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INTEGRATION OF MEROMORPHIC COHOMOLOGY CLASSES AND APPLICATIONS *

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En hommage à Yum Tong Siu pour ses soixante ans

Abstract. The main purpose of this article is to increase the efficiency of the tools introduced in [B.Mg. 98] and [B.Mg. 99], namely integration of meromorphic cohomology classes, and to generalize the results of [B.Mg. 99]. They describe how positivity conditions on the normal bundle of a compact complex submanifold Y of codimension $n+1$ in a complex manifold Z can be transformed into positivity conditions for a Cartier divisor in a space parametrizing $n-$cycles in Z .

As an application of our results we prove that the following problem has a positive answer in many cases :

Let Z be a compact connected complex manifold of dimension $n+p$. Let $Y \subset Z$ a submanifold of Z of dimension $p-1$ whose normal bundle $N_{Y|Z}$ is (Griffiths) positive. We assume that there exists a covering analytic family $(X_s)_{s\in S}$ of compact $n-$cycles in Z parametrized by a compact normal complex space S.

Is the algebraic dimension of Z $\geq p$?

0. Introduction. In the present paper we will develop further some of the methods introduced in [B.Mg.98] and [B.Mg.99] concerning integration of meromorphic cohomology classes. We will equally generalize the results of [B.Mg.99] which describe how positivity conditions on the normal bundle of a compact complex submanifold Y of codimension $n+1$ in a complex manifold Z can be transformed into positivity conditions for a Cartier divisor in a space parametrizing $n-$cycles in Z .

Applications. We shall motivate this work by an application to the following problem:

(0.1) PROBLEM. Let Z be a compact connected complex manifold of dimension $n+p$ in Fujiki's class $\mathcal{C}$[1] and denote the algebraic dimension of Z by $a(Z)$[2]. Let $Y \subset Z$ be a closed submanifold of dimension $p-1>0$ whose normal bundle $N_{Y|Z}$ is (Griffiths) positive[3].

Does this imply that $a(Z) \geq \dim(Y)+1 = p$ **?**

(0.2) REMARKS.

1) There exists a compact connected three-dimensional manifold Z of algebraic dimension 0 containing a smooth rational curve with normal bundle $\mathcal{O}(1)\oplus\mathcal{O}(1)$ (see for instance [C.91]). Thus the hypothesis $Z \in \mathcal{C}$ is not superfluous.
2) We assume $p>1$ (so Y is not a finite set of points) because the case $p=1$ is trivially false (let Z be any torus with $a(Z)=0$).

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[1]In [V.85], J. Varouchas proved that this is equivalent to Z being bimeromorphic to a Kähler manifold.

[2]The algebraic dimension of Z is the transcendence degree of the field of global meromorphic functions on Z.

[3]We learnt from this problem from a talk given by Th. Peternell in Nancy; see [O.P.01].

3) The case $\operatorname{codim}_Z Y = 1$ is well known because the line bundle associated with the divisor Y is of maximal Kodaira-dimension and therefore Z is a Moishezon manifold (i.e. $a(Z) = \dim Z$).
4) We shall also discuss the following two variants of the problem (0.1):
(a) the case where Y is a locally complete intersection of codimension $n+1$ in Z. In this case the normal bundle is again well defined and the hypothesis $N_{Y|Z} > 0$ still makes sense.
(b) the case when the hypothesis $N_{Y|Z} > 0$ is replaced by the assumption that $N_{Y|Z}$ is ample .
We shall also combine the variants (a) and (b) .

(0.3) A Necessary Condition. Assume that Z satisfies $a(Z) \geq \dim Y + 1 = p$. Then by choosing another bimeromorphic model $\tilde{Z}$ of Z given by a modification $\tau : \tilde{Z} \to Z$ if necessary, we can find an equidimensional map $\pi : \tilde{Z} \to W$ on a normal projective variety W such that $a(W) = p$. Then the fibres of π define, using the direct image by τ, an analytic family of $n-$cycles in Z which covers Z[4].
Remark that the covering family constructed in this way is not related to the submanifold Y. The inequality $a(Z) \geq p$ is now reflected in the fact that the normal compact complex space parametrizing this covering family is a Moishezon space.
This leads naturally to the following problem :

(0.4) Problem. Let Z be a compact connected complex manifold of dimension $n+p$. Let $Y \subset Z$ a submanifold of Z of dimension $p-1$ whose normal bundle $N_{Y|Z}$ is (Griffiths) positive. We assume that there exists a covering analytic family $(X_s)_{s\in S}$ of compact $n-$cycles in Z parametrized by a compact normal complex space S.

Is $a(Z) \geq p$ **?**

(0.5) A Special Case. Let Z be a compact connected complex manifold of dimension $n+p$. Let $\pi : Z \to S$ be a holomorphic equidimensional map onto a normal compact complex space S of dimension p. As before we assume that there exists a complex submanifold Y of dimension $p-1$ in Z whose normal bundle is (Griffiths) positive.

Is S **a Moishezon space ?**

(0.6) Remarks.
0) As the fibers of π give a covering analytic family of $n-$cycles of Z which is parametrized by the compact normal complex space S problem (0.5) is a special case of problem (0.4). We shall see below that in this case we have equivalence between "S is Moishezon" and "$a(Z) \geq p$".
1) In problem (0.4) (and (0.5)) the case $p = 1$ is allowed, and Y may be taken to be any point in Z. The conclusion $a(Z) \geq 1$ is then a consequence of [K.75] because $(X_s)_{s\in S}$ is a covering family of divisors in Z which is not compatible with $a(Z) = 0$. (In problem (0.5), $\dim S = 1$, so S is projective !)
2) Consider the classifying map $c : S \to \mathcal{C}_n(Z)$ for the analytic family $(X_s)_{s\in S}$ (see [B.75]); we can always replace S by the normalisation of its image

[4] See theorems 2 and 4 in [B.75] .

by c without changing the $n-$cycles which appear in the family. Thus, without any loss of generality, we shall always assume in the sequel that the map $c : S \to \mathcal{C}_n(Z)$ is the normalization of its image.

3) Now define

$$\Sigma := \{s \in S \mid |X_s| \cap Y \neq \emptyset\}. \tag{1}$$

As Y is projective (because it carries a positive bundle), Σ is a Moishezon space thanks to [C.80]. Moreover, if

$$Z = \bigcup_{s \in \Sigma} |X_s|,$$

then we obtain that $a(Z) \geq p$ by a"classical" argument (see for instance[C.80]).

4) Of course we shall also consider variants a) and b) or a) + b) for the problems (0.4) and (0.5) (see remark 4 in (0.2)).

Now we shall state our main result concerning these questions [5]:

(0.7) Theorem. *Assuming the hypotheses of problem (0.4) with the weaker assumption that Y is a locally complete intersection of codimension $n+1$ with an ample normal bundle. We assume that for each generic point of Σ the corresponding cycle meets Y in a finite set. We assume also that our covering family of cycles is locally separated along Y[6]. Then we have $a(Z) \geq p$.*

In the case of problem (0.5) we obtain a statement without referring to the previous separation condition on the covering family:

(0.8) Corollary. *In the situation of problem (0.5), if $Y \subset Z$ is a locally complete intersection of codimension $n+1$ with a positive normal bundle, then S is a Moishezon space.*

For some comments on the non-equidimensional case see the remark (4.2) following the proof of the corollary.

(0.9). We show moreover that, in general, the problem (0.4) can be reduced to the case where $\dim S = p$ (which is the lowest possible dimension) and we obtain in this case a transcendental analogue of the result using the convexity property of the cycle space (see [B.78]), namely that there exists a divisor in Z and p algebraically independent holomorphic functions on the complement of this divisor (but they may have essential singularities along the divisor)[7].

This shows that our filtered integration of meromorphic cohomology classes (see below) may be seen as an algebraic analogue of the convexity reduction via cycle space (see [A.N. 67] , [N.S.77] , [B.78] etc...)

[5]See section 4 for more results and a detailed discussion.

[6]See the precise definition in section 4 .

[7]The proof of this is explained in remark (4.4) 2).

Main tools and generalizations.

(0.10). In this paper we will consider the following setting. Let Z be a complex manifold and let $(X_s)_{s\in S}$ be an analytic family of $n-$cycles in Z. Let $\mathcal{X}$ be the graph of the family and let $p:\mathcal{X}\to Z$ and $\pi:\mathcal{X}\to S$ be the canonical projections. A locally complete intersection Y of codimension $n+1$ in Z will be called a **pole** for the family if the (analytic) incidence set $\{s\in S \mid X_s\cap Y\neq\emptyset\}$ is nowhere dense in S. We will say that a pole Y is **proper and generically finite** (respectively **finite**) if the mapping $\pi:p^{-1}(Y)\to S$ is proper and generically finite (respectively finite).

The same setting is under consideration in our papers [B.Mg.98] and [B.Mg.99] and we will now describe how results from them are generalized in the present paper.

In the former paper we show that if the pole Y is finite then the incidence set carries a natural structure of a Cartier divisor, called the incidence divisor and denoted by Σ_Y or simply by Σ. In the latter paper we assume the pole Y to be finite and we also assume that, for generic σ in Σ_Y, the cycle X_σ cuts Y in a unique point and at that point X_σ is smooth and not tangent to Y. Then in the case where Y is ample and S is compact we show that if the family of cycles satisfies certain **separation conditions** along Y then the incidence divisor Σ_Y is of maximal Kodaira dimension. This is proved by "integrating" the cohomology classes in $H^{n+1}_{[Y]}(\Omega^n_Z)$ on the cycles and thus producing enough holomorphic functions on $S\setminus\Sigma_Y$ having polar singularities along Σ_Y.

We generalize this result in two different ways. First we allow the pole Y to be proper and generically finite; so $X_\sigma\cap Y$ may be of positive dimension for all σ in a nowhere dense analytic subset of Σ_Y. In this more general setting the incidence set still carries a natural structure of a Cartier divisor thanks to [B.K.03], but the integration of the cohomology classes needs special attention.

The second generalization is much more delicate and consists in dropping all conditions on the intersection of the cycles with the pole. This means that, for generic σ in Σ_Y, the cycle X_σ can cut the pole in several points (tangentially or not). Under this weaker assumption the main problem is to get a precise bound – in fact an optimal bound – for the pole order of the meromorphic functions obtained by integrating the cohomology classes. In order to get such a bound we have to modify the definition of pole order for the cohomology classes. This new order will be referred to as the **conormal order** and it is defined by replacing the "naive" filtration on $\underline{H}^{n+1}_{[Y]}(\Omega^n_Z)$ by the so called **conormal filtration** described in (2.13). We then prove the following result:

Let C be an irreducible component of $|\Sigma_Y|$ and let q'_C denote its multiplicity in Σ_Y. There exists a rational number $\varkappa'_C\in]0,1]$ such that the pole order along C of a meromorphic function, obtained by integrating a cohomology class of conormal order ν, is bounded by $q'_C.\varkappa'_C.\nu$. Moreover this bound is optimal.

Of course we will have $\varkappa'_C=1$ when for a generic $\sigma\in C$ we have only one point in $X_\sigma\cap Y$ with non-tangential intersection .

(0.11) Theorem. *Let $(X_s)_{s\in S}$ be an analytic family of $n-$cycles in a complex manifold Z, parametrized by a reduced complex space S and let Y be a proper and* *generically finite pole for this family. Then there exists a quasi-filtered* [8] *integration map*

$$\pi_* p^* \underline{H}^{n+1}_{[Y]}(\Omega^n_Z) \to \underline{H}^1_{[\Sigma_Y]}(\mathcal{O}_S)$$

when we endow the sheaf $\underline{H}^{n+1}_{[Y]}(\Omega^n_Z)$ *with the conormal filtration and the sheaf* $\underline{H}^1_{[\Sigma_Y]}(\mathcal{O}_S)$ *with the* $\varkappa'$*−filtration*[9].

(0.12) Remark.We want to emphasize the fact that Z is **not** assumed to be compact in the previous theorem. Moreover **neither** the cycles X_s **nor** the divisor Σ_Y (nor *a fortiori* the space S) are supposed to be compact. The only compactness assumption we make is that Y is a proper and generically finite pole; meaning that $\pi : p^{-1}(|Y|) \to S$ is proper and generically finite on its image $|\Sigma_Y|$ which has empty interior in S.

(0.13). Our final step is to describe, for a generic point in an irreducible component C of $|\Sigma|$, how to control the poles of maximal order via a "normal quasi-cone", which determines the initial behaviour of the family $(X_s)_{s\in S}$ in the normal direction to Σ_Y. This allows us to define conditions for the family $(X_s)_{s\in S}$ to be locally separated along C, which generalizes the idea used in [B.Mg. 99] for the generically non-tangential intersection case.
The simplest case is when we get enough information on a generic cycle from its intersection points with Y (the order 0 case). This case is enough to prove corollary (0.8).
Two other cases are also treated. First, when the generic cycle in C cuts Y in only one point with multiplicity $k \geq 2$. A sufficient condition in this case to have local separation along Y is that the cycle X_s is determined (locally up to a finite set) by the couple $(y(s), C_{X_s,y(s)})$ where $\{y(s)\} = X_s \cap Y$ and where $C_{X_s,y(s)}$ is the image (assumed to be of codimension 1) of the Zariski tangent cone at $y(s)$ of X_s in $N_{Y/Z,y(s)}$.
The second simple situation is when, at the generic point of C, there exists at least one point with non-tangential intersection in $X_s \cap Y$. In this case the "normal quasi-cone" reduces to the collection of the contact elements given by the tangent spaces to the cycle at intersection points. It is interesting to note that this case cannot be deduced from our results in [B.Mg. 99]. Even in the case when the generic cycle is smooth and not tangent to Y at all of its intersection points with Y, we cannot reach this conclusion unless the intersection consists of a unique point.
With these tools we obtain that if there exists a component $C_{\max}$ for which we have
(i) $\varkappa'_{C_{\max}} = \varkappa'_{\max} := \sup_C \varkappa'_C$
(ii) the family $(X_s)_{s\in S}$ is locally separated along Y on $C_{\max}$,
then, assuming that the normal bundle $N_{Y/Z}$ is ample, the Cartier divisor Σ_Y has maximal Kodaira dimension in S[10]. In fact the ampleness of the normal bundle of Y allows us to exhibit enough global sections on S of some power of Σ_Y to separate the generic points of $C_{\max}$.

[8] Quasi-filtered means that locally on Σ_Y there exists $m_0 \in \mathbb{N}$ such that the map respects filtrations in degrees $\geq m_0$.
[9] These filtrations are defined in section 3.
[10] Here S and Y are assumed to be compact.

(0.14) Theorem. *Let* Z *be a complex manifold and let* $(X_s)_{s\in S}$ *be an analytic family of* $n-$*cycles in* Z *with* S *compact and connected. Let* Y *be a proper and generically finite pole with respect to* $(X_s)_{s\in S}$ *and let* Σ *denote the associated incidence divisor. If* Y *is an ample subvariety of* Z *and if the family is separated at order* 0, 1, *or by tangent cones along* Y, *then* Σ *is of maximal Kodaira dimension.*

(0.15) Remark. Here we assume S and Y compact. But the statement is local around $|Y|$ in Z and neither Z nor the cycles X_s are assumed to be compact.

(0.16) Final Remark. It is important to note that, even if the main application given here of this study is outside the context of complex (projective) algebraic geometry, our results lead to precise information on the Chow variety of a complex quasi-projective manifold. Even if the compact normal space S is already known to be projective it is interesting to build explicitly from geometric data on Z a Cartier divisor of S, and then produce from cohomology classes on Z enough global sections of the powers of the associated line bundle to ensure that its Kodaira dimension is maximal.

1. The hypersurface case.

(1.1). Let V be an open neighbourhood of the origin in $\mathbb{C}^{n+1}$ and consider an analytic family $(X_s)_{s\in S}$ of hypersurfaces in V parametrized by a reduced complex space S.
Let $P : S \times V \to \mathbb{C}$ be a (holomorphic) defining function for the graph of the family; thus X_s is the hypersurface defined by $P(s, z) = 0$ in V. Denote by $\sigma : S \to \mathbb{C}$ the holomorphic function on S given by $\sigma(s) := P(s, 0)$ and let Σ be the (Cartier) divisor defined by σ. We shall assume that $|\Sigma|$ is nowhere dense in S. Then Σ is the incidence divisor in S of the family $(X_s)_{s\in S}$ and the pole $Y := \{0\}$ (reduced in V).
Of course, we have $s \in \Sigma$ if and only if the origin belongs to X_s. Let Φ be the family of all closed sets F in V which are $\mathcal{X}-$proper[11]. Then we have an integration map (see [B.Mg. 98])

$$H^n_\Phi(V \setminus 0, \Omega^n_V) \to H^0(S \setminus \Sigma, \mathcal{O}_S)$$

which induces a filtered map (which we will also call an integration map)

$$\int_{\mathcal{X}} : H^{n+1}_{[0]}(V, , \Omega^n_V) \to H^1_{[\Sigma]}(S, \mathcal{O}_S),$$

where the filtrations are given respectively by the maximal ideal $\mathcal{M}_0$ of $\mathcal{O}_{V,0}$ and the ideal $\mathcal{I}_\Sigma = (\sigma)$ in $\mathcal{O}_S$. This means that the map $\int_{\mathcal{X}}$ sends $\mathrm{Ann}(\mathcal{M}_0^\nu)$ to $\mathrm{Ann}(\sigma^\nu)$ for each $\nu \in \mathbb{N}$, where $\mathrm{Ann}(\mathcal{M}_0^\nu)$ and $\mathrm{Ann}(\sigma^\nu)$ consist of those elements that are annihilated by $\mathcal{M}_0^\nu$ and σ^ν as global sections of the sheaves $\underline{H}^{n+1}_{[0]}(\Omega^n_V)$ and $\underline{H}^1_{[\Sigma]}(\mathcal{O}_S)$ respectively.

[11] This means that the pull-back of F on the graph $\mathcal{X}$ of the family $(X_s)_{s\in S}$ is proper on S. (See(2.3) and (2.4) for the general definition).

The aim of this section is to define a finer filtration on $\underline{H}^1_{[\Sigma]}(\mathcal{O}_S)$ in order to get an optimal bound for the pole order of the integral of a cohomology class of a given order.
Now using Grothendieck's local duality we may identify $H^{n+1}_{\{0\}}(V,\Omega^n_V)$ with the strong dual of the dual Fréchet space $\Omega^1_{V,0}$. Then the map $\int_{\mathcal{X}}$ is explicitly given by

$$\int_{\mathcal{X}}(\eta) = \partial\left\langle \frac{d_{|}P}{P}(s,z),\eta\right\rangle$$

where $d_{|}P$ is the $S-$relative differential of P (so we differentiate only with respect to z for fixed $s\in S\setminus\Sigma$ and $\frac{d_{|}P}{P}\in\Omega^1_{V,0}$), and where

$$\partial : H^0(S\setminus\Sigma,\mathcal{O}_S)\to H^1_{|\Sigma|}(S,\mathcal{O}_S)$$

is the “polar part ”map .
Let us consider a point $s_0\in\Sigma$ such that s_0 is a smooth point of S and of $|\Sigma|$ (after normalization of S, a generic point of $|\Sigma|$ always satisfies these two conditions because Σ is a Cartier divisor in S). For $\eta\in H^{n+1}_{[0]}(V,\Omega^n_V)$ denote by $w(\eta)$ the order of η relative to the $\mathcal{M}_0-$filtration and by $\varphi(\eta)$ the pole order of $\int_{\mathcal{X}}(\eta)$ along $|\Sigma|$ near s_0. Let q be the multiplicity of Σ near s_0 (so that $\Sigma = q.|\Sigma|$ near s_0) and define

$$\varkappa := \frac{1}{q}\sup\left\{\frac{\varphi(\eta)}{w(\eta)}\Big|\ \eta\in H^{n+1}_{[0]}(V,\Omega^n_V), \eta\neq 0\right\}$$

Then the following theorem holds:

(1.2) Theorem. *In the situation described above, where $|\Sigma|$ is a smooth hypersurface in a smooth S near s_0, define $k:=\mathrm{mult}_0(X_{s_0})$.*
Then for any integer $\mu\geq 1$ there exist an integer $m\in[\mu,\mu+k[$ and a class $\eta\in H^{n+1}_{[0]}(V,\Omega^n_V)$ such that $w(\eta)=m$ and $\varphi(\eta)=q.m.\varkappa$.

Proof of (1.2). The space S and its hypersurface $|\Sigma|$ are both smooth near s_0 so we can choose a coordinate system $(s_1,\cdots,s_p)$ centred at s_0 such that $|\Sigma|$ is defined by $s_1=0$. Since we have $\Sigma=q.|\Sigma|$ near s_0, it is sufficient to prove the theorem in the case where S is the unit disc in $\mathbb{C}$, $s_0=0$ and Σ is defined by $s^q=0$.
By definition of $k:=\mathrm{mult}_0(X_0)$ we can write

$$P(s,z) = P(s,0)+\sum_{j=1}^{k}P_j(s,z)\ -\ R(s,z)$$

where P_j is homogeneous of degree j in $z=(z_0,\cdots,z_n)$ and where R is of order $\geq k+1$ in z, with the following properties:

$$P_j(0,z)\equiv 0 \quad \text{for } 1\leq j\leq k-1, \quad P_k(0,z)\not\equiv 0 \quad \text{and} \quad P(s,0)=s^q.$$

For each $j\in[1,k]$ such that $P_j(s,z)$ is not identically zero, write

$$P_j(s,z) = -s^{\alpha_j}.Q_j(s,z) \quad \text{where} \quad Q_j(0,z)\not\equiv 0$$

and put $\alpha_j = +\infty$ when $P_j(s,z)$ is identically zero. Observe that

$$\alpha_j \geq 1 \ , \ \text{for} \quad j \in [1, k-1] \quad \text{and} \quad \alpha_k = 0.$$

We shall denote by $J_0 = \{j \in [1,k]/\alpha_j < \infty\}$. Remark that we have $k \in J_0$ so $J_0 \neq \emptyset$. We then have

$$P(s,z) = s^q - \left(\sum_{j \in J_0} s^{\alpha_j}.Q_j(s,z) + R(s,z) \right)$$

and to simplify the writing we put

$$T(s,z) = \sum_{j \in J_0} s^{\alpha_j}.Q_j(s,z) + R(s,z),$$

so $P(s,z) = s^q - T(s,z)$ or $P = s^q - T$ for short. Put

$$\tau := \frac{1}{q}.\max\left\{ \frac{q - \alpha_j}{j} \ \Big| \ 1 \leq j \leq k \right\} \tag{2}$$

and note that we obviously have

$$\frac{1}{k} \leq \tau \leq 1.$$

We shall prove the theorem by showing:
(i) For any $\eta \in H^{n+1}_{[0]}(V, \Omega^n_V)$ we have $\varphi(\eta) \leq q.\tau.w(\eta)$.
(ii) For any $\mu \geq 1$ there exists $\eta \in H^{n+1}_{[0]}(V, \Omega^n_V)$ of order $m \in [\mu, \mu + k[$ such that $\varphi(\eta) = q.\tau.w(\eta)$.
In particular this gives the equality $\tau = \varkappa$ and thereby formula (2) gives an easy way to compute $\varkappa$.

Proof of (i). Let $\eta \in H^{n+1}_{[0]}(V, \Omega^n_V)$ and put $m = w(\eta)$. Outside the zero locus of P we have

$$\frac{d_| P}{P} = -\frac{d_| T}{s^q - T} = -\sum_{j=0}^{m-1} \frac{T^j d_| T}{s^{q(j+1)}} - \frac{T^m d_| T}{(s^q - T).s^{q.m}}$$

where we use the $S-$relative differential $d_|$.
As T^m is of order $\geq m$ in z we have

$$\left\langle \frac{T^m d_| T}{(s^q - T).s^{q.m}} \, , \, \eta \right\rangle = 0$$

and consequently

$$\left\langle \frac{d_| P}{P}, \eta \right\rangle = -\sum_{j=1}^{m} \frac{1}{j.s^{q.j}} \left\langle d_| T^j, \eta \right\rangle = -\sum_{j=1}^{m} \frac{1}{j.s^{q.j}} \left\langle T^j, d\eta \right\rangle. \tag{3}$$

Note that $d\eta$ has order $\leq m+1$. For every $l \in [1,m]$ we have

$$T(s,z)^l \ = \ \left[\sum_{j \in J_0} s^{\alpha_j}.Q_j(s,z) + R(s,z) \right]^l \ = \sum_{|\gamma| + \nu = l} s^{\sum_{j=1}^{k}(\alpha_j \gamma_j)} \binom{l}{\gamma,\nu}.Q^\gamma.R^\nu$$

where $\nu \in \mathbb{N}$, $\gamma = (\gamma_1, \cdots, \gamma_k) \in \mathbb{N}^k$, $|\gamma| = \sum_{j=1}^k \gamma_j$,
$Q^\gamma = Q_1^{\gamma_1} \cdots Q_k^{\gamma_k}$ and $\binom{l}{\gamma,\nu}$ is the multinomial coefficient ; and consequently

$$\langle T^l, d\eta \rangle = \sum_{|\gamma|+\nu=l} s^{\sum_{j=1}^k (\alpha_j \gamma_j)} \binom{l}{\gamma,\nu} \langle Q^\gamma . R^\nu, d\eta \rangle . \tag{4}$$

By combining (3) and (4) we get

$$\left\langle \frac{d_{|}P}{P}, \eta \right\rangle = -\sum_{l=1}^m \frac{1}{l} \sum_{|\gamma|+\nu=l} s^{-e(\gamma,\nu,l)} \binom{l}{\gamma,\nu} \langle Q^\gamma . R^\nu, d\eta \rangle \tag{5}$$

where $e(\gamma,\nu,l) = q.l - \sum_{j \in J_0} \alpha_j \gamma_j$.
The order of $Q^\gamma . R^\nu$ in z is at least

$$\sum_{j=1}^k j.\gamma_j + \nu(k+1)$$

so that $\langle Q^\gamma . R^\nu, d\eta \rangle = 0$ if

$$\sum_{j=1}^k j.\gamma_j + \nu(k+1) > m.$$

Thus to prove that the function

$$s \mapsto \left\langle \frac{d_{|}P}{P}(s,z), \eta \right\rangle$$

has a pole of order at most $q.m.\tau$ at $s = 0$ it is sufficient to prove that for every $(\gamma,\nu) \in \mathbb{N}^k \times \mathbb{N}$ satisfying

$$\sum_{j=1}^k \gamma_j + \nu \; = \; l \qquad \text{and} \qquad \sum_{j=1}^k j.\gamma_j + \nu(k+1) \; \leq \; m \tag{6}$$

we have

$$q.l \; - \; \sum_{j=1}^k \alpha_j.\gamma_j \; \leq \; q.m.\tau \; .$$

But under the conditions in (6) we have

$$\begin{aligned}
2q.l \; - \; \sum_{j=1}^k \alpha_j.\gamma_j \; &= \sum_{j=1}^k q.\gamma_j + q.\nu - \sum_{j=1}^k \alpha_j.\gamma_j \\
&= \sum_{j=1}^k \frac{q-\alpha_j}{j}.j.\gamma_j + q.\nu \\
&\leq \; q.\tau.\sum_{j=1}^k j.\gamma_j + q.\nu && \text{(by definition of } \tau) \\
&\leq \; q.\tau.(m - \nu.(k+1)) + q.\nu \\
&= q.m.\tau + q.\nu.(1 - \tau.(k+1)) \\
&\leq q.m.\tau && \left(\text{because} \quad \tau \geq \frac{1}{k}\right)
\end{aligned}$$

and this completes the proof of (i).

Proof of (ii). First we note that in the previous computation the inequality

$$q.l \ - \ \sum_{j=1}^{k} \alpha_j.\gamma_j \ \leq \ q.m.\tau$$

is an equality if and only if the following three conditions are satisfied:

- $\nu = 0$,
- $\sum_{j=1}^{k} j.\gamma_j = m$,
- $\gamma_j = 0$ if $q - \alpha_j < j.q.\tau$.

Put $J := \{j \in [1, k] \mid q - \alpha_j = j.q.\tau\}$ and $\Gamma_m := \left\{\gamma \in \mathbb{N}^k \;\Big|\; \sum_{j \in J} j.\gamma_j = m\right\}$.

From the decomposition in (5) we then see that the meromorphic function $s \mapsto \left\langle \frac{d_{|}P}{P}, \eta \right\rangle$ has a pole of order $q.m.\tau$ at $s = 0$ if and only if the holomorphic function

$$s \mapsto \left\langle \sum_{\gamma \in \Gamma_m} \binom{|\gamma|}{\gamma} \frac{1}{|\gamma|} Q^{\gamma}(s, z) \ , \ d\eta \right\rangle$$

does not vanish at $s = 0$, that is to say if and only if

$$\left\langle \sum_{\gamma \in \Gamma_m} \binom{|\gamma|}{\gamma} \frac{1}{|\gamma|} Q^{\gamma}(0, z) \ , \ d\eta \right\rangle \neq 0.$$

We observe that $\sum_{\gamma \in \Gamma_m} \binom{|\gamma|}{\gamma} \frac{1}{|\gamma|} Q^{\gamma}(0, z)$ is the homogeneous component of degree m of the polynomial

$$\sum_{l=1}^{m} \frac{1}{l} \Big(\sum_{j \in J} Q_j(0, z) \Big)^{l}. \tag{7}$$

Claim. Write $J = \{ j_1, \ldots, j_r \}$ with $j_1 < \cdots < j_r$. Then in any interval in $\mathbb{N}^*$ of length $> j_r$ there exists an integer m such that the homogeneous component of degree m of the polynomial in (7) is not identically 0. Moreover such an integer m is a multiple of $\gcd\{j_1, \cdots, j_r\}$.

Proof of the claim. Clearly if $m \notin \mathbb{N}j_1 + \cdots + \mathbb{N}j_r$ there is no homogeneous component of degree m for (7) , so any solution is a multiple of $\gcd\{j_1, \ldots, j_r\}$. Let $z^0 \in \mathbb{C}^{n+1}$ such that $Q_j(0, z^0) \neq 0$ for all $j \in J$, and let $\pi : \mathbb{C} \to \mathbb{C}$ be defined by

$$\pi(t) := \sum_{j \in J} Q_j(0, t.z^0)$$

It is enough to show that for any $\mu \in \mathbb{N}^*$ there exists $m \in [\mu, \mu + j_r[$ such that the coefficient of t^m in

$$\sum_{l=1}^{m} \frac{1}{l} [\pi(t)]^{l}$$

is not 0. The formal power series

$$F(t) := \sum_{l=1}^{+\infty} \frac{1}{l}[\pi(t)]^l \tag{8}$$

is well defined because $\pi(0) = 0$ and it clearly has the same coefficient of t^m as the polynomial $\sum\limits_{l=1}^{m} \frac{1}{l}[\pi(t)]^l$. On differentiating equation (8) we find that $F'(t).[1-\pi(t)] = \pi'(t)$. Hence if we write

$$1-\pi(t) = 1 + \sum_{j\in J} b_j.t^j \quad \text{and} \quad F'(t) = \sum_{0}^{+\infty} c_j.t^j$$

it follows that

$$c_j = -(b_1.c_{j-1} + \cdots + b_{j_r}.c_{j-j_r}) \quad \text{for all} \quad j \geq j_r.$$

Therefore if $c_{j-j_r} = \cdots = c_{j-1} = 0$ for some $j \geq j_r$ then F' is a polynomial. But this is never the case because the degree of π is $j_r \geq 1$, which thus proves the claim.

End of the proof of (ii). Fix an integer $\mu \in \mathbb{N}^*$ and choose any $m \in [\mu, \mu + j_r[\subset [\mu, \mu + k[$ such that the homogeneous component of degree m of the polynomial in (7) is not identically zero. Let ∂_0 denote the analytic Dirac functional at the origin in $\mathbb{C}^{n+1}$, defined by:

$$\langle f.dz_0 \wedge \cdots \wedge dz_n, \partial_0 \rangle = f(0) \ .$$

For every $\alpha \in \mathbb{N}^{n+1}$ put

$$\partial_0^{(\alpha)} := \frac{1}{\alpha!}\frac{\partial^{|\alpha|}(\partial_0)}{\partial z^\alpha}. \tag{9}$$

Pick $\alpha \in \mathbb{N}^{n+1}$ with $|\alpha| = m$ such that the coefficient of z^α in the polynomial (7) is not zero. Let i be such that $\alpha_i \neq 0$ and put

$$\eta = (-1)^i \partial_0^{(\alpha')} dz_0 \wedge \cdots \widehat{dz_i} \wedge \cdots dz_n,$$

where $\alpha' = (\alpha_0, \ldots, \alpha_i - 1, \ldots, \alpha_n)$. Clearly η is an element of order $|\alpha'| + 1 = |\alpha| = m$ in $H^{n+1}_{[0]}(V, \Omega^n_V)$ such that

$$\langle z^\beta, d\eta \rangle = \begin{cases} 0 & \text{for} \quad \beta \neq \alpha \\ 1 & \text{for} \quad \beta = \alpha. \end{cases}$$

Hence $\left\langle \sum\limits_{\gamma \in \Gamma_m} \binom{|\gamma|}{\gamma} \frac{1}{|\gamma|}.Q^\gamma(0,z) \ , \ d\eta \right\rangle \neq 0$
and this completes the proof of (ii) . $\square$

(1.3) Remarks.

1) We actually proved a stronger result than announced in the theorem. Namely, for any given defining function γ for $|\Sigma|$ near s_0 we constructed a cohomology class η of order m satisfying $q.\varkappa.m \in \mathbb{N}$ and

$$\lim_{s\to s_0} \gamma^{q.\varkappa.m}(s) \int_{\mathcal{X}} (\eta)(s) \neq 0.$$

This will be used later on.

2) Formula (2) gives an explicit way to compute $\varkappa$ and from this formula it is easy to deduce the following:

- For $k = 1$ we always have $\varkappa = 1$.
- We have $\Sigma = q.|\Sigma|$ near s_0, in the case $q = 1$ we get $\varkappa = \frac{1}{k}$.
- In general ($k \geq 2$, $q \geq 2$) we have the inequalities $\frac{1}{k} \leq \varkappa \leq \frac{q-1}{q}$.

The geometric significance of $\varkappa$ will become clear in (2.9) and its algebraic meaning is explained in section 3.

2. The general case.

(2.1). The main results of this section are theorem (2.23) and its corollary (2.26). They give a generalization of theorem (1.2) to the case where the cycles are of arbitrary codimension. These results are considerably more difficult than theorem (1.2) and require some changes of the setting. Among other things we have to change the definition of pole order for the differential forms and cohomology classes in question.

In the sequel we will frequently make use of the fact that for a Stein open neigbourhood V of the origin in $\mathbb{C}^{n+1}$ with $n \geq 1$ the canonical mapping :

$$H^n(V \setminus \{0\}, \Omega^n_V) \to H^{n+1}_{\{0\}}(V, \Omega^n_V)$$

is an isomorphism. For an element η in $H^{n+1}_{[0]}(V, \Omega^n_V)$ we will refer to any $\bar\partial-$closed $(n, n)-$form in $V \setminus \{0\}$ that defines η via this isomorphism as a **Martinelli representative of** η. As before we let ∂_0 denote the Dirac functional at the origin in $\mathbb{C}^{n+1}$ and put

$$\partial_0^{(\alpha)} := \frac{1}{\alpha!} \frac{\partial^{|\alpha|}(\partial_0)}{\partial z^\alpha}.$$

for all α in $\mathbb{N}^{n+1}$.

(2.2) Lemma. Let $z = (z_0, \cdots, z_n)$ be the standard coordinates in $\mathbb{C}^{n+1}$. Then for every $\alpha \in \mathbb{N}^{n+1}$ there exists a Martinelli representative ψ_α of $\partial_0^{(\alpha)}$ in $\mathbb{C}^{n+1}$ such that the $(0, n)-$form

$$||z||^{2n+1+\varepsilon+|\alpha|}\psi_\alpha$$

has a continuous extension at 0, for every $\varepsilon > 0$.

Proof. First we observe that the standard Martinelli representative of ∂_0

$$\psi_0 = ||z||^{-(2n+2)} \sum_{j=0}^{n} (-1)^j \bar{z}_j \wedge_{k\neq j} d\bar{z}_k$$

has the required property, more precisely the $(0,n)$-form $||z||^{2n+1+\varepsilon}\psi_0$ has a continuous extension at 0, for every $\varepsilon > 0$.

Obviously the $(0,n)$-form $\psi_\alpha := \dfrac{1}{\alpha!}\dfrac{\partial^{|\alpha|}\psi_0}{\partial z^\alpha}$ is a Martinelli representative of $\partial_0^{(\alpha)}$ for all $\alpha \in \mathbb{N}^{n+1}$. We claim that the $(0,n)$-form $||z||^{2n+1+\varepsilon+|\alpha|}\psi_\alpha$ has a continuous extension at 0, for every $\varepsilon > 0$. To prove this we only have to note that $\dfrac{\partial^{|\alpha|}}{\partial z^\alpha}(||z||^{-(2n+2)})$ is a constant multiple of $\dfrac{\bar{z}^\alpha}{||z||^{2n+2+2|\alpha|}}$. □

(2.3) Definition. Let Z be a complex manifold and let $(X_s)_{s\in S}$ be an analytic family of $n-$cycles in Z. Let $\mathcal{X}$ denote the graph of the family and fix a point $s_0 \in S$.

- We say that a closed set F in Z is **$\mathcal{X}$-proper near** s_0 if there exists a fixed compact set $K \subset Z$ such that for all s near s_0 we have
$$F \cap |X_s| \subset K.$$
A closed set in Z is said to be **$\mathcal{X}$-proper** if it is $\mathcal{X}$- proper near every point in S.
- We say that a $\bar\partial-$closed (n,n) $\mathcal{C}^\infty-$form ψ on an open subset U of Z is **$\mathcal{X}$-proper near** s_0 (respectively **$\mathcal{X}$-proper**) if its support is $\mathcal{X}$-proper near s_0 (respectively $\mathcal{X}$-proper).

(2.4) Remark. Let $p : \mathcal{X} \to Z$ and $\pi : \mathcal{X} \to S$ be the canonical projections of the graph of the family $(X_s)_{s\in S}$ and let F be a closed subset of Z. Then F is $\mathcal{X}-$proper near s_0 (respectively $\mathcal{X}-$proper) if and only if the subset $p^{-1}(F)$ of $\mathcal{X}$ is $\pi-$proper in a neighbourhood of s_0 (respectively $\pi-$proper).

(2.5). In the sequel we will frequently refer to the following situation:

Let Z be a Stein open neighbourhood of the origin in $\mathbb{C}^{n+p}$ with $p \geq 1$ and let $(X_s)_{s\in S}$ be an analytic family of $n-$cycles in Z with graph $\mathcal{X}$. Let $f_0,\dots,f_n$ be a regular sequence of holomorphic functions in Z vanishing at 0 and let W be the image of Z by the associated mapping into $\mathbb{C}^{n+1}$. Denote by $f : Z \to W$ the corresponding flat surjective map and put $Y := f^{-1}(0)$. Suppose that

- the map f is finite on $|X_s|$ for every $s \in S$,
- Y is a finite pole for the family $(X_s)_{s\in S}$.

Let Σ denote the incidence divisor of Y and $(X_s)_{s\in S}$. Then $(f_*X_s)_{s\in S}$ is a well defined analytic family of hypersurfaces in W, and the (reduced) point 0 is a (finite) pole for this family. By the direct image invariance proved in [B.Mg. 98] we know that Σ is equally the incidence divisor of $(f_*X_s)_{s\in S}$ and $\{0\}$.

(2.6) Lemma. Assume the hypothesis in (2.5). Then for every $\eta \in H^{n+1}_{\{0\}}(\Omega^n_{\mathbb{C}^{n+1}})$ there exists an open neighbourhood Z_1 of 0 in Z and a Martinelli representative ψ of η on $f(Z_1)\setminus\{0\}$ such that $(f|_{Z_1})^*\psi$ is $\mathcal{X}|_{Z_1}-$proper .

Proof. We split the proof into three steps :

- (Step 1) Let S be a reduced complex space, U and B be open polydiscs in $\mathbb{C}^n$ and $\mathbb{C}^p$ respectively and let $f : S \times U \to \mathrm{Sym}^k(\mathbb{C}^p)$ be a holomorphic map defining an analytic family of $n-$cycles $(X_s)_{s\in S}$ in $U \times B$. Then for any compact set $K \subset U$ the closed set $K \times B$ is obviously $\mathcal{X}-$proper.

- (Step 2) Let $q : Z \to Z'$ be a proper finite surjective holomorphic map between two complex manifolds. Let $(X_s)_{s\in S}$ be an analytic family of $n-$cycles in Z and denote by $(q_*X_s)_{s\in S}$ the direct image family in Z'. Then a closed set $F \subset Z'$ is $q_*(\mathcal{X})-$proper if and only if the closed set $q^{-1}(F) \subset Z$ is $\mathcal{X}-$proper.
- (Step 3) Up to a change of the functions $f_0, \dots, f_n$ by generic linear combinations, we may assume that $\{f_1 = \cdots = f_n = 0\} \cap X_{s_0} = \{0\}$ near 0 in Z. Choose holomorphic functions $g_1, \dots, g_{p-1}$ near 0 such that $f_0 = \cdots = f_n = g_1 \cdots = g_{p-1} = 0$ define the origin (set theoretically) near 0. Then, shrinking Z around the origin, we may assume that the map

$$q := (f_0, \dots, f_n, g_1, \dots, g_{p-1}) : Z \to Z' := q(Z) \subset \mathbb{C}^{n+p}$$

is proper, finite and surjective. Now define a scale $U \times B$ adapted to the cycle $q_*(X_{s_0})$ near the origin in Z' using coordinates corresponding to $f_1, \dots, f_n$ for U and $g_1, \dots, g_{p-1}, f_0$ for B , and shrink S around s_0 in order to have an analytic family of cycles in $U \times B$. By step 1, for any compact set $K \subset U$ the set $K \times B$ is $q_*(\mathcal{X})-$proper. Now consider the map $1_U \times f_0 : U \times B \to U \times D$. The direct image of the family $(q_*(X_s))_{s\in S}$ in $U \times B$ by this map is simply the direct image by the map $f = (f_1, \dots, f_n, f_0)$ of our initial family $(X_s)_{s\in S}$. By step 1 it is then clear that for any compact set $K \subset U$, $K \times D$ is $f_*(\mathcal{X})-$proper. Thus from steps 1 and 2 we conclude that $f^{-1}(K \times D)$ is $\mathcal{X}-$proper. Choosing a Martinelli representative ψ on $U \times D \setminus \{0\}$ for η such that $\mathrm{Supp}(\psi) \subset K \times D$, where K is a compact neighbourhood of 0 in U[12], concludes the proof. ∎

(2.7) Lemma. Consider the situation in (2.5) and let s_0 be a point in Σ. Let ψ_α be a Martinelli representative of $\partial_0^{(\alpha)}$ in $W \setminus \{0\}$ such that $f^*\psi_\alpha$ is $\mathcal{X}-$proper near s_0 and put $\rho(X_s) := \mathrm{dist}(0, |f_*X_s|)$. Then for every $\varepsilon > 0$, for every holomorphic $n-$form ω on W and for every holomorphic function g on Z , the function

$$s \mapsto \rho(X_s)^{1+|\alpha|+\varepsilon} \int_{X_s} g.f^*(\psi_\alpha \wedge \omega)$$

is bounded on $S' \setminus S' \cap \Sigma$.

Proof. Let S_0 be a neighbourhood of s_0 such that $\mathrm{Supp} f^*\psi_\alpha \cap |X_s|$ is contained in a compact set L of Z for all s in S_0. Let D and U be relatively compact open polydiscs centred at the origin in $\mathbb{C}$ and $\mathbb{C}^n$ respectively and let S' be a relatively compact neighbourhood of s_0 in S_0 such that

$$\bar{D} \times \bar{U} \subset W, \quad |f_*X_s| \cap (D \times U) \subset \frac{1}{2}D \times U,$$

and such that f_*X_s induces a $k-$branched covering of U for all s in S', where $\frac{1}{2}D$ denotes the set D contracted by the factor $\frac{1}{2}$.
Now let $\varepsilon > 0$, $g \in \mathcal{O}(Z)$ and $\omega \in \Omega^n(W)$. Let $\tau \in \mathcal{C}_c^\infty(D \times U)$ be identically 1 in a neighbourhood of $\frac{1}{2}\bar{D} \times \{0\}$. Then the function

$$s \mapsto \int_{X_s} f^*(\psi_\alpha \wedge \omega) - \int_{X_s} f^*(\tau\psi_\alpha \wedge \omega) = \int_{X_s} f^*((1-\tau)\psi_\alpha \wedge \omega)$$

[12] This is possible by lemma 1 p.74 of [B.Mg. 99].

is continuous on S and thus bounded on S' since $\bar{S}'$ is compact. So it is enough to prove the lemma for $\tau.\psi_\alpha$ instead of ψ_α.
Denote by $z = (z_0, \dots, z_n)$ the coordinates in $\mathbb{C}^{n+1}$. From lemma (2.2) we know that the representative ψ_α can be chosen such that the $(n,n)-$form $||z||^{2n+1+|\alpha|+\varepsilon/2}\tau.\psi_\alpha \wedge \omega$ extends to a continuous form φ on W. By the "multi-projection trick" (see [B. 79]) one can write φ as a finite sum of forms of the type $r(t,z).dt \wedge d\bar{t}$ where $(z,t) = (z, t_1, \dots, t_n)$ is a coordinate system for $\mathbb{C} \times \mathbb{C}^n$, $dt \wedge d\bar{t} = dt_1 \wedge \dots \wedge dt_n \wedge d\bar{t}_1 \wedge \dots \wedge d\bar{t}_n$ and r is a continuous function with compact support in $D \times K$ where K is a compact subset of U. This is done by mappings of the type

$$(z, t_1, \dots, t_n) \to (t_1 - a_1.z, \cdots, t_n - a_n.z).$$

One should note that the condition $|f_*X_s| \cap (D \times U) \subset \frac{1}{2}D \times U$ ensures that there exists $\delta > 0$ such that for any $a = (a_1, \dots, a_n)$ with $||a|| < \delta$ the corresponding perturbed projection makes f_*X_s a $k-$branched covering in a neighbourhood of $\frac{1}{2}\bar{D} \times K$. Moreover, by shrinking S' if necessary, δ can be chosen uniformly with respect to s.
On the set $|f_*(X_s)|$ we have

$$\rho(X_s)^{1+|\alpha|+\varepsilon}.\tau.\psi_\alpha \wedge \omega = \left(\frac{\rho(X_s)}{||(z,t)||}\right)^{1+|\alpha|+\varepsilon} \frac{\varphi}{||(z,t)||^{2n-\varepsilon/2}}.$$

Hence it is enough to show that, for a continuous function r with compact support on $D \times U$, the function

$$s \mapsto \int_{X_s} g.f^*\left(\left(\frac{\rho(X_s)}{||(z,t)||}\right)^{1+|\alpha|+\varepsilon} . \frac{r(z,t)}{||(z,t)||^{2n-\varepsilon/2}} . dt \wedge d\bar{t}\right)$$

is bounded on $S' \setminus S' \cap \Sigma$.
Put

$$h := g.f^*\left(\left(\frac{\rho(X_s)}{||(z,t)||}\right)^{1+|\alpha|+\varepsilon} . r(z,t)\right).$$

Obviously we have $\mathrm{Supp}(h) \subset \mathrm{Supp}(f^*\psi_\alpha)$ so $\mathrm{Supp}(h) \cap |X_s| \subset L$ for all s in S'. Since $\rho(X_s) \leq ||(z,t)||$ for $(z,t) \in |f_*(X_s)|, \forall s \in S' \setminus S' \cap \Sigma$, and $g.f^*r$ is continuous on L, there exists a constant C such that $||h||_{|X_s|} \leq C$ for all s in $S' \setminus S' \cap \Sigma$. From this we deduce that

$$\left|\mathrm{trace}_{X_s}\left(h.f^*\left(\frac{1}{||(z,t)||^{2n-\varepsilon/2}}\right)\right)\right| \leq \frac{k.C}{||t||^{2n-\varepsilon/2}}$$

for all s in $S' \setminus S' \cap \Sigma$; here the trace is taken with respect to the (finite) map

$$(f_1, \dots, f_n) : X_s \to \mathbb{C}^n.$$

The support of the trace in the above formula is clearly contained in K so we finally get

$$\left|\int_{X_s} h.f^*\left(\frac{dt \wedge d\bar{t}}{\|(z,t)\|^{2n-\varepsilon/2}}\right)\right| = \left|\int_K \mathrm{trace}_{X_s}\left(h.f^*\left(\frac{1}{\|(z,t)\|^{2n-\varepsilon/2}}\right)\right) dt \wedge d\bar{t}\right|$$
$$\leq k.C\left|\int_K \frac{dt \wedge d\bar{t}}{\|t\|^{2n-\varepsilon/2}}\right|.$$

Since the last integral is convergent, this completes the proof. □

(2.8) Remark. The following proposition relates the rational number $\varkappa$ to the geometric behaviour of the family of cycles near the pole.

(2.9) Proposition. Let D be an open polydisc centred at the origin in $\mathbb{C}^l$ with coordinates $s = (s_1, \dots, s_l)$. Let $(X_s)_{s\in D}$ be an analytic family of hypersurfaces in an open neighbourhood of the origin in $\mathbb{C}^{n+1}$ such that the incidence divisor of the family with the origin is given by $s_1^q = 0$. Let $\varkappa$ be the rational number associated with the family at $s = 0$. Then, after shrinking D if necessary, there exists a constant $C > 0$ such that

$$\mathrm{dist}(0, |X_s|) \ \geq \ C.|s_1|^{q.\varkappa} \quad \text{for all} \quad s \in D \ .$$

Proof. The statement is local along Σ. Let U and Δ be polydiscs centred at the origin in $\mathbb{C}^n$ and $\mathbb{C}$ respectively, such that $U \times \Delta$ is an adapted scale for X_0 of degree $k = \mathrm{mult}_0(X_0)$. If $P(s,t,z) = 0$ is an equation of the graph of the family $(X_s)_{s\in D}$ in $D\times U\times\Delta$ we know that the initial form at (0,0) of $P(0,t,z)$ is homogeneous of degree k, and so we may write (compare with the proof of theorem (1.2)), shrinking D around $s = 0$ if necessary,

$$P(s,t,z) = P(s,0,0) + \sum_{j=1}^{k-1} s_1^{\alpha_j} Q_j(s,t,z) + R_k(s,t,z)$$

with $P(s,0,0) = s_1^q$, with Q_j homogeneous of degree j in (t,z) such that either $Q_j \equiv 0$ or $Q_j(0,s_2,\dots,s_l,t,z) \not\equiv 0$[13], $\alpha_j \in \mathbb{N}^*$ for $j \in [1,k-1]$ and

$$R_k(s,t,z) = \sum_{|\beta|=k} (t,z)^\beta G_\beta(s,t,z)$$

where $(t,z)^\beta = t_1^{\beta_1}.t_2^{\beta_2}\dots t_n^{\beta_n}.z^{\beta_0}$ for $\beta \in \mathbb{N}^{n+1}$ and where the functions G_β are holomorphic in $D\times U\times\Delta$.
Recall that $J_0 = \{j \in [1,k]/Q_j \not\equiv 0\}, \alpha_j = +\infty$ for $j \notin J_0$, and that $\alpha_k = 0$. Then we have

$$\varkappa = \frac{1}{q} \ . \max_{j\in[1,k]} \left(\frac{q-\alpha_j}{j}\right).$$

Now fix $U' \subset\subset U$ and $\Delta' \subset\subset \Delta$ relatively compact polydiscs centered at the origin in $\mathbb{C}^n$ and $\mathbb{C}$ respectively. Let $R > 0$ be such that for any $j \in [1,k-1]$ we have

$$R \geq ||Q_j||_{\frac{1}{2}\bar{D}\times\bar{U}'\times\bar{\Delta}'}$$
$$R \geq \sum_{|\beta|=k} ||G_\beta||_{\frac{1}{2}\bar{D}\times\bar{U}'\times\bar{\Delta}'}$$

Assuming now that the radius of D for the first variable is at most 1, we shall prove that $0 < \rho < \inf\{\frac{1}{4.R}, \frac{1}{2}\}$ implies

$$|X_s| \cap \Big(\{||t|| \leq \rho.|s_1|^{q.\varkappa}\} \times \{|z| \leq \rho.|s_1|^{q.\varkappa}\}\Big) = \emptyset,$$

[13] in (t,z) for $(0,s_2,\cdots,s_l)$ generic in $|\Sigma|$.

for all $s \in \frac{1}{2}\bar{D} \setminus \Sigma$. For any given $\rho > 0$ and any $(s,t,z) \in \frac{1}{2}\bar{D} \times \bar{U}' \times \bar{\Delta}' \setminus \Sigma$ define $\tau \in \mathbb{C}^n$ and $\xi \in \mathbb{C}$ as follows

$$t = \rho.|s_1|^{q.\varkappa}.\tau$$
$$z = \rho.|s_1|^{q.\varkappa}.\xi$$

If now we have $P(s,t,z) = 0$ for $(s,t,z) \in \frac{1}{2}\bar{D} \times \bar{U}' \times \bar{\Delta}' \setminus \Sigma$ then we obtain

$$s_1^q + \sum_{j \in J_0} s_1^{\alpha_j}.Q_j(s,t,z) + R_k(s,t,z) = 0.$$

So , using the definition of τ and ξ , homogeneity and the choice of R , we get

$$|s_1^q| \leq \sum_{j \in J_0} |s_1|^{\alpha_j}.\rho^j.R.|s_1|^{j.\varkappa.q} + \rho^k.R.|s_1|^{k.\varkappa.q}$$

and so

$$|s_1^q| \leq R.\sum_{j \in J_0} |s_1|^{\alpha_j + j.q.\varkappa}.\rho^j \quad .$$

But, by definition of $\varkappa$, we have for any $j \in [1,k]$

$$\alpha_j + j.q.\varkappa \geq q,$$

so for $s_1 \neq 0$ we obtain

$$1 \ \leq \ R.\sum_{j=1}^{j=k} \rho^j \ \leq \frac{R.\rho}{1-\rho} \ \leq \ \frac{1}{2}$$

which is absurd ! Thus we conclude that for any $s \in \frac{1}{2}\bar{D} \setminus \Sigma$ we have

$$|X_s| \cap \{\, (t,z) \in \bar{U}' \times \bar{\Delta}' \mid ||(t,z)|| \leq \rho.|s_1|^{q.\varkappa} \,\} = \emptyset.$$

This implies that there exists a constant $C > 0$ such that in $\bar{U}' \times \bar{\Delta}'$

$$\mathrm{dist}(0, |X_s|) \geq C.|s_1|^{q.\varkappa}$$

for any $s \in \frac{1}{2}\bar{D}$. $\square$

(2.10) Corollary. *Assume the hypothesis of (2.5) and let s_0 be a point in Σ where both $|\Sigma|$ and S are smooth. Denote by q the multiplicity of the divisor Σ near s_0 and denote by $\varkappa$ the rational number associated with the analytic family of hypersurfaces $(f_*X_s)_{s \in S}$ at s_0. Let ψ_α be a Martinelli representative of $\partial_0^{(\alpha)}$ in $W \setminus \{0\}$ such that $f^*\psi_\alpha$ is $\mathcal{X}-$proper near s_0. Then for every holomorphic $n-$form ω on W and for every holomorphic function g on Z the meromorphic function*

$$s \mapsto \int_{X_s} g.f^*(\psi_\alpha \wedge \omega)$$

has a pole of order $\leq q.\varkappa(1 + |\alpha|)$ along $|\Sigma|$ near s_0. In other words the pole order of the meromorphic function is bounded by $q.\varkappa.$(order of $\psi_\alpha \wedge \omega$).

Proof. This is a direct consequence of lemma (2.7) and proposition (2.9).

(2.11). The natural question to ask now, is whether we get the same kind of bound as in corollary (2.10), if we replace $g.f^*\omega$ by any holomorphic $n-$form on Z; in other words, does the meromorphic function

$$s \mapsto \int_{X_s} f^*(\psi_\alpha) \wedge \varphi$$

have a pole of order $\leq q.\varkappa(1+|\alpha|)$ for all holomorphic $n-$forms φ on Z ?
The following example shows that the answer to this question is **no !**

(2.12) Example. We consider $\mathbb{C}^3$ with coordinates (z_1, z_2, x) and let $(X_s)_{s\in\mathbb{C}}$ be the analytic family of $1-$cycles defined by

$$X_s := \{z_1 = x^k - s\} \cap \{z_2 = (z_1 + s).x\}$$

where $k \geq 2$ is a integer. We shall denote by $\mathcal{X}$ the graph of this family.
Let $Y = \{z_1 = z_2 = 0\}$ and let $pr : \mathbb{C}^3 \to \mathbb{C}^2$ be the projection $pr(z_1, z_2, x) = (z_1, z_2)$. For any s the direct image $pr_*(X_s)$ is a well-defined $1-$cycle in $\mathbb{C}^2$ and it is easy to see that it is the hypersurface given by

$$P_s(z_1, z_2) = z_2^k - (z_1 + s)^{k+1} = 0.$$

Thus the incidence divisor of Y is given by $\sigma(s) = -s^{k+1} = 0$ as a Cartier divisor in $\mathbb{C}$. So with the terminology introduced above we have $q = k+1$ and $k = k$ is the multiplicity of the origin in the cycle $pr_*(X_0)$. The integers α_j are given by $\alpha_j = k + 1 - j$ for $j \in [1, k-1]$ and $\alpha_k = 0$. Consequently

$$\varkappa \;=\; \frac{1}{q}\sup\left\{\frac{q-j}{j} \;\Big|\; j \in [1,k]\right\} \;=\; \frac{1}{k}\ .$$

Let $\psi_{(0,k)}$ be a $\mathcal{X}-$proper Martinelli representative of $\partial_0^{((0,k))}$ in the open set $\mathbb{C}^2 \setminus \{(0,0)\}$. The $\bar{\partial}-$closed $(1,1)-$form $w := pr^*\psi_{(0,k)} \wedge dx$ has order $k+1$ along Y and we shall show that the pole order of the meromorphic function $F(s) = \int_{X_s} w$ at $s = 0$ is $k+3$, which is strictly bigger than $q.\varkappa.(k+1) = \frac{(k+1)^2}{k} = k + 2 + \frac{1}{k}$ [14].
On X_s we have the identity $x.(z_1+s) = z_2$ from which we deduce $s.x = z_2 - x.z_1$ and thus

$$s^2.x = s.(z_2 - x.z_1) = s.z_2 - (z_2 - x.z_1).z_1 = s.z_2 - z_1.z_2 + x.z_1^2 \quad .$$

From this and Stokes' formula we deduce

$$\begin{aligned} s^2.F(s) =& s^2.\int_{X_s} pr^*\psi_{(0,k)} \wedge dx = s^2.\int_{X_s} x.pr^*d\psi_{(o,k)} \\ =& s.\int_{pr_*X_s} z_2.d\psi_{(0,k)} - \int_{pr_*X_s} z_1.z_2.d\psi_{(0,k)} + \int_{X_s} x.pr^*(z_1^2.d\psi_{(0,k)})\ . \end{aligned}$$

[14] Note that for the new order defined by the conormal filtration (see (2.13)) the order of w is $k+2$ and we have $q.\varkappa.(k+2) = \frac{(k+1)(k+2)}{k} \geq k+3,\ \forall k \in \mathbb{N}$.

Since the image of $z_1^2.d\psi_{(0,k)}$ in $H^2_{[0]}(\mathbb{C}^2, \Omega^1_{\mathbb{C}^2})$ is zero, the function $s \mapsto \int_{X_s} x.pr^*(z_1^2.d\psi_{(0,k)})$ is holomorphic near $s = 0$. The polar part of $s^2.F(s)$ at $s = 0$ is thus equal to the polar part of

$$\begin{aligned}
s^2.F(s) &= s.\left\langle \frac{dP_s}{P_s}, z_2.d\partial_0^{((0,k))} \right\rangle - \left\langle \frac{dP_s}{P_s}, z_1.z_2.d\partial_0^{((0,k))} \right\rangle \\
&= s.\left\langle \frac{dP_s}{P_s}, -\partial_0^{((1,k-1))}.dz_1 - k.\partial_0^{((0,k))}.dz_2 \right\rangle - \left\langle \frac{dP_s}{P_s}, -\partial_0^{((0,k-1))}.dz_1 \right\rangle \\
&= s.\left[\frac{1}{(k-1)!} \frac{\partial^{k+1}}{\partial z_1 \partial z_2^k}(\mathrm{Log}P_s)(0,0) - \frac{k+1}{k!} \frac{\partial^{k+1}}{\partial z_1 \partial z_2^k}(\mathrm{Log}P_s)(0,0) \right] - \\
&\qquad \frac{1}{(k-1)!} \frac{\partial^k}{\partial z_2^k}(\mathrm{Log}P_s)(0,0) \\
&= s.\left[\frac{k(k+1)}{s^{k+2}} - \frac{(k+1)^2}{s^{k+2}} \right] + \frac{k}{s^{k+1}} = \frac{-1}{s^{k+1}}.
\end{aligned}$$

We conclude that $F(s)$ has a pole of order $k+3$ at $s = 0$.

(2.13). Theorem (1.2) gives us the best possible bound for the pole order along the incidence divisor of a meromorphic function, obtained by integrating a meromorphic cohomology class on an analytic family of hypersurfaces. To get a similar kind of bound in the case of an analytic family of cycles of codimension greater than one, we will have to modify the definition of order for the meromorphic classes. Before we do that let us recall the usual definition:

Let Y be a locally complete intersection of codimension $n+1$ in a complex manifold Z defined by the $\mathcal{O}_Z-$ideal $\mathcal{I}_Y$. The usual order, which we shall from now on call the **naive order**, is derived from the filtration

$$\breve{\mathcal{F}}_1 \subset \breve{\mathcal{F}}_2 \subset \cdots \subset \underline{H}^{n+1}_{[Y]}(\Omega^n_Z),$$

where $\breve{\mathcal{F}}_k := \mathrm{Ann}(\mathcal{I}^k_Y)$ is the $\mathcal{O}_Z-$submodule of $\underline{H}^{n+1}_{[Y]}(\Omega^n_Z)$ consisting of all elements which are annihilated by $\mathcal{I}^k_Y$.
An element ξ in $H^{n+1}_{[Y]}(Z, \Omega^n_Z)$ is said to have **naive order** k if k is the smallest integer such that ξ defines a global section of the $\mathcal{O}_Z-$submodule $\breve{\mathcal{F}}_k$.

The new order that we are going to introduce now will be defined in the same way as the naive order but with respect to a different filtration.
We begin with the canonical identification of $\mathcal{O}_Z-$modules

$$\underline{H}^{n+1}_{[Y]}(\mathcal{O}_Z) \otimes_{\mathcal{O}_Z} \Omega^n_Z \simeq \underline{H}^{n+1}_{[Y]}(\Omega^n_Z) \tag{10}$$

and with the **naive** filtration on $\underline{H}^{n+1}_{[Y]}(\mathcal{O}_Z)$ formed by the submodules

$$\mathcal{F}_k := \mathrm{Ann}(\mathcal{I}^k_Y) \subset \underline{H}^{n+1}_{[Y]}(\mathcal{O}_Z), \quad k \geq 1. \tag{11}$$

Let $d\mathcal{I}_Y$ be the image of $\mathcal{I}_Y$ by the exterior differentiation map $d : \mathcal{O}_Z \to \Omega^1_Z$ and let $\mathcal{O}_Z.d\mathcal{I}_Y$ be the $\mathcal{O}_Z-$submodule of Ω^1_Z generated by $d\mathcal{I}_Y$. Then let $\mathcal{G}_k$ denote the image of $\Lambda^{n-k}(\mathcal{O}_Z.d\mathcal{I}_Y) \otimes \Omega^k_Z$ in Ω^n_Z.
Let us give a description of $\mathcal{G}_k$ in terms of local generators:

Let z_0 be a point in Y and let $(x_1,\dots,x_{n+p})$ be a system of local coordinates for Z centred at z_0. If $f_0,\dots,f_n$ generate $\mathcal{I}_Y$ near z_0 then the elements df_i and $f_i dx_j$ with $i\in[0,n]$ and $j\in[1,n+p]$ form a set of generators for $\mathcal{O}_Z.d\mathcal{I}_Y$ near z_0. Consequently the set of all elements of the form $f^\alpha dx_I\wedge df_J$, where $|J|\le n-k$, $|I|=n-|J|$ and $|\alpha|=|I|-k$, is a generating set for the $\mathcal{O}_Z-$module $\mathcal{G}_k$ near z_0. From this we get

$$\mathcal{G}_k=\sum_{m=k}^{n}\ \sum_{|J|=n-m}\mathcal{I}_Y^{m-k}.\Omega_Z^m\wedge df_J$$

and in particular

$$\mathcal{G}_0=\sum_{m=0}^{n}\ \sum_{|J|=n-m}\mathcal{I}_Y^{m}.\Omega_Z^m\wedge df_J$$

in a neighbourhood of the point z_0.
Then we define a (increasing) filtration $(\Phi_m)_{m\in\mathbb{N}}$ on $\underline{H}^{n+1}_{[Y]}(\Omega^n_Z)$ as the image of the tensor-product filtration $(\mathcal{F}_\bullet\otimes_{\mathcal{O}_Z}\mathcal{G}_\bullet)$ on $\underline{H}^{n+1}_{[Y]}(\mathcal{O}_Z)\otimes_{\mathcal{O}_Z}\Omega^n_Z$ by the previous identification (10); in other words, for all $m\ge 0$ we put

$$\Phi_m:=\sum_{k=0}^{m}\mathcal{F}_k\otimes_{\mathcal{O}_Z}\mathcal{G}_{m-k}.$$

(2.14) Definition. The filtration $(\Phi_m)_{m\in\mathbb{N}}$ is called the **conormal filtration** on $\underline{H}^{n+1}_{[Y]}(\Omega^n_Z)$. An element ξ in $H^{n+1}_{[Y]}(Z,\Omega^n_Z)$ is said to have **conormal order** m if m is the smallest integer such that ξ defines a global section of the $\mathcal{O}_Z-$submodule Φ_m.

(2.15) Lemma. Assume the hypotheses of (2.5). Then the $\mathcal{O}_{Z,0}-$module $\underline{H}^{n+1}_{[Y]}(\mathcal{O}_Z)_0$ is generated by $f^*(H^{n+1}_{[0]}(\mathcal{O}_{\mathbb{C}^{n+1}}))$. Moreover $f^*(\mathrm{Ann}(\mathcal{M}^k))$ generates $\mathrm{Ann}(\mathcal{I}^k_Y)$ for every $k\in\mathbb{N}$, where $\mathcal{M}$ denotes the maximal ideal of $\mathcal{O}_{\mathbb{C}^{n+1},0}$.

Proof. To simpify the writing we put $\mathcal{O}:=\mathcal{O}_{\mathbb{C}^{n+1}}$. We recall that $\mathrm{Ann}(\mathcal{M}^k)$ and $\mathrm{Ann}(\mathcal{I}^k_Y)$ can be naturally identified with $\mathrm{Ext}^{n+1}(\mathcal{O}_0/\mathcal{M}^k,\mathcal{O}_0)$ and $\mathrm{Ext}^{n+1}(\mathcal{O}_{Z,0}/\mathcal{I}^k_{Y,0},\mathcal{O}_{Z,0})$ respectively. The ring $\mathcal{O}_{Z,0}$ is a $\mathcal{O}_0-$module via the morphism f and since the morphism f is flat we get for each $k\ge 0$ an isomorphism

$$\mathrm{Ext}^{n+1}(\mathcal{O}_0/\mathcal{M}^k,\mathcal{O}_0)\otimes_{\mathcal{O}_0}\mathcal{O}_{Z,0}\simeq\mathrm{Ext}^{n+1}(\mathcal{O}_{Z,0}/\mathcal{I}^k_{Y,0},\mathcal{O}_{Z,0}).$$

This means that $f^*(\mathrm{Ann}(\mathcal{M}^k))$ generates $\mathrm{Ann}(\mathcal{I}^k_Y)$ for every $k\in\mathbb{N}$, so to conclude the proof we only have to use the facts that

$$\underline{H}^{n+1}_{[0]}(\mathcal{O})=\lim_{\substack{\rightarrow\\ k}}\mathrm{Ext}^{n+1}(\mathcal{O}_0/\mathcal{M}^k,\mathcal{O}_0)\qquad\text{and}$$

$$(\underline{H}^{n+1}_{[Y]}(\mathcal{O}_Z))_0=\lim_{\substack{\rightarrow\\ k}}\mathrm{Ext}^{n+1}(\mathcal{O}_{Z,0}/\mathcal{I}^k_{Y,0},\mathcal{O}_{Z,0})\ .$$

□

(2.16). Let Z be an open neighbourhood of the origin in $\mathbb{C}^{n+p}$ and let $x_1, \dots, x_{n+p}$ be coordinates $(\mathcal{O}_{Z,0})^{n+1}$. Let $\mathcal{I}$ be the ideal generated by $f_0, \dots, f_n$ and endow it with the canonical topology of a finitely generated $\mathcal{O}_{Z,0}-$module. We shall define elementary transformations $T^{\varepsilon}_{i,j,k}$ from $\mathcal{I}^{\oplus n+1}$ into $\mathcal{I}^{\oplus n+1}$ as follows : Fix $\varepsilon \geq 0$, $i \in [1, n+p]$ and $(j,k) \in [0,n]^2$. For $g = (g_0, \dots, g_n) \in \mathcal{I}^{\oplus n+1}$ define the elementary transform $T^{\varepsilon}_{i,j,k}(g)$[15] by the rule

$$T_{j,k,i}(g)_l = \begin{cases} g_l & \text{if} \quad l \neq j \\ g_j + \varepsilon.x_i.g_k & \text{if} \quad l = j. \end{cases}$$

For a fixed ε we shall only consider elements g that can be obtained from f by a finite number of elementary transforms. When an element g can be obtained from f by at most r elementary transforms we shall write $\langle g \rangle \leq r$.
Let r be a fixed integer. Since the mapping $\mathbb{R}_+ \to \mathcal{I}^{\oplus n+1}$ defined by $\varepsilon \mapsto T^{\varepsilon}_{i,j,k}(f)$ is continuous, lemma A.1 in the Appendix implies that for all sufficiently small $\varepsilon > 0$, every g in $\mathcal{I}^{\oplus n+1}$ with $\langle g \rangle \leq r$ generates $\mathcal{I}$.
We shall always assume that $r \leq 100.n$ and that $\varepsilon > 0$ is small enough to ensure that every g with $\langle g \rangle \leq r$ generates the ideal $\mathcal{I}$.
For any g such that $\langle g \rangle \leq r$, we define the $\mathcal{O}_{Z,0}-$submodule $\mathcal{E}_g$ of $\Omega^n_{Z,0}$ by

$$\mathcal{E}_g := \sum_{j \in [0,n]} \mathcal{O}_{Z,0}.dg_0 \wedge \cdots \widehat{dg_j} \cdots \wedge dg_n \quad .$$

(2.17) Proposition. Let $\mathcal{V}$ be an open neighbourhood of f in $\mathcal{I}^{\oplus n+1}$ endowed with its canonical topology. Then for small enough $\varepsilon > 0$ we have

- if g is such that $\langle g \rangle \leq n$, then $g \in \mathcal{V}$,
- $\sum_{\langle g \rangle \leq n} \mathcal{E}_g = \mathcal{G}_0$,

where $\mathcal{G}_0$ is the image of the $n-th$ exterior product of $\mathcal{O}_{Z,0}.d\mathcal{I}$ in $\Omega^n_{Z,0}$.

Proof. We can choose $\varepsilon > 0$ small enough to ensure that $\langle g \rangle \leq n$ implies $g \in \mathcal{V}$ (see the above). Let such a number $\varepsilon > 0$ be fixed for the rest of the proof.

To prove the second assertion note first that from the description of $\mathcal{G}_0$ in (2.13) we clearly get the inclusion $\sum_{\langle g \rangle \leq n} \mathcal{E}_g \subset \mathcal{G}_0$.
To prove the reverse inclusion we begin with an easy formula. Fix $j \in [0,n]$ and $i \in [1, n+p]$. Then

$$\varepsilon.g_j.dx_i = d((1+\varepsilon.x_i).g_j) - (1+\varepsilon.x_i).dg_j.$$

Let $J \subset [0,n]$ with $|J| \leq n$. For $j \in J$ and any $l \in [0,n]$ this formula implies

$$\varepsilon.g_j.dx_i \wedge dg_J = d(g_l + \varepsilon.x_i.g_j) \wedge dg_J - dg_l \wedge dg_J.$$

If we put $g' := T_{l,j,i}(g)$, then this equation can be written, when $l \notin J$, as

$$\varepsilon.g_j.dx_i \wedge dg_J = \pm(dg'_{J \cup l} - dg_{J \cup l}). \tag{12}$$

[15] We shall omit the subscript ε when there is no ambiguity.

For $j \notin J$ we see in the same way that the formula implies, with $g' = T_{j,j,i}(g)$:

$$\varepsilon.g_j.dx_i \wedge dg_J = \pm(dg'_{J\cup j} - (1+\varepsilon.dx_i).dg_{J\cup j}). \tag{13}$$

Let us now show, by induction on $m_0 \in [0,n]$, that for any integer r [16] we have

$$\sum_{\langle g\rangle \leq r} \sum_{m\leq m_0} \sum_{|J|=n-m} \mathcal{I}^m.\Omega^m_{Z,0} \wedge dg_J \subset \sum_{\langle g\rangle \leq m_0+r} \mathcal{E}_g. \tag{14}$$

Once this has been proved we finish the proof of the proposition by taking $r = 0$ and $m_0 = n$.

The statement is obvious for $m_0 = 0$. Assume that it is true for $m_0 \in [0, n-1]$ and consider g with $\langle g\rangle \leq r$, $\alpha = (\alpha_0,\dots,\alpha_n) \in \mathbb{N}^{n+1}$ with $|\alpha| = m_0+1$, $J \subset [0,n]$ with $|J| = n-(m_0+1)$ and $I \subset [1, n+p]$ with $|I| = m_0+1$. Then we shall produce $l \in [0,n]\setminus J$ and g' such that $\langle g'\rangle \leq r+1$ having the following property

$$g^\alpha dx_I \wedge dg_J \in \mathcal{I}^{m_0}.\Omega^{m_0}_{Z,0} \wedge dg_{J\cup l} + \mathcal{I}^{m_0}.\Omega^{m_0}_{Z,0} \wedge dg'_{J\cup l}. \tag{15}$$

Consider first the case where for some $j \in J$ we have $\alpha_j > 0$. Pick any $i \in I$, put $I' := I \setminus i$ and write

$$g^\alpha dx_I \wedge dg_J = \pm g^\beta.dx_{I'} \wedge g_j.dx_i \wedge dg_J$$

with $\beta + 1_j = \alpha$, where 1_j is the element in $\mathbb{N}^{n+1}$ which has 1 in the $j-$th place and 0 in all the others. Now choose any $l \in [0,n]\setminus J$ and put $g' = T_{l,j,i}(g)$. Then formula (12) gives (15).
If $\alpha_j = 0$, for all $j \in J$, then pick any $l \in [0,n]$ with $\alpha_l > 0$ and any $i \in I$, put $I' := I \setminus i$ and write

$$g^\alpha dx_I \wedge dg_J = \pm g^\beta.dx_{I'} \wedge g_l.dx_i \wedge dg_J$$

where $\beta + 1_l = \alpha$. Then by putting $g' = T_{l,l,i}(g)$ we get (15) from (13).
Since $\langle g\rangle \leq r$ implies $\langle g'\rangle \leq r+1$ our induction hypothesis combined with (15) then gives (14) for $m_0 + 1$.
Thus (14) is true for every $m_0 \in [0,n]$ and every r and the proof is completed. $\square$

(2.18). Assume the hypotheses of (2.5) and let s_0 be a point in Σ where both $|\Sigma|$ and S are smooth. Suppose further that the cycle X_{s_0} only intersects Y at the origin.
Put $k_f := \mathrm{mult}_0 f_\star X_0$ and let $\varkappa_f$ denote the positive rational number associated with the family $(f_* X_s)_{s\in S}$ at s_0 [see theorem (1.2)]. In section 3 we shall see that $\varkappa_f$ is independent of f so from now on we will simply denote it by $\varkappa$.
Evidently, the number k_f only depends on the germ of f at the origin.
Put $k := \min\limits_f k_f$, where the minimum is taken over all generating regular sequences that define the germ of Y at the origin. This minimum is attained on an open dense set of regular sequences at the origin[17].

[16] $r \leq 99.n$; in fact we are doing an induction on $m_0 + r \leq 100.n$.
[17] See A.4 in the Appendix.

(2.19) LEMMA. Consider the situation described above and let $(z_0, \ldots, z_n)$ be the standard coordinates in $\mathbb{C}^{n+1}$. Then every germ in $\underline{H}^{n+1}_{[Y]}(\Omega^n_Z)_0$ can be written as a finite $\mathcal{O}_{Z,0}-$linear combination of elements of the form

$$g^*(\varphi \wedge dz_0 \wedge \cdots \widehat{dz_i} \cdots \wedge dz_n) = (g^*\varphi) \wedge dg_0 \wedge \cdots \widehat{dg_i} \cdots \wedge dg_n$$

where φ is in $H^{n+1}_{[0]}(\mathcal{O}_{\mathbb{C}^{n+1}})$ and g is a flat morphism defined by a generating regular sequence for $\mathcal{I}_Y$ at 0, with $k_g = k$.
Moreover the conormal order of each element in the linear combination is less than or equal to the conormal order of the original element .

Proof. Let $(x_1, \cdots, x_{n+p})$ be the standard coordinates of $\mathbb{C}^{n+p}$. Then every element in $\underline{H}^{n+1}_{[Y]}(\Omega^n_Z)_0$ is a finite $\mathcal{O}_{Z,0}-$linear combination of elements of the type $\psi \wedge dx_I$ with $|I| = n$ and $\psi \in \underline{H}^{n+1}_{[Y]}(\mathcal{O}_Z)_0$. Let f be a flat morphism defined by a generating regular sequence for $\mathcal{I}_Y$ at 0 , with $k_f = k$. Then by lemma (2.15) each ψ is a finite $\mathcal{O}_{Z,0}-$linear combination of elements of the form $f^*(\eta)$ where $\eta \in H^{n+1}_{[0]}(\mathcal{O}_{\mathbb{C}^{n+1}})$ and such that the orders of these elements are less than or equal to the order of ψ. Since any element of order m in $H^{n+1}_{[0]}(\mathcal{O}_{\mathbb{C}^{n+1}})$ is a finite $\mathbb{C}-$linear combination of the family $(\partial^{(\alpha)}_0)_{0 \leq |\alpha| \leq m}$, it is sufficient to prove that $(f^*\partial^{(\alpha)}_0) \wedge dx_I$ can be written in the desired way.
From the identity $\partial^{(\alpha)}_0 = z_0^n.\partial^{(\beta)}_0$ where $\beta = (\alpha_0 + n, \alpha_1, \cdots, \alpha_n)$ we deduce that

$$(f^*\partial^{(\alpha)}_0) \wedge dx_I = f^*(z_0^n.\partial^{(\beta)}_0) = f^*(\partial^{(\beta)}_0) \wedge f_0^n dx_I.$$

Choose an open neigbourhood $\mathcal{V}$ of f in $\mathcal{I}^{\oplus n+1}_{Y,0}$ such that $k_g = k$ for every $g \in \mathcal{V}$[18] and take $\varepsilon > 0$ such that the two conditions in proposition (2.17) are satisfied. Then every element of the form $f_0^n dx_I$ with $|I| = n$ is a finite $\mathcal{O}_{Z,0}-$linear combination of elements of type $dg_0 \wedge \cdots \widehat{dg_i} \cdots \wedge dg_n$ where g is a flat morphism defined by a regular sequence generating $\mathcal{I}_{Y,0}$ with $k_g = k$. Thus every element in $\underline{H}^{n+1}_{[Y]}(\Omega^n_Z)_0$ can be written as a finite $\mathcal{O}_{Z,0}-$linear combination of elements of type $f^*(\partial^{(\beta)}_0) \wedge g^*(dz_0 \wedge \cdots \widehat{dz_i} \cdots \wedge dz_n)$ with g in $\mathcal{V}$.
Now consider a certain element of the form $f^*(\partial^{(\beta)}_0) \wedge g^*(dz_0 \wedge \cdots \widehat{dz_i} \cdots \wedge dz_n)$ with g fixed. Then by lemma (2.15), we can write $f^*(\partial^{(\beta)}_0)$ as a finite $\mathcal{O}_{Z,0}-$linear combination of elements of the form $g^*(\varphi)$ where φ is an element of order $\leq |\beta|$ in $H^{n+1}_{[0]}(\mathcal{O}_{\mathbb{C}^{n+1}})$. This completes the proof. □

(2.20) REMARK. In the terminology of (2.13) (see also definition (2.14)) the above lemma implies in particular the identity $\Phi_m = \mathcal{F}_m \otimes \mathcal{G}_0$, for all m.

(2.21) DEFINITION. Let S be a (reduced) complex space, let Σ be a divisor in S, let $s_0 \in \Sigma$ and let f be a continuous function on $S \setminus \Sigma$. We say that f has an **analytic singularity** along Σ near s_0 if there exists an open set $V \subset S$ containing s_0, a continuous function g in V and a holomorphic function $h \in \mathcal{O}(V \setminus V \cap \Sigma)$ such that on $V \setminus V \cap \Sigma$ we have

$$f = g + h.$$

[18] See A.4 in the Appendix.

If moreover h can be choosen to be meromorphic along Σ we say that f is **almost meromorphic** along Σ near s_0.

(2.22) Remark. Let f be an almost meromorphic function and let $f = g + h$ and $f = g_1 + h_1$ be two decompositions of f on $V \setminus V \cap \Sigma$ as in the above definition. Then $h_1 - h$ is a continuous extension of the holomorphic function $g - g_1$ to the set V. In the case where S is a normal space this means that $g - g_1$ has a holomorphic extension to the set V. Consequently f has a well-defined polar part in $H^1_{V\cap\Sigma}(V, \mathcal{O}_S)$. In particular f has a well-defined pole order along each component of $|\Sigma|$.

(2.23) Theorem. *Let Z be a Stein open neighbourhood of the origin in $\mathbb{C}^{n+p}$ (where $p \geq 1$) and let $(X_s)_{s\in S}$ be an analytic family of $n-$cycles in Z with graph $\mathcal{X}$. Let Y be a finite pole for the family and let Σ be the associated incidence divisor. Let s_0 be a point in Σ where both S and $|\Sigma|$ are smooth and such that X_{s_0} only intersects Y at the origin. Let q denote the multiplicity of Σ near s_0 and let k and $\varkappa$ be defined as in (2.18). Then for every $\bar\partial-$closed $(n,n)-$form φ on $Z \setminus Y$ of finite order along Y and for every $\rho \in \mathcal{C}^\infty_c(Z)$ such that $\rho \equiv 1$ in a neighbourhood of the origin the function $s \mapsto \int_{X_s} \rho.\varphi$ is almost meromorphic along Σ near s_0.*
Moreover, if m is the (conormal) order of φ then the order of this almost meromorphic function is $\leq q.\varkappa.m$ along $|\Sigma|$ near s_0.

Proof. Let φ be a $\bar\partial-$closed $(n,n)-$form on $Z \setminus Y$ of (conormal) order m along Y. First we note that the assertion is independant of the choice of ρ. For if ρ_1 is another such function, then $(\rho - \rho_1).\varphi$ is a $\mathcal{C}^\infty$ form with compact support which does not intersect $|X_s| \cap |Y|$ for all s close enough to s_0. Thus the function $s \mapsto \int_{X_s}(\rho - \rho_1).\varphi$ is continuous in a neighbourhood of s_0 .
By lemma (2.19) we know that the image of φ in $\underline{H}^{n+1}_{[Y]}(\Omega^n_Z)_0$ can be written as

$$\sum_{\nu=1}^{\nu=N} h_\nu.(f_\nu)^*\eta_\nu$$

where $h_\nu \in \mathcal{O}_{Z,0}$, η_ν is an element of order less than or equal to m in $H^{n+1}_{[0]}(\Omega^n_{\mathbb{C}^{n+1}})$ and f_ν is a germ of a flat morphism $(\mathbb{C}^{n+p}, 0) \to (\mathbb{C}^{n+1}, 0)$. Shrinking Z around the origin, we may assume that for each ν the germs h_ν and f_ν have representatives in Z, which will also be denoted by h_ν and f_ν.
By lemma (2.6) there exists, for each ν, an open neighbourhood U_ν of the origin in Z and Martinelli representatives ψ_ν for η_ν on $f_\nu(U_\nu) \setminus \{0\}$ such that $(f_\nu|_{U_\nu})^*\psi_\nu$ is $\mathcal{X}|_{U_\nu}-$proper. So in a Stein open neighbourhood Z' of the origin in $U_1 \cap \cdots \cap U_N$ we obtain

$$\varphi = \sum_{\nu=1}^{\nu=N} h_\nu.(f_\nu|_{U_\nu})^*\psi_\nu + \bar\partial\omega$$

where ω is a $\mathcal{C}^\infty-$form on $Z' \setminus Z' \cap Y$ of type $(n, n-1)$. For a function $\rho \in \mathcal{C}^\infty_c(Z')$, with $\rho \equiv 1$ near the origin we thus get

$$\int_{X_s} \rho.\varphi = \sum_{\nu=1}^{\nu=N} \int_{X_s} \rho.h_\nu.(f_\nu|_{U_\nu})^*\psi_\nu + \int_{X_s} \rho.\bar\partial\omega.$$

Write

$$\int_{X_s} \rho.h_\nu(f_\nu|_{U_\nu})^*\psi_\nu = -\int_{X_s} (1-\rho).h_\nu(f_\nu|_{U_\nu})^*\psi_\nu + \int_{X_s} h_\nu(f_\nu|_{U_\nu})^*\psi_\nu \ ,$$

$$\int_{X_s} \rho.\bar{\partial}\omega = \int_{X_s} \bar{\partial}(\rho.\omega) - \int_{X_s} \bar{\partial}\rho \wedge \omega$$

and choose an open neighbourhood S' of s_0 in S such that

$$X_s \cap Y \subset \{\rho \equiv 1\}$$

for all $s \in S'$. Then the functions $s \mapsto \int_{X_s} (1-\rho).h_\nu.(f_\nu|_{U_\nu})^*\psi_\nu$ and $s \mapsto \int_{X_s} \bar{\partial}\rho \wedge \omega$ are continuous in S' because the forms $(1-\rho).h_\nu.(f_\nu|_{U_\nu})^*\psi_\nu$ and $\bar{\partial}\rho \wedge \omega$ have $\mathcal{X}-$proper supports such that, for all $s \in S'$, they do not intersect $|X_s| \cap |Y|$. The function $s \mapsto \int_{X_s} \bar{\partial}(\rho.\omega)$ is identically zero on $S' \setminus S' \cap \Sigma$ and by corollary (2.10) the function $s \mapsto \int_{X_s} h_\nu(f_\nu)^*\psi_\nu$ is holomorphic on $S' \setminus S' \cap \Sigma$ having a pole of order $\leq k.\varkappa.m$ along $|\Sigma|$ near s_0. This completes the proof.

(2.24) Remark. If φ is a $\bar{\partial}-$closed $(n,n)-$form on $Z \setminus Y$ of (conormal) order m and with an $\mathcal{X}-$proper support near s_0, then the function $s \mapsto \int_{X_s} \varphi$ is holomorphic on $Z \setminus Y$ and the preceding theorem implies that it has a pole of order $\leq q.\varkappa.m$ along $|\Sigma|$ near s_0.

(2.25). Assume the hypothesis of the theorem and let η be an element of (conormal) order m in $\left(\underline{H}^{n+1}_{[Y]}(\Omega^n_Z)\right)_0$. Then there exists an open neighbourhood Z' of the origin in Z and a $\bar{\partial}-$closed $(n,n)-$form φ on $Z' \setminus Y \cap Z'$ of (conormal) order m which represents η. Let $\rho \in \mathcal{C}^\infty_c(Z')$ such that $\rho \equiv 1$ in a neighbourhood of the origin.Then it is clear from the proof of theorem (2.23) that the polar part in $\left(\underline{H}^1_{[\Sigma]}(\mathcal{O}_S)\right)_{s_0}$ defined by the function $s \mapsto \int_{X_s} \rho.\varphi$ is independent of the choice of φ. Hence we get a well defined integration map

$$\int_{\mathcal{X}} : \left(\underline{H}^{n+1}_{[Y]}(\Omega^n_Z)\right)_0 \to \left(\underline{H}^1_{[\Sigma]}(\mathcal{O}_S)\right)_{s_0} .$$

(2.26) Corollary. *Assume the hypotheses of theorem (2.23) . Let γ be the defining function for $|\Sigma|$ near s_0 and let $\int_{\mathcal{X}}$ be the integration map defined above. Then for every integer $\mu \geq 1$ there exists an integer m in $[\mu, \mu+k[$ and an element ξ of (conormal) order m in $\left(\underline{H}^{n+1}_{[Y]}(\Omega^n_Z)\right)_0$ such that $q.\varkappa.m$ is an integer and such that*

$$\lim_{s \to s_0} \gamma^{q.\varkappa.m}(s) \int_{\mathcal{X}} (\xi)\ (s) \neq 0 \ .$$

In particular $\int_{\mathcal{X}}(\eta)$ is of order $q.\varkappa.m$ in $\left(\underline{H}^1_{[\Sigma]}(\mathcal{O}_S)\right)_{s_0}$.

Proof. Let μ be a positive integer and let f be a flat morphism defined by a generating regular sequence for $\mathcal{I}_Y$ at the origin with $k_f = k$. Then by theorem

(1.2) and remark (1.3) 1) there exist an integer m and an element η of order m in $H^{n+1}_{[0]}(\Omega^n_{\mathbb{C}^{n+1}})$ such that $q.\varkappa.m$ is an integer and

$$\lim_{s\to s_0} \gamma^{q.\varkappa.m}(s)\int_{\mathcal{X}}(\eta)\ (s) \neq 0\ .$$

But $\int_{f_*\mathcal{X}}(\eta) = \int_{\mathcal{X}}(f^*\eta)$ so the element $\xi := f^*\eta$ has the desired properties.

3. Global results. We want to deduce a more general and more precise statement for the filtered integration of meromorphic cohomology classes on an analytic family of cycles, including the case of a proper and generically finite pole.

(3.1) THE MULTIPLICITY OF THE LOCAL INCIDENCE DIVISOR. Let $(X_s)_{s\in S}$ be an analytic family of n-cycles in a complex manifold Z, let Y be a proper, generically finite pole for $(X_s)_{s\in S}$ and let Σ be the corresponding incidence divisor.
Here we shall only assume that Y is a cycle of codimension $n+1$ in Z without any given subspace structure. Let $\mathcal{X}\subset S\times Z$ be the graph (as an analytic subset) of the family $(X_s)_{s\in S}$ and let $p:\mathcal{X}\to Z$ and $\pi:\mathcal{X}\to S$ be the canonical projections. Then by definition $p^{-1}(|Y|)$ is proper and generically finite via π on its image $|\Sigma_Y|$ in S, which is the support of the Cartier incidence divisor Σ_Y by [B.K.03] .
Let U be the dense Zariski open subset of $|\Sigma_Y|$ consisting of all points σ where $|\Sigma_Y|$ is smooth and such that $\pi^{-1}(\sigma)\cap p^{-1}(Y)$ is finite. Then for $(\sigma,y)\in\pi^{-1}(U)\cap p^{-1}(Y)$ we define $q(\sigma,y)$ as the multiplicity of $|\Sigma|$ in the incidence divisor defined by Y for the family $(X_s\cap V_y)_{s\in S}$ where V_y is an open neighbourhood of y in Z such that

$$|Y|\cap|X_\sigma|\cap V_y=\{y\}.$$

Note that, by additivity of the relative trace, if we have

$$\pi^{-1}(\sigma)\cap p^{-1}(Y)=\{(\sigma,y_1),\cdots,(\sigma,y_l)\}$$

then $\Sigma_Y = \left(\sum\limits_{j=1}^{l} q(\sigma,y_j)\right).|\Sigma_Y|$ near σ.
We define a locally constant function $q' : U\to\mathbb{N}^*$ by

$$q'(\sigma)=\sum_{j=1}^{l} q(\sigma,y_j)\ .$$

Hence, if $(C_\alpha)_{\alpha\in A}$ are the irreducible components of $|\Sigma_Y|$, then

$$\Sigma_Y=\sum_{\alpha\in A} q'_\alpha.C_\alpha \quad\text{with}\quad q'_\alpha = q'|_{U\cap C_\alpha}.$$

We shall also use the notation q'_C to denote the value of the constant function $q'|_{U\cap C}$.

(3.2) THE $\varkappa-$FUNCTION. From now on we shall assume that Y is given by a coherent ideal $\mathcal{I}_Y$ in $\mathcal{O}_Z$. In general we shall also assume that Y is a locally

complete intersection in Z, but we do not need this additional hypothesis for the definition of the function $\varkappa$.
We begin with the following general setting:
Let V be a reduced complex space and let s and x be holomorphic functions on V. Denote by (x) the ideal generated by the function x and suppose that $s \in \sqrt{(x)}$. Let $W = \{v \in V \mid x(v) = 0\}$ and assume that W is nowhere dense in V and that the function s is not identically 0 in a neighbourhood of any point of W. We then define a function

$$\varkappa : W \to \mathbb{Q}_+^*$$

in the following way

$$\varkappa(w) := \sup\{\varkappa \in \mathbb{Q}_+^* \mid \exists U(w), \exists C > 0 \quad |s(v)| \leq C.|x(v)|^{\varkappa} \quad \forall v \in U(w)\}$$

where $U(w)$ is an open neighbourhood of w in V .
Note that our hypothesis implies that there exists, locally around each point w in W, a positive integer $q(w)$ such that $s^{q(w)} \in (x)_w$.
Hence $\varkappa(w) \geq \frac{1}{q(w)} > 0$.

(3.3) LEMMA. In the setting described above let

$$\lambda(w) = \sup\{b/a \in \mathbb{Q}_+^* \mid s^a \in \mathrm{Intg}(x^b)_w\}$$

where $\mathrm{Intg}(x^b)$ denotes the integral closure of the ideal (x^b). Then we have :
1) $\lambda(w) \in \mathbb{Q}_+^*$ and $\lambda(w) = \frac{\beta}{\alpha}$ implies $s^\alpha \in \mathrm{Intg}(x^\beta)_w$.
2) $\lambda(w) = \varkappa(w)$.
3) The function $\varkappa : W \to \mathbb{Q}_+^*$ is constructible on W[19].

Proof. Consider the normalization map $\nu : \tilde{V} \to V$. Outside of an analytic subset of codimension ≥ 2 the space $\tilde{V}$ is smooth and there exists locally a holomorphic function z such that $dz \neq 0$ and such that $\nu^*(x) = z^p$ and $\nu^*(s) = h.z^q$ with h invertible . The conclusion follows easily. □

Now we may extend our setting :

(3.4) PROPOSITION. Let V be a reduced complex space and $\mathcal{I}$, $\mathcal{J}$ be two coherent ideals in $\mathcal{O}_V$ such that $\mathcal{I} \subset \sqrt{\mathcal{J}}$. Put $W := \mathrm{Supp}(\mathcal{O}_V/\mathcal{I})$ and assume that W is nowhere dense in V and that $\mathrm{Supp}(\mathcal{I}) = V$. For every $w \in W$ we put

$$\varkappa(w) := \sup\{b/a \in \mathbb{Q}_+^* \mid \mathcal{I}_w^a \subset \mathrm{Intg}(\mathcal{J}_w^b)\}$$

Then the function $\varkappa$ takes its values in $\mathbb{Q}_+^*$ and is constructible on W. For any $w \in W$ with $\varkappa(w) = \frac{\beta}{\alpha}$ we have $\mathcal{I}_w^\alpha \subset \mathrm{Intg}(\mathcal{J}_w^\beta)$.

Proof. By blowing up $\mathcal{I}$ and $\mathcal{J}$ in V we can reduce the proposition to the previous lemma; we just have to use the inequalities to go down thanks to the following standard equivalence on integral dependence :

[19] This means that for any $\varkappa_0 \in \mathbb{R}$ the set $\{w \in W \mid \varkappa(w) \geq \varkappa_0\}$ is a closed analytic subset of W.

The germ $s \in \mathcal{O}_{V,w}$ belongs to the integral closure of the ideal in $\mathcal{O}_{V,w}$ generated by $g_1, \cdots, g_p$ if and only if there exits an open neighbourhood $U(w)$ of w in V and a constant $C > 0$ such that

$$|s(v)| \leq C.\sum_{j=1}^{p} |g_j(v)| \quad \forall v \in U(w).$$

□

This $\varkappa-$function associated to a pair of ideals $\mathcal{I}$ and $\mathcal{J}$ in $\mathcal{O}_V$ satisfying the condition $\mathcal{I} \subset \sqrt{\mathcal{J}}$ has the following nice functorial property.

(3.5) Lemma. In the situation of the previous proposition, assume that we have a holomorphic map $f : V' \to V$ and consider the pull-backs $f^*(\mathcal{I})$ and $f^*(\mathcal{J})$. Define $W' = \mathrm{Supp}(\mathcal{O}'_V/f^*(\mathcal{I}))$ and assume that W' is nowhere dense in V' and that $\mathrm{Supp}(f^*(\mathcal{I})) = V'$. Then for any $w' \in W'$ we have the inequality

$$\varkappa(w') \leq \varkappa(f(w')).$$

Moreover this inequality is an equality as soon as the map f is open in a neighbourhood of w'.

Proof. This is an easy exercise using the defining inequality for $\varkappa$. □

(3.6). Now let us come back to the original situation in (3.1) and put $V := \mathcal{X} \subset S \times Z$ and $W := p^{-1}(|Y|)$. Fix $(\sigma, y) \in \pi^{-1}(U)$ and let V_y be an open neighbourhood of y in Z such that

$$|X_s| \cap Y \cap V_y = \{y\}.$$

Let $\mathcal{I}$ be the (locally) principal ideal in $\mathcal{O}_{\mathcal{X}}$ which is obtained by pulling back the defining ideal of the (Cartier) incidence divisor associated with the pole Y and the analytic family of $n-$cycles $(X_s \cap V_y)_{s \in S}$ to its graph. Put $\mathcal{J} := p^*(\mathcal{I}_Y)|_{\pi^{-1}(U)}$ and define $\varkappa(\sigma, y)$ with respect to the ideals $\mathcal{I}$ and $\mathcal{J}$ as in the previous proposition[20]. Now this function $\varkappa$ is well defined on $\pi^{-1}(U)$, takes its values in $\mathbb{Q}^*_+$ and is locally constructible[21].
Let $U_0 \subset U$ be a dense Zariski open set in $|\Sigma|$ such that the map π restricted to $p^*(|Y|)$ induces an unramified finite covering on U_0. Then $\varkappa$ is globally constructible on $\pi^{-1}(U_0)$ and so there exists a dense Zariski open set $U_1 \subset U_0$ such that $\varkappa$ is locally constant on $\pi^{-1}(U_1)$.
Now on $\pi^{-1}(U_1)$ the functions q and $\varkappa$ are locally constant .

(3.7) Definition. Assume the hypotheses in (3.6) and suppose also that S is compact[22]. Put

$$p^{-1}(Y)_{\max} := \Big\{(\sigma, y') \in \pi^{-1}(U_1) \mid q(\sigma, y).\varkappa(\sigma, y) \text{ is maximal in } p^{-1}(Y) \cap \pi^{-1}(\sigma)\Big\}$$

Then $\overline{p^{-1}(Y)_{\max}}$ is a union of irreducible components of $p^{-1}(|Y|)$ and $p(\overline{p^{-1}(Y)_{\max}}) =: Y_{\max}$ is a compact analytic subset of Y .

[20] Note that on the graph $\mathcal{X}$ the relation $\mathcal{I} \subset \sqrt{\mathcal{J}}$ is satisfied .
[21] From the previous proposition we only obtain the constructibility on V_y .
[22] This is to ensure that $|\Sigma|$ has only a finite number of irreducible components.

Of course when Y is a proper and generically finite pole, the restriction

$$\pi : \overline{p^{-1}(Y)_{\max}} \to |\Sigma|$$

is proper, generically finite and surjective.

(3.8) THE $\varkappa'-$FILTRATION ON $\underline{H}^1_{[\Sigma_Y]}(\mathcal{O}_S)$. For any irreducible component C of $|\Sigma|$ define the rational number $\varkappa'_C \in]0,1]$ as follows : let $\sigma \in C$ be a generic point (in the open set $U_1 \cap C$), and let $|X_\sigma| \cap |Y| = \{y_1, \cdots, y_l\}$. Using the functions q and $\varkappa$ defined above on $p^{-1}(|Y|) \cap \pi^{-1}(U_1)$ we define

$$\varkappa'_C := \frac{1}{q'(\sigma)} \sup_{j \in [1,l]} \Big\{ q(\sigma, y_j).\varkappa(\sigma, y_j) \Big\}.$$

Of course this has a meaning because the functions $q(\sigma, y)$ and $\varkappa(\sigma, y)$ are locally constant on $p^{-1}(|Y|) \cap \pi^{-1}(U_1)$.
Recall that $q'(\sigma) = \sum_{j=1}^{l} q(\sigma, y_j) = q'_C$.
Now define the $\varkappa'-$filtration on the sheaf $\underline{H}^1_{[\Sigma_Y]}(\mathcal{O}_S)$ as follows :
Let $\nu : \tilde{S} \to S$ be the normalization map .
We shall say that a section $\tau \in \underline{H}^1_{[\Sigma_Y]}(\mathcal{O}_S)$ has order $\leq m$ for the $\varkappa'-$filtration if and only if the pole order of $\nu^*(\tau)$ is bounded by $q'_C.\varkappa'_C.m$ along the generic point of $\nu^*(C)$ for any irreducible component C of $|\Sigma_Y|$ which meets the open set where τ is defined .
It will be clear from our local theorem (2.23) above (see the first claim in the proof of the theorem below) that for a finite pole we have a filtered integration map near the generic points of $|\Sigma_Y|$

$$\pi_* p^* \underline{H}^{n+1}_{[Y]}(\Omega^n_Z) \to \underline{H}^1_{[\Sigma_Y]}(\mathcal{O}_S)$$

relative to the conormal filtration defined on $\underline{H}^{n+1}_{[Y]}(\Omega^n_Z)$ (see (2.14)) and the $\varkappa'-$filtration of $\underline{H}^1_{[\Sigma_Y]}(\mathcal{O}_S)$.

Let us restate and prove theorem (0.11) of our introduction :

(3.9) THEOREM. *Let $(X_s)_{s \in S}$ be an analytic family of $n-$cycles in a complex manifold Z, parametrized by a reduced complex space S and let Y be a proper and generically finite pole for this family.*
Then there exists a quasi-filtered [23] *integration map :*

$$\pi_* p^* \underline{H}^{n+1}_{[Y]}(\Omega^n_Z) \to \underline{H}^1_{[\Sigma_Y]}(\mathcal{O}_S)$$

when we endow the sheaf $\underline{H}^{n+1}_{[Y]}(\Omega^n_Z)$ with the conormal filtration and the sheaf $\underline{H}^1_{[\Sigma_Y]}(\mathcal{O}_S)$ with the $\varkappa'-$filtration .

Proof. We begin by proving the following claim.

CLAIM. The integration map is filtered at the generic points of Σ.

[23] Quasi-filtered means that locally on Σ_Y there exits $m_0 \in \mathbb{N}$ such that the map respects filtrations in degrees $\geq m_0$.

Proof of the claim. Let C be an irreducible component of $|\Sigma|$. Let $\sigma_0 \in C$ be generic and put $E_{\sigma_0} := |Y| \cap |X_{\sigma_0}|$. For each y in E_{σ_0} we choose an open neighbourhood U_y of y and a $\bar{\partial}-$closed $(n,n)-$form φ_y on $U_y \setminus Y \cap U_y$ which represents ξ. For each y we also take $\rho_y \in \mathcal{C}^\infty_c(U_y)$ such that $\rho \equiv 1$ in a neighbourhood of y in U_y. Then the order of $\int_{\mathcal{X}}(\xi)$ along $|\Sigma|$ near σ_0 is equal to the order of the almost meromorphic function

$$s \mapsto \sum_{y \in E_{\sigma_0}} \int_{X_s} \rho_y . \varphi_y$$

along $|\Sigma|$ near σ_0 and from theorem (2.23) we know that this order is $\leq q'_C . \varkappa'_C . m$. Hence the claim is proved because of our definition of $\varkappa'-$filtration.

Proof of theorem continued. Now let us take into account the non generic points in $|\Sigma|$.
Let $T \subset |\Sigma_Y|$ be the closed analytic subset corresponding to positive dimensional fibers of the map (1). Then we have $\mathrm{codim}_S(T) \geq 2$ by assumption[24].

Claim. The sheaf $\underline{H}^0_T(\underline{H}^1_{[\Sigma]}(\mathcal{O}_S)) = \underline{H}^1_{[T]}(\mathcal{O}_S)$ is coherent.

Proof of the claim. Let $\nu : \tilde{S} \to S$ be the normalization map for S and put $\tilde{T} := \nu^*(T)$.Then we have

$$\underline{H}^1_{[T]}(\mathcal{O}_S) \simeq \underline{H}^0_T(\nu_* \mathcal{O}_{\tilde{S}} / \mathcal{O}_S)$$

because $\underline{H}^0_{\tilde{T}}(\nu_* \mathcal{O}_{\tilde{S}})$ and $\underline{H}^1_{\tilde{T}}(\nu_* \mathcal{O}_{\tilde{S}})$ vanish. Since $\nu_* \mathcal{O}_{\tilde{S}}$ is $\mathcal{O}_S-$coherent this completes the proof of the claim.

Proof of theorem continued. Now from [B.K.03] we have an integration map

$$\pi_* p^* \underline{H}^{n+1}_{|Y|}(\Omega^n_Z) \to \underline{H}^1_{|\Sigma_Y|}(\mathcal{O}_S) \tag{2}$$

defined by cup-product with the relative fundamental class $C^Z_{X/S}$ and $S-$relative trace. This is compatible with the local integration map defined in [B.Mg.98] in the case of a finite pole. So on $|\Sigma_Y| \setminus T$ we may apply the theorem (2.23) . But the coherency of $\underline{H}^1_{[T]}(\mathcal{O}_S)$ allows us to find, locally on $|\Sigma_Y|$, an integer l such that $\mathcal{I}^l_{|\Sigma_Y|} . \underline{H}^1_{[T]}(\mathcal{O}_S) = 0$.
This gives the fact that (2) induces a quasi-filtered map for the required filtrations. $\square$

(3.10) Remark. We shall use this precise "quasi-filtered" integration theorem in order to produce global sections of powers of the line bundle associated with the (Cartier) incidence divisor Σ_Y assuming both S and Y compact. But there is no reason for the $\varkappa'-$filtration used in the previous theorem to correspond to a $\mathbb{Q}-$Cartier divisor (with support in $|\Sigma_Y|$). Consequently we shall have to ignore the components of $|\Sigma_Y|$ where $\varkappa'$ is not maximal, and we shall only produce (with suitable hypothesis) global sections of powers of the line bundle associated to the Cartier incidence divisor Σ_Y which separate points around generic points of the $\varkappa'-$maximal components !

[24]In fact we know from [B.Ka.03] that $|\Sigma|$ is the support of a Cartier divisor in S .

(3.11) REMARK. We have $\varkappa'_C = 1$ if and only if the generic cycle in C meets Y in a single point and at that point the cycle is smooth and not tangent to Y.

(3.12) LEMMA. Let $(X_s)_{s\in S}$ be an analytic family of $n-$cycles in a complex manifold Z. Let Y be a proper, generically finite pole for this family and denote by Σ the corresponding incidence divisor. Let $\sigma \in \Sigma$ be such that both S and $|\Sigma|$ are smooth near σ. Let C be the irreducible component of $|\Sigma|$ containing σ and let γ be a defining function for $|\Sigma|$ near σ. Put $E := |X_\sigma| \cap Y$ and let $\mathcal{M}(\sigma)$ be the defining ideal for E in Z. Then there exists an integer ν such that for all ξ in $\Gamma(E, \underline{H}^{n+1}_{[Y]}(\Omega^n_Z))$ for all g in $\Gamma(E, \mathcal{M}^\nu(\sigma))$ we have

$$\lim_{s\to\sigma} \gamma^j(s) \int_{X_s} (g.\xi) = 0 \quad \text{if} \quad j \geq q'_C.\varkappa'_C.(\text{conormal order of } \xi).$$

Proof. We note first that in the limit above we have chosen representatives for the germs g, ξ and $\int_{\mathcal{X}}(g\xi)$, but the assertion is clearly independent of the choices. Let $p : \mathcal{X} \to Z$ and $\pi : \mathcal{X} \to S$ be the canonical projections and let $m(\sigma)$ denote the defining ideal of σ in S. Then we have

$$\begin{aligned}\operatorname{Supp}(\pi^* m(\sigma)) + p^*\mathcal{I}_Y) &= \operatorname{Supp}(\pi^* m(\sigma)) \ \cap \ \operatorname{Supp}(p^*\mathcal{I}_Y) \\ &= |\pi^{-1}(\sigma)| \ \cap \ |p^{-1}(Y)| = \{\sigma\} \times E.\end{aligned}$$

Hence every element of $\mathcal{M}(\sigma)$, considered as a function on $\mathcal{X}$, is identically zero on $\{\sigma\} \times E$. It then follows from the Nullstellensatz that there exists an integer ν such that $\mathcal{M}(\sigma)^\nu \subset (\pi^* m(\sigma) + p^*\mathcal{I}_Y)$. This implies that every g in $\mathcal{M}(\sigma)^\nu$ can be written $g(z) = f(s,z) + h(z)$ with $f(\sigma, z) = 0$ and h in $\mathcal{I}_Y$. Let ξ be an element of (conormal) order m in $\Gamma(E, \underline{H}^{n+1}_{[Y]}(\Omega^n_Z))$. Then $h.\xi$ is of (conormal) order $\leq m-1$ and consequently

$$\lim_{s\to\sigma} \gamma^j(s) \int_{\mathcal{X}} (h\xi)(s) = 0$$

because $j > q'_C.\varkappa'_C.(m-1)$. We also see that

$$\lim_{s\to\sigma} \gamma^j(s) \int_{\mathcal{X}} (h\xi)(s) = 0$$

because $f(\sigma, z) = 0$. This completes the proof of the lemma. $\square$

(3.13) REMARK. In the situation of the previous lemma, a class ξ of conormal order m such that $j := q'_C.\varkappa'_C.m \in \mathbb{N}$ defines by the principal polar part of its integral on the cycles a section of the $j-$th power of the line bundle associated to Σ near σ . The value at σ of this section is zero if and only if

$$\lim_{s\to\sigma} \gamma^j(s). \int_{X_s} (\xi) = 0 \ .$$

(3.14) DEFINITION. Let Z be a compact complex manifold and let Y be a compact locally complete intersection in Z. We say that Y is an **ample subvariety** of Z if its normal bundle is ample, or equivalently if the zero section of its conormal bundle is an exceptional analytic subset.

(3.15) THEOREM. *Let Z be a complex manifold and let $(X_s)_{s\in S}$ be an analytic family of $n-$cycles in Z . Let Y be an ample locally complete intersection in Z which is a proper and generically finite pole for the family and let Σ be the corresponding incidence divisor. Then for every irreducible component C of $|\Sigma|$ there exist an arbitrarily big integer m and an element ξ in $H^{n+1}_{[Y]}(Z,\Omega^n_Z)$ of (conormal) order m such that $j := q'_C.\varkappa'_C.m$ is an integer and such that for generic σ in C we have $\lim_{s\to\sigma}\gamma^j(s).\int_{X_s}(\xi)\neq 0$, where γ is a defining function for $|\Sigma|$ near σ.*

Proof. We observe first that the generic point σ in C satisfies the conditions in lemma (3.12). Let σ be such a point in C and let E, $\mathcal{M}(\sigma)$, ν and γ be defined as in lemma (3.12). Put $\mathcal{O}_\nu(s_0) := \mathcal{O}_Z/\mathcal{M}^\nu(s_0)$ and consider the conormal filtration

$$\Phi_1 \subset \Phi_2 \subset \cdots \subset \underline{H}^{n+1}_{[Y]}(\Omega^n_Z).$$

Let ξ be an element of (conormal) order m in $H^{n+1}_{[Y]}(Z,\Omega^n_Z)$. Then lemma (3.12) implies that for all $j\geq q'_C.\varkappa'_C.m$ the limit $\lim_{s\to\sigma}\gamma^j(s)\int_{\mathcal{X}}(\xi)(s)\neq 0$ only depends on the image of ξ in $\Big(\Phi_m/\Phi_{m-1}\Big)\otimes\mathcal{O}_\nu(\sigma)$.

Let f be a flat morphism defined by a generating regular sequence for I_Y in a neighbourhood V of E and consider the image family of hypersurfaces $(f_*X_s)_{s\in S}$ in $f(V)$, which is defined for all s near σ. Then from lemma (3.5) we know that the rational number associated with the image family (with the origin as a pole) is equal to $\varkappa'_C$[25], and by theorem (1.2) we know that in every interval of length k in $\mathbb{N}$ there is an integer m and an element η of order m in $H^{n+1}_{[0]}(Z,\Omega^n_{\mathbb{C}^{n+1}})$ such that $j := q'_C.\varkappa'_C.m$ is an integer and such that $\lim_{s\to\sigma}\gamma^j(s)\int_{f_*\mathcal{X}}(\eta)(s)\neq 0$. Since $\int_{f_*\mathcal{X}}(\eta)=\int_{\mathcal{X}}(f^*\eta)$ we see that the element $f^*\eta$ belongs to $\Gamma(E,\Phi_m)$ and $\lim_{s\to\sigma}\gamma^j(s)\int_{\mathcal{X}}(f^*\eta)(s)\neq 0$.

Thus to finish the prove of the theorem it is sufficient to show that the canonical morphism

$$\Gamma(Z,\Phi_m)\to\Gamma(E_{s_0},\Big(\Phi_m/\Phi_{m-1}\Big)\otimes\mathcal{O}_\nu(s_0))$$

is surjective for $m>>0$.

From the local description of $\mathcal{G}_0$ in (2.13) and the fact that $\mathcal{I}_Y\mathcal{F}_m=\mathcal{F}_{m-1}$ for all $m\geq 1$ we deduce that $\mathcal{I}_Y\Phi_m=\Phi_{m-1}$ and consequently that Φ_m/Φ_{m-1} is isomorphic to $\mathcal{F}_m/\mathcal{F}_{m-1}\otimes\mathcal{G}_0/\mathcal{I}_Y\mathcal{G}_0$ for all $m\geq 1$. From lemme 6 in [B.Mg. 99] we know that $\mathcal{F}_m/\mathcal{F}_{m-1}$ is isomorphic to $\mathrm{S}^{m-1}(\mathcal{N}_{Y|Z})\otimes\underline{\mathrm{Ext}}^{n+1}_{\mathcal{O}_Z}(\mathcal{O}_Y,\mathcal{O}_Z)$ where $\mathcal{N}_{Y|Z}$ is the locally free normal bundle sheaf of Y in Z. From the facts that the sheaf $\mathcal{G}_0/\mathcal{I}_Y\mathcal{G}_0$ is coherent on Y and that the sheaf $\mathcal{N}_{Y|Z}$ is ample on Y we then deduce (by the same argument as in lemme 6 in [B.Mg. 99]) that the canonical morphism

$$\Gamma(Z,\Phi_m)\to\Gamma(E_{s_0},\Big(\Phi_m/\Phi_{m-1}\Big)\otimes\mathcal{O}_\nu(s_0))$$

[25] We use the map induced by $id_S\times f$ between the graphs of $(X_s)_{s\in S}$ and of $(f_*X_s)_{s\in S}$. Then we let $\mathcal{I}$ be the ideal of definition of Σ and $\mathcal{J}$ be the ideal of definition for the origin in $\mathbb{C}^{n+1}$.

is surjective for $m >> 0$. $\square$

(3.16) REMARKS.

1) To obtain a representative $\hat{\xi}$ in $H^n(Z \setminus Y, \Omega^n_Z)$ for the cohomology class ξ in the previous theorem (for an arbitrarily big m), it would be enough to have $\dim\Big(H^{n+1}(Z, \Omega^n_Z)\Big) < +\infty$. This is of course the case (see [A-G]) when Z is a $n-$convex manifold (*a fortiori* when Z is compact!). In this case we obtain a holomorphic function on $S \setminus \Sigma$ with a well controlled pole order along the irreducible component C without assuming S compact.
2) If $\dim Y \leq n$ then there exists an arbitrarily small $n-$complete neighbourhood V of Y in Z (see [B. 80]) and consequently the class ξ can be represented by a class $\hat{\xi}$ in $H^n(V \setminus Y, \Omega^n_Z)$ in this case. Of course this gives meromorphic functions defined in a neighbourhood of Σ_Y with controlled pole order along Σ_Y.

(3.17). Let $\pi_m : \overline{p^{-1}(Y)_{\max}} \to |\Sigma|$ be induced by the canonical mapping π. Clearly it is proper, surjective and generically finite (see 3.7). For each irreducible component C of $|\Sigma|$ we denote by l_C the degree of π_m over C. Then there exist an open dense subset C' of C and a holomorphic multisection $\varphi^0_C : C' \to \mathrm{Sym}^{l_C}(Y_{\max})$ that associates to each point in C' the points in its π_m-fiber.
For a generic point in C the divisor Σ has a well defined multiplicity which we shall denote by q_C. Then for a generic point σ in C we have (see 3.8)

$$q'_C = \sum_{(\sigma,y)\in\pi^{-1}(\sigma)} q(\sigma, y).$$

(3.18) DEFINITION. We say that the analytic family $(X_s)_{s\in S}$ is **separated at order** 0 **along** Y if there exists an irreducible component C of Σ with $\varkappa'_C = \varkappa'$ such that the mapping φ^0_C is finite.

(3.19) REMARK. If the cycles of the family $(X_s)_{s\in S}$ are the fibres of an (equidimensional) morphism $\pi : Z \to S$ then it is separating of order 0 along Y [26].

(3.20). Let C be an irreducible component of $|\Sigma|$ and let l_C, C' and φ^0_C be defined as before. Let $\sigma \in C'$ and let y be a point in $|X_\sigma| \cap |Y|$ such that y is a smooth point of X_σ where X_σ is not tangent to Y. Then the image of $T_{X_\sigma,y}$ in $N_{Y|Z,y}$ is a hyperplane[27] and, as such, defines an element in $\mathbb{P}(N_{Y|Z,y}) = \mathbb{P}(N_{Y|Z})_y$. Suppose that we have a dense open subset of C, which we shall also denote by C', such that for every σ in C' every y in $|X_s| \cap Y_{\max}$ the cycle X_σ is smooth at y and not tangent to Y at y. Then we obtain a well defined holomorphic mapping

$$\varphi^1_C : C' \to \mathrm{Sym}^{l_C}(\mathbb{P}(N_{Y|Z})|_{Y_{\max}})$$

over $\mathrm{Sym}^{l_C}(Y_{\max})$.

[26] Assuming, of course, that Y is a proper, generically finite pole for the family !
[27] This is precisely the definition of "non tangent".

(3.21) Definition. We say that the analytic family $(X_s)_{s\in S}$ is **separated at order** 1 **along** Y if there exists an irreducible component C of $|\Sigma|$ with $\varkappa'_C = \varkappa'$ where the mapping

$$\varphi^1_C : C' \to \mathrm{Sym}^{l_C}(\mathbb{P}(N_{Y|Z})|_{Y_{\max}})$$

is well defined and generically finite.

(3.22). Let C be an irreducible component of $|\Sigma|$ with $l_C = 1$ and let $\sigma \in C$ such that $\{y\} = |X_\sigma| \cap Y_{\max}$. Let $C_{X_\sigma,y}$ denote the Zariski tangent cone of X_σ at y[28]. If the restriction of the canonical mapping $T_{Z,y} \to N_{Y|Z,y}$ to $|C_{X_s,y}|$ is injective then $C_{X_s,y}$ has a well defined image in $N_{Y|Z,y}$ and hence defines a hypersurface of degree k_σ in $\mathbb{P}(N_{Y|Z,y})$.
Let us denote by $k_C = k$ the generic value of k_σ for $\sigma \in C$. Now let $H_k(\mathbb{P}(N_{Y|Z})|_{Y_{\max}})$ be the bundle space over $Y_{\max}$ whose fibre over a point y is the set of hypersurfaces of degree k in $\mathbb{P}(N_{Y|Z,y})$, and let $\varphi^T_C(\sigma)$ be the element defined by $C_{X_\sigma,y}$ in $H_k(\mathbb{P}(N_{Y|Z})|_{Y_{\max}})$ for a generic $\sigma \in C$.

(3.23) Definition. We say that the analytic family $(X_s)_{s\in S}$ is **separated by tangent cones along** Y if there exists an irreducible component C of $|\Sigma|$ with $\varkappa'_C = \varkappa'$ and $l_C = 1$ such that C contains an open dense subset C' having the following properties :

- $\gcd(q'_C, k_C) = 1$
- the mapping $\varphi^T_C : C' \to H_k(\mathbb{P}(N_{Y|Z})|_{Y_{\max}})$ is well defined and has finite fibres.

(3.24) Theorem. *Let Z be a complex manifold and let $(X_s)_{s\in S}$ be an analytic family of $n-$cycles in Z with S compact and connected. Let Y be a compact and generically finite pole with respect to $(X_s)_{s\in S}$ and let Σ denote the associated incidence divisor. If Y is an ample subvariety* [29] *of Z and if the family is separated at order 0 , 1 or by tangent cones along Y then Σ is big.*

Proof. Let L be the line bundle on S defined by the Cartier divisor Σ. We observe first that in each of the three cases it is sufficient to prove the following :

For any two points σ_0 and σ_1 belonging to different fibres of the mapping in question (φ^0_C, φ^1_C or φ^Z_C) there exist an arbitrarily big integer j and a global section τ of L^j such that $\tau(\sigma_0) \neq 0$ and $\tau(\sigma_1) = 0$.

This is sufficient because in each case it implies the existence of an integer j such that the Kodaira mapping of L^j has fibres with isolated points. Hence it is generically finite and consequently L is big.

From now on $\mathcal{X}$ will denote the graph of the family $(X_s)_{s\in S}$.

Claim. Let σ_0 and σ_1 be two points belonging to the same irreducible component C of $|\Sigma|$ with $\varkappa'_C = \varkappa'$ such that both C and $|\Sigma|$ are smooth near $\{\sigma_0, \sigma_1\}$ and such that the corresponding cycles have only finitely many points of intersection with Y. Put $E := (|X_{\sigma_0}| \cup |X_{\sigma_1}|) \cap Y$ and let γ be a (reduced) defining

[28] $C_{X_\sigma,y}$ is a $n-$cycle in $T_{Z,y}$.
[29] Recall that this implies that Y is a l.c.i. in Z such that its normal bundle is ample on Y .

function for $|\Sigma|$ in a neighbourhood of $\{\sigma_0, \sigma_1\}$. If there exist an arbitrarily big integer m and an element ξ of conormal order m in $\Gamma(E, \underline{H}^{n+1}_{[Y]}(\Omega^n_Z))$ such that $\nu := q'_C.\varkappa'.m$ is an integer and such that

$$\lim_{s\to\sigma_0} \gamma^\nu(s).\int_{\mathcal{X}}(\xi)(s) \neq 0 \quad \text{and} \quad \lim_{s\to\sigma_1} \gamma^\nu(s).\int_{\mathcal{X}}(\xi)(s) = 0,$$

then there exist an arbitrarily big integer j and a global section τ of L^j such that $\tau(\sigma_0) \neq 0$ and $\tau(\sigma_1) = 0$.

Proof of the claim. Let $\mathcal{M}(E)$ denote the defining ideal of the set E in Z and put $\mathcal{O}_l(E) := \mathcal{O}_Z/\mathcal{M}^l(E)$ for every positive integer l. By lemma (3.12) there exists an integer l such that for every element ξ of (conormal) order m in $\Gamma(E, \underline{H}^{n+1}_{[Y]}(\Omega^n_Z))$ the limit $\lim\limits_{s\to\sigma_i} \gamma^j(s)\int_{\mathcal{X}}(\xi)(s)$ only depends on the image of ξ in $\Phi_m/\Phi_{m-1}\otimes\mathcal{O}_l(E)$, for all $j \geq q'_C.\varkappa'.m$ and $i = 0,\ 1$.
Then by the same argument as in lemme 6 in [B.Mg. 99] (already used in the proof of theorem (3.15)) we show that the canonical morphism

$$\Gamma(Y, \Phi_m/\Phi_{m-1}) \to \Gamma\Big(Y, (\Phi_m/\Phi_{m-1})\otimes\mathcal{O}_l(E)\Big) \tag{16}$$

is surjective for all $m >> 0$. Let $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \dots$ denote the $|\Sigma|-$filtration on $\underline{H}^1_{[Y]}(\mathcal{O}_S)$ and choose an integer m_0 such that both the canonical morphism above and the canonical mapping

$$H^0_{alg}(S\setminus\Sigma, \mathcal{O}_S) \to H^1_{[\Sigma]}(S, \mathcal{O}_S)/\mathcal{B}_{[q'_C.\varkappa'.m]-1} \tag{17}$$

are surjective for all $m \geq m_0$ [30], where $[\,]$ denotes the integral part of a real number. Let $m \geq m_0$ and let ξ be an element of (conormal) order m in $\Gamma(E, \underline{H}^{n+1}_{[Y]}(\Omega^n_Z))$ such that $q'_C.\varkappa'.m \in \mathbb{N}$ and such that

$$\lim_{s\to\sigma_0} \gamma^{q'_C.\varkappa'.m}(s).\int_{\mathcal{X}}(\xi)(s) \neq 0 \quad \text{and} \quad \lim_{s\to\sigma_1} \gamma^{q'_C.\varkappa'.m}(s).\int_{\mathcal{X}}(\xi)(s) = 0, \tag{18}$$

Since (16) is surjective there exists an element η in $\Gamma(Y, \Phi_m/\Phi_{m-1})$ having the same image as ξ in $\Gamma\Big(Y, (\Phi_m/\Phi_{m-1})\otimes\mathcal{O}_l(E)\Big)$, and since (17) is surjective there exists a holomorphic function g on $S\setminus\Sigma$ having a pole of order $q'_C\varkappa'.m$ along $|\Sigma|$ near $\{\sigma_0, \sigma_1\}$ such that g and η have the same image in $H^1_{[\Sigma]}(S, \mathcal{O}_S)/\mathcal{B}_{[q'_C.\varkappa'.m]-1}$.
Let γ be a defining function for $|\Sigma|$ in a neighbourhood of $\{\sigma_0, \sigma_1\}$. Then

$$\lim_{s\to\sigma_0} \gamma^{q'_C.\varkappa'.m}(s).g(s) = \lim_{s\to\sigma_0} \gamma^{q'_C.\varkappa'.m}(s).\int_{\mathcal{X}}(\xi)(s) \neq 0$$

and

$$\lim_{s\to\sigma_1} \gamma^{q'_C.\varkappa'.m}(s).g(s) = \lim_{s\to\sigma_1} \gamma^{q'_C.\varkappa'.m}(s).\int_{\mathcal{X}}(\xi)(s) \ = 0.$$

Let j be an integer such that $\varkappa'.m.j \in \mathbb{N}^*$. Then we obtain

$$\lim_{s\to\sigma_0} (\gamma^{q'_C}(s))^{\varkappa'.m.j}.g^j(s) \neq 0 \quad \text{and} \quad \lim_{s\to\sigma_1} (\gamma^{q'_C}(s))^{\varkappa'.m.j}.g^j(s) = 0. \tag{19}$$

[30] $H^0_{alg}(S\setminus\Sigma, \mathcal{O}_S)$ is the inverse image of $H^1_{[\Sigma]}(S, \mathcal{O}_S)$ by the morphism $H^0(S\setminus\Sigma, \mathcal{O}_S) \to H^1_{\Sigma}(S, \mathcal{O}_S)$.

Since $\gamma^{q'_C}$ is a defining function for Σ in a neighbourhood of $\{\sigma_0, \sigma_1\}$ we conclude that g^j is a holomorphic function on $S \setminus \Sigma$ having a pole of order $\varkappa'.m.j$ along Σ. Let τ be the global section of $L^{\varkappa'.m.j}$ induced by g^j. Then (19) implies that $\tau(\sigma_0) \neq 0$ and $\tau(\sigma_1) = 0$. Thus the claim is proved.

Proof of the theorem continued. In each case we will denote by C' the open dense subset of C where the relevant mapping (φ^0_C, φ^1_C or φ^T_C) is defined.
Let σ_0 and σ_1 be two points belonging to different fibres of the mapping in question and define E as in the above claim. We want to show that there exist an arbitrarily big integer m and an element ξ in $\Gamma(E, \underline{H}^{n+1}_{[Y]}(\Omega^n_Z))$ such that $q'_C.\varkappa'.m \in \mathbb{N}$ and such that $\int_{\mathcal{X}}(\xi)$ satisfies (18). We will treat the three cases separately.

(i) If $\varphi^0_C(\sigma_0) \neq \varphi^0_C(\sigma_1)$ then there exists a point y in $|X_{\sigma_0}| \cap Y_{\max}$ such that $y \notin |X_{\sigma_1}| \cap Y_{\max}$. Then $q(\sigma_0, y).\varkappa(\sigma_0, y) = q'_C.\varkappa'$ and by corollary (2.26) there exist an arbitrarily big integer m and an element $\xi(y)$ of (conormal) order m in $\left(\underline{H}^{n+1}_{[Y]}(\Omega^n_Z)\right)_y$ such that $\int_{\mathcal{X}}(\xi(y))$ is of $|\Sigma|-$order $q'_C.\varkappa'.m$ near σ_0. Let ξ be the element in $\Gamma(E, \underline{H}^{n+1}_{[Y]}(\Omega^n_Z))$ defined by $\xi_x = 0$ if $x \neq y$ and $\xi_y = \xi(y)$. Then ξ has the desired properties.

(ii) If $\varphi^1_C(\sigma_0) \neq \varphi^1_C(\sigma_1)$ then we have either $|X_{\sigma_0}| \cap Y_{\max} \neq |X_{\sigma_1}| \cap Y_{\max}$ in which case we get the the result directly from (i) or there exists a point y in E such that $T_{X_{\sigma_0}, y} \neq T_{X_{\sigma_1}, y}$. In the latter case we choose a flat morphism f defined by a generating sequence of $\mathcal{I}_Y$ in a neighbourhood V of y in Z and consider the image family $(f_*(X_s|_V)_s)$ in $f(V)$ defined for s near $\{\sigma_0, \sigma_1\}$. Let

$$P(s,z) = P(s,0) + P_1(s,z) + P_2(s,z) + \cdots$$

be a defining function for the image family, where $P_j(s,z)$ is homogeneous of degree j in z. Then by hypothesis $P_1(\sigma_0, z)$ and $P_1(\sigma_1, z)$ are linearly independant. Now, using the remark (1.3) 2) , we see that in this case we have

$$\varkappa(\sigma_0, z) = \varkappa((\sigma_1, z) = 1$$

and thus $q(\sigma_0, z) = q(\sigma_1, z) = q'_C.\varkappa'$. Let γ be a defining function for $|\Sigma|$ near $\{\sigma_0, \sigma_1\}$. Then by theorem (1.2) and remark (1.3) 1) we know that for any element η of order m in $H^{n+1}_{[0]}(\Omega^n_{\mathbb{C}^{n+1}})$ we have

$$\lim_{s \to \sigma_i} \gamma^{q'_C.\varkappa' m}(s). \int_{\mathcal{X}} (f^*\eta)(s) = 0 \quad \text{if and only if} \quad \langle P_1(\sigma_i, z)^m, d\eta \rangle = 0,$$

for $i = 0, 1$. Since the polynomials $P_1(\sigma_0, z)$ and $P_1(\sigma_1, z)$ are linearly independent $P_1^m(\sigma_0, z)$ and $P_1^m(\sigma_1, z)$ are also linearly independent for all $m \geq 1$. It follows that for every integer $m \geq 1$ there exists an element η of order m in $H^{n+1}_{[0]}(\Omega^n_{\mathbb{C}^{n+1}})$ such that

$$\langle P_1(\sigma_0, z)^m, d\eta \rangle \neq 0 \quad \text{and} \quad \langle P_1(\sigma_1, z)^m, d\eta \rangle = 0. \tag{20}$$

Then for any integer $m \geq 1$ we pick an element η which satisfies (20) and define ξ in $\Gamma(E, \underline{H}^{n+1}_{[Y]}(\Omega^n_Z))$ by $\xi_y = f^*\eta$ and $\xi_x = 0$ for $x \neq y$.

(iii) Let y be the only point in E. We proceed in the same manner as in case (ii) and consider a defining function for the image family

$$P(s,z) = P(s,0) + P_1(s,z) + P_2(s,z) + \cdots \qquad .$$

We observe that $\varkappa' = \dfrac{1}{k}$ and $\gamma(s) = P(s,0)$ is a defining function for Σ near $\{\sigma_0, \sigma_1\}$. By theorem (1.2) and remark (1.3) 1) it follows that for any integer $m \geq 1$ and any element η of order $m.k$ in $H^{n+1}_{[0]}(\Omega^n_{\mathbb{C}^{n+1}})$ we have

$$\lim_{s\to\sigma_i} \gamma^m(s).\int_{\mathcal{X}} (f^*\eta)(s) = 0 \quad \text{if and only if} \quad \langle P_k(\sigma_i, z)^m, d\eta\rangle = 0,$$

for $i = 0, 1$. By hypothesis the cycles $f_*X_{\sigma_0}$ and $f_*X_{\sigma_1}$ have different Zariski tangent cones at the origin and these cones are given by $P_k(\sigma_0, z) = 0$ and $P_k(\sigma_1, z) = 0$. Consequently the homogeneous polynomials $P_k(\sigma_0, z)$ and $P_k(\sigma_1, z)$ are linearly independent. Hence for any integer of the form $m.k$ we can pick an element η of order $m.k$ in $H^{n+1}_{[0]}(\Omega^n_{\mathbb{C}^{n+1}})$ that satisfies the conditions

$$\langle P_k(\sigma_0, z)^m, d\eta\rangle \neq 0 \quad \text{and} \quad \langle P_k(\sigma_1, z)^m, d\eta\rangle = 0$$

and then $\xi := f^*\eta$ is an element of (conormal) order $m.k$ in

$$\Big(\underline{H}^{n+1}_{[Y]}(\Omega^n_Z)\Big)_y = \Gamma(\{y\}, \underline{H}^{n+1}_{[Y]}(\Omega^n_Z))$$

having the desired properties. $\square$

(3.25) Remarks.

1) What is actually proved in the theorem is that the fibres of the mappings φ^0_C, φ^1_C and φ^T_C can be separated by global sections of some powers of L. Hence without assuming finite fibres we get the following inequality:

$$\text{Kodaira dimension of } L \;\geq\; \dim S \;-\; (\text{fibre dimension of } \varphi^i_C)$$

for $i = 0,\ 1$ or Z[31].

2) Let Z be a complex manifold and let $\pi : Z \to S$ be an $n-$equidimensional morphism where S is a normal complex space. Let Y be a subvariety of Z whose dimension is equal to $\dim S - 1$. If the restriction of the morphism π to Y is proper and generically finite onto its image $|\Sigma|$, then Y is a proper, generically finite pole for the analytic family of cycles given by the fibres of π (see [B.75]) and this family is obviously separated of order 0 along Y.

4. Applications.

(4.1). In this last section we shall prove the results announced in the introduction. The theorem (0.7) is an immediate consequence of theorem (3.24). But this is not as simple in the case of corollary (0.8).

Proof of corollary (0.8). We begin by producing a covering analytic family of $n-$cycles in Z. As the map π is $n-$equidimensional and S is normal, there exists an analytic map (see [B.75])

$$f : S \to \mathcal{C}_n(Z)$$

[31] The mappings φ^0_C φ^1_C φ^Z_C are holomorphic on some connected open and dense set in C and consequently have a well defined fibre dimension.

such that for s generic in S we have $f(s) = |\pi^{-1}(s)|$. The map f defines a covering analytic family $(X_s)_{s\in S}$ of $n-$cycles in Z such that for the generic s we have $X_s = |\pi^{-1}(s)|$.

The next step is to prove that Y is a proper and generically finite pole for this covering family. So we have to prove that the map

$$\pi : |Y| \to \pi(|Y|) = |\Sigma_Y|$$

is generically finite on its image. As $\dim(|Y|) = p-1$ and $\dim(S) = p$, it is enough to prove that each irreducible component of $|\Sigma_Y|$ is of pure codimension 1 in S.
We shall use the $n-$convexity of $Z \setminus |Y|$ to prove this. As Y is a locally complete intersection of codimension $n+1$ with a (Griffiths) positive normal bundle, the $n-$convexity of $Z \setminus |Y|$ is a consequence of [F. 76] . Now we may deduce from [B.78] that $S \setminus \Sigma$ is holomorphically convex : There is no Kähler metric on $Z \setminus Y$ to allow us to follow the technique of loc.cit. but we know that for any hemitian $\mathcal{C}^0$ metric on Z the corresponding volume function $S \to \mathbb{R}_+$ is bounded (by continuity and compacity of S !)
Then we conclude that each irreducible component of Σ has codimension 1 in S . Hence Y is a proper, generically finite pole for the covering family $(X_s)_{s\in S}$. Now it is easy to use theorem (3.24) because generically the cycles are locally separated along Y (at order 0) for any irreducible component of $|\Sigma_Y|$ by remark (3.19). $\square$

(4.2) Remark. When the surjective map π is not assumed to be equidimensional, there exist a compact normal complex space $\tilde{S}$, a modification $\tau : \tilde{S} \to S$ along the analytic set T where the fibre dimension of π jumps and a covering analytic family of compact $n-$cycles $(X_{\tilde{s}})_{\tilde{s}\in\tilde{S}}$ in Z such that for generic $\tilde{s}$ we have $X_{\tilde{s}} = |\pi^{-1}(\tau(\tilde{s})|$[32].
Now to conclude that S is Moishezon it is enough to show that Y is a proper, generically finite pole for this family and that the local separation condition holds along $\tilde{\Sigma} := \{\tilde{s} \in \tilde{S} / X_{\tilde{s}} \cap Y \neq \emptyset\}$ holds.
This is clear as soon as it is shown that no irreducible component of $\tilde{\Sigma}$ is contained in $\tau^{-1}(T)$:
the same proof shows that $\tilde{S} \setminus \tilde{\Sigma}$ is holomorphically convex , so $\tilde{\Sigma}$ is of pure codimension 1 in $\tilde{S}$. Then our assumption implies that the image by τ of any irreducible component C of $\tilde{S}$ is an irreducible component of $\pi(|Y|)$ that has codimension 1 in S. It follows that we have the equalities of dimensions $\dim C = \dim \tau(C) = p-1 = \dim Y$; thus Y is a generically finite pole.
The separation (at order 0) is again obvious from remark (3.19) because for any C the generic point of C is mapped by τ outside of T.

We show now, using results on holomorphic convexity of cycle spaces, that problem (0.4) can reduced to a rather special parameter space S and give for these cases a transcendental analogue of what is expected .

(4.3) Theorem. *Let Z a compact connected complex manifold of dimension $n+p$ and let $Y \subset Z$ be a compact locally complete intersection of dimension $p-1$.*

[32]Take for $\tilde{S}$ the normalization of the graph of the meromorphic map $S - -- \to \mathcal{C}_n(Z)$ given by the generic fibres of π .

We assume that

1) $N_{Y|Z}$ *is positive*
2) *there exists a compact normal complex space* S *parametrizing a covering analytic family of compact* $n-$*cycles* $(X_s)_{s\in S}$ *in* Z *such that the corresponding classifying map* $c : S \to \mathcal{C}_n(Z)$ *is the normalization of its image, and such that the generic cycle of this family is irreducible and without multiple components* [33].

Then, either $a(Z) \geq p$ *or* $\dim S = p$, S *does not contain a compact covering family of algebraic cycles of positive dimension* [34], Σ *is of pure codimension* 1 *in* S *and* $S \setminus \Sigma$ *is a proper modification of a Stein space.*

Proof. Denote by d the dimension of S , and assume $a(S) < p$; of course $d \geq p$ because the graph of the covering family has dimension $d + n \geq n + p$. Assume now $d \geq p + 1$. Then the strong $n-$convexity of $Z \setminus Y$, which is a consequence of the positivity of $N_{Y|Z}$ thanks to [F. 76] , implies the holomorphic convexity of $S \setminus \Sigma$ by the same argument as in the proof of corollary (0.8)[35]. Since $S \setminus \Sigma$ is not compact[36] we conclude that Σ is of pure codimension 1 in S and that $S \setminus \Sigma$ is a proper modification of its Remmert's reduction, which is a Stein space of dimension d[37].
Now consider the graph $S \bowtie Z$ of our analytic family of cycles and its projections p and π onto Z and S respectively. The generic fibre of p has dimension $d+n-(n+p) = d-p \geq 1$ by our assumption, so for generic $z \in Z$ its fibre $p^{-1}(z)$ is a compact analytic subset of S of dimension ≥ 1. If, for a generic $z \in Z$ we have $\Sigma \cap \pi(p^{-1}(z)) \neq \emptyset$ this means that for such a z there exists $s(z) \in \Sigma$ with $z \in X_{s(z)}$. Thus we have $Z = \bigcup_{s\in\Sigma} X_s$ which gives $a(Z) \geq p$ thanks to remark (0.6) 3) in the introduction .
Thus we may assume that $\Sigma \cap \pi(p^{-1}(z)) = \emptyset$ for generic $z \in Z$. This means that $\Gamma(z) := \cup_{X_s \ni z} X_s$ is a compact analytic set of dimension $> n$ in $Z \setminus Y$. But care is required because this set could have some irreducible component of dimension n. For a generic z there exists $s \in \pi(p^{-1}(z))$ such that X_s is irreducible . This implies that at least one irreducible component of $\Gamma(z)$ containing z has dimension $> n$ (because the classifying map c is finite and generically injective). Fix a smooth exhaustion function on $Z \setminus Y$ which is strongly $n-$convex outside a compact set K. Then for a generic $z \in Z \setminus Y \setminus K$ we have a compact irreducible complex subset of dimension $> n$ of $Z \setminus Y$ not contained in K and this gives a contradiction .
So we conclude that $d \geq p + 1$ implies $a(Z) \geq p$.
Now the only case we are left to consider is the case where $d = p$.
Assume that S has a covering family $(C_t)_{t\in T}$ of compact algebraic cycles of dimension $k \geq 1$ parametrized by a compact normal complex space T . As before, without loss of generality, we may assume that for t generic the cycle C_t is irreducible and has multiplicity 1 and that the corresponding classifying map $T \to \mathcal{C}_k(S)$ is finite and generically injective. We define $\Gamma_t = \cup_{s\in C_t} X_s$ for each $t \in T$.

[33] We can modify the family in order to satisfy these properties when we begin with any covering family.

[34] Algebraic means that each component of the cycle is a (reduced) Moishezon complex space.

[35] Actually the volume of the cycles in our family is uniformly bounded (for any continuous hermitian metric on Z) and the method of [B.78] applies here as well.

[36] The case $S = \Sigma$ is excluded because it implies $a(Z) \geq p$ as explained in remark 3 of (0.6) in the introduction.

[37] We can separate generic cycles near Σ by global holomorphic functions on $S \setminus \Sigma$ coming from integration of (n, n) cohomology classes using our assumption that the generic cycle in the family is irreducible.

As we assume that T is normal , the family $(\Gamma_t)_{t\in T}$ is an analytic family of compact $(n+k)-$cycles in Z [38] such that the generic cycle is irreducible and with multiplicity 1 . If $C_t \cap \Sigma \neq \emptyset$ for all t in T , then T is a Moishezon space because we know that Σ is a Moishezon space using [C.80]. But $T \bowtie S$ has algebraic fibers over S (C_t is algebraic by assumption) and there exists an "algebraic section" $T \bowtie \Sigma$ of this fibration (because T and Σ are Moishezon !) so , thanks again to [C.80] we conclude that S is Moishezon .
Thus we may assume $C_t \cap \Sigma = \emptyset$ for generic t . Then $\Gamma_t \subset Z \setminus Y$ for generic t . As the family $(\Gamma_t)_{t\in T}$ covers Z , for a generic t, $\Gamma_t \not\subset K$. This gives a contradiction , because a compact irreducible analytic subset of dimension $> n$ of $Z \setminus Y$ is contained in K . ☐

(4.4) Remarks.

1) When $\dim S = p$ and $a(S) = p-1$ we can cover S with a compact analytic family of compact complex curves (using the algebraic reduction of S). So the proposition applies and for $a(S) = p-1$ we have $a(Z) \geq p$.
2) Let $D := p(\pi^{-1}(\Sigma))$ and assume that $\dim S = p$ and $a(Z) < p$. Then the projection of the graph $p : S \bowtie Z \to Z$ is generically finite because it is surjective and $\dim(S \bowtie Z) = n + p = \dim Z$. The composed map

$$S \bowtie Z \setminus \pi^{-1}(\Sigma) \to S \setminus \Sigma \to R$$

where the second map is the Remmert's reduction, can be used to construct p holomorphic functions on $S \bowtie Z \setminus \pi^{-1}(\Sigma)$ that are algebraically independent. Using traces of powers of these, it is easy to obtain p holomorphic functions on $Z \setminus D$ that are also algebraically independent. So we have $\mathrm{codim}_Z(D) = 1$ and $Y \subset D$.
This is a transcendental analogue of what we are looking for here (i.e. $a(Z) \geq p$).

(4.5) Corollary. *Assume that in problem (0.4) we have at least one cycle in the covering family $(X_s)_{s\in S}$ that intersects Y in a finite non-empty set. Then $a(S) \geq 1$ and consequently $a(Z) \geq 1$.*

(4.6) Remark. For $p = 2$ (in which case Y is a curve) and $n = 1$ the problem (0.4) is solved by the previous corollary subject to a very weak condition involving the covering family and Y . Compare with remark (4.4) 1) above.

The proof of the corollary is an immediate consequence of remark (3.16) 1) .

Appendix. Let Y be a locally complete intersection of codimension $n+1$ in a complex manifold Z .

(A.1) Lemma. Let y be a point in Y. Then the set of all $(n+1)$-tuples $(g_0, \dots, g_n)$ in $\mathcal{I}_{Y,y}^{\oplus n+1}$ such that $g_0, \dots, g_n$ generate $\mathcal{I}_{Y,y}$ is an open dense subset of $\mathcal{I}_{Y,y}^{\oplus n+1}$.

[38] This means that we can choose multiplicities , generically equal to 1 , to have an analytic family of cycles ; see [B.75] th.1 .

Proof. Let m be the defining ideal of y in Z. Since the minimal length of a generating sequence for $\mathcal{I}_{Y,y}$ is $n+1$, Nakayama's lemma tells us that the $\mathbb{C}$ vector space $V := \mathcal{I}_{Y,y}/m\mathcal{I}_{Y,y}$ is of dimension $n+1$. By the same lemma we know that $n+1$ elements $g_0, \ldots, g_n$ in $\mathcal{I}_{Y,y}$ generate $\mathcal{I}_{Y,y}$ if and only if their images $\overline{g_0}, \ldots, \overline{g_n}$ in V form a basis for V. It is clear that the set of all bases $(v_0, \ldots, v_n)$ of V is a Zariski open dense subset of V^{n+1}, so to finish the proof we only have to note that $\mathcal{I}_{Y,y}$ and $m\mathcal{I}_{Y,y} \oplus V$ are isomorphic as topological vector spaces. $\square$

(A.2) Remark. It is clear from the proof of the preceeding lemma that the generating $(n+1)$-tuples in a finite-dimensional affine subspace $\mathcal{A}$ of $\mathcal{I}_{Y,y}^{\oplus n+1}$ form a Zariski open subset of $\mathcal{A}$. In particular it is connected.

We shall assume from now on that Y is a finite pole for an analytic family $(X_s)_{s\in S}$ of $n-$cycles in Z , and we shall denote by Σ the incidence divisor of Y in S .

(A.3). For each s in Σ we put $E_s := |X_s| \cap |Y|$. We will say that a finite sequence of elements in $\Gamma(E_s, \mathcal{I}_Y)$ is a generating sequence for $\Gamma(E_s, \mathcal{I}_Y)$ if it generates $\mathcal{I}_{Y,y}$ for every y in E_s; in other words if it generates $\Gamma(E_s, \mathcal{I}_Y)$ as a $\Gamma(E_s, \mathcal{O}_Y)$-module. Let $g_0, \ldots, g_n$ be a generating sequence for $\Gamma(E_s, \mathcal{I}_Y)$. Then $g_0, \ldots, g_n$ is a regular sequence and thus defines a flat morphism $g = (g_0, \ldots, g_n)$ from an open neighbourhood U of E_s onto an open neighbourhood of the origin in $\mathbb{C}^{n+1}$. Let $k_g(s)$ denote the multiplicity of the image cycle $g_\star(X_s|U)$ at the origin in $\mathbb{C}^{n+1}$, i.e. $k_g(s) = \mathrm{mult}_0\, g_\star(X_s|U)$. Then we put

$$k(s) := \min_g\, k_g(s),$$

where the minimum is taken over all generating sequences of length $n+1$ in $\Gamma(E_s, \mathcal{I}_Y)$.

(A.4) Lemma. Let $s \in S$. The set of all $g = (g_0, \ldots, g_n)$ in $\Gamma(E_s, \mathcal{I}_Y)^{n+1}$ such that $g_0, \ldots, g_n$ generate $\Gamma(E_s, \mathcal{I}_Y)$ and such that $k_g(s) = k(s)$ is an open dense subset of $\Gamma(E_s, \mathcal{I}_Y)^{n+1}$.

Proof. Let $\mathcal{U}$ be the set of all $g = (g_0, \ldots, g_n)$ in $\Gamma(E_s, \mathcal{I}_Y)^{n+1}$ such that $g_0, \ldots, g_n$ generate $\Gamma(E_s, \mathcal{I}_Y)$ and put

$$\mathcal{U}' := \{g \in \mathcal{U} \,|\, k_g(s) = k(s)\}.$$

From lemma (3.3) we know that $\mathcal{U}$ is a dense open subset of $\Gamma(E_s, \mathcal{I}_Y)^{n+1}$ so it is enough to show that $\mathcal{U}'$ is a dense open subset of $\mathcal{U}$. Let $f \in \mathcal{U}$ and let $\mathcal{V}$ be an open neighbourhood of f in $\mathcal{U}$ such that $(g_\star X_s)_{g\in\mathcal{V}}$ is an analytic family of (germs of) hypersurfaces at the origin in $\mathbb{C}^{n+1}$. Let

$$P : \mathcal{V} \times (\mathbb{C}^{n+1}, 0) \to \mathbb{C}$$

be a defining function for the graph of this family and write

$$P(g, x) = P_0(g, x) + P_1(g, x) + P_2(g, x) + \cdots$$

where $P_j(g, x)$ is homogeneous of degree j in x. Then by the definition of $k(s)$ we have $P_j(g, \cdot) = 0$ for all g if $j < k(s)$ and the set of all g in $\mathcal{V}$ with $k_g(s) > k(s)$ is given by the analytic equation $P_{k(s)}(g, \cdot) = 0$; in other words

$$\mathcal{V} \setminus \mathcal{V} \cap \mathcal{U}' = \{g \in \mathcal{V} \mid P_{k(s)}(g, \cdot) = 0\}.$$

This shows in particular that $\mathcal{U}'$ is an open subset of $\mathcal{U}$. Now, suppose that f belongs to the boundary of $\mathcal{U}'$ in $\mathcal{U}$ and suppose that the neighbourhood $\mathcal{V}$ is convex. We will show that $\mathcal{V}$ is contained in the closure of $\mathcal{U}'$ in $\mathcal{U}$ and thereby prove that $\mathcal{U}'$ is dense in $\mathcal{U}$. Fix an element g_1 in $\mathcal{V} \cap \mathcal{U}'$ and let g_2 be any other element in $\mathcal{V}$. Let $\mathcal{L}$ be the affine line through g_1 and g_2 in $\Gamma(E_s, \mathcal{I}_Y)^{n+1}$. Then $\mathcal{V} \cap \mathcal{L}$ is an open connected subset of $\mathcal{L}$ that contains g_1 and g_2. Since

$$\mathcal{L} \cap (\mathcal{V} \setminus \mathcal{V} \cap \mathcal{U}') = \{g \in \mathcal{V} \cap \mathcal{L} \mid P_{k(s)}(g, \cdot) = 0\}$$

is an analytic subset of $\mathcal{L} \cap \mathcal{V}$ that does not contain g_1, it is a discrete subset of $\mathcal{L} \cap \mathcal{V}$ and consequently it is contained in the closure of $\mathcal{U}'$. □

REFERENCES

[A.G.62] Andreotti, A and Grauert, H., *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France, 90 (1962), pp. 193–259.

[A.N.67] Andreotti, A and Norguet, F., *La convexité holomorphe dans l'espace analytique des cycles d'une variété algébrique*, Ann. Scuola Norm. Pisa, 21 (1967), pp. 31–82.

[B.75] Barlet, D., *Espace analytique réduit des cycles analytiques complexes compacts d'un espace analytique complexe de dimension finie*, Séminaire F.Norguet, Lecture Notes in Math., 482, pp. 1–158.

[B.78] Barlet, D., *Convexité de l'espace des cycles*, Bull. Soc. Math. France, 106 (1978), pp. 373–397.

[B.78.b] Barlet, D., *Majoration du volume des fibres génériques et forme géométrique du théorème d'aplatissement*, Séminaire Pierre Lelong-Henri Skoda (Analyse) 78/79 pp. 1–17, Lecture Notes in Math., 822, Springer, Berlin-New York, 1980.

[B.79] Barlet, D., *Le théorème d'intégration sur un ensemble analytique complexe de P. Lelong*, Revue de l'Institut E.Cartan vol.5, Séminaire de Géométrie Analytique 77/79, 2-ième partie pp. 1–11 (1982).

[B.80] Barlet, D., *Convexité au voisinage d'un cycle* Séminaire F.Norguet, Lecture Notes in Math., 807, pp. 1–158.

[B.Mg.98] Barlet, D. and Magnusson, J., *Intégration de classes de cohomologie méromorphes et diviseurs d'incidence*, Ann. Sc. Ec. Norm. Sup., 31 (1998), pp. 811–841.

[B.Mg.99] Barlet, D. and Magnusson, J., *Transfert de l'amplitude du fibré normal au diviseur d'incidence*, J. reine angew. Math., 513 (1999), pp. 71–95.

[B.Ka.03] Barlet, D. and Kaddar, M., *Incidence divisor*, Int. Math. J., 14:4 (2003), pp. 339–359.

[C.80] Campana, F., *Compacité et algébricité dans l'espace des cycles*, Math. Ann., 251 (1980), pp. 7–18.

[C.91] Campana, F., *The class C is not stable by small deformations*, Math. Ann., 290 (1991), pp. 19–30.

[F.76] Fritzsche, K., *q-convex restmengen in kompakten komplexen mannigfaltigkeiten*, Math. Ann., 221 (1976), pp. 251–273.

[K.75] Krasnov, V.A., *Compact complex manifolds without meromorphic functions*, Mat. Zameki, 17 (1975), pp. 119–122 (russe) English Transl. Math. Notes, 17 (1975), pp. 69–75.

[N.S.77] Norguet, F. and Siu, Y.T., *Holomorphic convexity of spaces of cycles*, Bull. Soc. Math. France, 105 (1977), pp. 191–223.

[O.P.01] Oguiso, K. and Peternell, Th., *The dual Kähler cone of compact Kähler threefolds*, preprint ArXiv.math. AG/0107224 v2 7/9/2001.

[V.85] Varouchas, J., *Sur l'image d'une variété kahlérienne compacte* , Seminaire F. Norguet, Lecture Notes in Math.,1188, Springer, Berlin-New York, 1986.