ASIAN J. MATH.
Vol. 8, No. 2, pp. 233–258, June 2004

MAPS BETWEEN $\mathbf{B}^n$ AND $\mathbf{B}^N$ WITH GEOMETRIC RANK $k_0 \leq n-2$ AND MINIMUM N *

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Dedicated to Professor Yum-Tong Siu on the Occasion of his 60th Birthday

1. Introduction. Let $\mathbf{B}^n = \{z \in \mathbf{C}^n : |z| < 1\}$ be the unit ball in $\mathbf{C}^n$. The problem of classifying proper holomorphic mappings between $\mathbf{B}^n$ and $\mathbf{B}^N$ has attracted considerable attention (see [Fo 1992] [DA 1988] [DA 1993] [W 1979] [H 1999][HJ 2001] for extensive references) since the work of Poincare [P 1907][T 1962] and Alexander [A 1977]. Let us denote by $Prop(\mathbf{B}^n, \mathbf{B}^N)$ the collection of proper holomorphic mappings from $\mathbf{B}^n$ to $\mathbf{B}^N$. It is known [A 1977] that any map $F \in Prop(\mathbf{B}^n, \mathbf{B}^n)$ must be biholomorphic and must be equivalent to the identity map. Here we say that $f, g \in Prop(\mathbf{B}^n, \mathbf{B}^N)$ are *equivalent* if there are automorphisms $\sigma \in Aut(\mathbf{B}^n)$ and $\tau \in Aut(\mathbf{B}^N)$ such that $f = \tau \circ g \circ \sigma$. For general $N > n$, the discovery of inner functions indicates that $Prop(\mathbf{B}^n, \mathbf{B}^N)$ is too complicated to be classified. Hence we may focus on $Rat(\mathbf{B}^n, \mathbf{B}^N)$, the collection of all rational proper holomorphic mappings from $\mathbf{B}^n$ to $\mathbf{B}^N$. We first recall the following results:

THEOREM 1.0. *Let $2 \leq n \leq N$.*

(1) [We 1979][Fa 1986] When $N < 2n-1$, $Rat(\mathbf{B}^n, \mathbf{B}^N)$ has only one equivalent class.
(2a) [Fa 1982] When $N = 2n-1$ and $n = 2$, $Rat(\mathbf{B}^2, \mathbf{B}^3)$ has four equivalent classes.
(2b) [HJ 2001] When $N = 2n-1$ and $n > 2$, $Rat(\mathbf{B}^n, \mathbf{B}^{2n-1})$ has exactly two equivalent classes. One is the linear map and another one is Whitney map.
(3) [DA 1988] When $N = 2n$, $Rat(\mathbf{B}^n, \mathbf{B}^{2n})$ has infinitely many equivalent classes. In particular, $\{F_t(z_1 \cdots, z_n) = (z_1, \cdots, z_{n-1}, \cos(t) z_n, \sin(t) z_n z) : t \in (0, \pi/2)\}$ is a family of mutually inequivalent polynomial proper embeddings.

However a puzzle remains: Why is the case of $n = 2$ in Theorem 1.0 (2a) more complicated than the one of $n \geq 3$ in Theorem 1.0(2b)?

This puzzle can be solved by the following new formulation, which is crucially based on a notion, *geometric rank*, introduced recently by Huang [H 2003]. For any $2 \leq n \leq N$, any $F \in Rat(\mathbf{B}^n, \mathbf{B}^N)$ can be associated an invariant integer $\kappa_0 \in \{1, 2, ..., n-1\}$, called its *geometric rank* (see § 2 for the definition). It is known that for any $F \in Rat(\mathbf{B}^n, \mathbf{B}^N)$, its geometric rank $\kappa_0 = 0$ if and only if F is equivalent to the linear map ([H 1999, Theorem 4.2] cf. [HJ 2001, Propostion 2.2]). Therefore, to study maps in $Rat(\mathbf{B}^n, \mathbf{B}^N)$, it is sufficient to study maps with geometric rank $\kappa_0 \geq 1$.

If $F \in Rat(\mathbf{B}^n, \mathbf{B}^N)$ with the geometric rank κ_0, it is known [H 2003, lemma 3.2] that $N \geq n + \frac{(2n-\kappa_0-1)\kappa_0}{2}$ must hold, namely, the least dimension of the target space is $n + \frac{(2n-\kappa_0-1)\kappa_0}{2}$. Therefore, to understand the simplest case, given an integer $\kappa_0 \in \{1, ..., n-1\}$, we are interested in studying maps $F \in Rat(\mathbf{B}^n, \mathbf{B}^N)$ with the

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geometric rank κ_0 and with the minimum dimension of the target space:

$$F : \mathbf{B}^n \to \mathbf{B}^N, \quad N = n + \frac{(2n - \kappa_0 - 1)\kappa_0}{2}. \tag{1}$$

When $\kappa_0 = 1$, (1) becomes $F : \mathbf{B}^n \to \mathbf{B}^{2n-1}$, which is the case covered by Theorem 1.0 (2a) and (2b).

It is also known from a recent deep and important result by Huang [H 2003] that there is a significant difference between the case $1 \leq \kappa_0 \leq n - 2$ and the case $\kappa_0 = n - 1$. More precisely, when $1 \leq \kappa_0 \leq n - 2$, the maps F have so-called semi-linear property while the maps with $\kappa_0 = n - 1$ may not have such property. This gives a philosophy that maps F with $\kappa_0 = n - 1$ are comparatively much more complicated than the ones with $1 \leq \kappa_0 \leq n - 2$. From this philosophy, the following two problems are naturally formulated.

Problem A. Study and classify maps $F \in Rat(\mathbf{B}^n, \mathbf{B}^N)$ with $N = n + \frac{(2n-\kappa_0-1)\kappa_0}{2}$ and $1 \leq \kappa_0 \leq n - 2$.

Problem B. Study and classify maps $F \in Rat(\mathbf{B}^n, \mathbf{B}^N)$ with $N = n + \frac{(2n-\kappa_0-1)\kappa_0}{2}$ and $\kappa_0 = n - 1$.

When $\kappa_0 = 1$, Problem A is solved in Theorem 1.0 (2b), and beyond this case, the next simplest unsolved case is $Rat(\mathbf{B}^4, \mathbf{B}^9)$ with $\kappa_0 = 2$. When $n = 2$, Problem B is solved in Theorem 1.0 (2a), and beyond this case, the next simplest unsolved case is $Rat(\mathbf{B}^3, \mathbf{B}^6)$ with $\kappa_0 = 2$.

In fact, the formulation of Problems A and B explains why $Rat(\mathbf{B}^2, \mathbf{B}^3)$ is more complicated than $Rat(\mathbf{B}^n, \mathbf{B}^{2n-1})$ with $n \geq 3$: Each of Theorem 1.0 (2a) and (2b) is an initial case of Problem A and Problem B.

In this paper, we study Problem A and we first estimate the degree of such maps F. As the main result, we investigate Problem A by studying maps $F \in Rat(\mathbf{B}^4, \mathbf{B}^9)$ with $\kappa_0 = 2$ and $deg(F) = 2$.

THEOREM 1.1. *Let $F \in Rat(\mathbf{B}^n, \mathbf{B}^N)$ with geometric rank κ_0, $1 \leq \kappa_0 \leq n - 2$, and with $N = n + \frac{(2n-\kappa_0-1)\kappa_0}{2}$. Then $deg(F) \leq \kappa_0 + 2$.*

THEOREM 1.2. *Let $F \in Rat(\mathbf{B}^4, \mathbf{B}^9)$ with the geometric rank 2 and with $deg(F) = 2$. Then F is equivalent to Whitney map $W_{4,2}$ of rank 2.*

The paper is organized as follows. We first prove Theorem 1.1, by using the same technique in the proof of Lemma 5.2 in [HJ 2001]. Here we would like to mention a conjecture by D'Angelo [DA 1993, p.189] which is open: if $F \in Rat(\mathbf{B}^n, \mathbf{B}^N)$, then $deg(F) \leq 2N - 3$. Next we introduce the definition of Whitney map of rank κ_0 (see (9)) and prove a criterion for such maps. In Section 5, we determine the form of $F(z, 0)$, in which the semi-linearity property of F by Huang [H 2003] will be crucially used. In Section 6, we further determine the form of $F(z, w)$. Theorem 6.1 tells us what F looks like when $F \in Rat(\mathbf{H}_4, \mathbf{H}_9)$ satisfies the normalization condition in Theorem 2.2: F has three complex parameters $b^{(11)}_{1001}, b^{(13)}_{1001}, E_{0001}$ and one real parameter $\mu_2 \geq 1$ that are related by certain equations. To prove such map F is equivalent to Whitney map, one key step is to change the parameter μ_2 into 1. However, the difficulty is that such μ_2 is invariant under any equivalent change that fix the origin (see (4)(5)). If we consider F^{***}_p, which is equivalent to F (see Theorem 2.2), the calculation of $\mu_{2,F}(p)$ is too complicated to be handled. Our idea is to calculate only the linear part of $\mu_{2,F}(p)$, which dramatically reduces our computation. As a result, we see that $\mu_{2,F}(p)$ decreases when p moves along a certain direction. Then we are able to show that $\mu_{2,F}(p)$ can reach the minimum and hence it must be 1 and the resulting map F is exactly Whitney map.

Acknowledgments. We wish to thank Xiaojun Huang for helpful discussions and valuable suggestion for our work in this paper. We also thank the referees who gave very useful suggestions and comments.

2. Preliminaries. Let $\mathbf{H}_n := \{(z,w) \in C^{n-1} \times C, Im(w) > |z|^2\}$ be the *Siegel upper-half space* in $\mathbf{C}^n$ where $n \ge 2$. By using the Cayley transformation:

$$\rho_n : \mathbf{H}_n \to \mathbf{B}^n,\ \ \rho_n(z,w) = \left(\frac{2z}{1-iw}, \frac{1+iw}{1-iw}\right),\ \rho_n^{-1}(z^*,w^*) = \left(\frac{z^*}{w^*+1}, \frac{1}{i}\cdot\frac{w^*-1}{w^*+1}\right),$$

$\mathbf{H}_n$ is biholomorphic equivalent to the unit ball $\mathbf{B}^n$ and $\partial\mathbf{H}_n$ is equivalent to the unit sphere $\partial\mathbf{B}^n$. For any $F \in Rat(\mathbf{B}^n, \mathbf{B}^N)$, $\rho_N^{-1} \circ F \circ \rho_n \in Rat(\mathbf{H}_n, \mathbf{H}_N)$. Then we can identify a map $F \in Rat(\mathbf{B}^n, \mathbf{B}^N)$ with the one in $Rat(\mathbf{H}_n, \mathbf{H}_N)$, and we shall still denote it as F for simplicity. By the work of Cima and Suffridge[CS 1990], F extends holomorphically up to the boundary. Hence the map F induces a non-constant CR mapping from $\partial\mathbf{H}_n$ to $\partial\mathbf{H}_N$. As before, we also denote it as F.

Write $L_j = 2i\overline{z_j}\frac{\partial}{\partial w} + \frac{\partial}{\partial z_j}$ for $j = 1, \cdots, n-1$ and $T = \frac{\partial}{\partial u}$ with $w = u + iv$. $\{L_1, \cdots, L_{n-1}\}$ forms a basis for the complex tangent bundle $T^{(1,0)}\partial\mathbf{H}_n$, and T is the tangent vector field of $\partial\mathbf{H}_n$ transversal to $T^{(1,0)}\partial\mathbf{H}_n \cup T^{(0,1)}\partial\mathbf{H}_n$. $\partial\mathbf{H}_n$ can be parameterized by $(z,\overline{z},u)$ through the map $(z,\overline{z},u) \to (z, u+i|z|^2)$. Assign the weight of z and u to be 1 and 2 respectively. If m is a non-negative integer, a function h defined over a neighborhood U of 0 in $\partial\mathbf{H}_n$ is said to be of quantity $o_{wt}(m)$ if $\frac{h(tz,t\overline{z},t^2u)}{|t|^m} \to 0$ uniformly for (z,u) on any compact subset of U as $t(\in \mathbf{R}) \to \mathbf{0}$. For this case, we write $h = o_{wt}(m)$.

Let $F = (f,\phi,g) = (\widetilde{f},g) = (f_1,\cdots,f_{n-1},\phi_1,\cdots,\phi_{N-n},g)$ be a map in $Rat(\mathbf{H}_n,\mathbf{H}_N)$. For any $p = (z_0,w_0) \in \partial\mathbf{H}_n$, we consider automorphisms $\sigma^0_{(z_0,w_0)} \in Aut(\mathbf{H}_n)$ and $\tau^F_{(z_0,w_0)} \in Aut(\mathbf{H}_N)$ given by

$$\sigma^0_{(z_0,w_0)}(z,w) = (z+z_0, w+w_0+2i\langle z,\overline{z_0}\rangle),$$

$$\tau^F_{(z_0,w_0)}(z^*,w^*) = (z^* - \tilde{f}(z_0,w_0), w^* - \overline{g(z_0,w_0)} - 2i\langle z^*, \overline{\tilde{f}(z_0,w_0)}\rangle).$$

Then $F_p = \tau^F_p \circ F \circ \sigma^0_p \in Rat(\mathbf{H}_n,\mathbf{H}_N)$ is equivalent to F with $F_p(0) = 0$. By the work of Huang [H 1999], F_p is equivalent to another new map $F^{**}_p = (f^{**}_p, \phi^{**}_p, g^{**}_p)$ that satisfies the following normalization condition.

THEOREM 2.1. [H 1999, Lemma 5.3] *Let F be a C^2-smooth CR map from a connected open subset $M \subset \partial\mathbf{H}_n$ into $\partial\mathbf{H}_N$ with $N \ge n \ge 2$. Then for each $p \in M$, $F^{**}_p = (f,\phi,g)$ satisfies the normalization condition:*

$$f_p = z + \frac{i}{2}a^{(1)}_p(z)w + o_{wt}(3),\ \ \phi_p = \phi^{(2)}_p(z) + o_{wt}(2),\ \ g_p = w + o_{wt}(4) \qquad (2)$$

with $\langle \overline{z}, a_p^{(1)}(z)\rangle |z|^2 = |\phi_p^{(2)}(z)|^2$, *where we denote by* $h^{(j)}(z)$ *a polynomial of* z *with homogeneous degree* j.

Here $a_p^{(1)}(z) = z\mathcal{A}(p)$ where $\mathcal{A}(p)$ is a certain $(n-1)\times(n-1)$ semi-positive Hermitian matrix. The rank of $\mathcal{A}(p)$ is said to be the *geometric rank* of F at the point of p. We denote it by $Rk_F(p)$. We define the *geometric rank* of F to be $\kappa_0 = max_{p\in\partial\mathbf{H_n}} Rk_F(p)$. Notice that $0 \le \kappa_0 \le n-1$.

Theorem 2.2. [H 2003, Lemma 3.2, 3.3, Corollary 4.2, 5.2, (3.6.4) and Claim 4.4] *Let* F *be a* C^3*-smooth CR map from a connected open subset* $M \subset \partial\mathbf{H}_n$ *into* $\partial\mathbf{H}_N$ *with* $F(0) = 0$, $1 \le \kappa_0 = Rk_F(0) \le n-2$ *and* $N = n + \frac{(2n-\kappa_0-1)\kappa_0}{2}$. *Then for* $\forall p(\approx 0) \in M$, F_p *is equivalent to another map* F_p^{***}, *still denote it by* (f,ϕ,g), *from* $\partial\mathbf{H}_n$ *to* $\partial\mathbf{H}_N$, *with the following conditions:*

$$\begin{aligned}
&f_j = z_j + \tfrac{i\mu_j}{2} z_j w + o_{wt}(3),\\
&f_j = z_j + o_{wt}(3), \tfrac{\partial^2 f_j}{\partial w^2}(0) = 0,\ 1 \le j \le n-1;\\
&\phi_{jl} = \mu_{jl} z_j z_l + o_{wt}(2),\ \tfrac{\partial^2 \phi_{jl}}{\partial z_k \partial w}(0) = 0 \text{ for } k > \kappa_0,\ \tfrac{\partial^2 \phi_{jl}}{\partial w^2}(0) = 0,\quad \forall (j,l) \in S_0;\\
&g = w + o_{wt}(5),\quad g(0,w) = w + o(w^3),
\end{aligned} \tag{3}$$

where $S_0 = \{(jl), 1 \le j \le \kappa_0,\ 1 \le l \le n-1,\ j \le l\}$, $\mu_j \ge \mu_1 = 1$ *for* $j \le \kappa_0$ *and* $\mu_j = 0$ *for* $j > \kappa_0$; $\mu_{jl} = \sqrt{\mu_j + \mu_l}$ *for* $1 \le j \le \kappa_0$, $1 \le l \le n-1$, $j \ne l$ *and* $\mu_{jj} = \sqrt{\mu_j}$ *for* $1 \le j \le \kappa_0$.

We notice that S_0 is the finite index set of $\{\phi_{jl}\}$ and $|S_0| = \frac{(2n-\kappa_0-1)\kappa_0}{2}$. Also, by [H 2003, Corollary 5.2], the set $\{p \in \partial\mathbf{H}_n, Rk_F(p) = \kappa_0\}$ is an open dense subset of M. Therefor the assumption that $F(0) = 0, \kappa_0 = Rk_F(0)$ holds for almost $p \in M$.

For any rational holomorphic map $H = \frac{(P_1,\cdots,P_m)}{Q}$ on $\mathbf{C}^n$, where P_j, Q are holomorphic polynomials and $(P_1,\cdots,P_m,Q) = 1$, the *degree* of H is defined to be $deg(H) = max\{deg(P_j),\ 1 \le j \le m,\ deg(Q)\}$.

Lemma 2.3. [HJ 2001, Lemma 5.3 and 5.4] *Let* $F \in Rat(\mathbf{H}_n, \mathbf{H}_N)$. *If* $deg(F_p(z,0)) \le l$ *for any* p *in an open subset of* $\partial\mathbf{H}_n$, *then* $deg(F) \le l$.

Consider $\sigma \in Aut_0(\mathbf{H}_n)$ and $\tau^* \in Aut_0(\mathbf{H}_N)$ given by

$$\sigma = \frac{(\lambda(z+aw)\cdot U,\ \lambda^2 w)}{q(z,w)}, \tag{4}$$

where $q(z,w) = 1 - 2i\langle\overline{a}, z\rangle + (r - i|a|^2)w$, $\lambda > 0$, $r \in \mathbf{R}$, $a \in \mathbf{C}^{n-1}$ and U is an $(n-1)\times(n-1)$ unitary matrix, and

$$\tau^*(z^*,w^*) = \frac{\lambda^*(z^* + a^* w^*)\cdot U^*, \lambda^{*2} w^*}{q^*(z^*,w^*)}, \tag{5}$$

where $q^*(z^*,w^*) = 1 - 2i\langle\overline{a^*}, z^*\rangle + (r^* - i|a^*|^2)w^*$, $\lambda^* > 0$, $r^* \in \mathbf{R}$, $a^* \in \mathbf{C}^{N-1}$ and U^* is an $(N-1)\times(N-1)$ unitary matrix.

Lemma 2.4. ([H 2003, Lemma 2.2]) *Let* $F = (f,\phi,g)$ *and* $F^* = (f^*,\phi^*,g^*)$ *be* C^2 *CR map from a neighborhood of* 0 *in* $\partial\mathbf{H}_n$ *into* $\partial\mathbf{H}_N$ $(N \ge n > 1)$ *such that both* *satisfy the normalization condition (2). Suppose that $F^* = \tau^* \circ F \circ \sigma$ where σ and τ^* are as in (4) and (5) respectively. Then it holds that*

$$\lambda^* = \lambda^{-1},\ a_1^* = -\lambda^{-1} a \cdot U,\ a_2^* = 0,\ r^* = -\lambda^{-2} r,\ U^* = \begin{pmatrix} U^{-1} & 0 \\ 0 & U_{22}^* \end{pmatrix}, \tag{6}$$

where $a^* = (a_1^*, a_2^*)$ with a_1^* its first $(n-1)$ components, U_{22}^* is an $(N-n)\times(N-n)$ unitary matrix. Conversely, suppose τ^* and σ, given in (4) and (5) respectively, are related by (6). Suppose that F satisfies (2). Then $F^* := \tau^* \circ F \circ \sigma$ also satisfies (2).

3. Proof of Theorem 1.1 . Assume that F satisfies (3). Let $\mathcal{L}_j = 2i\zeta_j \frac{\partial}{\partial w} + \frac{\partial}{\partial z_j}$ be the complexfication of L_j. We apply $\mathcal{L}_j, \mathcal{L}_k\mathcal{L}_j (k \le \kappa_0,\ k \le j)$ to the equation: $\frac{g(z,w)-\overline{g(\zeta,\eta)}}{2i} = f(z,w)\overline{f(\zeta,\eta)} + \phi(z,w)\overline{\phi(\zeta,\eta)}$ for any $w - \eta = 2i(z\cdot\zeta)$. Let $(z,w) = 0$, $\eta = 0$. Then we get

$$\overline{\widetilde{f}(\zeta,0)^t} = \begin{pmatrix} I & 0 \\ -B^{-1}A & B^{-1} \end{pmatrix} \begin{pmatrix} \overline{\zeta}^t \\ 0 \end{pmatrix} = \begin{pmatrix} \overline{\zeta}^t \\ -B^{-1}A\overline{\zeta}^t \end{pmatrix}, \quad \forall \zeta \in \mathbf{C}^{n-1}, \tag{7}$$

where

$$A = \begin{pmatrix} \mathcal{L}_1\mathcal{L}_1(f_1) & \mathcal{L}_1\mathcal{L}_1(f_2) & \cdots & \mathcal{L}_1\mathcal{L}_1(f_{n-1}) \\ \mathcal{L}_1\mathcal{L}_2(f_1) & \mathcal{L}_1\mathcal{L}_2(f_2) & \cdots & \mathcal{L}_1\mathcal{L}_2(f_{n-1}) \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{L}_1\mathcal{L}_{n-1}(f_1) & \mathcal{L}_1\mathcal{L}_{n-1}(f_2) & \cdots & \mathcal{L}_1\mathcal{L}_{n-1}(f_{n-1}) \\ \mathcal{L}_2\mathcal{L}_2(f_1) & \mathcal{L}_2\mathcal{L}_2(f_2) & \cdots & \mathcal{L}_2\mathcal{L}_2(f_{n-1}) \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{L}_2\mathcal{L}_{n-1}(f_1) & \mathcal{L}_2\mathcal{L}_{n-1}(f_2) & \cdots & \mathcal{L}_2\mathcal{L}_{n-1}(f_{n-1}) \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{L}_{\kappa_0}\mathcal{L}_{n-1}(f_1) & \mathcal{L}_{\kappa_0}\mathcal{L}_{n-1}(f_2) & \cdots & \mathcal{L}_{\kappa_0}\mathcal{L}_{n-1}(f_{n-1}) \end{pmatrix}\Bigg|_{(0,0,\zeta,0)} = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_{\kappa_0} \end{pmatrix}$$

and C_i are $(n-i)\times(n-1)$ matrices with the following forms:

$$C_1 = \begin{pmatrix} -2\overline{\zeta}_1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ -\overline{\zeta}_2 & -\mu_2\overline{\zeta}_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ -\overline{\zeta}_3 & 0 & -\mu_3\overline{\zeta}_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\overline{\zeta}_{\kappa_0} & 0 & 0 & \cdots & -\mu_{\kappa_0}\overline{\zeta}_1 & 0 & \cdots & 0 \\ -\overline{\zeta}_{\kappa_0+1} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\overline{\zeta}_{n-1} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 0 & -2\mu_2\overline{\zeta}_2 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & -\mu_2\overline{\zeta}_3 & -\mu_3\overline{\zeta}_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -\mu_2\overline{\zeta}_{\kappa_0} & 0 & \cdots & -\mu_{\kappa_0}\overline{\zeta}_2 & 0 & \cdots & 0 \\ 0 & -\mu_2\overline{\zeta}_{\kappa_0+1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -\mu_2\overline{\zeta}_{n-1} & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$\cdots, \quad C_{\kappa_0} = \begin{pmatrix} 0 & 0 & 0 & \cdots & -2\mu_{\kappa_0}\overline{\zeta}_{\kappa_0} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & -\mu_{\kappa_0}\overline{\zeta}_{\kappa_0+1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\mu_{\kappa_0}\overline{\zeta}_{n-1} & 0 & \cdots & 0 \end{pmatrix},$$

and B is equal to

$$\begin{pmatrix} \mathcal{L}_1\mathcal{L}_1(\phi_{11}) & \cdots & \mathcal{L}_1\mathcal{L}_1(\phi_{1(n-1)}) & \mathcal{L}_1\mathcal{L}_1(\phi_{22}) & \cdots & \mathcal{L}_1\mathcal{L}_1(\phi_{\kappa_0(n-1)}) \\ \mathcal{L}_1\mathcal{L}_2(\phi_{11}) & \cdots & \mathcal{L}_1\mathcal{L}_2(\phi_{1(n-1)}) & \mathcal{L}_1\mathcal{L}_2(\phi_{22}) & \cdots & \mathcal{L}_1\mathcal{L}_2(\phi_{\kappa_0(n-1)}) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathcal{L}_1\mathcal{L}_{n-1}(\phi_{11}) & \cdots & \mathcal{L}_1\mathcal{L}_{n-1}(\phi_{1(n-1)}) & \mathcal{L}_1\mathcal{L}_{n-1}(\phi_{22}) & \cdots & \mathcal{L}_1\mathcal{L}_{n-1}(\phi_{\kappa_0(n-1)}) \\ \mathcal{L}_2\mathcal{L}_2(\phi_{11}) & \cdots & \mathcal{L}_2\mathcal{L}_2(\phi_{1(n-1)}) & \mathcal{L}_2\mathcal{L}_2(\phi_{22}) & \cdots & \mathcal{L}_2\mathcal{L}_2(\phi_{\kappa_0(n-1)}) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathcal{L}_2\mathcal{L}_{n-1}(\phi_{11}) & \cdots & \mathcal{L}_2\mathcal{L}_{n-1}(\phi_{1(n-1)}) & \mathcal{L}_2\mathcal{L}_{n-1}(\phi_{22}) & \cdots & \mathcal{L}_2\mathcal{L}_{n-1}(\phi_{\kappa_0(n-1)}) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathcal{L}_{\kappa_0}\mathcal{L}_{n-1}(\phi_{11}) & \cdots & \mathcal{L}_{\kappa_0}\mathcal{L}_{n-1}(\phi_{1(n-1)}) & \mathcal{L}_{\kappa_0}\mathcal{L}_{n-1}(\phi_{22}) & \cdots & \mathcal{L}_{\kappa_0}\mathcal{L}_{n-1}(\phi_{\kappa_0(n-1)}) \end{pmatrix}\Bigg|_{(0,0,\zeta,0)}.$$

Hence $A\overline{\zeta}^t = \begin{pmatrix} D_1 \\ D_2 \\ \vdots \\ D_{\kappa_0} \end{pmatrix}$ with

$$D_1 = \begin{pmatrix} -2\overline{\zeta}_1^2 \\ -(1+\mu_2)\overline{\zeta}_1\overline{\zeta}_2 \\ -(1+\mu_3)\overline{\zeta}_1\overline{\zeta}_3 \\ \vdots \\ -(1+\mu_{\kappa_0})\overline{\zeta}_1\overline{\zeta}_{\kappa_0} \\ -\overline{\zeta}_1\overline{\zeta}_{\kappa_0+1} \\ \vdots \\ -\overline{\zeta}_1\overline{\zeta}_{n-1} \end{pmatrix}, \quad D_2 = \begin{pmatrix} -2\mu_2\overline{\zeta}_2^2 \\ -(\mu_2+\mu_3)\overline{\zeta}_2\overline{\zeta}_3 \\ \vdots \\ -(\mu_2+\mu_{\kappa_0})\overline{\zeta}_2\overline{\zeta}_{\kappa_0} \\ -\mu_2\overline{\zeta}_2\overline{\zeta}_{\kappa_0+1} \\ \vdots \\ -\mu_2\overline{\zeta}_2\overline{\zeta}_{n-1} \end{pmatrix}, \ldots, D_{\kappa_0} = \begin{pmatrix} -2\mu_{\kappa_0}\overline{\zeta}_{\kappa_0}^2 \\ -\mu_{\kappa_0}\overline{\zeta}_{\kappa_0}\overline{\zeta}_{\kappa_0+1} \\ \vdots \\ -\mu_{\kappa_0}\overline{\zeta}_{\kappa_0}\overline{\zeta}_{n-1} \end{pmatrix}.$$

Notice that $\mathcal{L}_i\mathcal{L}_j(\phi_{st}(0,0)) = 2i\overline{\zeta}_i\frac{\partial^2\phi_{st}(0,0)}{\partial w\partial z_j} + 2i\overline{\zeta}_j\frac{\partial^2\phi_{st}(0,0)}{\partial w\partial z_i} + \frac{\partial^2\phi_{st}(0,0)}{\partial z_i\partial z_j}$ from (3). If we denote b_{ij}^k to be $\frac{\partial^2\phi_{ij}}{\partial w\partial z_k}|_0$, then we get

$$\begin{aligned} &\mathcal{L}_i\mathcal{L}_i(\phi_{ii}) = 2\sqrt{\mu_i} + 4i\overline{\zeta}_i b_{ii}^i; \quad \mathcal{L}_i\mathcal{L}_i(\phi_{ij}) = 4i\overline{\zeta}_i b_{ij}^i, \quad i \neq j; \\ &\mathcal{L}_i\mathcal{L}_j(\phi_{ij}) = \sqrt{\mu_i+\mu_j} + 2i\overline{\zeta}_i b_{ij}^j + 2i\overline{\zeta}_i b_{ij}^i, \text{for } \ i < j \leq \kappa_0; \\ &\mathcal{L}_i\mathcal{L}_j(\phi_{st}) = 2i\overline{\zeta}_i b_{st}^j + 2i\overline{\zeta}_i b_{st}^i, \text{for } \ (ij) \neq (st), \quad i < j \leq \kappa_0; \\ &\mathcal{L}_i\mathcal{L}_j(\phi_{ij}) = \sqrt{\mu_i+\mu_j} + 2i\overline{\zeta}_j b_{ij}^i, \text{for } \ j > \kappa_0; \\ &\mathcal{L}_i\mathcal{L}_j(\phi_{st}) = 2i\overline{\zeta}_j b_{st}^i, \text{for } \ j > \kappa_0. \end{aligned}$$

Recalling $\mu_1 = 1, \mu_j = 0$ for $j > \kappa_0$, we can write $B = D + \tilde{B}$ with

$$D = \begin{pmatrix} 2 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1+\mu_2} & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sqrt{1+\mu_{\kappa_0}} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 2\sqrt{\mu_2} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \sqrt{\mu_2+\mu_3} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \sqrt{\mu_{\kappa_0}} \end{pmatrix},$$

$$\tilde{B} = \begin{pmatrix} E_1 \\ \vdots \\ E_{\kappa_0} \end{pmatrix} \begin{pmatrix} b_{11}^1 & b_{12}^1 & \cdots & b_{1(n-1)}^1 & b_{22}^1 & \cdots & b_{\kappa_0(n-1)}^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{11}^{\kappa_0} & b_{12}^{\kappa_0} & \cdots & b_{1(n-1)}^{\kappa_0} & b_{22}^{\kappa_0} & \cdots & b_{\kappa_0(n-1)}^{\kappa_0} \end{pmatrix},$$

where

$$E_1 = \begin{pmatrix} 4i\overline{\zeta}_1 & 0 & 0 & \cdots & 0 \\ 2i\overline{\zeta}_2 & 2i\overline{\zeta}_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2i\overline{\zeta}_{\kappa_0} & 0 & 0 & \cdots & 2i\overline{\zeta}_1 \\ 2i\overline{\zeta}_{\kappa_0+1} & 0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2i\overline{\zeta}_{n-1} & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 4i\overline{\zeta}_2 & 0 & \cdots & 0 \\ 0 & 2i\overline{\zeta}_3 & 2i\overline{\zeta}_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 2i\overline{\zeta}_{\kappa_0} & 0 & \cdots & 2i\overline{\zeta}_2 \\ 0 & 2i\overline{\zeta}_{\kappa_0+1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 2i\overline{\zeta}_{n-1} & 0 & \cdots & 0 \end{pmatrix},$$

$$\ldots, \quad E_{\kappa_0} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 4i\overline{\zeta}_{\kappa_0} \\ 0 & 0 & 0 & \cdots & 2i\overline{\zeta}_{\kappa_0+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2i\overline{\zeta}_{n-1} \end{pmatrix}.$$

Denote $B^* = D^{-1}\tilde{B} = 2i\vec{\zeta} \cdot \vec{b}$ where

$$D^{-1}\begin{pmatrix} E_1 \\ \vdots \\ E_{\kappa_0} \end{pmatrix} = 2i\vec{\zeta}, \quad \textit{and} \quad \vec{b} = \begin{pmatrix} b_{11}^1 & b_{12}^1 & \cdots & b_{1(n-1)}^1 & b_{22}^1 & \cdots & b_{\kappa_0(n-1)}^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{11}^{\kappa_0} & b_{12}^{\kappa_0} & \cdots & b_{1(n-1)}^{\kappa_0} & b_{22}^{\kappa_0} & \cdots & b_{\kappa_0(n-1)}^{\kappa_0} \end{pmatrix}.$$

Then $(B^*)^2 = (2i)^2\vec{\zeta} \cdot (\vec{b} \cdot \vec{\zeta}) \cdot \vec{b}, \cdots, (B^*)^k = (2i)^k\vec{\zeta} \cdot (\vec{b} \cdot \vec{\zeta})^{k-1} \cdot \vec{b}, \cdots$. Hence

$$\begin{aligned} &-B^{-1}A\overline{\zeta}^t = -(I+B^*)^{-1}D^{-1}A\overline{\zeta}^t = -(I + \textstyle\sum_{j=1}^{\infty}(-1)^j B^{*j})D^{-1}A\overline{\zeta}^t \\ &= -[I - 2i\vec{\zeta}\big(\textstyle\sum_{j=1}^{\infty}(-1)^{j-1}(2i)^{j-1}(\vec{b}\cdot\vec{\zeta})^{j-1}\big) \cdot \vec{b}]D^{-1}A\overline{\zeta}^t \\ &= -[I - 2i\vec{\zeta}\big(I + 2i\vec{b}\cdot\vec{\zeta}\big)^{-1} \cdot \vec{b}]D^{-1}A\overline{\zeta}^t. \end{aligned} \tag{8}$$

Notice that $\vec{b}\cdot\vec{\zeta}$ is a $\kappa_0\times\kappa_0$ matrix. Each entry of this matrix is a polynomial of $\overline{\zeta}$ with degree 1. We also notice that $D^{-1}A\overline{\zeta}^t$ is a vector of polynomial about $\overline{\zeta}$ with degree 2, we conclude $\deg\phi\le\kappa_0+2$. If we let $z=w=\eta=0$ in $\frac{g(z,w)-\overline{g(\xi,\eta)}}{2i}=\tilde{f}(z,w)\overline{\tilde{f}(\xi,\eta)}$, we get $g(\zeta,0)=0$. Hence $deg(F_p^{***})\le\kappa_0+2$ for any p in $\partial\mathbf{H}_n$ that is closed to 0. By Lemma 2.3, $deg(F)\le\kappa_0+2$. The proof of Theorem 1.1 is completed. □

4. Whitney Maps of Rank κ_0. Let $1\le\kappa_0\le n-1$, $N=n+\frac{(2n-\kappa_0-1)\kappa_0}{2}$. A map $W_{n,\kappa_0}=(\Gamma_1,\cdots,\Gamma_{\kappa_0+1})$ in $Rat(\mathbf{B}^n,\mathbf{B}^N)$ is called *Whitney map of rank* κ_0 if it is of the following form:

$$\begin{aligned}&\Gamma_1=(z_1^2,\sqrt{2}z_1z_2,\cdots,\sqrt{2}z_1z_{\kappa_0},z_1z_{\kappa_0+1},\cdots,z_1z_{n-1},z_1w),\\&\Gamma_2=(z_2^2,\sqrt{2}z_2z_3,\cdots,\sqrt{2}z_2z_{\kappa_0},z_2z_{\kappa_0+1},\cdots,z_2z_{n-1},z_2w),\\&\vdots\\&\Gamma_{\kappa_0}=(z_{\kappa_0}^2,z_{\kappa_0}z_{\kappa_0+1},\cdots,z_{\kappa_0}z_{n-1},z_{\kappa_0}w),\\&\Gamma_{\kappa_0+1}=(z_{\kappa_0+1},\cdots,z_{n-1},w).\end{aligned}\tag{9}$$

Notice that when $\kappa_0=1$, $\Gamma_1=(z_1^2,z_1z_2,...,z_1z_{n-1},z_1w)$ and $\Gamma_2=(z_2,\cdots,z_{n-1},w)$ give the classical Whitney map. By Cayley transformation, W_{n,κ_0} can be identified as a map in $Rat(\mathbf{H}_n,\mathbf{H}_N)$. As an example, $W_{4.2}\in Rat(\mathbf{H}_4,\mathbf{H}_9)$ is of the form

$$\begin{aligned}&f_1=\frac{z_1+\frac{i}{4}z_1w}{1-\frac{i}{4}w},\ f_2=\frac{z_2+\frac{i}{4}z_2w}{1\frac{i}{4}w},\ f_3=\frac{z_3-\frac{i}{4}z_3w}{1-\frac{i}{4}w},\\&\phi_{11}=\frac{z_1^2}{1-\frac{i}{4}w},\ \phi_{12}=\frac{\sqrt{2}z_1z_2}{1-\frac{i}{4}w},\ \phi_{13}=\frac{z_1z_3}{1-\frac{i}{4}w},\ \phi_{22}=\frac{z_2^2}{1-\frac{i}{4}w},\ \phi_{23}=\frac{z_2z_3}{1-\frac{i}{4}w},\ g=w.\end{aligned}\tag{10}$$

We want to prove a criterion for Whitney map which will be used to prove Theorem 1.2.

THEOREM 4.1. *Let $F\in Rat(\mathbf{H}_n,\mathbf{H}_N)$ with the geometric rank κ_0, $1\le\kappa_0\le n-2$, and with $N=n+\frac{(2n-\kappa_0-1)\kappa_0}{2}$. Then F is equivalent to W_{n,κ_0} if and only if $deg(F)=2$ and F is equivalent to another map $\widetilde{F}=(f,\phi,g)$ that satisfies (3) and $\frac{\partial^2\phi_{jl}}{\partial z_k\partial w}(0,0)=0$ for all j,l and k.*

Proof. It suffices to show that if F satisfies (3) and $\frac{\partial^2\phi_{jl}}{\partial z_k\partial w}(0,0)=0$ for all j,l and k, then F is equivalent to W_{n,κ_0}.

Step 1. Determine $F(z,0)$**.** As we did in §3, apply $\mathcal{L}_j,\mathcal{L}_k\mathcal{L}_j(k\le\kappa_0,\ k\le j)$ to

$$\frac{g(z,w)-\overline{g(\zeta,\eta)}}{2i}=f(z,w)\overline{f(\zeta,\eta)}+\phi(z,w)\overline{\phi(\zeta,\eta)},\tag{11}$$

for any $w-\eta=2i(z\cdot\zeta)$.

By (7), we have

$$\begin{pmatrix}\overline{f_1(\zeta,0)}\\ \vdots\\ \overline{f_{n-1}(\zeta,0)}\\ \overline{\phi(\zeta,0)}\end{pmatrix}=\begin{pmatrix}I&0\\-B^{-1}A&B^{-1}\end{pmatrix}\begin{pmatrix}\overline{\zeta_1}\\ \vdots\\ \overline{\zeta_{n-1}}\\ 0\end{pmatrix},$$

where A is as in Section 3, and

$$B = \begin{pmatrix} D_{(n-1)\times|S_0|} \\ D_{(n-2)\times|S_0|} \\ \vdots \\ D_{(n-\kappa_0)\times|S_0|} \end{pmatrix}$$

is an $|S_0| \times |S_0| = \frac{(2n-\kappa_0-1)\kappa_0}{2} \times \frac{(2n-\kappa_0-1)\kappa_0}{2}$ diagonal matrix with

$$D_{(n-1)\times|S_0|} = \begin{pmatrix} 2\mu_{11} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \mu_{12} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \mu_{1(n-1)} & 0 & \cdots & 0 \end{pmatrix},$$

$$\vdots$$

$$D_{(n-\kappa_0)\times|S_0|} = \begin{pmatrix} 0 & \cdots & 0 & 2\mu_{\kappa_0\kappa_0} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & \mu_{\kappa_0(\kappa_0+1)} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \mu_{\kappa_0(n-1)} \end{pmatrix}.$$

Hence $\overline{f(\zeta,0)} = \overline{\zeta}$, $\overline{\phi_{kl}(\zeta,0)} = \frac{\mu_k+\mu_l}{\mu_{kl}}\overline{\zeta_k\zeta_l}$ for $k \leq \kappa_0$ and $k < l < n$, $\overline{\phi_{kk}(\zeta,0)} = \frac{\mu_k}{\mu_{kk}}\overline{\zeta_k}^2$ for $k \leq \kappa_0$. Putting $(z,w) = 0$ and $\eta = 0$ in (11), we get $\overline{g(\zeta,0)} = 0$. Since $\mu_{kl} = \sqrt{\mu_k + \mu_l}$, $\mu_{kk} = \sqrt{\mu_k}$ for $k < l < n$ and $k \leq \kappa_0$ (see Theorem 2.2), from the above argument, we have proved the following.

$$\begin{aligned} & f(z,0) = z; \\ & \phi_{kl}(z,0) = \sqrt{\mu_k + \mu_l} z_k z_l, \quad 1 \leq k \leq k_0, \ k < l \leq n-1; \\ & \phi_{kk}(z,0) = \sqrt{\mu_k} z_k^2, \ 1 \leq k \leq \kappa_0; \\ & g(z,0) = 0. \end{aligned} \tag{12}$$

Step 2. Determine $F(z,w)$. We claim:

$$\begin{aligned} & f_j = \frac{z_j + (b+\frac{i}{4})z_j w}{1+(b-\frac{i}{4})w}, \ 1 \leq j \leq \kappa_0; \\ & f_j = z_j, \ \kappa_0 < j \leq n-1; \\ & \phi_{jl} = \frac{\sqrt{2} z_j z_l}{1+(b-\frac{i}{4})w}, \quad 1 \leq j < l \leq \kappa_0; \\ & \phi_{jl} = \frac{z_j z_l}{1+(b-\frac{i}{4})w}, \quad 1 \leq j \leq \kappa_0, \ \kappa_0 + 1 \leq l \leq n-1; \\ & \phi_{jj} = \frac{z_j^2}{1+(b-\frac{i}{4})w}, \ 1 \leq j \leq \kappa_0; \\ & g = w, \end{aligned} \tag{13}$$

where $b \in \mathbf{R}$ is a real number.

In fact, Since $deg(F) = 2$, by (12), we can write F in the form

$$
\begin{aligned}
f_j &= \frac{A_j^{(1)}(z)+A_j^{(2)}(z)+\widetilde{A_j^{(1)}(z)}w+A'_jw+A''_jw^2}{1+E^{(1)}(z)+E^{(2)}(z)+\widetilde{E^{(1)}(z)}w+e_1w+e_2w^2}, \quad 1\le j\le n-1;\\
\phi_{jl} &= \frac{B_{jl}^{(1)}(z)+B_{jl}^{(2)}(z)+\widetilde{B_{jl}^{(1)}(z)}w+B'_{jl}w+B''_{jl}w^2}{1+E^{(1)}(z)+E^{(2)}(z)+\widetilde{E^{(1)}(z)}w+e_1w+e_2w^2}, \quad (j,l)\in S_0;\\
g &= \frac{C^{(1)}(z)+C^{(2)}(z)+\widetilde{C^{(1)}(z)}w+C'w+C''w^2}{1+E^{(1)}(z)+E^{(2)}(z)+\widetilde{E^{(1)}(z)}w+e_1w+e_2w^2}.
\end{aligned} \tag{14}
$$

Here we use notation $h^{(k)}(z)$ to denote a homogeneous polynomial of z with total degree k. We write ϕ_{jl} as a Taylor series at 0 and compare the expression with (12). Then we get

$$B_{jl}^{(2)}(z)=\mu_{jl}z_jz_l,\ B_{jl}^{(1)}(z)=B'_{jl}=B''_{jl}=\widetilde{B_{jl}^{(1)}(z)}=0,\ \forall (j,l)\in S_0.$$

Applying (12) to $\phi_{jl}(z,0)$, we obtain $E^{(1)}(z)=E^{(2)}(z)=0$. Similarly, writing f_j as a Taylor series at 0 and compare it with (3), we have $A_j^{(1)}(z)=z_j$, $A'_j=A''_j=0$, $\widetilde{A_j^{(1)}(z)}=\frac{i}{2}\mu_jz_j+e_1z_j$ for $j\le\kappa_0$, and $\widetilde{A_j^{(1)}(z)}=e_1z_j$ for $\kappa_0+1\le j\le n-1$. By using (12) and the fact that $E^{(1)}(z)=E^{(2)}(z)=0$ to $f_j(z,0)$, we get $A_j^{(2)}(z)=0$.

For g, we similarly obtain $C'=1$, $C^{(1)}(z)=\widetilde{C^{(1)}(z)}=C^{(2)}(z)=0$. Since $deg(F)\le 2$, as the proof of [HJ 2001, Lemma 6.1], using the last two equations of (3), we get $g(z,w)\equiv w$. Therefore from (14), we find $C'=1$, $C''=e_1$, $\widetilde{E^{(1)}(z)}=0$ and $e_2=0$. Combining the above results, we get

$$
\begin{aligned}
&f_j=\frac{z_j+(\frac{i}{2}\mu_jz_j+e_1z_j)w}{1+e_1w},\ 1\le j\le\kappa_0;\ f_j=\frac{z_j+e_1z_jw}{1+e_1w},\ \kappa_0<j\le n-1;\\
&\phi_{jl}=\frac{\mu_{jl}z_jz_l}{1+e_1w},\ \forall (j,l)\in S_0;\ g=w.
\end{aligned} \tag{15}
$$

Since F maps $\partial\mathbf{H}_n$ into $\partial\mathbf{H}_\mathbf{N}$, we have $Im(g)=|f|^2+|\phi|^2$ on $\partial\mathbf{H}_n$. Notice $g(z,w)=w$, this equation can be written into

$$|z|^2=|f(z,w)|^2+|\phi(z,w)|^2,\quad \forall (z,w)\in\partial\mathbf{H}_n. \tag{16}$$

Replacing f,ϕ by the ones in (15), we can write (16) as

$$|z|^2|1+e_1w|^2=\sum_{j=1}^{\kappa_0}\left|z_j+(\frac{i}{2}\mu_jz_j+e_1z_j)w\right|^2+\sum_{j=\kappa_0+1}^{n-1}|z_j+e_1z_jw|^2+\sum_{(jl)\in S_0}|\mu_{jl}z_jz_l|^2$$

for any $(z,w)\in\partial\mathbf{H}_n$. Since $w=u+i|z|^2$, we obtain several equations:

$$|z|^2|1+e_1(u+i|z|^2)|^2=|z|^2\Big(1+\overline{e_1}u+e_1u+ie_1|z|^2-i\overline{e_1}|z|^2+|e_1|^2u^2+|e_1|^2|z|^4\Big),$$

$$
\begin{aligned}
&\sum_{j=1}^{\kappa_0}\left|z_j+(\frac{i}{2}\mu_jz_j+e_1z_j)(u+i|z|^2)\right|^2\\
&=\sum_{j=1}^{\kappa_0}|z_j|^2\Big(1+\overline{e_1}u+e_1u+ie_1|z|^2-i\overline{e_1}|z|^2+|e_1|^2u^2+|e_1|^2|z|^4-\mu_j|z|^2\Big)\\
&\quad+\sum_{j=1}^{\kappa_0}|z_j|^2\Big(\frac{1}{4}u^2\mu_j^2+\frac{i}{2}\mu_j\overline{e_1}u^2-\frac{i}{2}\mu_je_1u^2+\frac{1}{4}\mu_j^2|z|^4+\frac{i}{2}\mu_j\overline{e_1}|z|^4-\frac{i}{2}\mu_je_1|z|^4\Big),
\end{aligned}
$$

$$\sum_{j=\kappa_0+1}^{n-1} |z_j + e_1 z_j(u + i|z|^2)|^2$$
$$= \sum_{j=\kappa_0+1}^{n-1} |z_j|^2 \Big(1 + \overline{e_1}u + e_1 u + ie_1|z|^2 - i\overline{e_1}|z|^2 + |e_1|^2u^2 + |e_1|^2|z|^4\Big),$$

and

$$|\phi|^2 = \sum_{(j,l)\in S_0} \mu_{jl}^2 |z_j|^2|z_l|^2.$$

Substituting the above terms into (16), we get

$$0 = -\sum_{j=1}^{\kappa_0} \mu_j |z_j|^2|z|^2 + (u^2 + |z|^4)\Big(\frac{1}{4}\sum_{j=1}^{\kappa_0} |z_j|^2\mu_j^2 + \frac{i}{2}\overline{e_1}\sum_{j=1}^{\kappa_0}\mu_j|z_j|^2 - \frac{i}{2}e_1\sum_{j=1}^{\kappa_0}\mu_j|z_j|^2\Big)$$
$$+ \sum_{(jl)\in S_0} \mu_{j,l}^2|z_j|^2|z_l|^2, \quad \forall z \in \mathbf{C}^{n-1} \text{ and } u \in \mathbf{R}.$$

Since z, u are independent variables,

$$\frac{1}{4}\sum_{j=1}^{\kappa_0} |z_j|^2\mu_j^2 + \frac{i}{2}\overline{e_1}\sum_{j=1}^{\kappa_0}\mu_j|z_j|^2 - \frac{i}{2}e_1\sum_{j=1}^{\kappa_0}\mu_j|z_j|^2 \equiv 0.$$

This means

$$\sum_{j=1}^{\kappa_0}\Big(\frac{1}{4}\mu_j^2 + \frac{i}{2}\mu_j\overline{e_1} - \frac{i}{2}\mu_j e_1\Big)|z_j|^2 \equiv 0.$$

Therefore $\frac{1}{4}\mu_j^2 + \frac{i}{2}\mu_j\overline{e_1} - \frac{i}{2}\mu_j e_1 = 0$, $\forall j = 1, \cdots, \kappa_0$. Since $\mu_1 = 1$ (see Theorem 2.2), this implies $Im(e_1) = -\frac{1}{4}$ and $1 = \mu_1 = \mu_2 \cdots = \mu_{\kappa_0}$. Our claim (13) is proved.

Step 3. ***F* is equivalent to Whitney Map.** Let F be of the form (13). $F = (f, \phi, g)$ is equivalent to

$$F' = \Big(\lambda f(\frac{z}{\lambda}, \frac{w}{\lambda^2}), \lambda\phi(\frac{z}{\lambda}, \frac{w}{\lambda^2}), \lambda^2 g(\frac{z}{\lambda}, \frac{w}{\lambda^2})\Big).$$

Let $\lambda = \frac{1}{2}$. By permutating the components of F', we may assume that F' is of the following form $F'(z, w) = (\psi_1, \cdots, \psi_{\kappa_0+1})$, where

$$\psi_1 = \Big(\frac{2z_1^2}{1+(4b-i)w}, \frac{2\sqrt{2}z_1z_2}{1+(4b-i)w}, \cdots, \frac{2\sqrt{2}z_1z_{\kappa_0}}{1+(4b-i)w}, \frac{2z_1z_{\kappa_0+1}}{1+(4b-i)w}, \cdots,$$
$$\frac{2z_1z_{n-1}}{1+(4b-i)w}, \frac{z_1+(4b+i)z_1w}{1+(4b-i)w}\Big),$$
$$\psi_2 = \Big(\frac{2z_2^2}{1+(4b-i)w}, \frac{2\sqrt{2}z_2z_3}{1+(4b-i)w}, \cdots, \frac{2\sqrt{2}z_2z_{\kappa_0}}{1+(4b-i)w}, \frac{2z_2z_{\kappa_0+1}}{1+(4b-i)w}, \cdots,$$
$$\frac{2z_2z_{n-1}}{1+(4b-i)w}, \frac{z_2+(4b+i)z_2w}{1+(4b-i)w}\Big),$$

$$\vdots$$

$$\psi_{\kappa_0}=\left(\frac{2z_{\kappa_0}^2}{1+(4b-i)w},\frac{2z_{\kappa_0}z_{\kappa_0+1}}{1+(4b-i)w},\cdots,\frac{2z_{\kappa_0}z_{n-1}}{1+(4b-i)w},\frac{z_{\kappa_0}+(4b+i)z_{\kappa_0}w}{1+(4b-i)w}\right),$$
$$\psi_{\kappa_0+1}=(z_{\kappa_0+1},\cdots,z_{n-1},w).$$

Using the Cayley transformations, F' induces a proper holomorphic mapping $\widetilde{F}=\rho_N\circ F'\circ\rho_n^{-1}$ in $Rat(\mathbf{B}^n,\mathbf{B}^n)$ given by $\widetilde{F}=(\widetilde{\psi_1},\cdots,\widetilde{\psi_{\kappa_0+1}})$, where

$$\widetilde{\psi_1}=\Bigg(\frac{z_1^2}{1+2bi-2biw},\frac{\sqrt{2}z_1z_2}{1+2bi-2biw},\cdots,\frac{\sqrt{2}z_1z_{\kappa_0}}{1+2bi-2biw},\frac{z_1z_{\kappa_0+1}}{1+2bi-2biw},\cdots,$$
$$\frac{z_1z_{n-1}}{1+2bi-2biw},\frac{z_1w(1-2bi)+2biz_1}{1+2bi-2biw}\Bigg),$$
$$\widetilde{\psi_2}=\Bigg(\frac{z_2^2}{1+2bi-2biw},\frac{\sqrt{2}z_2z_3}{1+2bi-2biw},\cdots,\frac{\sqrt{2}z_2z_{\kappa_0}}{1+2bi-2biw},\frac{z_2z_{\kappa_0+1}}{1+2bi-2biw},\cdots,$$
$$\frac{z_2z_{n-1}}{1+2bi-2biw},\frac{z_2w(1-2bi)+2biz_2}{1+2bi-2biw}\Bigg),$$

$$\vdots$$

$$\widetilde{\psi_{\kappa_0}}=\left(\frac{z_{\kappa_0}^2}{1+2bi-2biw},\frac{z_{\kappa_0}z_{\kappa_0+1}}{1+2bi-2biw},\cdots,\frac{z_{\kappa_0}z_{n-1}}{1+2bi-2biw},\frac{z_{\kappa_0}w(1-2bi)+2biz_{\kappa_0}}{1+2bi-2biw}\right),$$
$$\widetilde{\psi_{\kappa_0+1}}=(z_{\kappa_0+1},\cdots,z_{n-1},w).$$

Consider

$$\sigma(z,w)=\left(z,\frac{i(1+2ib)}{\sqrt{1+4b^2}}w\right),\ and\ \tau(z^*,w^*)=\left(\frac{1+2ib}{\sqrt{1+4b^2}}z^*,\frac{(1+2ib)i}{\sqrt{1+4b^2}}w^*\right),$$

which are elements in $Aut(\mathbf{B}^n)$ and $Aut(\mathbf{B}^N)$ respectively. By definition, $\widetilde{F}$ is equivalent to $\widetilde{F}'=\tau\circ\widetilde{F}\circ\sigma$. Replacing b by $-b$, as before, we write $\widetilde{F}'=(\widetilde{\psi_1}',\cdots,\widetilde{\psi_{\kappa_0+1}}')$ where

$$\widetilde{\psi_1}'=\Bigg(\frac{z_1^2}{\sqrt{1+4b^2}(1-\frac{2b}{\sqrt{1+4b^2}}w)},\frac{\sqrt{2}z_1z_2}{\sqrt{1+4b^2}(1-\frac{2b}{\sqrt{1+4b^2}}w)},\cdots,\frac{\sqrt{2}z_1z_{\kappa_0}}{\sqrt{1+4b^2}(1-\frac{2b}{\sqrt{1+4b^2}}w)},$$
$$\frac{z_1z_{\kappa_0+1}}{\sqrt{1+4b^2}(1-\frac{2b}{\sqrt{1+4b^2}}w)},\cdots,\frac{z_1z_{n-1}}{\sqrt{1+4b^2}(1-\frac{2b}{\sqrt{1+4b^2}}w)},-iz_1\frac{\frac{2b}{\sqrt{1+4b^2}}-w}{1-\frac{2b}{\sqrt{1+4b^2}}w}\Bigg),$$

$$\widetilde{\psi_2}'=\Bigg(\frac{z_2^2}{\sqrt{1+4b^2}(1-\frac{2b}{\sqrt{1+4b^2}}w)},\frac{\sqrt{2}z_2z_3}{\sqrt{1+4b^2}(1-\frac{2b}{\sqrt{1+4b^2}}w)},\cdots,\frac{\sqrt{2}z_2z_{\kappa_0}}{\sqrt{1+4b^2}(1-\frac{2b}{\sqrt{1+4b^2}}w)},$$
$$\frac{z_2z_{\kappa_0+1}}{\sqrt{1+4b^2}(1-\frac{2b}{\sqrt{1+4b^2}}w)},\cdots,\frac{z_2z_{n-1}}{\sqrt{1+4b^2}(1-\frac{2b}{\sqrt{1+4b^2}}w)},-iz_2\frac{\frac{2b}{\sqrt{1+4b^2}}-w}{1-\frac{2b}{\sqrt{1+4b^2}}w}\Bigg),$$

$$\vdots$$

$$\widetilde{\psi_{\kappa_0}}' = \Big(\frac{z_{\kappa_0}^2}{\sqrt{1+4b^2}(1-\frac{2b}{\sqrt{1+4b^2}}w)}, \frac{z_{\kappa_0}z_{\kappa_0+1}}{\sqrt{1+4b^2}(1-\frac{2b}{\sqrt{1+4b^2}}w)}, \cdots, \frac{z_{\kappa_0}z_{n-1}}{\sqrt{1+4b^2}(1-\frac{2b}{\sqrt{1+4b^2}}w)}, -iz_{\kappa_0}\frac{\frac{2b}{\sqrt{1+4b^2}}-w}{1-\frac{2b}{\sqrt{1+4b^2}}w}\Big),$$

$$\widetilde{\psi_{\kappa_0+1}}' = \Big(\frac{1-2ib}{\sqrt{1+4b^2}}z_{\kappa_0+1}, \cdots, \frac{1-2ib}{\sqrt{1+4b^2}}z_{n-1}, -\Big(\frac{1-2ib}{\sqrt{1+4b^2}}\Big)^2 w\Big).$$

Define $\beta = \frac{2b}{\sqrt{1+4b^2}}$ and $S_\beta = \sqrt{1-\beta^2}$. Noticing $|\frac{1-2ib}{\sqrt{1+4b^2}}| = 1$, we multiply i to the last components of $\widetilde{\psi_1}', \cdots, \widetilde{\psi_{\kappa_0}}'$, $-\Big(\frac{\sqrt{1+4b^2}}{1-2ib}\Big)^2$ to the last component of $\widetilde{\psi_{\kappa_0+1}}'$ and $\frac{\sqrt{1+4b^2}}{1-2ib}$ to the other components of $\widetilde{\psi_{\kappa_0+1}}'$, $\widetilde{F}'$ is equivalent to a new map, still denote it as $\widetilde{F}' = (\widetilde{\psi_1}', \cdots, \widetilde{\psi_{\kappa_0+1}}')$:

$$\begin{aligned}
\widetilde{\psi_1}' &= \Big(\tfrac{S_\beta z_1^2}{1-\beta w}, \tfrac{\sqrt{2}S_\beta z_1 z_2}{1-\beta w}, \cdots, \tfrac{\sqrt{2}S_\beta z_1 z_{\kappa_0}}{1-\beta w}, \tfrac{S_\beta z_1 z_{\kappa_0+1}}{1-\beta w}, \cdots, \tfrac{S_\beta z_1 z_{n-1}}{1-\beta w}, z_1\tfrac{\beta-w}{1-\beta w}\Big),\\
\widetilde{\psi_2}' &= \Big(\tfrac{S_\beta z_2^2}{1-\beta w}, \tfrac{\sqrt{2}S_\beta z_2 z_3}{1-\beta w}, \cdots, \tfrac{\sqrt{2}S_\beta z_2 z_{\kappa_0}}{1-\beta w}, \tfrac{S_\beta z_2 z_{\kappa_0+1}}{1-\beta w}, \cdots, \tfrac{S_\beta z_2 z_{n-1}}{1-\beta w}, z_2\tfrac{\beta-w}{1-\beta w}\Big),\\
&\vdots\\
\widetilde{\psi_{\kappa_0}}' &= \Big(\tfrac{S_\beta z_{\kappa_0}^2}{1-\beta w}, \tfrac{S_\beta z_{\kappa_0} z_{\kappa_0+1}}{1-\beta w}, \cdots, \tfrac{S_\beta z_{\kappa_0} z_{n-1}}{1-\beta w}, z_{\kappa_0}\tfrac{\beta-w}{1-\beta w}\Big),\\
\widetilde{\psi_{\kappa_0+1}}' &= (z_{\kappa_0+1}, \cdots, z_{n-1}, w).
\end{aligned} \tag{17}$$

On the other hand, for any $a \in R$, we define

$$\varphi_a := \Big(\frac{S_a z}{1-aw}, \frac{a-w}{1-aw}\Big), \quad S_a = \sqrt{1-a^2},$$

which is an automorphism of $\mathbf{B}^n$. Similarly, we can define $\varphi_a^* \in Aut(\mathbf{B}^N)$. For Whitney map $W_{n,\kappa_0} = (\Gamma_1, \cdots, \Gamma_{\kappa_0+1})$ from $\mathbf{B}^n$ to $\mathbf{B}^N$ defined in (9), we see that $\varphi_a^* \circ W_{n,\kappa_0} \circ \varphi_a$ has the following form

$$\begin{aligned}
\widetilde{\Gamma_1} &= \Big(\tfrac{S_a z_1^2}{1-aw}, \tfrac{\sqrt{2}S_a z_1 z_2}{1-aw}, \cdots, \tfrac{\sqrt{2}S_a z_1 z_{\kappa_0}}{1-aw}, \tfrac{S_a z_1 z_{\kappa_0+1}}{1-aw}, \cdots, \tfrac{S_a z_1 z_{n-1}}{1-aw}, z_1\tfrac{a-w}{1-aw}\Big),\\
\widetilde{\Gamma_2} &= \Big(\tfrac{S_a z_2^2}{1-aw}, \tfrac{\sqrt{2}S_a z_2 z_3}{1-aw}, \cdots, \tfrac{\sqrt{2}S_a z_2 z_{\kappa_0}}{1-aw}, \tfrac{S_a z_2 z_{\kappa_0+1}}{1-aw}, \cdots, \tfrac{S_a z_2 z_{n-1}}{1-aw}, z_2\tfrac{a-w}{1-aw}\Big),\\
&\vdots\\
\widetilde{\Gamma_{\kappa_0}} &= \Big(\tfrac{S_a z_{\kappa_0}^2}{1-aw}, \tfrac{S_a z_{\kappa_0} z_{\kappa_0+1}}{1-aw}, \cdots, \tfrac{S_a z_{\kappa_0} z_{n-1}}{1-aw}, z_{\kappa_0}\tfrac{a-w}{1-aw}\Big),\\
\widetilde{\Gamma_{\kappa_0+1}} &= (z_{\kappa_0+1}, \cdots, z_{n-1}, w).
\end{aligned} \tag{18}$$

Comparing (17) with (18), if we put $a = \frac{2b}{\sqrt{1+4b^2}}$, $\widetilde{F}' = \varphi_a^* \circ W_{n,\kappa_0} \circ \varphi_a$ and this proves Theorem 4.1. □

5. Determining $F(z,0)$**.** From now on, we always consider $F \in Rat(\partial\mathbf{H}_4, \partial\mathbf{H}_9)$ with geometric rank 2 and degree 2 as in Theorem 1.2. In order to determine $F(z,w)$, we need to determine $F(z,0)$ first. Let us denote $\phi_{jl}(z,w) = \sum_{u,v,s,t} b^{(jl)}_{uvst} z_1^u z_2^v z_3^s w^t$.

LEMMA 5.1. *Let* $F \in Rat(\partial\mathbf{H}_4, \partial\mathbf{H}_9)$ *satisfying (3) with* $\kappa_0 = 2$ *and* $deg(F) = 2$. *Then*

$$
\begin{aligned}
&f_1(z,0) = z_1, \ f_2(z,0) = z_2, \quad f_3(z,0) = z_3,\\
&\phi_{11}(z,0) = \frac{z_1^2}{1 - 2i\overline{b^{(11)}_{1001}}z_1 - 2i\sqrt{1+\mu_2}\overline{b^{(12)}_{1001}}z_2 - 2i\overline{b^{(13)}_{1001}}z_3},\\
&\phi_{12}(z,0) = \frac{\sqrt{1+\mu_2}z_1z_2}{1 - 2i\overline{b^{(11)}_{1001}}z_1 - 2i\sqrt{1+\mu_2}\overline{b^{(12)}_{1001}}z_2 - 2i\overline{b^{(13)}_{1001}}z_3},\\
&\phi_{13}(z,0) = \frac{z_1z_3}{1 - 2i\overline{b^{(11)}_{1001}}z_1 - 2i\sqrt{1+\mu_2}\overline{b^{(12)}_{1001}}z_2 - 2i\overline{b^{(13)}_{1001}}z_3},\\
&\phi_{22}(z,0) = \frac{\sqrt{\mu_2}z_2^2}{1 - 2i\overline{b^{(11)}_{1001}}z_1 - 2i\sqrt{1+\mu_2}\overline{b^{(12)}_{1001}}z_2 - 2i\overline{b^{(13)}_{1001}}z_3},\\
&\phi_{23}(z,0) = \frac{\sqrt{\mu_2}z_2z_3}{1 - 2i\overline{b^{(11)}_{1001}}z_1 - 2i\sqrt{1+\mu_2}\overline{b^{(12)}_{1001}}z_2 - 2i\overline{b^{(13)}_{1001}}z_3},
\end{aligned}
$$

and $b^{(22)}_{1001} = b^{(23)}_{1001} = b^{(11)}_{0101} = b^{(13)}_{0101} = 0$, $b^{(12)}_{0101} = \frac{\mu_2}{\sqrt{1+\mu_2}} b^{(11)}_{1001}$, $b^{(23)}_{0101} = \sqrt{\mu_2} b^{(13)}_{1001}$, $b^{(22)}_{0101} = \sqrt{\mu_2(1+\mu_2)} b^{(12)}_{1001}$, *where* $\mu_2 \geq 1$.

Proof. As we did in §3 and by the same notation, we have

$$
A = \begin{pmatrix} -2\overline{\zeta}_1 & 0 & 0 \\ -\overline{\zeta}_2 & -\mu_2\overline{\zeta}_1 & 0 \\ -\overline{\zeta}_3 & 0 & 0 \\ 0 & -2\mu_2\overline{\zeta}_2 & 0 \\ 0 & -\mu_2\overline{\zeta}_3 & 0 \end{pmatrix}, \quad \text{and} \quad A\overline{\zeta}^t = \begin{pmatrix} -2\overline{\zeta_1}^2 \\ -(1+\mu_2)\overline{\zeta_1\zeta_2} \\ -\overline{\zeta_1\zeta_3} \\ -2\mu_2\overline{\zeta_2}^2 \\ -\mu_2\overline{\zeta_2\zeta_3} \end{pmatrix}.
$$

We can write the 5×5 matrix $B = D + \widetilde{B}$ where

$$
D = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1+\mu_2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{\mu_2} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\mu_2} \end{pmatrix},
$$

$$
\widetilde{B} = \begin{pmatrix} 4i\overline{\zeta_1} & 0 \\ 2i\overline{\zeta_2} & 2i\overline{\zeta_1} \\ 2i\overline{\zeta_3} & 0 \\ 0 & 4i\overline{\zeta_2} \\ 0 & 2i\overline{\zeta_3} \end{pmatrix} \begin{pmatrix} b^{(11)}_{1001} & b^{(12)}_{1001} & b^{(13)}_{1001} & b^{(22)}_{1001} & b^{(23)}_{1001} \\ b^{(11)}_{0101} & b^{(12)}_{0101} & b^{(13)}_{0101} & b^{(22)}_{0101} & b^{(23)}_{0101} \end{pmatrix}.
$$

By (8), we have $-B^{-1}A\overline{\zeta}^t = -[I - 2i\vec{\zeta}\big(I + 2i\vec{b}\cdot\vec{\zeta}\big)^{-1}\cdot\vec{b}]D^{-1}A\overline{\zeta}^t$. Writing

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} := 2i\vec{b}\cdot\vec{\zeta}$$

$$= 2i\begin{pmatrix} b^{(11)}_{1001} & b^{(12)}_{1001} & b^{(13)}_{1001} & b^{(22)}_{1001} & b^{(23)}_{1001} \\ b^{(11)}_{0101} & b^{(12)}_{0101} & b^{(13)}_{0101} & b^{(22)}_{0101} & b^{(23)}_{0101} \end{pmatrix}\begin{pmatrix} \overline{\zeta_1} & 0 \\ \frac{1}{\sqrt{1+\mu_2}}\overline{\zeta_2} & \frac{1}{\sqrt{1+\mu_2}}\overline{\zeta_1} \\ \overline{\zeta_3} & 0 \\ 0 & \frac{1}{\sqrt{\mu_2}}\overline{\zeta_2} \\ 0 & \frac{1}{\sqrt{\mu_2}}\overline{\zeta_3} \end{pmatrix}$$

$$= 2i\begin{pmatrix} b^{(11)}_{1001}\overline{\zeta_1} + \frac{b^{(12)}_{1001}}{\sqrt{1+\mu_2}}\overline{\zeta_2} + b^{(13)}_{1001}\overline{\zeta_3} & \frac{b^{(12)}_{1001}}{\sqrt{1+\mu_2}}\overline{\zeta_1} + \frac{b^{(22)}_{1001}}{\sqrt{\mu_2}}\overline{\zeta_2} + \frac{b^{(23)}_{1001}}{\sqrt{\mu_2}}\overline{\zeta_3} \\ b^{(11)}_{0101}\overline{\zeta_1} + \frac{b^{(12)}_{0101}}{\sqrt{1+\mu_2}}\overline{\zeta_2} + b^{(13)}_{0101}\overline{\zeta_3} & \frac{b^{(12)}_{0101}}{\sqrt{1+\mu_2}}\overline{\zeta_1} + \frac{b^{(22)}_{0101}}{\sqrt{\mu_2}}\overline{\zeta_2} + \frac{b^{(23)}_{0101}}{\sqrt{\mu_2}}\overline{\zeta_3} \end{pmatrix},$$

we have

$$-B^{-1}\cdot A\cdot\overline{\zeta} = \left[I - 2i\frac{\vec{\zeta}\cdot\begin{pmatrix} 1+C_{22} & -C_{12} \\ -C_{21} & 1+C_{11} \end{pmatrix}\cdot\vec{b}}{det\begin{pmatrix} 1+C_{11} & C_{12} \\ C_{21} & 1+C_{22} \end{pmatrix}}\right]\begin{pmatrix} \overline{\zeta_1}^2 \\ \sqrt{1+\mu_2}\overline{\zeta_1\zeta_2} \\ \overline{\zeta_1\zeta_3} \\ \sqrt{\mu_2}\overline{\zeta_2}^2 \\ \sqrt{\mu_2}\overline{\zeta_2\zeta_3} \end{pmatrix}.$$

Denoting by Δ the determinant $det\begin{pmatrix} 1+C_{11} & C_{12} \\ C_{21} & 1+C_{22} \end{pmatrix}$ and by (7), by direct computation, we obtain

$$\Delta(\overline{\zeta},0)\overline{\phi_{11}(\zeta,0)} = \overline{\zeta_1}^2 + \frac{2ib^{(12)}_{0101}}{\sqrt{1+\mu_2}}\overline{\zeta_1}^3 + \left(\frac{2ib^{(12)}_{1001}}{\sqrt{1+\mu_2}} + \frac{2ib^{(22)}_{0101}}{\sqrt{\mu_2}} - 2i\sqrt{1+\mu_2}b^{(12)}_{1001}\right)\overline{\zeta_1}^2\overline{\zeta_2}$$
$$+\frac{2ib^{(23)}_{0101}}{\sqrt{\mu_2}}\overline{\zeta_1}^2\overline{\zeta_3} - 2i\sqrt{\mu_2}b^{(22)}_{1001}\overline{\zeta_1\zeta_2}^2 - 2i\sqrt{\mu_2}b^{(23)}_{1001}\overline{\zeta_1\zeta_2\zeta_3}. \tag{19}$$

Since F is of degree 2, the numerator of $\overline{\phi_{11}(\zeta,0)}$ must be $\overline{\zeta}_1^2$ by (3) so that from (19) we get $b^{(22)}_{1001} = b^{(23)}_{1001} = 0$ and then

$$\phi_{11} = \frac{\overline{\zeta_1}^2}{\Delta}\left[1 + \frac{2ib^{(12)}_{0101}}{\sqrt{1+\mu_2}}\overline{\zeta_1} + \left(\frac{2ib^{(12)}_{1001}}{\sqrt{1+\mu_2}} + \frac{2ib^{(22)}_{0101}}{\sqrt{\mu_2}} - 2i\sqrt{1+\mu_2}b^{(12)}_{1001}\right)\overline{\zeta_2} + \frac{2ib^{(23)}_{0101}}{\sqrt{\mu_2}}\overline{\zeta_3}\right]. \tag{20}$$

Similarly we calculate ϕ_{22} to obtain $b^{(11)}_{0101} = b^{(13)}_{0101} = 0$ and at the value $(\zeta,0)$

$$\overline{\phi_{22}} = \frac{\sqrt{\mu_2}\overline{\zeta_2}^2}{\Delta}\left[1 + 2i\left(b^{(11)}_{1001} - \frac{1}{\mu_2\sqrt{1+\mu_2}}b^{(12)}_{0101}\right)\overline{\zeta_1} + \frac{2ib^{(12)}_{1001}}{\sqrt{1+\mu_2}}\overline{\zeta_2} + 2ib^{(13)}_{1001}\overline{\zeta_3}\right], \tag{21}$$

$$\overline{\phi_{13}} = \frac{\overline{\zeta_1\zeta_3}}{\Delta}\left[1 + \frac{2ib^{(12)}_{0101}}{\sqrt{1+\mu_2}}\overline{\zeta_1} + \left(\frac{2ib^{(12)}_{1001}}{\sqrt{1+\mu_2}} + \frac{2ib^{(22)}_{0101}}{\sqrt{\mu_2}} - 2i\sqrt{1+\mu_2}b^{(12)}_{1001}\right)\overline{\zeta_2} + \frac{2ib^{(23)}_{0101}}{\sqrt{\mu_2}}\overline{\zeta_3}\right], \tag{22}$$

$$\overline{\phi_{23}} = \frac{\sqrt{\mu_2}\overline{\zeta_2\zeta_3}}{\Delta}\left[1 + 2i\left(b^{(11)}_{1001} - \frac{1}{\mu_2\sqrt{1+\mu_2}}b^{(12)}_{0101}\right)\overline{\zeta_1} + \frac{2ib^{(12)}_{1001}}{\sqrt{1+\mu_2}}\overline{\zeta_2} + 2ib^{(13)}_{1001}\overline{\zeta_3}\right], \tag{23}$$

$$\phi_{12}=\frac{\sqrt{1+\mu_2}\overline{\zeta_1\zeta_2}}{\Delta}\Big[1+\frac{2i\mu_2}{1+\mu_2}b^{(11)}_{1001}\overline{\zeta_1}+\frac{2ib^{(22)}_{0101}}{\sqrt{\mu_2}(1+\mu_2)}\overline{\zeta_2}+2i\Big(\frac{\mu_2 b^{(13)}_{1001}}{1+\mu_2}+\frac{\sqrt{\mu_2}b^{(23)}_{0101}}{\mu_2(1+\mu_2)}\Big)\overline{\zeta_3}\Big]. \tag{24}$$

With the fact that $b^{(22)}_{1001}=b^{(23)}_{1001}=b^{(11)}_{0101}=b^{(13)}_{0101}=0$, we also calculate

$$\begin{aligned}\Delta(\overline{\zeta},0)= & det(1+2i\vec{b}\cdot\vec{\zeta})=1+C_{11}+C_{22}+C_{11}C_{22}-C_{12}C_{21}\\ = & \ 1+\overline{\zeta_1}\Big(2ib^{(11)}_{1001}+2i\frac{b^{(12)}_{0101}}{\sqrt{1+\mu_2}}\Big)+\overline{\zeta_2}\Big(2i\frac{b^{(12)}_{1001}}{\sqrt{1+\mu_2}}+2i\frac{b^{(22)}_{0101}}{\sqrt{\mu_2}}\Big)\\ + & \ \overline{\zeta_3}\Big(2ib^{(13)}_{1001}+2i\frac{b^{(23)}_{0101}}{\sqrt{\mu_2}}\Big)\\ - & \ 4\overline{\zeta_1}^2\frac{b^{(11)}_{1001}b^{(12)}_{0101}}{\sqrt{1+\mu_2}}-4\overline{\zeta_2}^2\frac{b^{(12)}_{1001}b^{(22)}_{0101}}{\sqrt{1+\mu_2}\sqrt{\mu_2}}-4\overline{\zeta_3}^2\frac{b^{(13)}_{1001}b^{(23)}_{0101}}{\sqrt{\mu_2}}-4\overline{\zeta_1\zeta_2}\frac{b^{(11)}_{1001}b^{(22)}_{0101}}{\sqrt{\mu_2}}\\ + & \ \overline{\zeta_1\zeta_3}\Big(-4\frac{b^{(11)}_{1001}b^{(23)}_{0101}}{\sqrt{\mu_2}}-4\frac{b^{(13)}_{1001}b^{(12)}_{0101}}{\sqrt{1+\mu_2}}\Big)+\overline{\zeta_2\zeta_3}\Big(-4\frac{b^{(12)}_{1001}b^{(23)}_{0101}}{\sqrt{(1+\mu_2)\mu_2}}-4\frac{b^{(13)}_{1001}b^{(22)}_{0101}}{\sqrt{\mu_2}}\Big).\end{aligned}$$

The factors in the right hand sides of (20)(21)...(24) must be the same, and it also must be a factor of $\Delta(\overline{\zeta},0)$. Then Lemma 5.1 follows immediately. □

6. Determining $F(z,w)$.

THEOREM 6.1. *Let $F:\partial\mathbf{H}_4\to\partial\mathbf{H_9}$ satisfies (3) with $\kappa_0=2$ and $deg(F)=2$. Then F must be of the form:*

$$f_1=\frac{z_1-2i\overline{b^{(11)}_{1001}}z_1^2-2i\overline{b^{(13)}_{1001}}z_1z_3+(E_{0001}+\frac{i}{2})z_1w}{1-2i\overline{b^{(11)}_{1001}}z_1-2i\overline{b^{(13)}_{1001}}z_3+E_{0001}w},$$

$$f_2=\frac{z_2-2i\overline{b^{(11)}_{1001}}z_1z_2-2i\overline{b^{(13)}_{1001}}z_2z_3+(E_{0001}+\frac{i\mu_2}{2})z_2w}{1-2i\overline{b^{(11)}_{1001}}z_1-2i\overline{b^{(13)}_{1001}}z_3+E_{0001}w},$$

$$f_3=\frac{z_3-2i\overline{b^{(13)}_{1001}}z_3^2-2i\overline{b^{(11)}_{1001}}z_1z_3+E_{0001}z_3w}{1-2i\overline{b^{(11)}_{1001}}z_1-2i\overline{b^{(13)}_{1001}}z_3+E_{0001}w},$$

$$\phi_{11}=\frac{z_1^2+b^{(11)}_{1001}z_1w}{1-2i\overline{b^{(11)}_{1001}}z_1-2i\overline{b^{(13)}_{1001}}z_3+E_{0001}w},\ \phi_{12}=\frac{\sqrt{1+\mu_2}z_1z_2+\frac{\mu_2}{\sqrt{1+\mu_2}}b^{(11)}_{1001}z_2w}{1-2i\overline{b^{(11)}_{1001}}z_1-2i\overline{b^{(13)}_{1001}}z_3+E_{0001}w},$$

$$\phi_{13}=\frac{z_1z_3+b^{(13)}_{1001}z_1w}{1-2i\overline{b^{(11)}_{1001}}z_1-2i\overline{b^{(13)}_{1001}}z_3+E_{0001}w},\ \phi_{22}=\frac{\sqrt{\mu_2}z_2^2}{1-2i\overline{b^{(11)}_{1001}}z_1-2i\overline{b^{(13)}_{1001}}z_3+E_{0001}w},$$

$$\phi_{23}=\frac{\sqrt{\mu_2}z_2z_3+\sqrt{\mu_2}b^{(13)}_{1001}z_2w}{1-2i\overline{b^{(11)}_{1001}}z_1-2i\overline{b^{(13)}_{1001}}z_3+E_{0001}w},\quad g=w,$$

where $b^{(11)}_{1001}, b^{(13)}_{1001}, E_{0001} \in \mathbf{C}$ *and* $\mu_2 \ge 1$ *with*

$$|b^{(11)}_{1001}|^2 = \frac{\mu_2^2 - 1}{4}, \quad Im(E_{0001}) = -\frac{\mu_2^2}{4} - |b^{(13)}_{1001}|^2. \tag{25}$$

Proof. By the normalization condition (3) and Lemma 5.1, we can write the map F in the following form:

$$f_1 = \left[z_1 - 2i\overline{b^{(11)}_{1001}}z_1^2 - 2i\sqrt{1+\mu_2}\overline{b^{(12)}_{1001}}z_1z_2 - 2i\overline{b^{(13)}_{1001}}z_1z_3 + (E_{0001} + \frac{i}{2})z_1w\right] \cdot \left[1 - 2i\overline{b^{(11)}_{1001}}z_1 - 2i\sqrt{1+\mu_2}\overline{b^{(12)}_{1001}}z_2 - 2i\overline{b^{(13)}_{1001}}z_3 + E_{0001}w + E_{1001}z_1w + +E_{0101}z_2w + E_{0011}z_3w + E_{0002}w^2\right]^{-1},$$

$$f_2 = \left[z_2 - 2i\sqrt{1+\mu}\overline{b^{(12)}_{1001}}z_2^2 - 2i\overline{b^{(11)}_{1001}}z_1z_2 - 2i\overline{b^{(13)}_{1001}}z_2z_3 + (E_{0001} + \frac{i\mu_2}{2})z_2w\right] \cdot \left[1 - 2i\overline{b^{(11)}_{1001}}z_1 - 2i\sqrt{1+\mu_2}\overline{b^{(12)}_{1001}}z_2 - 2i\overline{b^{(13)}_{1001}}z_3 + E_{0001}w + E_{1001}z_1w + +E_{0101}z_2w + E_{0011}z_3w + E_{0002}w^2\right]^{-1},$$

$$f_3 = \left[z_3 - 2i\overline{b^{(13)}_{1001}}z_3^2 - 2i\overline{b^{(11)}_{1001}}z_1z_3 - 2i\sqrt{1+\mu_2}\overline{b^{(12)}_{1001}}z_2z_3 + E_{0001}z_3w\right] \cdot \left[1 - 2i\overline{b^{(11)}_{1001}}z_1 - 2i\sqrt{1+\mu_2}\overline{b^{(12)}_{1001}}z_2 - 2i\overline{b^{(13)}_{1001}}z_3 + E_{0001}w + E_{1001}z_1w + +E_{0101}z_2w + E_{0011}z_3w + E_{0002}w^2\right]^{-1},$$

$$\phi_{11} = \left[z_1^2 + b^{(11)}_{1001}z_1w\right]\left[1 - 2i\overline{b^{(11)}_{1001}}z_1 - 2i\sqrt{1+\mu_2}\overline{b^{(12)}_{1001}}z_2 - 2i\overline{b^{(13)}_{1001}}z_3 + E_{0001}w + E_{1001}z_1w + E_{0101}z_2w + E_{0011}z_3w + E_{0002}w^2\right]^{-1},$$

$$\phi_{12} = \left[\sqrt{1+\mu_2}z_1z_2 + b^{(12)}_{1001}z_1w + \frac{\mu_2}{\sqrt{1+\mu_2}}b^{(11)}_{1001}z_2w\right]\left[1 - 2i\overline{b^{(11)}_{1001}}z_1 - 2i\sqrt{1+\mu_2}\overline{b^{(12)}_{1001}}z_2 - 2i\overline{b^{(13)}_{1001}}z_3 + E_{0001}w + E_{1001}z_1w + E_{0101}z_2w + E_{0011}z_3w + E_{0002}w^2\right]^{-1},$$

$$\phi_{13} = \left[z_1z_3 + b^{(13)}_{1001}z_1w\right]\left[1 - 2i\overline{b^{(11)}_{1001}}z_1 - 2i\sqrt{1+\mu_2}\overline{b^{(12)}_{1001}}z_2 - 2i\overline{b^{(13)}_{1001}}z_3 + E_{0001}w + E_{1001}z_1w + E_{0101}z_2w + E_{0011}z_3w + E_{0002}w^2\right]^{-1},$$

$$\phi_{22}= \left[\sqrt{\mu_2}z_2^2+\sqrt{\mu_2(1+\mu_2)}b^{(12)}_{1001}z_2w\right]\left[1-2i\overline{b^{(11)}_{1001}}z_1-2i\sqrt{1+\mu_2}\overline{b^{(12)}_{1001}}z_2\right.$$
$$\left.-2i\overline{b^{(13)}_{1001}}z_3+E_{0001}w+E_{1001}z_1w+E_{0101}z_2w+E_{0011}z_3w+E_{0002}w^2\right]^{-1},$$

$$\phi_{23}= \left[\sqrt{\mu_2}z_2z_3+\sqrt{\mu_2}b^{(13)}_{1001}z_2w\right]\left[1-2i\overline{b^{(11)}_{1001}}z_1-2i\sqrt{1+\mu_2}\overline{b^{(12)}_{1001}}z_2\right.$$
$$\left.-2i\overline{b^{(13)}_{1001}}z_3+E_{0001}w+E_{1001}z_1w+E_{0101}z_2w+E_{0011}z_3w+E_{0002}w^2\right]^{-1},$$

$$g= \left[w+C_{1001}z_1w+C_{0101}z_2w+C_{0011}z_3w+E_{0001}w^2\right]$$
$$\left[1-2i\overline{b^{(11)}_{1001}}z_1-2i\sqrt{1+\mu_2}\overline{b^{(12)}_{1001}}z_2\right.$$
$$\left.-2i\overline{b^{(13)}_{1001}}z_3+E_{0001}w+E_{1001}z_1w+E_{0101}z_2w+E_{0011}z_3w+E_{0002}w^2\right]^{-1}.$$

When $z_2=z_3=0$ and $Im(w)=z_1$, we get $f_2=f_3=\phi_{22}=\phi_{23}=0$, and

$$f_1 = \frac{z_1-2i\overline{b^{(11)}_{1001}}z_1^2+(E_{0001}+\frac{i}{2})z_1w}{1-2i\overline{b^{(11)}_{1001}}z_1+E_{0001}w+E_{1001}z_1w+E_{0002}w^2},$$
$$\phi_{11}= \frac{z_1^2+b^{(11)}_{1001}z_1w}{1-2i\overline{b^{(11)}_{1001}}z_1+E_{0001}w+E_{1001}z_1w+E_{0002}w^2},$$
$$\phi_{12}= \frac{b^{(12)}_{1001}z_1w}{1-2i\overline{b^{(11)}_{1001}}z_1+E_{0001}w+E_{1001}z_1w+E_{0002}w^2},$$
$$\phi_{13}= \frac{b^{(13)}_{1001}z_1w}{1-2i\overline{b^{(11)}_{1001}}z_1+E_{0001}w+E_{1001}z_1w+E_{0002}w^2},$$
$$g = \frac{w+C_{1001}z_1w+E_{0001}w^2}{1-2i\overline{b^{(11)}_{1001}}z_1+E_{0001}w+E_{1001}z_1w+E_{0002}w^2}.$$

Consider the basic equation $Im(g) = |f|^2+|\phi|^2$ for any $Im(w) = |z_1|^2$. By considering the u^2 terms, we see that E_{0002} is real. Considering the u^4 terms, we obtain $E_{0002}(E_{0001}-\overline{E_{0001}})=0$. Considering the uz_1 terms, $C_{1001}+2i\overline{b^{(11)}_{1001}}=0$. Considering the u^2z_1 terms, we get $E_{1001}=0$. Considering the u^3z_1 terms, we get $C_{1001}\overline{E_{0002}}=0$. Similarly, when $z_1=z_3=0$ and $Im(w)=|z_2|^2$, we get $C_{0101}=-2i\sqrt{1+\mu_2}\overline{b^{(12)}_{1001}}$, $E_{0101}=0, C_{0101}\overline{E_{0002}}=0$; when $z_1=z_2=0$ and $Im(w)=|z_3|^2$, we get $C_{0011}=-2i\overline{b^{(13)}_{1001}}$, $E_{0011}=0, C_{0011}\overline{E_{0002}}=0$. Therefore we can distinguish two cases.

Case i: $C_{1001} = C_{0101} = C_{0011} = 0$. Then $b^{(11)}_{1001} = b^{(12)}_{1001} = b^{(13)}_{1001} = 0$ and we apply Claim (13) in the proof of Theorem 4.1 to know that F is of the form (13). We are done.

Case ii: $(C_{1001}, C_{0101}, C_{0011}) \neq (0,0,0)$. In this case $E_{0002} = 0$. By the basic equation $Im(g) = |\widetilde{f}|^2$ on $\partial\mathbf{H}_4$, we have

$$\begin{aligned}
&|z|^2|1 - 2i\overline{b^{(11)}_{1001}}z_1 - 2i\sqrt{1+\mu_2}\overline{b^{(12)}_{1001}}z_2 - 2i\overline{b^{(13)}_{1001}}z_3 + E_{0001}w|^2\\
&= |z_1|^2|1 - 2i\overline{b^{(11)}_{1001}}z_1 - 2i\sqrt{1+\mu_2}\overline{b^{(12)}_{1001}}z_2 - 2i\overline{b^{(13)}_{1001}}z_3 + (E_{0001} + \frac{i}{2})w|^2\\
&+|z_2|^2|1 - 2i\sqrt{1+\mu}\overline{b^{(12)}_{1001}}z_2 - 2i\overline{b^{(11)}_{1001}}z_1 - 2i\overline{b^{(13)}_{1001}}z_3 + (E_{0001} + \frac{i\mu_2}{2})w|^2\\
&+|z_3|^2|1 - 2i\overline{b^{(13)}_{1001}}z_3 - 2i\overline{b^{(11)}_{1001}}z_1 - 2i\sqrt{1+\mu_2}\overline{b^{(12)}_{1001}}z_2 + E_{0001}w|^2\\
&+|z_1|^2|z_1 + b^{(11)}_{1001}w|^2 + |\sqrt{1+\mu_2}z_1z_2 + b^{(12)}_{1001}z_1w + \frac{\mu_2}{\sqrt{1+\mu_2}}b^{(11)}_{1001}z_2w|^2\\
&+|z_1|^2|z_3 + b^{(13)}_{1001}w|^2 + |z_2|^2|\sqrt{\mu_2}z_2 + \sqrt{\mu_2(1+\mu_2)}b^{(12)}_{1001}w|^2\\
&+|z_2|^2|\sqrt{\mu_2}z_3 + \sqrt{\mu_2}b^{(13)}_{1001}w|^2.
\end{aligned}$$

By considering the $z_1\overline{z_2}u^2$ terms, we get $b^{(12)}_{1001}\overline{b^{(11)}_{1001}} = 0$. We consider $z_2 = z_3 = 0$ and $Im(w) = |z_1|^2$, divided by $|z_1|^2$, to get

$$\begin{aligned}
&|1 - 2i\overline{b^{(11)}_{1001}}z_1 + E_{0001}w|^2\\
&= |1 - 2i\overline{b^{(11)}_{1001}}z_1 + (E_{0001} + \frac{i}{2})w|^2 + |z_1 + b^{(11)}_{1001}w|^2 + |b^{(12)}_{1001}w|^2 + |b^{(13)}_{1001}w|^2.
\end{aligned}$$

By considering the u^2 terms, we have

$$Im(E_{0001}) = -\frac{1}{4} - |b^{(11)}_{1001}|^2 - |b^{(12)}_{1001}|^2 - |b^{(13)}_{1001}|^2. \tag{26}$$

We also consider $z_1 = z_3 = 0$ and $Im(w) = |z_2|^2$, divided by the $|z_2|^2$ terms, to get:

$$\begin{aligned}
&|1 - 2i\sqrt{1+\mu_2}\overline{b^{(12)}_{1001}}z_2 + E_{0001}w|^2 = |1 - 2i\sqrt{1+\mu}\overline{b^{(12)}_{1001}}z_2 + (E_{0001} + \frac{i\mu_2}{2})w|^2\\
&+|\frac{\mu_2}{\sqrt{1+\mu_2}}b^{(11)}_{1001}w|^2 + |\sqrt{\mu_2}z_2 + \sqrt{\mu_2(1+\mu_2)}b^{(12)}_{1001}w|^2 + |\sqrt{\mu_2}b^{(13)}_{1001}w|^2.
\end{aligned}$$

By considering the u^2 terms, divided by μ_2, we have $Im(E_{0001}) = -\frac{\mu_2}{4} - \frac{\mu_2}{1+\mu_2}|b^{(11)}_{1001}|^2 - (1+\mu_2)|b^{(12)}_{1001}|^2 - |b^{(13)}_{1001}|^2$. From above two formulas, we get $0 = \frac{\mu_2 - 1}{4} - \frac{1}{1+\mu_2}|b^{(11)}_{1001}|^2 + \mu_2|b^{(12)}_{1001}|^2$ so that $|b^{(11)}_{1001}|^2 = \frac{\mu_2^2-1}{4} + \mu_2(1+\mu_2)|b^{(12)}_{1001}|^2$. Recalling $b^{(12)}_{1001}\overline{b^{(11)}_{1001}} = 0$ and $\mu_2 \ge 1$, we obtain the equation $|b^{(12)}_{1001}|^2\left(\frac{\mu_2^2-1}{4} + \mu_2(1+\mu_2)|b^{(12)}_{1001}|^2\right) = 0$. Then either $b^{(12)}_{1001} = 0$ or $\frac{\mu_2^2-1}{4} + \mu_2(1+\mu_2)|b^{(12)}_{1001}|^2 = 0$. Since $\mu_2 \ge 1$, the second possibility implies $b^{(12)}_{1001} = \mu_2 = 0$. We have proved $b^{(12)}_{1001} = 0$ and $|b^{(11)}_{1001}|^2 = \frac{\mu_2^2-1}{4}$. Therefore our basic equation becomes

$$\begin{aligned}
&|z|^2|1-2i\overline{b^{(11)}_{1001}}z_1-2i\overline{b^{(13)}_{1001}}z_3+E_{0001}w|^2\\
=&|z_1|^2|1-2i\overline{b^{(11)}_{1001}}z_1-2i\overline{b^{(13)}_{1001}}z_3+(E_{0001}+\frac{i}{2})w|^2\\
&+|z_2|^2|1-2i\overline{b^{(11)}_{1001}}z_1-2i\overline{b^{(13)}_{1001}}z_3+(E_{0001}+\frac{i\mu_2}{2})w|^2\\
&+|z_3|^2|1-2i\overline{b^{(13)}_{1001}}z_3-2i\overline{b^{(11)}_{1001}}z_1+E_{0001}w|^2+|z_1|^2|z_1+b^{(11)}_{1001}w|^2\\
&+|z_2|^2|\sqrt{1+\mu_2}z_1+\frac{\mu_2}{\sqrt{1+\mu_2}}b^{(11)}_{1001}w|^2+|z_1|^2|z_3+b^{(13)}_{1001}w|^2\\
&+|z_2|^2|\sqrt{\mu_2}z_2|^2+|z_2|^2|\sqrt{\mu_2}z_3+\sqrt{\mu_2}b^{(13)}_{1001}w|^2.
\end{aligned}$$

Considering all terms involving u^2, we get

$$\begin{aligned}
&|z|^2|E_{0001}|^2=|z_1|^2|E_{0001}+\frac{i}{2}|^2+|z_2|^2|E_{0001}+\frac{i\mu_2}{2}|^2+|z_3|^2|E_{0001}|^2\\
&+|z_1|^2|b^{(11)}_{1001}|^2+|z_2|^2|\frac{\mu_2}{\sqrt{1+\mu_2}}b^{(11)}_{1001}|^2+|z_1|^2|b^{(13)}_{1001}|^2+|z_2|^2|\sqrt{\mu_2}b^{(13)}_{1001}|^2.
\end{aligned}$$

By recalling $|b^{(11)}_{1001}|^2=\frac{\mu_2^2-1}{4}$, we have

$$\begin{aligned}
0=&|z_1|^2\left(\frac{1}{4}+Im(E_{0001})\right)+|z_2|^2\left(\frac{\mu_2^2}{4}+\mu_2 Im(E_{0001})\right)\\
&+\frac{\mu_2^2-1}{4}|z_1|^2+\frac{\mu_2^2(\mu_2-1)}{4}|z_2|^2+\left(|z_1|^2+\mu_2|z_2|^2\right)|b^{(13)}_{1001}|^2.
\end{aligned}$$

Comparing the $|z_1|^2$ and $|z_2|^2$ respectively, we find out

$$\begin{aligned}
0&=\left(\frac{1}{4}+Im(E_{0001})\right)+\frac{\mu_2^2-1}{4}+|b^{(13)}_{1001}|^2,\\
0&=\left(\frac{\mu_2}{4}+Im(E_{0001})\right)+\frac{\mu_2(\mu_2-1)}{4}+|b^{(13)}_{1001}|^2.
\end{aligned}$$

Thus we have proved that $|b^{(11)}_{1001}|^2=\frac{\mu_2^2-1}{4}$ and $Im(E_{0001})=-\frac{\mu_2^2}{4}-|b^{(13)}_{1001}|^2$. ∎

Corollary 6.2. *Let $F\in Rat(\mathbf{H}_4,\mathbf{H}_9)$ be as in Theorem 6.1. Then F is equivalent to another map that is of the form as in Theorem 6.1 with the same μ_2 value and satisfies the additional property:*

$$b^{(13)}_{1001}=0 \text{ and } Re(E_{0001})=0. \tag{27}$$

Proof. Let F be as in Theorem 6.1. We take $\sigma\in Aut_0(\mathbf{H}_4)$ and $\tau^*\in Aut_0(\mathbf{H}_9)$ in the forms of (4) and (5) with $U=Id,\lambda=1,U^*_{22}=Id$, $r=0$ and $a=\begin{pmatrix}0\\0\\-b^{(13)}_{1001}\end{pmatrix}$. Then by Lemma 2.4, we can verify that $\widetilde{F}=\tau^*\circ F\circ\sigma$ still is of the form as in Theorem 6.1, with the same μ_2 value, and satisfies $b^{(13)}_{1001}=0$.

Next fixing F that is of the form as in Theorem 6.1 with $b^{(13)}_{1001}=0$. We again take $\sigma\in Aut_0(\mathbf{H}_4)$ and $\tau^*\in Aut_0(\mathbf{H}_9)$ in the forms of (4) and (5) with $U=Id,\lambda=$

$1, a = 0, U_{22}^* = Id$ and $r = -Re(E_{0001})$. Then by Lemma 2.4, we can verify that $\widetilde{F} = \tau^* \circ F \circ \sigma$ is of the form as in Theorem 6.1, with the same μ_2 value, and satisfies (27). ∎

7. The Proof of Theorem 1.2. Let $F \in Rat(\mathbf{H}_4, \mathbf{H}_9)$ be as in Corollary 6.2. For any $p \in \partial\mathbf{H}_4$, let $\widetilde{\mu}_{1,F}(p)$ and $\widetilde{\mu}_{2,F}(p)$, with $\widetilde{\mu}_{1,F}(p) \le \widetilde{\mu}_{2,F}(p)$, be the eigenvalues of the semipositive matrix $\mathcal{A}(p) : = (a_{jl}(p)) = (-2i\frac{\partial^2 f_{j,p}^{**}}{\partial z_l \partial w}|_0)$. Define $\mu_{1,F}(p) = 1$ and $\mu_{2,F}(p) = \frac{\widetilde{\mu}_{2,F}(p)}{\widetilde{\mu}_{1,F}(p)} \ge 1$. Recall that $\mu_{1,F}(p)$ and $\mu_{2,F}(p)$ are the coefficients $\mu_1 = 1$ and μ_2 in Theorem 2.2, respectively, which are the eigenvalues of the semipositive matrix $(-2i\frac{\partial^2 f_{j,p}^{***}}{\partial z_l \partial w}|_0)$. Write $p = (p_1, ..., p_7) = (z_1, \overline{z}_1, z_2, \overline{z}_2, z_3, \overline{z}_3, u)$ that is identified as a point in $\partial\mathbf{H}_4$, and write $a_{kk}(p) = a_0^{(kk)} + \sum_{j=1}^7 a_j^{(kk)} p_j + o(|p|)$, $k = 1, 2, 3$.

Lemma 7.1. *Let F be as in Corollary 6.2. for any p that is closed to* 0

$$
\begin{aligned}
&a_{11}(p) = 1 - 2ReE_{0001}u + 4i(\overline{b_{1001}^{(11)}}z_1 - b_{1001}^{(11)}\overline{z_1}) + o(|p|),\\
&a_{22}(p) = \mu_2 - 2\mu_2 ReE_{0001}u + 2i\mu_2(\overline{b_{1001}^{(11)}}z_1 - b_{1001}^{(11)}\overline{z_1}) + o(|p|),\\
&a_{33}(p) = o(|p|).
\end{aligned}
$$

We shall assume this lemma, which will be proved in the next section, to prove Theorem 1.2. We first study the real analytic function $\mu_{2,F}(p)$ of p.

Lemma 7.2. *Let F be as in Corollary 6.2. Then*

$$\mu_{2,F}(p) = \mu_2 + 4\mu_2 Im\left(\overline{b_{1001}^{(11)}}z_1\right) + o(|p|)$$

for any such p near 0*, where μ_2 is the one as in (25).*

Proof. Recall that $\widetilde{\mu}_{1,F}(p)$ and $\widetilde{\mu}_{2,F}(p)$ must be the eigenvalues of the equation $det(\lambda I - \mathcal{A}(p)) = 0$, i.e.,

$$det\begin{bmatrix} \lambda - a_{11}(p) & -a_{12}(p) & -a_{13}(p) \\ -a_{21}(p) & \lambda - a_{22}(p) & -a_{23}(p) \\ -a_{31}(p) & -a_{32}(p) & \lambda - a_{33}(p) \end{bmatrix} = 0.$$

Since the geometric rank is 2, we must have $det(\lambda I - \mathcal{A}(p)) = \lambda(\lambda - \widetilde{\mu}_{1,F}(p))(\lambda - \widetilde{\mu}_{2,F}(p))$. For simplicity, we denote $a_{ij} = a_{ij}(p)$. Then

$$
\begin{aligned}
&det(\lambda I - \mathcal{A}(p)) = \lambda\Big[\lambda^2 - (a_{11} + a_{22} + a_{33})\lambda + (a_{11}a_{22} + a_{11}a_{33}\\
&+a_{22}a_{33} - a_{21}a_{12} - a_{23}a_{32} - a_{31}a_{13})\Big],
\end{aligned}
$$

$$
\begin{aligned}
&\widetilde{\mu}_{1,F}(p) = \frac{1}{2}\Big[(a_{11} + a_{22} + a_{33}) - \Big((a_{11} + a_{22} + a_{33})^2 - 4(a_{11}a_{22}\\
&+a_{11}a_{33} + a_{22}a_{33} - a_{21}a_{12} - a_{23}a_{32} - a_{31}a_{13})\Big)^{1/2}\Big],
\end{aligned}
$$

$$\widetilde{\mu}_{2,F}(p) = \frac{1}{2}\Bigg[(a_{11}+a_{22}+a_{33}) + \bigg((a_{11}+a_{22}+a_{33})^2 - 4(a_{11}a_{22} \\ +a_{11}a_{33}+a_{22}a_{33}-a_{21}a_{12}-a_{23}a_{32}-a_{31}a_{13})\bigg)^{1/2}\Bigg].$$

Then

$$\mu_{2,F}(p) = \frac{\widetilde{\mu}_2(p)}{\widetilde{\mu}_1(p)} = \frac{1+\sqrt{1-N(p)+M(p)}}{1-\sqrt{1-N(p)+M(p)}}$$

where, by Lemma 7.1,

$$N(p) = 4\frac{a_{11}a_{22}}{(a_{11}+a_{22})^2} + o(|p|), \quad M(p) = 4\frac{a_{21}a_{12}+a_{13}a_{31}+a_{23}a_{32}}{(a_{11}+a_{22}+a_{33})^2} = o(|p|).$$

By considering the Tayler seriese of $a_{jj} = a_0^{(jj)} + \sum_k a_k^{(jj)} p_k + o(|p|)$ for $k = 1,2,3$, we obtain

$$\mu_{2,F}(p) = \frac{a_0^{(22)}}{a_0^{(11)}} - \frac{1}{(a_0^{(11)})^2}\sum_{j=1}^{7}\left(a_0^{(22)}a_j^{(11)} - a_0^{(11)}a_j^{(22)}\right)p_j + o(|p|).$$

Then from Lemma 7.1, the desired equality is proved. ∎

Proof of Theorem 1.2. Let F be as in Corollary 6.2 and fixed . If its $\mu_2 = 1$, then F must be Whitney map (10). Suppose that $\mu_2 > 1$. Then by Lemma 7.2, we can choose $p = (-ib_{1001}^{(11)}r,\ 0,0,0,0,\ 0,0)$ or identify $p = p(r) = (-ib_{1001}^{(11)}r,\ 0,0,\ i|b_{1001}^{(11)}|^2 r) \in \partial\mathbf{H}_4$, where $r > 0$, to conclude that

$$\mu_{2,F}(p(r)) = \mu_2 - 4\mu_2|b_{1001}^{(11)}|^2 r + o(|r|).$$

Therefore, there is a constant $\sigma > 0$ such that for any $0 < r < \sigma$, the derivative $\frac{d\mu_{2,F}(p(r))}{dr} < 0$. Therefore for such p, $\mu_{2,F_{p(r)}}(0)$ is decreasing as r increases. Hence $\mu_{2,F_{p(r)}}(0) < \mu_{2,F}(0)$. In other words, we find a new map that has smaller μ_2 value. By Corollary 6.2, we can assume that this new map is of the form as in Theorem 6.1, with the same μ_2 value, $b_{1001}^{(13)} = 0$ and $Re(E_{0001}) = 0$. Let us denote this map as F_1. Repeating this process, we obtain a sequence of maps $\{F_k\}_{k=1}^{\infty}$ such that each map F_k is of the form as in Theorem 6.1 with $b_{1001}^{(13)} = 0$ and $Re(E_{0001}) = 0$ and that each F_k is equivalent to the F and that $\mu_{2,F_{k+1}} < \mu_{2,F_k}$ holds for all k. Then the limit map $\hat{F} = \lim_k F_k$ must be of the form as in Theorem 6.1 with $b_{1001}^{(13)} = 0$ and $Re(E_{0001}) = 0$, and with the minimum $\mu_{2,\hat{F}}$ value.

We want to prove that this map is the desired one. In fact, suppose that $\mu_{2,\hat{F}} > 1$. Then $p = 0$ must be a critical point of the real analytic function $\mu_{2,\hat{F}}(p)$. By Lemma 7.2, $p = 0$ is a critical point if and only if $b_{1001}^{(11)} = 0$. This implies, by (25), that $\mu_{2,\hat{F}} = 1$, which is a contradiction to the assumption that $\mu_{2,\hat{F}} > 1$.

Finally, since we can assume that the map $\hat{F}$ is of the form as in Corollary 6.2 with $\mu_2 = 1$, it is the Whiteny map (10). ∎

To finish the proof of Theorem 1.2, it remains to prove Lemma 7.1.

8. The Proof of Lemma 7.1. We are going to use the following formula (see cf. [H 2003, § 2]) to prove Lemma 7.1:

$$a_{jl}(p) = -2i\frac{p^2 f^{**}_{p,j}}{\partial z_l \partial w}|_0 = -2i\Big\{\frac{1}{\lambda(p)}L_j T\widetilde{f}(p)\cdot\overline{L_l\widetilde{f}(p)}^t \tag{28}$$
$$-\frac{2i}{\lambda(p)^2}\Big(T\widetilde{f}(p)\cdot\overline{L_l\widetilde{f}(p)}^t\Big)\Big(T\widetilde{f}(p)\cdot\overline{L_j\widetilde{f}(p)}^t\Big)-\frac{\delta_{jl}}{2\lambda(p)}\Big(T^2g(p)-2iT^2\widetilde{f}(p)\cdot\overline{\widetilde{f}(p)}^t\Big)\Big\}.$$

Since F is of the form as in Corollary 6.2, we have

$$f_1 = \frac{z_1 - 2i\overline{b^{(11)}_{1001}}z_1^2 + (E_{0001}+\frac{i}{2})z_1 w}{1-2i\overline{b^{(11)}_{1001}}z_1 + E_{0001}w} = z_1 + o(|(z,w)|).$$

Then

$$\begin{aligned} Tf_1 &= \frac{(E_{0001}+\frac{i}{2})z_1}{1-2i\overline{b^{(11)}_{1001}}z_1+E_{0001}w} - \frac{E(z_1-2i\overline{b^{(11)}_{1001}}z_1^2+(E_{0001}+\frac{i}{2})z_1w)}{(1-2i\overline{b^{(11)}_{1001}}z_1+E_{0001}w)^2} \\ &= \frac{i}{2}z_1 + o(|(z,w)|), \end{aligned}$$

Similarly, by direct computation, we obtain the following results. For simplicity, we use $o(1)$ to denote $o(|(z,w)|)$.

$$\begin{aligned} &T^2 f_1 = -iE_{0001}z_1 + o(1),\ L_1 f_1 = 1 + \frac{i}{2}w + o(1), \\ &L_1 T f_1 = \frac{i}{2} - 2\overline{b^{(11)}_{1001}}z_1 - iE_{0001}w + o(1),\ L_2 f_1 = o(1),\ L_2 T f_1 = o(1),\ L_3 f_1 = o(1), \end{aligned}$$

$$\begin{aligned} &f_2 = z_2 + o(1),\ Tf_2 = \frac{i\mu_2}{2}z_2 + o(1),\ T^2 f_2 = -i\mu_2 E_{0001}z_2 + o(1), \\ &L_1 f_2 = o(1),\ L_1 T f_2 = -\mu_2\overline{b^{(11)}_{1001}}z_2 + o(1),\ L_2 f_2 = 1 + \frac{i\mu_2}{2}w + o(1), \\ &L_2 T f_2 = \frac{i\mu_2}{2} - \mu_2\overline{b^{(11)}_{1001}}z_1 - i\mu_2 E_{0001}w + o(1),\ L_3 f_2 = o(1) \end{aligned}$$

$$\begin{aligned} &f_3 = z_3 + o(1),\ Tf_3 = o(1),\ T^2 f_3 = o(1),\ L_1 f_3 = o(1),\ L_1 T f_3 = o(1), \\ &L_2 f_3 = 0,\ L_2 T f_3 = 0,\ L_3 f_3 = 1 + o(1),\ L_3 T f_3 = o(1), \end{aligned}$$

$$\begin{aligned} &\phi_{11} = o(1),\ T\phi_{11} = b^{(11)}_{1001}z_1 + o(1), \\ &T^2\phi_{11} = -2E_{0001}b^{(11)}_{1001}z_1 + o(1),\ L_1\phi_{11} = 2z_1 + b^{(11)}_{1001}w + o(1), \\ &L_1 T\phi_{11} = b^{(11)}_{1001} + (-2E_{0001} + 4i|b^{(11)}_{1001}|^2)z_1 - 2b^{(11)}_{1001}E_{0001}w + o(1), \\ &L_2\phi_{11} = 0,\ L_2 T\phi_{11} = 0,\ L_3\phi_{11} = o(1), \end{aligned}$$

$$
\begin{aligned}
&\phi_{12} = o(1),\ T\phi_{12} = \frac{\mu_2}{\sqrt{1+\mu_2}} b^{(11)}_{1001} z_2 + o(1),\\
&T^2\phi_{12} = -2E_{0001}\frac{\mu_2}{\sqrt{1+\mu_2}} b^{(11)}_{1001} z_2 + o(1),\ L_1\phi_{12} = \sqrt{1+\mu_2} z_2 + o(1),\\
&L_1 T\phi_{12} = -E_{0001}\sqrt{1+\mu_2} z_2 + 2i|b^{(11)}_{1001}|^2 \frac{\mu_2}{\sqrt{1+\mu_2}} z_2 + o(1),\\
&L_2\phi_{12} = \sqrt{1+\mu_2} z_1 + \frac{\mu_2}{\sqrt{1+\mu_2}} b^{(11)}_{1001} w + o(1),\\
&L_2 T\phi_{12} = \frac{\mu_2 b^{(11)}_{1001}}{\sqrt{1+\mu_2}} + (\frac{2i\mu_2}{\sqrt{1+\mu_2}}|b^{(11)}_{1001}|^2 - E_{0001}\sqrt{1+\mu_2}) z_1 - \frac{2\mu_2 E_{0001} b^{(11)}_{1001}}{\sqrt{1+\mu_2}} w + o(1),\\
&L_3\phi_{12} = o(1),
\end{aligned}
$$

$$
\begin{aligned}
&\phi_{13} = o(1),\ T\phi_{13} = o(1),\ T^2\phi_{13} = o(1),\ L_1\phi_{13} = z_3 + o(1),\ L_1 T\phi_{13} = o(1),\\
&L_2\phi_{13} = 0,\ L_2 T\phi_{13} = 0,\ L_3\phi_{13} = z_1 + o(1),\ L_3 T\phi_{13} = -E_{0001} z_1 + o(1),
\end{aligned}
$$

$$
\begin{aligned}
&\phi_{22} = o(1),\ T\phi_{22} = o(1),\ T^2\phi_{22} = o(1),\\
&L_1\phi_{22} = o(1),\ L_1 T\phi_{22} = o(1),\ L_2\phi_{22} = 2\sqrt{\mu_2} z_2 + o(1),\\
&L_2 T\phi_{22} = -2E_{0001}\sqrt{\mu_2} z_2 + o(1),\ L_3\phi_{22} = o(1),
\end{aligned}
$$

$$
\begin{aligned}
&\phi_{23} = o(1),\ T\phi_{23} = o(1),\ T^2\phi_{23} = o(1),\ L_1\phi_{23} = o(1),\\
&L_1 T\phi_{23} = o(1),\ L_2\phi_{23} = \sqrt{\mu_2} z_3 + o(1),\ L_2 T\phi_{23} = o(1),\\
&L_3\phi_{23} = \sqrt{\mu_2} z_2,\ L_3 T\phi_{23} = -E_{0001}\sqrt{\mu_2} z_2 + o(1).
\end{aligned}
$$

By (29) and all above formulas, the desired formulas in Lemma 7.1 are obtained. □

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