \in VH_W$. Thus we have (ics).

(ics) ⇒ (ct): This follows from Lemma 9.3.

(ct) ⇒ (owu) : It is proved in Proposition 9.4.

Thus we see conditions (owu), (ca), (ics), (hcs) and (ct) are equivalent to each other.

(ct) ⇒ (cs) : If $j^r f$ is transversal to the contactomorphism class of $j^r f(x_0)$ for $r \geq \dfrac{n}{2} + 1$, then, for any slight perturbation f' of f, there exists a point x_0' near x_0 such that $j^r f'$ intersects to the contactomorphism class of $j^r f(x_0)$ at x_0'. Since f is an open Whitney umbrella, f is r-determined by contactomorphisms. Therefore we see $f' : (N, x_0') \to W$ is contactomorphic to $f : (N, x_0) \to W$. Therefore f is contact stable.

(cs) ⇒ (ct) : Take a representative $f : U \to W$ of f. Then f is approximated by an integral mapping $f' : U \to W$ such that $j^r f' : U \to J_I^r(N, W)$ is transverse to the contactomorphism-orbit $[j^r f(x_0)]$. Since f is contact stable, there exists $x_0' \in U$ such that $f' : (N, x_0') \to W$ and $f : (N, x_0) \to W$ are contactomorphism. Then $j^r f'$ is transverse to $[j^r f(x_0)] = [j^r f'(x_0')]$ at x_0', and therefore $j^r f$ is transverse to $[j^r f(x_0)]$ at x_0.

Thus (ct) is equivalent to (cs).

The implication (ics) ⇒ (fics) is obvious.

(fics) ⇒ (ct) : If f is infinitesimally contact stable on the level of formal series, then, for any r,

$$VI_f = tf(V_N) + wf(VH_W) + VI_f^r.$$

Therefore, by Lemma 9.3, we have (ct).

Thus all conditions are equivalent to each other. □

Based on Theorem 3.1, now we prove the main result Theorem 4.1.

Proof of Theorem 4.1.

First we show the equivalence of (hls) and (ils).

(hls) ⇒ (ils): Let $v \in VI_f$. Then there exists an integral deformation (f_t) of f with $(df_t/dt)|_{t=0} = v$. Since f is homotopically Legendre stable, f_t is trivialised under Legendre equivalence: $f_t = \tau_t^{-1} \circ f \circ \sigma_t$. Differentiating both sides by t and setting $t = 0$, we have $v = (df_t/dt)|_{t=0} = tf(\xi) + wf(\eta)$, for some $\xi \in V_N, \eta \in VL_W$. Thus we have (ils).

(ils) ⇒ (hls): Since f is infinitesimally Legendre stable, f is infinitesimally contact stable. So f is an open Whitney umbrella and thus f is finite. Therefore R_f is a finite $\mathcal{E}_N$-module. Then $VI_f/wf(VL_W)$ is a finite $\mathcal{E}_Z$-module. Let f_t be an integral deformation of f. Set $F = (f_t, t)$. Then $VI_{F/J}/(wF/J)(VL_W)$ is also a finite $\mathcal{E}_{Z\times J}$-module. Thus, by Nakayama's lemma, we have $VI_{F/J}/((wF/J)(VL_W) + (tF/J)(V_N)) = 0$, similarly to the proof of Theorem 3.1. Therefore f is homotopically Legendre stable.

Second we show (hls) (⇔ (ils)) ⇒ (lt_r) ⇒ (la_r'') ⇒ (la') ⇒ (la) ⇒ (ils). (Therefore these conditions are equivalent to each other).

(hls) ⇒ (lt_r): It is clear since the condition (lt_r) is equivalent to that

$$VI_f = tf(V_N) + wf(VL_W) + VI_f^r,$$

by Lemma 9.2.

(lt_r) ⇒ (la_r''): Taking generating functions of both sides of the equality

$$VI_f = tf(V_N) + wf(VL_W) + VI_f^r,$$

we have

$$R_f = (\pi \circ f)^*\mathcal{E}_Z + \sum_{i=1}^{n} (\pi \circ f)^*\mathcal{E}_Z(p_i \circ f) + R_f \cap m_N^{r+2}.$$

Remarking $R_f = f^*\mathcal{E}_W$, we have (a_r'').

(a_r'') ⇒ (a'): Since $R_f \cap m_N^{r+2} \subset m_W^{n+3}R_f$, we see the condition ($\mathrm{a}_r''$) implies that $R_f/(m_Z R_f + m_W^{n+3}R_f)$ is generated by $1, p_1 \circ f, \ldots, p_n \circ f$ over $\mathbf{K}$. Then we have $m_W^{n+1}R_f \subset m_Z R_f + m_W^{n+3}R_f$, therefore, by Nakayama's lemma, $m_W^{n+1}R_f \subset m_Z R_f$. Then $m_Z R_f + m_W^{n+3}R_f = m_Z R_f$, so we have that $R_f/m_Z R_f$ is generated by $1, p_1 \circ f, \ldots, p_n \circ f$ over $\mathbf{K}$, namely, the condition (a').

(la') ⇒ (la): By the assumption, and by the Malgrange's preparation theorem of differentiable algebras ([25]), we see R_f is generated by 1 and $p_1 \circ f, \ldots, p_n \circ f$ over $\mathcal{E}_Z$.

(la) ⇒ (ils): Since f is an open Whitney umbrella, we have $R_f = f^*\mathcal{E}_W$, and we have an $\mathcal{E}_Z$-isomorphism $VI_f/(tf(V_N) + wf(VL_W)) \cong R_f/f^*(\mathcal{E}_Z + \sum_{i=1}^{n} \mathcal{E}_Z \cdot p_i)$. Thus we see (la) implies (ils).

Lastly we show (ls) ⇒ (lt_r)(⇔ (ils)) ⇒ (ls).

(ls) ⇒ (lt_r): Take a representative $f : U \to W$ of f. Then f is approximated by an integral mapping $f' : U \to W$ such that $j^r f' : U \to J_I^r(N, W)$ is transverse to the Legendre orbit $[j^r f(x_0)]$. Since f is Legendre stable, there exists $x_0' \in U$ such

that $f' : (N, x'_0) \to W$ and $f : (N, x_0) \to W$ are Legendre equivalent. Then $j^r f'$ is transverse to $[j^r f(x_0)] = [j^r f'(x'_0)]$ at x'_0, and therefore $j^r f$ is transverse to $[j^r f(x_0)]$ at x_0.

(lt_r) & (ils) $\Rightarrow$ (ls): If $j^r f$ is transverse to $[j^r f(x_0)]$ at x_0, then there exists a neighborhood $\Omega \subseteq C^\infty_I(N, W)^1$ of an integral representative $f : U \to W$ such that, for any $f' \in \Omega$, $j^r f'$ is transverse to $[j^r f(x_0)]$ at a point $x'_0 \in U$. Since $j^r f'(x'_0) \in [j^r f(x_0)]$, there exists an integral map-germ $f'' : (X, x_0) \to W$ which is Legendre equivalent to $f'_{x'_0}$ with respect to π and $j^r f''(x_0) = j^r f(x_0)$. On the other hand, since f is infinitesimally Legendre stable, f is Legendre r-determined (Lemma 10.3(4)). Therefore (f'', π) and (f, π) are Legendre equivalent. Thus $(f'_{x'_0}, \pi)$ and (f, π) are Legendre equivalent, and f is Legendre stable.

Moreover (ils) $\Rightarrow$ (fils) $\Rightarrow$ (lt_r), by Lemma 9.2.

Thus we have proved Theorem 4.1.

12. Versality. The basic singularity theory originated by H. Whitney, R. Thom, J. Mather, J. Martinet, C.T.C. Wall and other people, are, in particular, unified into the theory of geometric subgroups of $\mathcal{A}$ or $\mathcal{K}$ due to J. Damon [7][8][9]. Naturally we try to apply the theory of differentiable mappings to our situation. The Damon's theory guarantees the unfolding theorem (the versality theorem) and the determinacy theorem for a subgroup $\mathcal{G}$ of $\mathcal{A}$ or $\mathcal{K}$ acting on a linear subspace $\mathcal{F}$ of map-germs $\mathcal{E}(n,p) = \{f : (\mathbf{K}^n, 0) \to (\mathbf{K}^p, 0),\ C^\infty\}$, provided that $\mathcal{G}$ and $\mathcal{F}$ together with their "unfolding spaces" $\mathcal{G}_{un}$, $\mathcal{F}_{un}$ satisfy several required conditions.

However our space

$$\mathcal{F} = \{f : (\mathbf{K}^n, 0) \to (\mathbf{K}^{2n+1}, 0) \mid f \text{ is integral of corank at most one.}\}$$

is not linear. Therefore, we can not apply directly the ordinary theory to our case.

There are two possibilities to overcome this difficulty.

One is the reduction to the linear situations case by case. For example, the method of generating families, due to Hörmander and Arnold, is successful for the study of singularities of Lagrange and Legendre immersions. Moreover the linear theory successfully is applied to certain non-linear spaces such as spaces of solutions to non-linear partial differential equations, e.g. Hamilton-Jacobi equations, non-linear diffusions, and so on [11]. Note that in [11] also results on the finite determinacy are given.

Another is the modifying of the original theory itself. It is useful to find a system of axioms which guarantees the versality theorem in non-linear cases, since then it is sufficient to just check the system of axioms. Then we observe, under an additional axiom, that the same proof of the versality theorem in the original theory works well for the generalisation (Theorem 9.3 of [8]). Thus we give a direct generalisation of Damon's theory to the non-linear situations. The generalisation is well-applied at least for Lagrange and Legendre singular immersions (isotropic and integral mappings) of corank ≤ 1.

We recall the theory on versal unfoldings: Groups of diffeomorphisms and spaces of mappings involve in the theory. Moreover we treat groups of unfoldings of diffeomorphisms and spaces of unfoldings of mappings.

Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, C^∞ or holomorphic. Take a group of diffeomorphisms $\mathcal{G} \subset \tilde{\mathcal{K}}$ where

$$\tilde{\mathcal{K}} := \{h : \mathbf{K}^n \times \mathbf{K}^p \to \mathbf{K}^n \times \mathbf{K}^p \mid \text{fiber-preserving diffeomorphism germs w.r.t. the fibration } \mathbf{K}^n \times \mathbf{K}^p \to \mathbf{K}^n\}$$

and a space of mappings $\mathcal{F} \subset \mathcal{E} := \{f : \mathbf{K}^n \to \mathbf{K}^p \mid \text{map-germs}\}$. Let $f \in \mathcal{E}$ and $h \in \tilde{\mathcal{K}}$. Then, $h(\text{graph}(f)) = \text{graph}(h(f))$ for the unique $h(f) \in \mathcal{E}$. We assume, for $f \in \mathcal{F}$ and $h \in \mathcal{G}, h(f) \in \mathcal{F}$.

Furthermore we assume there are given a group of unfoldings of diffeomorphisms $\mathcal{G}_{un}(r) \subseteq \tilde{\mathcal{K}}_{un}(r)$ which acts on a space of unfoldings of mappings $\mathcal{F}_{un}(r) \subseteq \mathcal{E}_{un}(r), r = 0, 1, 2, \ldots$, with $\mathcal{G}_{un}(0) = \mathcal{G}, \mathcal{F}_{un}(0) = \mathcal{F}$. Here $\tilde{\mathcal{K}}_{un}(r)$ is the space of r-parameter unfoldings of elements in $\tilde{\mathcal{K}}$, and $\mathcal{E}_{un}(r)$ is the space of r-parameter unfoldings of elements in $\mathcal{E}$.

First we assume $\mathcal{F}_{un}(r) \subset \mathcal{E}_{un}(r)$ is a linear subspace, relatively to the ordinary vector-space structure on $\mathcal{E}_{un}(r)$, $r = 0, 1, 2, \ldots$. Then, according to J. Damon, $(\mathcal{G}, \mathcal{G}_{un}; \mathcal{F}, \mathcal{F}_{un})$ is called a *geometric subgroups and subspaces* if it satisfies the axioms: (1) Naturality, (2) Tangent space structure, (3) Exponential map, (4) Filtration.
Then Damon has shown that axioms (1), (2), and (3) implies $\mathcal{G}$-versality theorem in $\mathcal{F}$, and axioms (1), (2), (3), and (4) implies $\mathcal{G}$-determinacy theorem in $\mathcal{F}$. See [8]. See also [9].

Remark that $(\mathcal{A}, \mathcal{A}_{un}; \mathcal{E}, \mathcal{E}_{un}), (\mathcal{K}, \mathcal{K}_{un}; \mathcal{E}, \mathcal{E}_{un})$ and $(\widetilde{\mathcal{K}}, \widetilde{\mathcal{K}}_{un}; \mathcal{E}, \mathcal{E}_{un})$ are geometric. Also equivariant diffeomorphisms and mappings provide examples of geometric subgroups and subspaces. Damon and all predecessors formulated the theory explicitly for linear spaces (of non-linear mappings). However naturally the theory works also for non-linear mapping spaces (non-linear spaces of non-linear mappings) in $\mathcal{E}$ where geometric subgroups of $\widetilde{\mathcal{K}}$ act.

Consider non-linear $\mathcal{F} \subset \mathcal{E}$ with non-linear $\mathcal{F}_{un} \subset \mathcal{E}_{un}$ with the action of a $\mathcal{G}_{un} \subset \tilde{\mathcal{K}}$.

Consider the restriction $\mathcal{F}_{un}(r+s) \to \mathcal{F}_{un}(r)$ to the first r-parameters (resp. $\mathcal{F}_{un}(r+s) \to \mathcal{F}_{un}(s)$ to the last s-parameters) and the restriction $\mathcal{F}_{un}(r) \to \mathcal{F}$ (resp. $\mathcal{F}_{un}(s) \to \mathcal{F}$) to the origin of the parameter space:

$$\begin{array}{ccc} \mathcal{F}_{un}(r+s) & \xrightarrow{\text{rest.}} & \mathcal{F}_{un}(r) \\ {\scriptstyle\text{rest.}}\downarrow & & {\scriptstyle\text{rest.}}\downarrow \\ \mathcal{F}_{un}(s) & \xrightarrow[\text{rest.}]{} & \mathcal{F} \end{array}$$

Now we pose:

(3') Extension axiom: The natural mapping

$$\mathcal{F}_{un}(r+s) \to \mathcal{F}_{un}(r) \times_{\mathcal{F}} \mathcal{F}_{un}(s)$$

to the fiber product is surjective, for any non-negative integers r, s.

The axiom (3') states that a deformation of a $f \in \mathcal{F}$ over $\mathbf{K}^r \times \{0\} \cup \{0\} \times \mathbf{K}^s$ extends to a deformation over $\mathbf{K}^{r+s}$ near 0.

By the same argument as in [8], we can show that, if $(\mathcal{G}, \mathcal{F})$ satisfies axioms (1), (2), (3) and (3') implies that $\mathcal{G}$-versality theorem in $\mathcal{F}$ holds, namely we have the infinitesimal characterization, the existence and uniqueness of $\mathcal{G}$-versal unfoldings in $\mathcal{F}$.

Here we show how to modify the conditions (1), (2) and (3) in [8] pp. 40–42.

For the naturality, we need no change: (1) For any $\Sigma \in \mathcal{G}_{\text{un}}(r), F \in \mathcal{F}_{\text{un}}(r)$ and for any map-germ $\varphi : (\mathbf{K}^s, 0) \to (\mathbf{K}^r, 0), \varphi \in \mathcal{E}(s, r)$, we have $\varphi^*\Sigma \in \mathcal{G}_{\text{un}}(s)$ and $\varphi^* F \in \mathcal{F}_{\text{un}}(s)$.

For the tangent space structure, since we can not suppose $T\mathcal{F}_{\rm un} = \mathcal{F}_{\rm un}$ in the non-linear case, we have to modify the condition slightly: First we define the extended tangent spaces $T_1\mathcal{G}_{{\rm un},e}(r)$ and $T_F\mathcal{F}_{{\rm un},e}(r)$ from $\mathcal{G}_{\rm un}(r+1), \mathcal{F}_{\rm un}(r+1)$ in the same way as [8], p.40. Then, (2) There exists an adequately ordered system of differentiable-analytic (DA) algebras $\{R_\alpha\}$ in $\mathcal{E}_{n+p}$ such that $T_1\mathcal{G}_{{\rm un},e}$ (resp. $T_F\mathcal{F}_{{\rm un},e}$) is a finitely generated $\{R_{\alpha,\lambda}\}$-module containing $T_1\mathcal{G}_{un}$ (resp. $T_F\mathcal{F}_{\rm un}$) as a finitely generated submodules, λ indicating the parameter and that, for the extended orbit mapping $\alpha_F : \mathcal{G}_{{\rm un},e} \to \mathcal{F}_{{\rm un},e}$, the differential mapping $(\alpha_F)_* : T_1\mathcal{G}_{{\rm un},e} \to T_F\mathcal{F}_{{\rm un},e}$ is an $\{R_{\alpha,\lambda}\}$-module homomorphism. The finiteness condition is required only when $f = F|_{\mathbf{K}^n\times 0}$ satisfies $\dim_{\mathbf{K}}(T_f\mathcal{F}/T\mathcal{G}_e\dot{f}) < \infty$. Moreover there exist isomorphisms

$$T_1\mathcal{G}_{{\rm un},e}/m_\lambda T_1\mathcal{G}_{{\rm un},e} \cong T_1\mathcal{G}_e,\ T_F\mathcal{F}_{{\rm un},e}/m_\lambda T_F\mathcal{F}_{{\rm un},e} \cong T_f\mathcal{F}_e,$$

as $\{R_\alpha\}$-modules, and that $\{m_\alpha\}T_1\mathcal{G}_e \subset T_1\mathcal{G}$, and $\{m_\alpha\}T_f\mathcal{F}_e \subset T_f\mathcal{F}$. About the generalities on DA-algebras see [8][9].

For the exponential property (3) we need no change.

Example 12.1. Let $\mathcal{I}$ be a differential system (namely, an ideal of differential forms that is d-closed) on $\mathbf{K}^p$. Set $\mathcal{F} := \{f : (\mathbf{K}^n, 0) \to (\mathbf{K}^p, 0) \mid f^*\mathcal{I} = 0\}$, the set of integral map-germs, and $\mathcal{G} := \{(\sigma, \tau) \in \mathcal{A} \mid \tau^*\mathcal{I} = \mathcal{I}\}$, the group of $\mathcal{A}$-equivalences preserving $\mathcal{I}$. Then $\mathcal{F} \subset \mathcal{E}$ is $\mathcal{G}$-invariant. Moreover we set $\mathcal{F}_{un}(r)$ as the space of r-parameter unfoldings

$$F = (f_\lambda, \lambda) : (\mathbf{K}^n \times \mathbf{K}^r, (0,0)) \to (\mathbf{K}^p \times \mathbf{K}^r, (0,0))$$

satisfying $f_\lambda^*\mathcal{I} = 0 (\lambda \in (\mathbf{K}^r, 0))$, and set

$$\mathcal{G}_{un}(r) := \{(\sigma_\lambda, \tau_\lambda, \lambda) \in \mathcal{A}_{un}(r) \mid \tau_\lambda^*\mathcal{I} = \mathcal{I}, (\lambda \in (\mathbf{K}^r, 0))\}.$$

Then $\mathcal{F}_{un}(r) \subset \mathcal{E}_{un}(r)$ is $\mathcal{G}_{un}(r)$-invariant. Remark that $\mathcal{F}$ and $\mathcal{F}_{un}$ are in general non-linear.

In particular we apply the above general theory to the singularity theory of integral mappings.

Set

$$\mathcal{F} = \{f : (\mathbf{K}^n, 0) \to (\mathbf{K}^{2n+1}, 0) \mid f \text{ is integral of corank at most one.}\}$$

and $\mathcal{G} = \{(\sigma, \tau)\}$ the group of contactomorphisms acting on $\mathcal{F}$. We set $\mathcal{F}_{un}(r)$ as the space of r-parameter integral unfoldings of integral germs in $\mathcal{F}$, and $\mathcal{G}_{un}(r)$ as the group of r-parameter unfoldings of contactomorphisms in $\mathcal{G}$. Then we have $T_f\mathcal{F}_e = VI_f, T_f\mathcal{F} = VI_f^0$ and $T_1\mathcal{G}_e = V_n \oplus VH_{2n}, T_1\mathcal{G} = m_nV_n \oplus (m_{2n}^{(2)} * VH_{2n})$.

Then $T_f\mathcal{F}_e$ is an $\mathcal{E}_{2n+1}$-module. If f is finite, then $T_f\mathcal{F}_e$ is a finite $\mathcal{E}_{2n+1}$-module.

We consider the system of algebras: $\mathcal{E}_{2n+1} \to \mathcal{E}_{2n+1}$ with the identity connection homomorphism. Note that $T_1\mathcal{G}_e = V_n \oplus VH_{2n}$ is an $\{\mathcal{E}_{2n+1}, \mathcal{E}_{2n+1}\}$-module by

$$H * \xi := (f^*H) \cdot \xi,\ H * X_K := X_{HK},$$

where X_H is the contact Hamilton vector field with Hamilton function H (§6). Moreover $T_f\mathcal{F}_e$ is an $\{\mathcal{E}_{2n+1}, \mathcal{E}_{2n+1}\}$-module by

$$\{H, K\} * v := H * v,$$

using the multiplication given in Proposition 7.3. (Here $\{H, K\}$ does not mean the Poisson product, but just comes from the notion on algebra-systems used in [8] §6).

Also for unfolding spaces, module structures are defined as in Corollary 7.13.

For the Legendre versality, we set $\mathcal{G}$ the group of Legendre equivalences $\{(\sigma, \tau)\}$ for the Legendre fibration $\pi : W = (\mathbf{K}^{2n+1}, 0) \to Z = (\mathbf{K}^{n+1}, 0)$. We read as

$$VI_f = T_f\mathcal{F}_e \text{ and } \ tf(V_N) + wf(VL_W) = T\mathcal{G}_e \cdot f.$$

The system of algebra we consider is

$$\mathcal{E}_{n+1} \to \mathcal{E}_{2n+1} \to \mathcal{E}_{2n+1}$$

with the connection homomorphisms π^* and the identity respectively.

In both case we can check the axioms (1), (2) and (3).

Now, based on the above general scheme due to Damon after the modification, we give alternative proof of Theorem 4.3

Proof of Theorems 3.3 and 4.3. The axioms (1), (2), (3) are easily checked. We need to show the extension axiom (3') is satisfied. for the category of integral unfoldings of an integral map-germs of corank at most one, in order to apply the Mather-Damon's machine to our situation. Note that the geometric group does not involve into the axiom (3').

Let $f : (N, x_0) \to (W, w_0)$ be an integral map-germ of corank ≤ 1. Let $F : (N \times \mathbf{K}^r, (x_0, 0)) \to (W, w_0)$ and $F' : (N \times \mathbf{K}^s, (x_0, 0)) \to (W, w_0)$ be integral deformations of f. We may set $(q, p_n) \circ f = (x_1, \ldots, x_{n-1}, u, v)$, for a function-germs $u = u(x', t), v = v(x', t), x' = (x_1, \ldots, x_{n-1}), t = x_n$, after a contactomorphism. Set

$$(q, p_n) \circ F = ((q \circ F)(x', t, \lambda), (p_n \circ F)(x', t, \lambda)),$$

$$(q, p_n) \circ F' = ((q \circ F')(x', t, \mu), (p_n \circ F')(x', t, \mu)).$$

Then there exist a coordinate change on the q-space depending on λ, μ such that we have

$$(q, p_n) \circ F = (x', U(x', t, \lambda), V(x', t, \lambda)),$$

$$(q, p_n) \circ F' = (x', U'(x', t, \mu), V'(x', t, \mu)),$$

with $U(x', t, 0) = u, V(x', t, 0) = v, U'(x', t, 0) = u, V'(x', t, 0) = v$. Then we can extend $(q, p_n) \circ F$ and $(q, p_n) \circ F'$ to $H : (N \times \mathbf{K}^r \times \mathbf{K}^s, (x_0, 0, 0)) \to (W, w_0)$ of form

$$H(x', t, \lambda, \mu) = (x', \widetilde{U}(x', t, \lambda, \mu), \widetilde{V}(x', t, \lambda, \mu))$$

by setting

$$\widetilde{U}(x', t, \lambda, \mu) := U(x', t, \lambda) + U'(x', t, \mu) - u(x', t),$$

$$\widetilde{V}(x', t, \lambda, \mu) := V(x', t, \lambda) + V'(x', t, \mu) - v(x', t).$$

Then we define $F'' : (N \times \mathbf{K}^r \times \mathbf{K}^s, (x_0, 0, 0)) \to (W, w_0)$ by $(q, p_n) \circ F'' = H$, $(r \circ F'')(x_0) = r \circ f(x_0)$, and by

$$d(r \circ F'') = \sum_{i=1}^{n-1} (p_i \circ F'') dx_i + \widetilde{U} d\widetilde{V}.$$

Here d means the exterior differential by $x_1, \ldots, x_{n-1}, x_n = t$. The last condition means that

$$\frac{\partial r \circ F''}{\partial x_i} = p_i \circ F'' + \widetilde{U} \frac{\partial \widetilde{V}}{\partial x_i}, \quad \frac{\partial r \circ F''}{\partial t} = \widetilde{U} \frac{\partial \widetilde{V}}{\partial t}.$$

We determine $r \circ F''$ by the latter condition and $(r \circ F'')(x_0) = r \circ f(x_0)$. Then $p_i \circ F'', (1 \leq i \leq n-1)$ are determined by the former condition. Thus we have the extension F'' of both F and F'.

Therefore the extension axiom (3') is satisfied. Then by the general framework we have the proof of contact and Legendre versality theorems. □

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