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ARMAND BOREL

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My topic is Armand Borel and the theory of automorphic forms. Borel's most important contributions to the area are undoubtedly those established in collaboration with Harish-Chandra [Œ 54, 58]. They include the construction and properties of approximate fundamental domains, the proof of finite volume of arithmetic quotients, and the characterization in terms of algebraic groups of those arithmetic subgroups that give compact quotients. These results created the opportunity for working in the context of general algebraic groups. They laid the foundations of the modern theory of automorphic forms that has flourished for the past forty years.

The classical theory of modular forms concerns holomorphic functions on the upper half plane $\mathcal{H}$ that transform in a certain way under the action of a discrete subgroup Γ of $SL(2,\mathbb{R})$. The multiplicative group $SL(2,\mathbb{R})$ consists of the 2×2 real matrices of determinant 1, and each element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $SL(2,\mathbb{R})$ acts on $\mathcal{H}$ by the linear fractional transformation $z \to \frac{az+b}{cz+d}$. For example, one can take Γ to be the subgroup $SL(2,\mathbb{R})$ of integral matrices or, more generally, the congruence subgroup

$$\Gamma(N) = \{\gamma \in SL(2,\mathbb{Z}) : \gamma \equiv I \text{ mod } N\}$$

attached to a positive integer N. The theory began as a branch of complex analysis. However, with the work of E. Hecke, it acquired a distinctive number theoretic character. Hecke introduced a commuting family of linear operators on any space of automorphic forms for $\Gamma(N)$, one for each prime not dividing N, with interesting arithmetic properties. We now know that eigenvalues of the Hecke operators govern how prime numbers p split in certain nonabelian Galois extensions of the field $\mathbb{Q}$ of rational numbers [Sh], [D]. Results of this nature are known as reciprocity laws and are in some sense the ultimate goal of algebraic number theory. They can be interpreted as a classification for the number fields in question. The Langlands program concerns the generalization of the theory of modular forms from the group of 2×2 matrices of determinant 1 to an arbitrary reductive group G. It is believed to provide reciprocity laws for all finite algebraic extensions of $\mathbb{Q}$.

Let us use the results of Borel and Harish-Chandra as a pretext for making a very brief excursion into the general theory of automorphic forms. In so doing, we can follow a path illuminated by Borel himself. The expository articles and monographs of Borel encouraged a whole generation of mathematicians to pursue the study of automorphic forms for general algebraic groups. Together with the mathematical conferences he organized, they have had extraordinary influence.

The general theory entails two reformulations of the classical theory of modular forms. The first is in terms of the unitary representation theory of the group $SL(2,\mathbb{R})$.

The action of $SL(2,\mathbb{R})$ on $\mathcal{H}$ is transitive. Since the stabilizer of the point $i = \sqrt{-1}$

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is the special orthogonal group

$$K_{\mathbb{R}} = SO(2,\mathbb{R}) = \left\{ k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right\},$$

one can identify $\mathcal{H}$ with the space of cosets $SL(2,\mathbb{R})/K_{\mathbb{R}}$. The space of orbits in $\mathcal{H}$ under a discrete group $\Gamma \subset SL(2,\mathbb{R})$ becomes the space of double cosets $\Gamma\backslash SL(2,\mathbb{R})/K_{\mathbb{R}}$. A *modular form* of weight $2k$ is a holomorphic function f on $\mathcal{H}$ such that[1]

$$f(\gamma z) = (cz+d)^{2k} f(z)$$

whenever $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in Γ. A modular form of weight 2, for instance, amounts to a holomorphic 1-form $f(z)dz$ on the Riemann surface $\Gamma\backslash\mathcal{H}$, since $d(\gamma z) = (cz+d)^{-2}dz$. For a given f, the function F on $SL(2,\mathbb{R})$ defined by $F(g) = (ci+d)^{2k} f(z)$ when $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $z = gi$ is easily seen to satisfy

$$F(\gamma g k_\theta) = F(g) e^{-2ki\theta},$$

for $\gamma \in \Gamma$. The requirement that f be holomorphic translates to the condition that F be an eigenfunction of a canonical biinvariant differential operator Δ on $SL(2,\mathbb{R})$ of degree 2, with eigenvalue a simple function of k. The theory of modular forms of any weight becomes part of the following more general problem:

> *Decompose the unitary representation of $SL(2,\mathbb{R})$ by right translation on $L^2(\Gamma\backslash SL(2,\mathbb{R}))$ into irreducible representations.*

That the problem is in fact more general is due to a variant of Schur's lemma. Namely, as an operator that commutes with $SL(2,\mathbb{R})$, Δ acts as a scalar on the space of any irreducible representation. To recover the modular forms of weight $2k$, one would collect the irreducible subspaces of $L^2(\Gamma\backslash SL(2,\mathbb{R}))$ with the appropriate Δ-eigenvalue, and from each of these, extract the smaller subspace on which the restriction to $K_{\mathbb{R}}$ of the corresponding $SL(2,\mathbb{R})$-representation equals the character $k_\theta \to e^{-2ki\theta}$.

This is all explained clearly in Borel's survey article [Œ 75] in the proceedings of the 1965 AMS conference at Boulder [1]. The Boulder conference was organized jointly by Borel and G. D. Mostow. It was a systematic attempt to make the emerging theory of automorphic forms accessible to a wider audience. Borel himself wrote four articles [Œ 73, 74, 75, 76] for the proceedings, each elucidating a different aspect of the theory.

The second reformulation is in terms of adeles. Though harder to justify at first, the language of adeles ultimately streamlines many fundamental operations on automorphic forms. The relevant Boulder articles are [T] and [K]. These were not written by Borel, but were undoubtedly commissioned by him as part of a vision for presenting a coherent background from the theory of algebraic groups.

The adele ring

$$\mathbb{A} = \mathbb{R} \times \mathbb{A}_{\text{fin}} = \mathbb{R} \times \left(\tilde{\prod_{p \text{ prime}}} \mathbb{Q}_p \right)$$

[1] There is also a mild growth condition that need not concern us here.

of $\mathbb{Q}$ is a locally compact ring that contains the real field $\mathbb{R}$, as well as completions $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to the p-adic absolute values

$$|x|_p = p^{-r}, \qquad x = p^r(ab^{-1}),\ (a,p) = (b,p) = 1,$$

on $\mathbb{Q}$. One constructs $\mathbb{Q}_p$ by a process identical to the completion $\mathbb{R}$ of $\mathbb{Q}$ with respect to the usual absolute value. In fact, one has an enhanced form of the triangle inequality,

$$|x_1 + x_2|_p \leq \max\left\{|x_1|_p\,,\, |x_2|_p\right\}, \qquad x_1, x_2 \in \mathbb{Q},$$

which has the effect of giving the compact "unit ball" $\mathbb{Z}_p = \left\{x_p \in \mathbb{Q}_p : |x_p|_p \leq 1\right\}$ the structure of a subring of $\mathbb{Q}_p$. The complementary factor $\mathbb{A}_{\text{fin}}$ of $\mathbb{R}$ in $\mathbb{A}$ is defined as the "restricted" direct product

$$\tilde{\prod_p}\mathbb{Q}_p = \{x = (x_p) : x_p \in \mathbb{Q}_p\,, \quad x_p \in \mathbb{Z}_p \text{ for almost all } p\}\,,$$

which becomes a locally compact (totally disconnected) ring under the appropriate direct limit topology. The diagonal image of $\mathbb{Q}$ in $\mathbb{A}$ is a discrete subring. This implies that the group $SL(2,\mathbb{Q})$ of rational matrices embeds into the locally compact group $SL(2,\mathbb{A})$ of unimodular adelic matrices as a discrete subgroup. The theory of automorphic forms on $\Gamma\backslash SL(2,\mathbb{R})$, for any congruence subgroup $\Gamma = \Gamma(N)$, becomes part of the following more general problem:

> *Decompose the unitary representation of $SL(2,\mathbb{A})$ by right translation on $L^2(SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A}))$ into irreducible representations.*

The reason that the last problem is more general than the previous one is provided by the theorem of strong approximation, which applies to the simply connected[2] group $SL(2)$. The theorem asserts that if K is any open compact subgroup of $SL(2,\mathbb{A}_{\text{fin}})$, then

$$SL(2,\mathbb{A}) = SL(2,\mathbb{Q}) \cdot (K \cdot SL(2,\mathbb{R})).$$

This implies that if $\Gamma = SL(2,\mathbb{R}) \cap SL(2,\mathbb{Q})K$, then there is a unitary isomorphism

$$L^2(SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A})/K) \xrightarrow{\sim} L^2(\Gamma\backslash SL(2,\mathbb{R}))$$

that commutes with right translation by $SL(2,\mathbb{R})$. For example, if we take K to be the group

$$K(N) = \{x = (x_p) : x_p \in SL(2,\mathbb{Z}_p), \quad x_p \equiv I \bmod (M_2(N\mathbb{Z}_p))\},$$

then Γ equals $\Gamma(N)$. To recover the decomposition of $L^2(\Gamma(N)\backslash SL(2,\mathbb{R}))$, one would collect the irreducible subspaces of $L^2(SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A}))$, and from each of these, extract the smaller subspace on which the restriction to K of the corresponding $SL(2,\mathbb{A})$-representation is trivial.

[2] "Simply connected" in this instance means that $SL(2,\mathbb{C})$ is simply connected as a topological space.

Given that the decomposition of $L^2(SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A}))$ includes the classical theory of modular forms, we can see reasons why the adelic formulation is preferable. It treats the theory simultaneously for all weights and all congruence subgroups. It is based on a discrete group $SL(2,\mathbb{Q})$ of rational matrices rather than a group $\Gamma(N)$ of integral matrices. Most significantly, perhaps, it clearly displays the supplementary structure given by right translation under the group $SL(2,\mathbb{A}_{\text{fin}})$. The unitary representation theory of the p-adic groups $SL(2,\mathbb{Q}_p)$ thus plays an essential role in the theory of modular forms. This is the source of the operators discovered by Hecke. Eigenvalues of Hecke operators are easily seen to parametrize irreducible representations of the group $SL(2,\mathbb{Q}_p)$ that are *unramified* in the sense that their restrictions to the maximal compact subgroup $SL(2,\mathbb{Z}_p)$ contain the trivial representation. It turns out that in fact *any* irreducible representation of $SL(2,\mathbb{Q}_p)$ that occurs in the decomposition of $L^2(SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A}))$ carries fundamental arithmetic information.

It is now straightforward to set up higher-dimensional analogs of the theory of modular forms. One replaces[3] the group $SL(2)$ by an arbitrary connected reductive algebraic group G defined over $\mathbb{Q}$. As in the special case of $SL(2)$, $G(\mathbb{Q})$ embeds as a discrete subgroup of the locally compact group $G(\mathbb{A})$. The Langlands program has to do with the irreducible constituents (known as *automorphic representations*) of the unitary representation of $G(\mathbb{A})$ by right translation on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. A series of conjectures of Langlands, dating from the mid-1960s through the 1970s, characterizes the internal structure of automorphic representations. The conjectures provide a precise description of the arithmetic data in automorphic representations, together with a formulation of deep and unexpected relationships among these data as G varies (known as the "principle of functoriality").

General automorphic representations are thus firmly grounded in the theory of algebraic groups. It seems safe to say that the many contributions of Borel to algebraic groups described by Springer and Tits in this article are all likely to have some role to play in the theory of automorphic forms. Borel did much to make the Langlands program more accessible. For example, his Bourbaki talk [Œ 103] in 1976 was one of the first comprehensive lectures on the Langlands conjectures to a general mathematical audience.

In 1977 Borel and W. Casselman organized the AMS conference in Corvallis on automorphic forms and L-functions, as a successor to the Boulder conference. It was a meticulously planned effort to present the increasingly formidable background material needed for the Langlands program. The Corvallis proceedings [2] are considerably more challenging than those of Boulder. However, they remain the best comprehensive introduction to the field. They also show evidence of Borel's firm hand. Speakers were not left to their own devices. On the contrary, they were given specific advice on exactly what aspect of the subject they were being asked to present. Conference participants actually had to share facilities with a somewhat unsympathetic football camp, led by Coach Craig Fertig of the Oregon State University Beavers. At the end of the four weeks, survivors were rewarded with orange T-shirts, bearing the inscription ARMAND BOREL MATH CAMP. Borel himself sported[4] a T-shirt with the further designation COACH.

Let us go back to the topic we left off earlier, Borel's work with Harish-Chandra. The action of $SL(2,\mathbb{Z})$ on $\mathcal{H}$ has a well-known fundamental domain, given by the

[3]Even in the classical case, one has to replace $SL(2)$ by the slightly larger group $GL(2)$ to obtain all the operators defined by Hecke.

[4]See the photograph on the previous page.

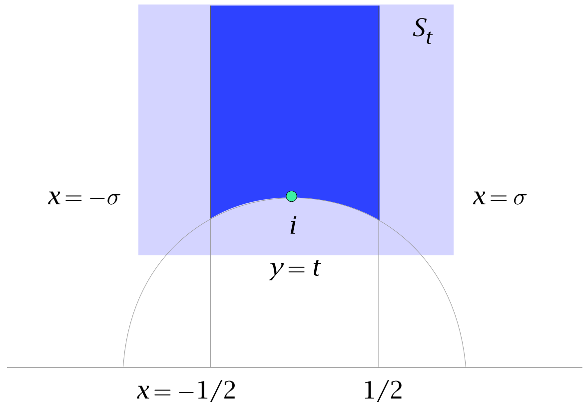

FIGURE 1. *Standard fundamental domain for the action of $SL(2,\mathbb{Z})$ on the upper half plane, together with a more tractable approximate fundamental domain. The standard fundamental domain, darkly shaded, is the semi-infinite region bounded by the unit circle and the vertical lines at $x = \pm 1/2$. The approximate fundamental domain S_t generalized by Borel and Harish-Chandra is the total shaded region.*

darker shaded region in Figure 1. This region is difficult to characterize in terms of the transitive action of $SL(2,\mathbb{R})$ on $\mathcal{H}$. The total shaded rectangular region S_t in the diagram is more tractable, for there is a topological decomposition $SL(2,\mathbb{R}) = P(\mathbb{R})K_{\mathbb{R}} = N(\mathbb{R})M(\mathbb{R})K_{\mathbb{R}}$, where P, N, and M are the subgroups of matrices in $SL(2)$ that are respectively upper triangular, upper triangular unipotent, and diagonal. The group $N(\mathbb{R})$ acts by horizontal translation on $\mathcal{H}$, while $M(\mathbb{R})$ acts by vertical dilation. We have already noted that $K_{\mathbb{R}}$ stabilizes the point i. We can therefore write

$$S_t = \omega A_t \cdot i,$$

where ω is the compact subset $\left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : |x| \leq \sigma\right\}$ of $N(\mathbb{R})$, and A_t is the one-dimensional cone $\left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0,\ a^2 \geq t\right\}$. The set S_t is an approximate fundamental domain for the action of $SL(2,\mathbb{Z})$ on $\mathcal{H}$, in the sense that it contains a set of representatives of the orbits, while there are only finitely many $\gamma \in SL(2,\mathbb{Z})$ such that S_t and γS_t intersect.

For a general group G, the results of Borel and Harish-Chandra provide an approximate fundamental domain for the action of $G(\mathbb{Q})$ by left translation on $G(\mathbb{A})$. To describe it, I have to rely on a few notions from the theory of algebraic groups. Let me write P for a minimal parabolic subgroup of G over $\mathbb{Q}$, with unipotent radical N and Levi component M. The adelic group $M(\mathbb{A})$ can be written as a direct product $M(\mathbb{A})^1 A_M(\mathbb{R})^0$, where A_M is the $\mathbb{Q}$-split part of the center of M, $A_M(\mathbb{R})^0$ is the connected component of 1 in $A_M(\mathbb{R})$, and $M(\mathbb{A})^1$ is a canonical complement of $A_M(\mathbb{R})^0$ in $M(\mathbb{A})$ that contains $M(\mathbb{Q})$. The *roots* of (P, A_M) are characters $a \longrightarrow a^\alpha$ on A_M that determine a cone

$$A_t = \{a \in A_M(\mathbb{R})^0 : a^\alpha \geq t, \text{ for every } \alpha\}$$

in $A_M(\mathbb{R})^0$ for any $t > 0$. Suppose that $K_{\mathbb{A}} = K_R K_{\text{fin}}$ is a maximal compact subgroup of $G(\mathbb{A})$. If Ω is a compact subset of $N(\mathbb{A})M(\mathbb{A})^1$, the product

$$S_t = \Omega A_t K_{\mathbb{A}}$$

is called a *Siegel set* in $G(\mathbb{A})$, following special cases introduced by C. L. Siegel. One of the principal results of [Œ 58] implies that for suitable choices of $K_{\mathbb{A}}$, Ω, and t, the set S_t is an approximate fundamental domain for $G(\mathbb{Q})$ in $G(\mathbb{A})$.

The obstruction to $G(\mathbb{Q})\backslash G(\mathbb{A})$ being compact is thus governed by the cone A_t in the group $A_M(\mathbb{R})^0 \cong \mathbb{R}^{\dim A_M}$. It follows that $G(\mathbb{Q})\backslash G(\mathbb{A})$ is compact if and only if A_M is trivial, which is to say that G has no proper parabolic subgroup over $\mathbb{Q}$ and no $\mathbb{Q}$-split central subgroup. This is essentially the criterion of [Œ 58].[5] Borel and Harish-Chandra obtained other important results from their characterization of approximate fundamental domains. For example, in the case of semisimple G, they proved that the quotient $G(\mathbb{Q})\backslash G(\mathbb{A})$ has finite volume with respect to the Haar measure of dx on $G(\mathbb{A})$. This is a consequence of a decomposition formula for Haar measures $dx = a^{-2\rho}\, d\omega\, da\, dk$, where ω is in Ω, a is in A_t, and k is in $K_{\mathbb{A}}$ and where 2ρ denotes the sum of the roots of (P, A_M).

The papers of Borel and Harish-Chandra were actually written for arithmetic quotients $\Gamma\backslash G(\mathbb{R})$ of real groups, as were the supplementary articles [Œ 59, 61] of Borel. Prodded by A. Weil [W, p. 25], Borel wrote two parallel papers [Œ 55, 60] that formulated the results in adelic terms and established many basic properties of adele groups.[6] His later lecture notes [Œ 79], written again in the setting of real groups, immediately became a standard reference.

Borel and Harish-Chandra were probably motivated by the 1956 paper [Sel] of A. Selberg. Selberg brought many new ideas to the study of the spectral decomposition of $L^2(\Gamma\backslash SL(2,\mathbb{R}))$, including a construction of the continuous spectrum by means of Eisenstein series and a trace formula for analyzing the discrete spectrum. A familiarity with the results of Siegel no doubt gave Borel and Harish-Chandra encouragement for working with general groups. Their papers were followed in the mid-1960s by Langlands's manuscript on general Eisenstein series (published only later in 1976 [L]). In the context of adele groups, Langlands's results give a complete description of the continuous spectrum of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. A starting point was the work of Borel and Harish-Chandra and, in particular, the properties of approximate fundamental domains. In recent years Borel lectured widely on the theory of Eisenstein series: in the three-year Hong Kong program mentioned by Bombieri, for example, and the 2002 summer school in Park City. One of his ambitions, alas unrealized, was to write an introductory volume on the general theory of Eisenstein series.

In attempting to give a sense of both the scope of the field and Borel's substantial influence, I have emphasized Borel's foundational work with Harish-Chandra and his leading role in making the subject more accessible. Borel made many other important contributions. These were often at the interface of automorphic forms with geometry, especially as it pertains to the locally symmetric spaces

$$X_\Gamma = \Gamma\backslash G(\mathbb{R})/K_{\mathbb{R}}.$$

Elements in the deRham cohomology group $H^*(X_\Gamma, \mathbb{C})$ are closely related to automorphic forms for G, as we have already noted in the special case of modular forms of weight 2. This topic was fully explored in Borel's monograph [Œ 115, 172] with N. Wallach. Borel collaborated in the creation of two very distinct compactifications of spaces X_Γ: one with W. Baily [Œ 63, 69], the other with J-P. Serre [Œ 90, 98]. The

[5]A similar result was established independently by Mostow and Tamagawa [MT].

[6]In his 1963 Bourbaki lecture [G], R. Godement presented an alternative argument, which he also formulated in adelic terms.

Baily-Borel compactification became the setting for the famous correspondence between intersection cohomology (discovered by Goresky and MacPherson in the 1970s) and L^2-cohomology (applied to square integrable differential forms on X_Γ), a relationship first conjectured by S. Zucker.[7] The L^2-cohomology of X_Γ is the appropriate analog of deRham cohomology in case X_Γ is noncompact. Its relations with automorphic forms were investigated by Borel and Casselman [Œ 126, 131]. In general, the cohomology groups of spaces X_Γ are very interesting objects, which retain many of the deepest properties of the corresponding automorphic representations. They bear witness to the continuing vitality of mathematics that originated with Borel.

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[7] The conjecture was later proved by L. Saper and M. Stern, and E. Looijenga.