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# ARMAND BOREL

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When I learned of Armand Borel's death in August 2003, my thoughts went back to our long friendship, to our joint work, and to our last exchange of correspondence, which had occurred around Armand's eightieth birthday in May 2003. I had sent him congratulations on that occasion, and in his response he mentioned his appreciation for the introduction I had given him for his Euler Lecture on May 19, 1995.

The Euler Lecture takes places in Sanssouci in Potsdam. By now this is a rather well-known yearly lecture that attracts mathematicians from Berlin, Potsdam, and also farther away. The lecture is in a small rococo theater built by Friedrich the Great, who had invited Euler to Potsdam, where Euler worked for many years before he went to St. Petersburg. In 1995 Borel was the person invited to deliver the Euler Lecture, and it is a custom that a kind of extended introduction about the speaker is given. The nature of this introduction is best rendered in a single word by "laudatio" in Latin or "encomium" in English. I did this for Borel in 1995. He liked this "laudatio", and this is what he was referring to in his final email to me on June 6, 2003:

> Dear Fritz,
>
> How nice of you to remember an old friend and send me such a warm letter on the occasion of my becoming an octogenarian. Already ten years of retirement. As I wrote to you at your "Emeritierung", I found this a pleasant situation and I hope you feel the same, especially since, as I gather from your letter, Inge and you are in good health.
>
> There are still projects of publications and trips but also a tendency at this age to reminisce. You speak of the Euler lecture. In retrospect, I was very glad that you insisted so much for me to give it (I was rather reluctant at first), since this is indeed one of our fond memories. I really enjoyed giving a lecture in these unique surroundings, especially after your (too) nice "laudatio". And of course, our two years in Princeton, our joint work and so much else, remain very much on my mind.
>
> We both travel, but our paths do not seem to cross anymore. I hope they will sometime.
>
> With best regards from both of us to both of you.
>
> Armand

The facts and sentiments in that "laudatio" still apply today. Charles Thomas[1] has kindly translated the "laudatio" into English, so that I can include it now:

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Armand Borel studied at the Eidgenossische Technische Hochschule (ETH) in Zürich and was an "assistant" there from 1947 to 1949. After this he worked in Paris while supported by a grant from the CNRS, served as a replacement for the professor of algebra in Geneva, and from 1952 to 1954 was a member of the renowned Institute for Advanced Study in Princeton, becoming a professor in 1957.

Borel was always a little ahead of me; already he was born a few years before me. When I studied in Zürich from 1949 to 1950, he was an assistant in Zürich or researcher in Paris. Forty-five years ago in Zürich was the first time we were able to discuss mathematics together. When in 1952 we both began two exciting years as members of the Institute for Advanced Study, he but not I was already married. My wife-to-be arrived somewhat later (November 1952) in Princeton, where we actually married, the Borels joining in the celebration. Hence today it is a great pleasure for me to welcome not only Armand to the "Neues Palais" but also Gaby Borel. When we began our joint work in 1953 in Princeton ("Characteristic classes and homogeneous spaces", first appearing 1958–1960), Borel's knowledge of Lie groups far outstripped mine, and it still does. Borel's impressive Paris thesis (in Germany it would be called his "Habilitationsschrift") with the title "Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts"[2] appeared in 1953 in the *Annals of Mathematics* and was for me a kind of bible. The thesis examiners were J. Leray, H. Cartan, and A. Lichnerowicz.

In 1992 Borel was awarded the Balzan Prize for mathematics. In his acceptance speech[3] he described the development of the notion of a group from Galois through Felix Klein to Sophus Lie. Groups appeared first as symmetry groups, as self-maps of a mathematical configuration, such that each map has an inverse and the composition of two maps still belongs to the set. Galois groups operate on the roots of an equation. Felix Klein suggested that a geometry should be studied with the help of its symmetry group. The Norwegian mathematician Sophus Lie, who served as Klein's successor in Leipzig from 1886 to 1898, there published his great work *Theorie der Trans-formationsgruppen*.[4] In the 1870s Lie had already considered finite-dimensional transformation groups. Examples of such Lie group actions are the group of motions of Euclidean space (rotations and translations) and the Lorentz group of special relativity. The three fundamental theorems of the Lie theory of transformation groups include the generation of local groups by infinitesimal transformations. Lie groups and Lie algebras were born.[5] Today groups and algebras are considered abstractly, divorced from the mathematical objects on which they act. The whole area of Lie groups and algebras is central for mathematical research; here many continually developing branches of mathematics come together, and modern physics is inconceivable without this. Armand Borel's great work, building on that of Élie Cartan and Hermann Weyl, belongs to this center and its numerous ramifications, and in recent decades has influenced many important developments in mathematics.

After these few general remarks let me return to our joint time in Princeton (1952–54). There Borel lectured frequently on the results and further development of

[2] "On the cohomology of principal fiber spaces and homogeneous spaces of compact Lie groups".

[3] "Quelques réflexions sur les mathématiques en général et la théorie des groupes en particulier", lecture on the occasion of the award of the 1992 Balzan Prize in mathematics, published in French in a booklet *Orientamenti e Attività dei Premi Balzan 1992* of the Fondazione Internazionale Balzan, Milan.

[4] Theory of Transformation Groups.

[5] This was the birth. The christening came in the 1930s with the introduction of the terms "Lie groups" and "Lie algebras".

the thesis already mentioned. In volume one of the three volumes of collected papers, published by Springer-Verlag[6] in 1983, may be found many important works, reaching back to the Princeton period. These demonstrate how much today's knowledge of the cohomology and homotopy of Lie groups and their homogeneous spaces is due to him.

On this transparency[7] one sees Borel during lectures at the summer school of the American Mathematical Society in 1953 in Maine. What is the significance of the first line on the blackboard

$$(B_T,\ B_G,\ G/T,\ \rho(T,G))\ ?$$

Well, $G$ is a connected compact Lie group, $T$ a maximal torus in $G$. For example $G$ might be the group $U(n)$ of all unitary matrices, i.e., the automorphisms of the $n$-dimensional Hermitian space $\mathbb{C}^n$, and $T$ the group of diagonal matrices. But what is $B_G$? Shortly before I came to Princeton, I had learned the theory of fiber bundles from Steenrod's book. For Borel the theory of fiber bundles was already something self-evident, as it is today for many mathematicians and physicists. $B_G$ is the classifying space for $G$, from which all fiber bundles with structure group $G$ can be induced. Properties of $G$ are fundamental for the study of $G$-bundles. For $G = U(n)$ the cohomology ring of $B_G$ is generated by the Chern classes, which are thus defined for each $U(n)$-bundle. We have arrived at the theory of characteristic classes, which at that time played such a major role for me and which I was able to learn from Borel. More precisely I ought to say that the classifying space $B_T$ of the maximal torus is fibered over $B_G$ with the homogeneous space $G/T$ as fiber, and that $\rho(T,G)$ is the projection map $B_T \to B_G$ of this fiber bundle. The Weyl group acts on this fibration. In the case $G = U(n)$ this is the symmetric group for $n$ symbols. The fibers are flag manifolds, consisting of all pairwise mutually perpendicular $n$-tuples of 1-dimensional subspaces of an $n$-dimensional Hermitian vector space, permuted by the Weyl group. The cohomology comes from the fibration and the theory of symmetric functions: the symmetric group acts on the $n$ variables in the polynomial ring over them (the cohomology ring of $B_T$), and the Chern classes appear as the elementary symmetric functions.

The manifold $G/T$ is algebraic. To it one can apply the Riemann-Roch theorem that I was developing at that time. I did this for $G = U(n)$, needed first to determine the Chern classes of the tangent bundle of $G/T$, and showed this to Borel, who recognized the roots of the Lie algebra in the formula. This held more generally for arbitrary $G$ and led to our joint work. The Riemann-Roch theorem applied to $G/T$ suddenly delivered Hermann Weyl's formula for the degree of the irreducible representations of $G$, from which it was only a short step to the Borel-Weil theorem in the representation theory of $G$. This one sentence hides a long story. Today a mathematics graduate student would certainly require a one-semester course of four hours per week in order to understand it.

Since the two years in Princeton were so important for me, I have devoted much of this short lecture to this time. Everything else will have to be much shorter. Immediately after the Princeton period Borel went to Chicago and worked there on algebraic groups. His important work "Groupes linéaires algébriques"[8] appeared in 1956 in the *Annals of Mathematics*. This work is essential for the classification of

[6] A fourth volume appeared in 2001, after this "laudatio" was given.

[7] This refers to the photograph that appeared as the frontispiece of Volume I of the collected papers.

[8] "Linear algebraic groups".

semisimple groups over algebraically closed fields (achieved by Chevalley) and for many further developments. The work of W. Baily and A. Borel ("Compactification of arithmetic quotients of bounded symmetric domains", *Annals of Mathematics*, 1966) is famous. It deals with arithmetically defined, discontinuous groups of automorphisms of a bounded homogeneous symmetric domain, with the compactification of the quotient space as a normal analytic space, and with the embedding of the compactification in a projective space by means of automorphic forms.

Many other deep results for arithmetic groups and the links with number theory cannot be mentioned here. Areas to which Borel has also made decisive contributions in the past twenty years can be characterized under the following headings: cohomology of arithmetic groups, applications to $K$-theory, automorphic functions, and infinite-dimensional representations of real and $p$-adic Lie groups.

From March 1 until June 30, 1994, Borel was supported by a Humboldt Research Award at the Max Planck Institute. Every morning he was the first to arrive at the Institute, having beforehand swum for one hour in the Ennert-Gebirge open-air pool. His final report on his time at the MPI shows that he still stands in the middle of active research:

1. "Joint seminar with D. Zagier on the higher regulators for algebraic number fields, which I introduced about twenty years ago, polylogarithms, the Zagier conjectures relating them and Goncharov's proof in two special cases. I gave about six lectures in that seminar."
2. He reported the completion of a survey article on "Values of zeta-functions at integers, cohomology and polylogarithms". Here it seems to me we are close to his topic for today.
3. He reported on new results in the homology of general linear groups over a number field.

As has been already said, since 1957 Borel has been a professor at the Institute for Advanced Study. Here he has always played a conspicuous role. His seminars on topics of current research interest have enriched the scientific life of young members of the Institute. Here new theories were worked through, and his own new research discoveries were introduced. From these seminars have come these books, all with contributions by other authors:

*Seminar on Transformation Groups* (1960),

*Seminar on Algebraic Groups and Related Finite Groups* (1970),

*Continuous Cohomology, Discrete Subgroups and Representation of Reductive Groups* (1980).

Between 1983 and 1986 he worked part-time at the Institute for Advanced Study and also held a chair at the ETH in Zürich. He organized a Pan-Swiss research seminar, meeting over the railroad station in Bern. From it the books *Intersection Cohomology* and *Algebraic D-modules* emerged. The success of this seminar can be gleaned from the observation that it still continues under the label "Borel Seminar", without Borel's participation.

The Borels have looked after members at the Institute for Advanced Study in many ways. My wife and I have attended many parties, for which Gaby Borel is to thank. On September 23, 1959, he collected us, three children, and nineteen pieces of luggage, by car and trailer from Hoboken, New Jersey, and brought us to 39 Einstein Drive in the Institute's housing complex in Princeton. Let this service, admittedly reserved for only a few members, be acknowledged and applauded at this time. In any event it was a good start to my first sabbatical as professor at the University of

Bonn.

Borel has given much thought to the place of mathematics in our culture. Many people were impressed by his lecture to the Siemens Foundation on "Mathematik, Kunst und Wissenschaft".[9] On the last page is the sentence:

> I hope that I have at least left you with the impression that mathematics is an extremely complex creation, which exhibits so many essentially common traits from Art and from both the experimental and theoretical Sciences. It reflects simultaneously all three of them and therefore must be distinguished from all three of them.

At the end of his acceptance lecture[10] upon being awarded the Balzan Prize, he warned us all of well-known dangers:

> More and more, people insist on usefulness. It is said that we have paid enough attention to fundamental research and that it is time to turn to its applications. If such a policy is adopted, people will doubtless realize at some stage that they have spoiled the very source of the practical applications that they are trying to favor. This may not happen immediately, because mathematical research possesses such verve that nothing can prevent it from continuing on its trajectory for a certain time. I could therefore console myself by saying that, if there is decline, I will not witness it. But that would be small consolation for someone who does not tire of admiring the wealth and beauty of mathematics in its present state and is convinced that what is to come will be in no way inferior.

Mathematicians and politicians can learn much from Borel.

[9] "Mathematics, Art and Science", lecture of May 7, 1981, to the Carl Friedrich von Siemens Foundation, Munich, published in German as number 35 in the series of lectures to the foundation.

[10] See footnote 3.