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ON THE DIRICHLET PROBLEMS FOR SYMMETRIC FUNCTION EQUATIONS OF THE EIGENVALUES OF THE COMPLEX HESSIAN *

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To Yum-Tong Siu, on his sixtieth birthday

1. Introduction. Let D be a bounded domain in $\mathbb{C}^n$, and let $u \in C^2(D)$ be a real valued-function. Then the complex Hessian of u

$$H[u](z) = \Big[\frac{\partial^2 u(z)}{\partial z_i \partial \bar{z}_j}\Big]_{n\times n} \tag{1.1}$$

is an $n \times n$ hermitian matrix at each point $z \in D$. Let $\lambda(H(u)) = (\lambda_1(z), \cdots, \lambda_n(z))$ be all eigenvalues of $H[u](z)$ as a vector in $\mathbb{R}^n$. Then the kth elementary symmetric function $\sigma^{(k)}$ is defined as follows:

$$\sigma^{(k)}(\lambda) = \sum_{i_1<\cdots<i_k} \lambda_{i_1} \cdots \lambda_{i_k} \tag{1.2}$$

In particular,

$$\det H(u) = \sigma^{(n)}(\lambda(H(u)), \quad \Delta u = \mathrm{tr}(H(u)) = \sigma^{(1)}(\lambda(H(u))). \tag{1.3}$$

It was proved in [8] that $\sigma^{(k)}(\lambda)^{1/k}$ is a concave strictly increasing function on the symmetric convex cone:

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma^{(j)}(\lambda) > 0,\ 1 \le j \le k\}, \tag{1.4}$$

and

$$\Gamma_n = \{\lambda \in \mathbb{R}^n : \lambda_j > 0, \quad 1 \le j \le n\}, \quad \Gamma_1 = \Big\{\lambda : \sum_{j=1}^n \lambda_j > 0\Big\}. \tag{1.5}$$

Also Γ_k is symmetric in $\lambda = (\lambda_1, \cdots, \lambda_n)$, which means that if $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma_k$, then $\tilde{\lambda} = (\lambda_{i_1}, \cdots, \lambda_{i_n}) \in \Gamma_k$ where $(i_1, i_2, \cdots, i_n)$ is any permutation of $1, 2, \cdots, n$. We say that u is plurisubharmonic in D if $\lambda(H(u)(z)) \in \overline{\Gamma_n}$, for all $z \in D$; we say that u is subharmonic in D if $\lambda(H(u)(z)) \in \overline{\Gamma_1}$ for all $z \in D$. We will let Γ be a convex cone which is symmetric in $\lambda \in \Gamma$, with vertex 0 so that $\Gamma_n \subseteq \Gamma \subseteq \Gamma_1$. Let $\mathcal{M}(n, \Gamma)$ be subset of all $n \times n$ hermitian matices H over $\mathbb{C}$ so that $\lambda(H) \in \Gamma$ where $\lambda(H)$ is a vector in $\mathbb{R}^n$ being formed by all eigenvalues of H. We will consider more general symmetric function than the kth symmetric function $\sigma^{(k)}$ on $\mathcal{M}(n, \Gamma)$. Let D be a bounded domain in $\mathbb{C}^n$ with smooth boundary ∂D. We say that a real-valued function u is Γ-subharmonic if $\lambda(H(u)(z)) \in \Gamma$ for all $z \in D$. we will consider the Dirichlet problem for a functional equation:

$$F(H(u)(z)) = \psi(z), \quad z \in D, \quad u = \phi \quad \text{on } \partial D. \tag{1.6}$$

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where $F(H)$ can be written as

$$F(H) = f(\lambda(H)), \quad H \in \mathcal{M}(n, \Gamma), \tag{1.7}$$

with f a symmetric function in $\lambda \in \Gamma$. In order that the equation (1.6) is elliptic, we assume that $f(\lambda)$ is a positive, strictly increasing, concave function on Γ. In other words, we assume that f is positive, concave and satisfies

$$\frac{\partial f}{\partial \lambda_j} > 0, \quad 1 \leq j \leq n. \tag{1.8}$$

In order to prove an existence theorem, we may assume more on function f. Let

$$\psi_0 = \min\{\psi(z) : z \in \overline{D}\}. \tag{1.9}$$

Let ψ^0 be any positive number so that $\psi^0 < \psi_0$. Then we assume that

$$\limsup_{\lambda \to \lambda_0} f(\lambda) \leq \psi^0, \quad \lambda_0 \in \partial\Gamma, \tag{1.10}$$

and for any compact subset K of Γ, we assume that

$$\lim_{R\to\infty} f(\lambda_1, \cdots, \lambda_{n-1}, \lambda_n + R) = \lim_{R\to\infty} f(R\lambda) = \infty, \tag{1.11}$$

uniformly for $\lambda \in K$.

It was proved in [8] that

$$f_k(\lambda) = \left(\sigma^{(k)}(\lambda)\right)^{1/k} \tag{1.12}$$

is a symmetric, positive, strictly increasing concave function on Γ_k satisfying (1.10) and (1.11).

Let D be a smoothly bounded domain in $\mathbb{C}^n$, we say that D is Γ–pseudoconvex if there is $C_D \geq 0$ so that $\lambda(-H(d) + CH(d^2)) \in \Gamma$ on ∂D where $d(z)$ is the distance function from z to ∂D.

The existence of a unique classical solution for the Dirichlet problem of symmetric function of the eigenvalues of real hessian matrix of a function u on a domain in $\mathbb{R}^n$ was proved by Caffarelli, Nirenberg and Spruck in [8]. Many other problems related to geometric problems were studied by B. Guan and P. Guan [16], P. Guan and X. Ma in [18], P. Guan, C. S. Lin and X. Ma in [19], P. Guan and Y. Li in [17] and J. Urbas in [34] with references therein and many others. The Dirichlet problem for the complex Monge-Ampere equations on strictly pseudoconvex domain in $\mathbb{C}^n$ has been studied many authors, we refer to [1], [2], [9], [7], [15], [25],[22], [26] and references therein. The existence of classical plurisubharmonic solution for complex Monge-Ampère equations on strictly pseudoconvex domain was proved by Caffarelli, Kohn, Nirenberg and Spruck in [7]. In [15], B. Guan proved the similar results hold on weakly pseudoconvex domain provided the bounday data has plurisubharmonic subsolution. Results on Hörlder continuity were proved by the author in weakly pseudoconvex domain of finity type in [26]. Based on the all above known results, we are proposed to prove the following theorems which are natural generalizations of the known results mentioned above.

THEOREM 1.1. *Let Γ be a convex symmetric cone in $\mathbb{R}^n$ with $\Gamma_n \subseteq \Gamma \subseteq \Gamma_1$. Let D be a smoothly bounded domain in $\mathbb{C}^n$. Let f be a strictly increasing, concave function* *on* Γ *satisfying (1.10) and (1.11). If* $\phi \in C^{\infty}(\partial D)$ *has an extension* $\underline{u} \in C^{\infty}(\overline{D})$ *so that*

$$f(\lambda(H(\underline{u}))) \geq \psi(z) + \epsilon, \ z \in D \quad \text{for some} \ \ \epsilon > 0, \tag{1.13}$$

then the Dirichlet problem

$$f(\lambda(H(u))) = \psi > 0 \quad \text{in} \ \ \overline{D}, \quad u = \phi \ \ \text{on} \ \ \partial D \tag{1.14}$$

admits a (unique) solution $u \in C^{\infty}(\overline{D})$ *with* $\lambda(H(u)) \in \Gamma$ *on* D.

THEOREM 1.2. *Let* Γ *be a convex symmetric cone in* $\mathbb{R}^n$ *with* $\Gamma_n \subseteq \Gamma \subseteq \Gamma_1$. *Let* D *be a smoothly bounded* Γ*–pseudoconvex domain in* $\mathbb{C}^n$. *Let* f *be a strictly increasing, concave function on* Γ *satisfying (1.10) and (1.11). Then the Dirichlet problem*

$$f(\lambda(H(u))) = \psi > 0 \quad \text{in} \ \ \overline{D}, \quad u = \phi \ \ \text{on} \ \ \partial D \tag{1.15}$$

admits a (unique) solution $u \in C^{\infty}(\overline{D})$ *with* $\lambda(H(u)) \in \Gamma$ *on* D.

In particular, for $f = \sigma^{(k)}$, we have the following theorem.

THEOREM 1.3. *For* $1 < k \leq n$, *the Dirichlet problem*

$$\sigma^{(k)}(\lambda(H(u))) = \psi > 0 \quad \text{in} \ \ \overline{D}, \quad u = \phi \ \ \text{on} \ \ \partial D \tag{1.16}$$

admits a (unique) admissible solution $u \in C^{\infty}(\overline{D})$ *provided that*

$$\partial D \ \ \text{is connected, and} \ \ \sigma^{(k-1)}(\lambda(\mathcal{L})) > 0 \ \text{on} \ \ \partial D \tag{1.17}$$

where $\mathcal{L}$ *is a Levi-form of* ∂D.

Moreover, when ϕ *=constant, condition (1.17) is also necessary for existence of a solution* $u \in C^2(\overline{D})$.

Applying the method of continuity, and by results proved by Caffarelli, Nirenberg and Spruck in [6] and Krylov in [23], as well as by the iterration or L^p theory given by Kohn and Nirenberg [21], it suffices to prove C^2 *a priori* estimates for solutions in Theorems 1.1–1.3. We will organized the rest of the paper as follows. In Section 2, we provide some preliminary results. In Section 3, we proved results of the subsolution. Finally, the main part of the paper, in Section 4, we provide C^2 *a priori* estimates.

2. Preliminary. From now on, we will always let Γ be a symmetric convex cone in $\mathbb{R}^n$ with $\Gamma_n \subset \Gamma \subset \Gamma_1$, and $f(\lambda(H)) = F(H)$ be an increasing, concave, non-negative function on $\mathcal{M}(n, \Gamma)$. Let $\Gamma' = \{(\lambda_1, \cdots, \lambda_{n-1}) \in \mathbb{R}^{n-1} : \lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma\}$ be the projection of Γ into $\mathbb{R}^{n-1}$. It is easy to see that

$$\Gamma'_n = \Gamma_{n-1}(\mathbb{R}^{n-1}) = \{(\lambda_1, \cdots, \lambda_{n-1}) : \lambda_j > 0\}, \quad \Gamma'_1 = \mathbb{R}^{n-1}. \tag{2.1}$$

For any $n \times n$ hermitian matrix $H = [h_{i\bar{j}}]$, we let $H' = [h_{\alpha\bar{\beta}}]_{(n-1)\times(n-1)}$. Let $\lambda(H')$ be a vector formed by all eigenvalues of H'. The following lemmas were proved in [8].

LEMMA 2.1. *(*Maximum Principle*). Let* D *be a bounded domain in* $\mathbb{C}^n$ *with* C^1 *boundary. Let* $u, v \in C^2(D) \cap C(\overline{D})$ *so that* $H(u)(z) \in \mathcal{M}(n, \Gamma)$ *for all* $z \in D$. *Assume that at every point* $z \in D$, $\lambda(H(v(z)))$ *lies outside the set* $\Gamma(z, u) = \{\lambda \in \Gamma : f(\lambda) > f(\lambda(H(u)(z))\}$. *If* $u \leq v$ *on* ∂D *then* $u \leq v$ *in* D.

LEMMA 2.2. *With the notation above, we have*

(2.2) $\lambda'(H) = (\lambda_1(H), \cdots, \lambda_{n-1}(H)) = \lambda(H') + o(1), \quad \lambda_n(H) = h_{n\overline{n}}(1 + O(1/h_{n\overline{n}})),$

as $h_{n\overline{n}} \to +\infty$.

LEMMA 2.3. *Let* Γ *be a convex cone in* $\mathbb{R}^n$ *so that* $\Gamma_n \subseteq \Gamma \subseteq \Gamma_1$. *Let* $\Gamma' = \{(\lambda_1, \cdots, \lambda_{n-1}) : \lambda \in \Gamma\}$. *If* $\lambda(H) \in \Gamma$ *then* $\lambda(H') \in \Gamma'$ *where* $H' = (h_{\alpha\overline{\beta}})_{(n-1)\times(n-1)}$.

Proof. Since $\lambda(H) \in \Gamma$, there is $\epsilon > 0$ so that $\lambda(H - \epsilon I_n) \in \Gamma$. Let E_{ij} be an $n \times n$ matrix with entries 1 at (i, j) position, others are zero. Let $H(t) = H + tE_{nn}$. Then

$$\lambda'(H(t)) = \lambda(H') + o(1) \in \Gamma', \quad \text{as } t \to \infty.$$

Let $t \to +\infty$, we have $\lambda(H') \in \overline{\Gamma'}$, and so $\lambda(H' - \epsilon I_{n-1}) \in \overline{\Gamma'}$. Thus $\lambda'(H) \in \Gamma'$. □

The following proposition is proved by M. Marcus in [31] by using knowledge of probability, we will provide a direct proof here.

PROPOSITION 2.4. *Let* $A = [a_{i\overline{j}}]$ *and* $B = [b_{i\overline{j}}]$ *be hermitian matrices over* $\mathbb{C}$. *Let* $\lambda_1 \leq \cdots \leq \lambda_n$ *be all eigenvalues of* A *and let* $\mu_1 \geq \cdots \geq \mu_n$ *be all eigenvalues of* B. *Then*

$$tr(AB) \geq \sum_{k=1}^{n} \lambda_k \mu_k. \tag{2.3}$$

Proof. Without loss of generality, we may assume A and B are positive definite, otherwise, we will add them by cI_n with large positive c. The proposition is true when $n = 1$. Assume that it is true for n, we will prove that it is ture for $n + 1$. Without loss of generality, we may assume that A is a diagonal matrix, say, $A = \text{Diag}(\mu_1, \cdots, \mu_n, \mu_{n+1})$. Since

$$\text{tr}(AB) = \text{tr}((A - \mu_{n+1} I_{n+1})B) + \mu_{n+1}\text{tr}(B). \tag{2.4}$$

Let $B' = [b_{\alpha\overline{\beta}}]_{1\leq\alpha,\beta\leq n}$, and let $\lambda'_1 \leq \lambda'_2 \cdots, \leq \lambda'_n$ be the all eigenvalues of B'. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \lambda_{n+1}$ be all eigenvalues of B. We claim that $\lambda_j \leq \lambda'_j$ for all $1 \leq j \leq n$. Write

$$B = \begin{bmatrix} B' & b' \\ (b')^T & b \end{bmatrix}, \quad b' = \begin{bmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_{n-1} \end{bmatrix}.$$

Hence

$$g_{n+1}(\lambda) = \det(\lambda I_{n+1} - B) = \prod_{j=1}^{n} (\lambda - \lambda'_j)(\lambda - b - \frac{1}{\lambda - \lambda'_j} \sum_{k=1}^{n} |b_k|^2) \tag{2.5}$$

we have

$$g_{n+1}(\lambda'_\ell) = -\prod_{j\neq\ell}^{n} (\lambda'_\ell - \lambda'_j) \sum_{k=1}^{n} |b_k|^2 = (-1)^{n+1-\ell} A_\ell, \quad A_\ell \geq 0. \tag{2.6}$$

If $\sum_{k=1}^{n}|b_k|^2 = 0$ then it is easy. We may assume that $\sum_{k=1}^{n}|b_k|^2 > 0$ or $A_\ell > 0$ for all ℓ. We will continue our discussion in the following two cases.
Case 1. If $n+1$ is even then $(-1)^\ell g_{n+1}(\lambda'_\ell) \geq 0$, then $g_{n+1}(\lambda)$ has zeros in $(\lambda'_j, \lambda'_{j+1}]$ for all $j = 1, \cdots, n-1$. Since $g_{n+1}(\lambda) \to +\infty$ as $|\lambda| \to +\infty$, this implies that $g_{n+1}(\lambda)$ has a zero $\lambda_1 \leq \lambda'_1$ and a zero $\lambda_{n+1} > \lambda'_n$. So $\lambda_j \leq \lambda'_j$ for all $j = 1, \cdots, n$.
Case 2. If $n+1$ is odd then $(-1)^\ell g_{n+1}(\lambda'_\ell) \leq 0$. Since $g_{n+1}(\lambda) \to -\infty$ as $\lambda \to -\infty$, with similar argument above, we have $g_{n+1}(\lambda)$ has zeros λ_j with $\lambda'_j \geq \lambda_j$ for $j = 1, 2, \cdots, n$. So the proof of the claim is complete. Now since

$$\operatorname{tr}((A - \mu_{n+1}I_{n+1})B) = \sum_{k=1}^{n}(\mu_k - \mu_{n+1})b_{k\bar{k}} \geq \sum_{k=1}^{n}(\mu_k - \mu_{n+1})\lambda'_k$$

Therefore

$$\begin{aligned}
\operatorname{tr}(AB) &\geq \sum_{k=1}^{n}(\mu_k - \mu_{n+1})\lambda'_k + \mu_{n+1}\operatorname{tr}(B) \\
&= \sum_{k=1}^{n}\mu_k\lambda'_k + \mu_{n+1}\lambda_{n+1} + \mu_{n+1}\sum_{k=1}^{n}(\lambda_k - \lambda'_k) \\
&= \sum_{k=1}^{n+1}\mu_k\lambda_k + \sum_{k=1}^{n}\mu_k(\lambda'_k - \lambda_k) + \mu_{n+1}\sum_{k=1}^{n}(\lambda_k - \lambda'_k) \\
&= \sum_{k=1}^{n+1}\mu_k\lambda_k + \sum_{k=1}^{n}(\mu_k - \mu_{n+1})(\lambda'_k - \lambda_k) \\
&\geq \sum_{k=1}^{n+1}\mu_k\lambda_k,
\end{aligned}$$

and the proof of the proposition is complete by mathematics induction. □

Next we gives a simple version of continous case of the previous proposition.

PROPOSITION 2.5. *Let $f(t)$ be decreasing on $[0,1]$ and $g(t)$ is increasing on $[0,1]$. Let $\phi(x,y) \geq 0$ be a measurable function on $[0,1]\times[0,1]$ so that $\phi(x,y) = \phi(y,x)$ and*

$$\int_0^1 \phi(x,y)dy = 1, \quad a.e. \quad x \in [0,1].$$

Then

$$\int_0^1\int_0^1 \phi(x,y)f(x)g(y)dxdy \geq \int_0^1 f(x)g(x)dx$$

Proof. For any $x, y \in [0,1]$, since $(f(x) - f(y))(g(x) - g(y)) \leq 0$, we have

$$f(x)g(x) + f(y)g(y) - f(x)g(y) - f(y)g(x) \leq 0.$$

Therefore

$$f(x)g(x) + f(y)g(y) \leq f(x)g(y) + f(y)g(x),$$

and

$$\int_0^1\int_0^1 \phi(x,y)(f(y)g(x)+f(x)g(y))dxdy$$
$$\geq \int_0^1\int_0^1 \phi(x,y)[f(x)g(x)+f(y)g(y)]dy$$
$$= 2\int_0^1 f(x)g(x)dx$$

Since

$$\int_0^1\int_0^1 \phi(x,y)f(y)g(x)dxdy = \int_0^1\int_0^1 \phi(y,x)f(x)g(y)dxdy$$

we have

$$\int_0^1\int_0^1 \phi(x,y)(f(y)g(x)+f(x)g(y))dxdy$$
$$= \int_0^1\int_0^1 \phi(y,x)f(x)g(y) + \int_0^1\int_0^1 \phi(x,y)f(x)g(y))dxdy$$
$$= \int_0^1\int_0^1 (\phi(y,x)+\phi(x,y))f(x)g(y)dxdy$$

Therefore

$$\int_0^1\int_0^1 \phi(x,y)f(x)g(y)dxdy = \int_0^1\int_0^1 \frac{\phi(y,x)+\phi(x,y)}{2}f(x)g(y)dxdy$$
$$\geq \int_0^1 f(x)g(x)dx,$$

and the proof of the lemma is complete. □

3. Existence of subsolutions. We first prove the following theorem on defining function.

THEOREM 3.1. *Let Γ be a convex cone in $\mathbb{R}^n$ with $\Gamma_n \subset \Gamma \subset \Gamma_1$. Let $\rho \in C^\infty(\overline{D})$ be a defining function for D so that $\lambda(H(\rho)(z)) \in \Gamma$ for all $z \in \partial D$. Then there is a defining function $\rho^0 \in C^\infty(\overline{D})$ for D so that $\lambda(H(\rho^0)) \in \Gamma$ on $\overline{D}$.*

Proof. Since Γ is an open set, and $\lambda(H(\rho)(z)) \in \Gamma$ for all $z \in \partial D$, and ∂D is compact. By continuity, there is a $\delta > 0$ so that $\lambda(H(\rho)(z)) \in \Gamma$ for all $\{z \in \overline{D} : \rho(z) \geq -\delta\}$. Now we choose a convex increasing function $g(t) \in C^\infty(-\infty, 0]$ so that $g(0) = 0$, $g(t) = -1$ when $t \leq -\delta$ and $g'(t) > 0$ for $t \in (-\delta, 0)$. Let

$$\rho^1(z) = g(\rho(z))$$

It is easy to see that $\rho^1 \in C^\infty(\overline{D})$ and is a defining function for D. Since

$$H(\rho^1)(z) = g'(\rho)H(\rho) + g''(\rho)\partial\rho\otimes\overline{\partial}\rho(z)$$

since $g'' \geq 0$, $\partial\rho\otimes\overline{\partial}\rho$ is positive semi-definite, and $g' \geq 0$, we have that $\lambda(H(\rho^1)(z)) \in \overline{\Gamma}$ for all $z \in \overline{D}$ and $\lambda(H(\rho^1) \in \Gamma$ for $\rho > -\delta$. We let $h \in C_0^\infty(D)$ so that $h(z) \geq 0$ on D and $h(z) = 1$ when $\rho(z) < -\delta/2$. Let

$$\rho^0(z) = ch(z)|z|^2 + \rho^1(z), \quad c > 0$$

It is obvious that ρ^0 is a defining function for D, and $\rho^0 < 0$ if $c > 0$ is small enough. Moreover,

$$H(\rho^0) = g'(\rho)H(\rho) + g''(\rho)\partial\rho \otimes \overline{\partial}\rho(z) + ch(z)I_n + c\partial h(z) \otimes z + c\overline{z} \otimes \overline{\partial}h(z) + c|z|^2 H(h)$$

It is easy to see that $\lambda(H(\rho^0)(z)) \in \Gamma$ when $\rho(z) < -\delta/2$ for any $c > 0$ since $h \equiv 1$ there. Now since $\lambda(H(\rho^1)) \in \Gamma$ for $\rho(z) \geq -3\delta/4$, those λ contained in a compact subset of Γ, there is a positive distance from it to the boundary $\partial\Gamma$. Therefore, there is $c > 0$ very small so that $\lambda(H(\rho^0)) = \lambda(H(\rho^1) + cH(h|z|^2)) \in \Gamma$. Thus, the proof of the theorem is complete. □

THEOREM 3.2. *Let Γ be a convex symmetric cone in $\mathbb{R}^n$. Let F be a concave increasing function on $\mathcal{M}(n,\Gamma)$ satisfying (1.7) –(1.11). Let D be a smoothly bounded domain in $\mathbb{C}^n$ so that there is a defining function $\rho \in C^\infty(\overline{D})$ for D with $H(\rho) \in \mathcal{M}(n,\Gamma)$ and $F(H(\rho)) > 0$ on $\overline{D}$. If $\phi \in C^\infty(\partial D)$ and $\psi \in C^\infty(\overline{D})$ is positive, then there is $\underline{u} \in C^\infty(\overline{D})$ so that*

$$F(H(\underline{u})) \geq \psi(z) + 1, \quad \text{in } D, \quad \text{and } \underline{u} = \phi \quad \text{on } \partial D. \tag{3.1}$$

Proof. Let $\phi \in C^\infty(\partial D)$, we still use ϕ to denote its harmonic extension to $\overline{D}$. Let $C_1 > 0$ so that $\phi_0 = \phi(z) + C_1\rho(z)$ satisfying $\lambda(H(\phi_0)(z)) \in \Gamma$ by assumption $H(\rho) \in \mathcal{M}(n,\Gamma)$ and (1.11). Let

$$\underline{u} = \phi_0(z) + C\rho(z). \tag{3.2}$$

Then

$$H(\underline{u}) = H(\phi_0(z)) + CH(\rho(z)). \tag{3.3}$$

Since F is concave on $\mathcal{M}(n,\Gamma)$ or f is concave on Γ, we have

$$\begin{aligned} F(H(\underline{u})) &\geq \frac{F(2H(\phi_0))}{2} + \frac{F(2CH(\rho))}{2} \\ &= \frac{F(2H(\phi_0))}{2} + \frac{f(2C\lambda(H(\rho)))}{2} \\ &\geq \frac{f(2C\lambda(H(\rho)))}{2}. \end{aligned}$$

By (1.11), for any compact subset K and a positive constant C_0, there is $R = R(K, C_0) > 0$ so that

$$f(R\lambda) \geq C_0, \quad \text{for all } \lambda \in K \tag{3.4}$$

Now let $K = \{\lambda(H(\rho(z))) : z \in \overline{D}\}$ is a compact set in Γ. Let

$$C_0 = 2(\psi_1 + 1), \quad \psi_1 = \max\{\psi(z) : z \in \overline{D}\} \tag{3.5}$$

and $2C = R = R(C_0, K)$. Then we have

$$F(H(\underline{u})(z)) \geq \psi_1 + 1 \quad \text{for } z \in \overline{D}, \tag{3.6}$$

and the proof is complete. □

Finally, in this section, we shall prove the following proposition.

PROPOSITION 3.3. *Let D be a bounded domain in $\mathbb{C}^n$ with ∂D is connected and C^∞. If there is a defining function ρ for D so that $\lambda(\mathcal{L}_\rho) \in \Gamma_{k-1}$ on ∂D then there is a defining function ρ_0 for D so that $\lambda(H(\rho_0) \in \Gamma_k$ on ∂D. Here $\mathcal{L}_\rho$ is the Levi-form of ρ on ∂D.*

Proof. Without loss of generality, we may assume that $|\partial\rho|^2 = 1/2$ on ∂D. Let

$$\rho_0(z) = \rho(z) + C\rho(z)^2, \quad z \in D.$$

Then

$$\mathcal{L}_\rho(z) = \mathcal{L}_{\rho_0}(z), \quad z \in \partial D$$

and

$$\lambda(H(\rho_0)(z)) = (\lambda'(z), \lambda_n) = (\lambda(\mathcal{L}_\rho(z)) + o(1), C + O(1)), \quad \text{as } C \to +\infty.$$

Since $\lambda(\mathcal{L}_\rho(z)) \in \Gamma_{k-1}$ on ∂D there is $a > 0$ and $C >> 1$ so that

$$\sigma^{(j)}(\lambda'(z) + o(1)) \geq a, \quad z \in \partial D, \quad \text{and } 1 \leq j \leq k-1.$$

This imples that when $C > 1$ is large enough we have $\sigma^{(j)}(\lambda(H(\rho)) > 0$ on ∂D for all $1 \leq j \leq k$, and so $\lambda(H(\rho)) \in \Gamma_k$. Therefore, the proof is complete. □

4. A priori estimates up to the second derivatives. Let D be a bounded domain in $\mathbb{C}^n$ with smooth boundary. Let u be a real-valued function on D. We consider the Dirichlet boundary value problem:

$$F(H(u)) = f(\lambda(H(u))) = \psi(z), \quad z \in D; \quad u = \phi \ \text{ on } \ \partial D \tag{4.1}$$

with the assumptions: $\psi \in C^\infty(\overline{D})$ and $\psi(z) > 0$ on $\overline{D}$. Moreover, we assume that $\phi \in C^\infty(\partial D)$ has an extension $\underline{u}$ on $\overline{D}$ so that

$$\lambda(H(\underline{u})(z)) \in \Gamma \quad \text{and} \quad F(H(\underline{u})(z)) \geq \psi(z) + \epsilon, \quad z \in D \tag{4.2}$$

for some $\epsilon > 0$. We say that $u \in C^2(D)$ is an admissible solution of (4.1) if $\lambda(H(u)) \in \Gamma$ on D. Then we shall prove the following lemma.

LEMMA 4.1. *Let $\phi \in C^\infty(\partial D)$ have an extension $\underline{u}$ satisfying (4.2). Let (4.1) have an admissible solution $u \in C^1(\overline{D})$. Then*

$$\|u\|_{C^1(\overline{D})} \leq C_1. \tag{4.3}$$

where C_1 is a constant depending only on $\|\phi\|_{C(\partial D)}, \|\underline{u}\|_{C^1(\overline{D})}, \|\psi\|_{C^1(\overline{D})}$ and ϵ.

Proof. We still use ϕ as its harmonic extension of ϕ from ∂D to $\overline{D}$. Then $u - \phi = 0$ on ∂D and $\Delta(u-\phi) = \Delta u > 0$ (since $\Delta u > 0$). By maximum principle, we have that

$$u(z) - \phi(z) \leq 0, \quad z \in \overline{D}. \tag{4.4}$$

Moreover,

$$D_\nu(u - \phi)(z) \geq 0, \quad z \in \partial D \tag{4.5}$$

where ν is the outer unit normal vector to ∂D.

Since $f(\lambda(H(\underline{u}))) \geq f(\lambda(H(u))$ and $\underline{u} = u$ on ∂D (of course, $\underline{u} \leq u$ on ∂D), by the Maximum Principle or Lemma 2.1, we have $\underline{u} \leq u$ on $\overline{D}$. Since $\underline{u} - u \leq 0$ on D and $\underline{u} - u = 0$ on ∂D, this implies that

$$D_\nu(\underline{u} - u) \geq 0, \quad \text{on } \partial D. \tag{4.6}$$

Therefore,

$$\|u\|_{C(\overline{D})} \leq \|\phi\|_{C(\overline{D})} + \|\underline{u}\|_{C(\overline{D})} \leq C(\|\underline{u}\|_{C(\overline{D})}, \|\phi\|_{C(\partial D)}). \tag{4.7}$$

where C is a constant depending only on $\|\phi\|_{C(\partial D)}$ and $\|\underline{u}\|_{C(\overline{D})}$ and ∂D. Moreover,

$$-\|\phi\|_{C^1(\partial D)} \leq D_\nu \phi(z) \leq D_\nu u(z) \leq D_\nu \underline{u}(z) \leq \|\underline{u}\|_{C^1(\overline{D})}, \quad z \in \partial D. \tag{4.8}$$

Since $u = \phi$ on ∂D, we have

$$|\nabla u(z)| \leq \|\phi\|_{C^1(\partial D)} + \|\underline{u}\|_{C^1(\overline{D})}, \quad z \in \partial D. \tag{4.9}$$

In order to estimate $|\nabla u|$ on D, we let

$$F^{i\bar{j}} = \frac{\partial F}{\partial u_{i\bar{j}}}(H(u)), \quad 1 \leq i, j \leq n. \tag{4.10}$$

Then $L = F^{i\bar{j}}\partial_{i\bar{j}}$ is an elliptic operator. Since F is concave function on $\mathcal{M}(n, \Gamma)$ and since (4.1) and (4.2), we have

$$\epsilon \leq F(H(\underline{u})) - F(H(u)) \leq F^{i\bar{j}}(H(u))(\underline{u}_{i\bar{j}} - u_{i\bar{j}}) = L(\underline{u} - u), \quad \text{on } D. \tag{4.11}$$

$F(H(u)(z)) = \psi(z)$ on D and the Chain Rules imply that

$$\frac{\partial \psi(z)}{\partial z_i} = \frac{\partial F(H(u)(z))}{\partial z_i} = F^{k\bar{\ell}}(H(u))\frac{\partial u_{k\bar{\ell}}}{\partial z_i} = L\partial_i u$$

Similarly, we have with $\psi_i = \frac{\partial \psi}{\partial z_i}$ and $\psi_{\bar{j}} = \frac{\partial \psi}{\partial \bar{z}_j}$,

$$Lu_i = \psi_i, \quad Lu_{\bar{j}} = \psi_{\bar{j}}, \quad L\frac{\partial u}{\partial x_j} = \frac{\partial \psi}{\partial x_j}, \quad L\frac{\partial u}{\partial y_j} = \frac{\partial \psi}{\partial y_j}, \quad 1 \leq i, j \leq n. \tag{4.12}$$

Therefore, by (4.11) and (4.12)

$$L(a\frac{\partial u}{\partial x_j} + b\frac{\partial u}{\partial y_j} + C(\underline{u} - u)) = a\frac{\partial \psi}{\partial x_j} + b\frac{\partial \psi}{\partial y_j} + CL(\underline{u} - u) \geq a\frac{\partial \psi}{\partial x_j} + b\frac{\partial \psi}{\partial y_j} + C\epsilon \geq 0 \tag{4.13}$$

where $C = \frac{2}{\epsilon}\|\nabla\psi\|_{C(\overline{D})}$ and $a^2 + b^2 = 1$. Thus, $a\frac{\partial u}{\partial x_j} + b\frac{\partial u}{\partial y_j} + C(\underline{u} - u)$ attains its maximum over $\overline{D}$ at some point on ∂D. Therefore, by taking $a = \pm 1,\ b = 0$ and $a = 0,\ b = \pm 1$, we have

$$|\nabla u(z)| \leq 2nC\|\underline{u} - u\|_{C(\overline{D})} + \|\nabla u\|_{C(\partial D)}, \quad z \in \overline{D}.$$

Therefore,

$$\|u\|_{C^1(\overline{D})} \leq C(\|\phi\|_{C(\partial D)}, \|\underline{u}\|_{C^1(\overline{D})}, \|\psi\|_{C^1(\overline{D})}, \epsilon). \tag{4.14}$$

where $C(\|\phi\|_{C(\partial D)}, \|\underline{u}\|_{C^1(\overline{D})}, \|\psi\|_{C^1(\overline{D})}, \epsilon)$ is a constant depending only on $\|\phi\|_{C^1(\partial D)}, \|\underline{u}\|_{C^1(\overline{D})}$, $\|\psi\|_{C^1(\overline{D})}$ and ϵ. Therefore, the proof of the lemma is complete. □

LEMMA 4.2. *With the assumption of Lemma 4.1, if $u \in C^2(\overline{D})$ is an admissible solution of (4.1), then*

$$\|u\|_{C^2(\overline{D})} \le C(\|u\|_{C^1(\overline{D})}, \|\psi\|_{C^2(\overline{D})}, \epsilon, \|u\|_{C^2(\partial D)}) \tag{4.15}$$

where C is a constant depending only on $\|u\|_{C^1(\overline{D})}, \|\psi\|_{C^2(\overline{D})}, \epsilon$ and $\|u\|_{C^2(\partial D)}$.

Proof. For any $\xi \in S^{2n-1}$, we consider

$$U(z) = D_{\xi\xi}u(z) + C(\underline{u} - u), \quad D_{\xi\xi} = D_\xi^2, \quad D_\xi = \sum_{j=1}^n \xi_{n+j}\frac{\partial u}{\partial x_j} + \sum_{j=1}^n \xi_j \frac{\partial u}{\partial y_j}. \tag{4.16}$$

Let

$$F^{i\bar{j},k\bar{\ell}} = \frac{\partial^2 F}{\partial u_{i\bar{j}}\partial u_{k\bar{\ell}}}. \tag{4.17}$$

Then, by the concavity of F, we have $F^{i\bar{j},k\bar{\ell}}a_{i\bar{j}}a_{k\bar{\ell}} \le 0$ for any hermitian matrix $A = [a_{i\bar{j}}]$. Therefore,

$$\begin{aligned} D_\xi D_\xi \psi(z) &= D_\xi D_\xi F(H(u)) \\ &= D_\xi\left(F^{i\bar{j}}D_\xi u_{i\bar{j}}\right) \\ &= F^{i\bar{j}k\bar{\ell}}D_\xi u_{k\bar{\ell}}D_\xi u_{i\bar{j}} + F^{i\bar{j}}D_\xi^2 u_{i\bar{j}} \\ &= F^{i\bar{j}k\bar{\ell}}D_\xi u_{k\bar{\ell}}D_\xi u_{i\bar{j}} + F^{i\bar{j}}\frac{\partial^2}{\partial z_i \partial \bar{z}_j}D_{\xi\xi}u \\ &\le LD_{\xi\xi}u. \end{aligned}$$

If we let $C = \frac{1}{\epsilon}\|D^2\psi\|_{C(\overline{D})}$ then

$$L(D_{\xi\xi}u + C(\underline{u} - u)) \ge D_{\xi\xi}\psi(z) + C\epsilon \ge 0. \tag{4.18}$$

Therefore $D_{\xi\xi}u + C(\underline{u} - u)$ attains its maximum over $\overline{D}$ at some point $z_0 \in \partial D$. Thus

$$D_{\xi\xi}u(z) \le C(\|\phi\|_{C^2(\partial D)}, \|\psi\|_{C^2(\overline{D})}, \epsilon, \|\underline{u}\|_{C^1(\overline{D})}) + \|D^2u\|_{C(\partial D)} = C_2 \tag{4.19}$$

Since

$$\Delta u(z) > 0, \quad z \in D,$$

we have

$$C_2 \ge u_{x_jx_j} \ge -(2n-1)C_2, \quad C_2 \ge u_{y_jy_j} \ge -(2n-1)C_2.$$

Thus

$$|u_{x_jx_j}|, \quad |u_{y_j,y_j}| \le (2n-1)C_2, \quad 1 \le j \le n. \tag{4.20}$$

Since

$$\pm 2D_{x_i}D_{x_j}u = D^2_{x_i\pm x_j}u(z) - (D^2_{x_i}u + D^2_{x_j}u) \leq (4n-1)C_2,$$

and

$$\pm 2D_{x_i}D_{y_j}u = D^2_{x_i\pm y_j}u(z) - (D^2_{x_i}u + D^2_{y_j}u) \leq (4n-1)C_2$$

and

$$\pm 2D_{y_i}D_{y_j}u = D^2_{y_i\pm y_j}u(z) - (D^2_{y_i}u + D^2_{y_j}u) \leq (4n-1)C_2.$$

We have

$$|D_{x_i}D_{x_j}u| + |D_{y_i}D_{y_j}u| + |D_{x_i}D_{y_j}u| \leq 6nC_2. \tag{4.21}$$

Thus

$$\|u\|_{C^2(\overline{D})} \leq 6n^2C_2 \tag{4.22}$$

and the proof of the lemma is complete. □

To complete the estimates for second derivatives of u, it suffices to estimate $|\nabla^2 u|$ on ∂D (or C_2). Since $u = \phi \in C^\infty(\partial D)$, we have

$$|D^2_T u(z)| \leq C(\|\phi\|_{C^2(\partial D)}, \|u\|_{C^1(\overline{D})}), \quad z \in \partial D \tag{4.23}$$

for any smooth unit tangent vector T to ∂D at $z \in \partial D$. What we need to estimate are $|D_T D_\nu u|$ on ∂D and $|D^2_\nu u|$ on ∂D. Which will be given by the following two lemmas.

LEMMA 4.3. *Let ρ be a defining function for D with $D_\nu \rho = 1$ on ∂D. With the assumptions of Lemma 4.1, we have*

$$|D_T D_\nu u(z)| \leq C(\|\phi\|_{C^3(\partial D)}, \|\psi\|_{C^2(\overline{D})}, \epsilon), \quad z \in \partial D. \tag{4.24}$$

Proof. Let $z_0 \in \partial D$ be an arbitrary point. Without loss of generality, by shift and rotation, we may assume that $z_0 = 0$, $\frac{\partial \rho}{\partial x_j}(0) = 0$ for $1 \leq j \leq n-1$ and $\frac{\partial \rho}{\partial y_j}(0) = 0$ for all $1 \leq j \leq n$. Moreover, nearby $z_0 = 0$, we can write

$$\rho(z) = -x_n + \operatorname{Re}\sum_{i,j=1}^n \rho_{ij}(0)z_iz_j + \sum_{i,j=1}^n \rho_{i\bar{j}}(0)z_i\overline{z}_j + P(z) + R(z) \tag{4.25}$$

where $P(z)$ is cubic polynomial in z and $\overline{z}$, and $|R(z)| \leq C|z|^4$.

Let $t_\alpha = y_\alpha$ if $1 \leq \alpha \leq n$ and $t_{\alpha+n} = x_\alpha$ if $1 \leq \alpha \leq n-1$. Let

$$a_\alpha(z) = -\frac{\partial \rho}{\partial t_\alpha}/\frac{\partial \rho}{\partial x_n}, \quad 1 \leq \alpha \leq 2n-1.$$

Then

$$a_\alpha(0) = 0, \quad T = \frac{\partial}{\partial t_\alpha} + a_\alpha \frac{\partial}{\partial x_n}$$

is a tangent vector to ∂D near $z = 0 \in \partial D$. We write

$$a_\alpha(z) = \sum_{\beta=1}^{2n-1} b_{\alpha\beta} t_\beta + b_\alpha x_n + O(|t|^2 + x_n^2), \quad z \in \overline{D} \text{ near } 0.$$

Let

$$T_\alpha = \frac{\partial}{\partial t_\alpha} + \sum_{\beta=1}^{2n-1} b_{\alpha\beta} t_\beta \frac{\partial}{\partial x_n}, \quad b_{\alpha\beta} = \frac{\partial a_\alpha}{\partial t_\beta}(0).$$

Then $T = T_\alpha + b_\alpha x_n \frac{\partial}{\partial x_n} + O(|z|^2)\frac{\partial}{\partial x_n}$ and

$$T_\alpha(u - \overline{u}) = O(|t|^2), \quad \text{on } \partial D.$$

Since

$$\partial_i t_\beta = \begin{cases} -\frac{\sqrt{-1}}{2}\delta_{i\,\beta} & \text{if } 1 \le \beta \le n, \\ \frac{1}{2}\delta_{i\,\beta-n}, & \text{if } \beta > n \end{cases}$$

and

$$\partial_{\bar{j}} t_\beta = \begin{cases} \frac{\sqrt{-1}}{2}\delta_{j\beta} & \text{if } 1 \le \beta \le n-1, \\ \frac{1}{2}\delta_{j\,\beta-n}, & \text{if } \beta > n. \end{cases}$$

Therefore,

$$\begin{aligned} LT_\alpha u &= T_\alpha \psi(z) + \sum_\beta b_{\alpha\beta} F^{i\bar{j}} (\partial_i t_\beta \partial_{\bar{j}} u_{x_n} + \partial_{\bar{j}} t_\beta \partial_i u_{x_n}) \\ &= T_\alpha \psi(z) + 2\sum_{\beta<2n} b_{\alpha\beta} F^{i\bar{j}} (\partial_i t_\beta \partial_{\bar{j}} u_n + \partial_{\bar{j}} t_\beta \partial_i u_{\bar{n}}) \\ &\quad + 2\sum_\beta b_{\alpha\beta} F^{i\bar{j}} (\sqrt{-1}\partial_i t_\beta \partial_{\bar{j}} u_{y_n} - \sqrt{-1}\partial_{\bar{j}} t_\beta \partial_i u_{y_n}) \\ &= T_\alpha \psi(z) + \sqrt{-1}\sum_{\beta=1}^{n} b_{\alpha\beta} F^{i\bar{j}} (-\delta_{i\beta} u_{n\bar{j}} + \delta_{j\beta} u_{i\bar{n}}) \\ &\quad + \sum_{\beta=1}^{n-1} b_{\alpha\,\beta+n} F^{i\bar{j}} (\delta_{i\beta} u_{n\bar{j}} + \delta_{j\beta} u_{i\bar{n}}) + 2\sqrt{-1}\sum_\beta b_{\alpha\beta} F^{i\bar{j}} (\partial_i t_\beta \partial_{\bar{j}} u_{y_n} - \partial_{\bar{j}} t_\beta \partial_i u_{y_n}) \\ &= T_\alpha \psi(z) + 2\sqrt{-1}\sum_{\beta=1}^{n} b_{\alpha\beta} \mathrm{Im}\,(F^{i\bar{\beta}} u_{i\bar{n}}) + 2\sum_{\beta=1}^{n-1} b_{\alpha\,\beta+n} \mathrm{Re}\,(F^{\beta\bar{j}} u_{n\bar{j}}) \\ &\quad + 2\sqrt{-1}\sum_\beta b_{\alpha\beta} F^{i\bar{j}} (\partial_i t_\beta \partial_{\bar{j}} u_{y_n} - \partial_{\bar{j}} t_\beta \partial_i u_{y_n}) \end{aligned}$$

Since $b_{\alpha\beta}$ are real number, we have

$$2\sqrt{-1}\sum_{i=1}^{n}\sum_{\beta=1}^{n} b_{\alpha\beta} \mathrm{Im}\,(F^{i\bar{\beta}} u_{i\bar{n}}) + 2\sqrt{-1}\mathrm{Re}\,\sum_\beta b_{\alpha\beta} F^{i\bar{j}} (\partial_i t_\beta \partial_{\bar{j}} u_{y_n} - \partial_{\bar{j}} t_\beta \partial_i u_{y_n}) = 0$$

By rotation of first $(n-1)$ variables, we may assume that

$$F^{\beta\overline{j}} = \delta_{j\beta}F^{\beta\overline{\beta}}, \quad \operatorname{Re} F^{\beta\overline{n}} = \operatorname{Re} u_{j\overline{n}} = 0, \quad 1 \leq \beta, j \leq n-1.$$

Thus

$$\begin{aligned}
2\sum_{\beta=1}^{n-1} b_{\alpha\,\beta+n}\operatorname{Re}(F^{\beta\overline{j}}u_{n\overline{j}}) &= 2\sum_{j,\beta=1}^{n-1} b_{\alpha\,\beta+n}\operatorname{Re}(F^{\beta\overline{j}}u_{n\overline{j}}) + 2\sum_{\beta=1}^{n-1} b_{\alpha\,\beta+n}\operatorname{Re}(F^{\beta\overline{n}}u_{n\overline{n}}) \\
&= 2\sum_{\beta=1}^{n-1} b_{\alpha\,\beta+n}\operatorname{Re}(F^{\beta\overline{\beta}}u_{n\overline{\beta}}) + 2\sum_{\beta=1}^{n-1} b_{\alpha\,\beta+n}u_{n\overline{n}}\operatorname{Re}(F^{\beta\overline{n}}) \\
&= 2\sum_{\beta=1}^{n-1} b_{\alpha\,\beta+n}F^{\beta\overline{\beta}}\operatorname{Re}(u_{n\overline{\beta}}) \\
&= 0.
\end{aligned}$$

Since

$$-2\operatorname{Im}\sum_{\beta} b_{\alpha\beta}F^{i\overline{j}}(\partial_i t_\beta \partial_{\overline{j}} u_{y_n} - \partial_{\overline{j}} t_\beta \partial_i u_{y_n}) \geq -C\operatorname{tr}(F^{i\overline{j}}) - \frac{1}{4}F^{i\overline{j}}\partial_i(u_{t_\beta} - \underline{u}_{t_\beta})\,\partial_{\overline{j}}(u_{t_\beta} - \underline{u}_{t_\beta}).$$

Therefore

$$\pm LT_\alpha(u-\underline{u}) \geq -C(\operatorname{tr}(F^{i\overline{j}})+1) - F^{i\overline{j}}\partial_i(u_{t_\beta} - \underline{u}_{t_\beta})\,\partial_{\overline{j}}(u_{t_\beta} - \underline{u}_{t_\beta}). \tag{4.26}$$

Since $\lambda(H(\underline{u}) \in \Gamma$ and $f(\lambda(H(\underline{u})) \geq \psi + \epsilon$, there is $\epsilon_1 > 0$ such that $\lambda(H(\underline{u}) - 2\epsilon_1 I_n)) \in \Gamma$ for all $z \in \overline{D}$, and $f(\lambda(H(\underline{u}) - 2\epsilon_1 I_n)) \geq \psi(z) + \epsilon_0(\epsilon_1) > 0$ on $\overline{D}$, where $\epsilon_0(\epsilon_1) \leq \epsilon$ is positive constant depending only on f ϵ and sufficiently small ϵ_1. Now we let

$$V^{\pm}(z) = \pm T_\alpha(u-\underline{u}) + \sum_{\beta=1}^{2n-1}(u_{t_\beta} - \underline{u}_{t_\beta})^2 + A(\underline{u} - u - \epsilon_1|z|^2)$$

Then

$$L\Big(A(\underline{u} - u - 2\epsilon_1|z|^2)\Big) \geq A[F(H(\underline{u} - 2\epsilon_1|z|^2)) - F(H(u))] = A\epsilon_0(\epsilon_1)$$

$$\begin{aligned}
&LV^{\pm}(z) \\
&\geq -C_1(\operatorname{tr}(F^{i\overline{j}})+1) - F^{i\overline{j}}\partial_i(u_{t_\beta} - \underline{u}_{t_\beta})\,\partial_{\overline{j}}(u_{t_\beta} - \underline{u}_{t_\beta}) \\
&\quad +2F^{i\overline{j}}\partial_i(u_{t_\beta} - \underline{u}_{t_\beta})\partial_{\overline{j}}(u_{t_\beta} - u_{t_\beta}) + 2(u_{t_\beta} - \underline{u}_{t_\beta})\Big(\frac{\partial\psi}{\partial t_\beta} + F^{i\overline{j}}\frac{\partial^3\underline{u}}{\partial z_i\partial\overline{z}_j\partial t_\beta}\Big) \\
&\quad +A\epsilon_0(\epsilon_1) + A\epsilon_1\operatorname{tr}(F^{i\overline{j}}) \\
&\geq (A\epsilon_0 - C_1 - 2C) + (A\epsilon_1 - C_1 - 2C\|\underline{u}\|_{C^3(\overline{D})})\operatorname{tr}(F^{i\overline{j}})
\end{aligned}$$

where $A \geq (C_1 + 2C)/\epsilon_0(\epsilon_1)$ and $\epsilon_0(\epsilon_1) \leq \epsilon_1 \leq \epsilon$ and $C = (\|u\|_{C^1(\overline{D})} + \|\underline{u}\|_{C^1(\overline{D})})$.

For $z \in \overline{D}$ near $z=0$, we have $\underline{u}(z) - u(z) \leq 0$. Furthermore, if $z \in \partial D$ we have

$$V^{\pm}(z) \leq C|z|^2 - A\epsilon_1|z|^2 \leq 0, \quad \text{when } A\epsilon_1 \geq C.$$

If $z \in D \cap \partial B(0,\delta)$ for some small $\delta > 0$ then

$$V^{\pm}(z) \le C(\delta) - A\epsilon_1\delta^2 \le 0, \quad \text{when } A\epsilon_1 \ge C(\delta)/\delta^2.$$

where $C, C(\delta)$ is a constant depending only on $\|u\|_{C^1(\overline{D})} + \|\underline{u}\|_{C^1(\overline{D})}$. Therefore, $V^{\pm}(z)$ attains its maximum over $\overline{D} \cap \overline{B}(0,\delta)$ at $z = 0$ (since $V^{\pm}(0) = 0$). Thus

$$D_\nu V^{\pm}(0) \ge 0.$$

This implies that

$$|D_{\nu T}u(z_0)| \le A + C(\|u\|_{C^1(\overline{D})} + \|\underline{u}\|_{C^3(\overline{D})} + \|\psi\|_{C^2(\overline{D})}).$$

where $A \ge C/\epsilon_0(\epsilon_1) + C/(\epsilon_1\delta^2)$. So, the proof of the lemma is complete. ∎

Lemma 4.4. *Let D, f ϕ satisfy conditions of Theorem 1.1. Then*

$$|D_{\nu\nu}u(z)| \le C, \quad z \in \partial D. \tag{4.27}$$

where C is a constant depending only on f, $\|\psi\|_{C^2(\overline{D})}$, $\|\phi\|_{C^4(\partial D)}$, ϵ and D.

Proof. Let $z_0 \in \partial D$ be arbitrary point in ∂D. By shift and rotation, we may assume that $z_0 = 0$ and $\rho(z)$ has expression of (4.25)

To prove (4.27), by (4.25) and Lemma 4.3, it suffices to prove

$$\Big|\frac{\partial^2 u}{\partial x_n^2}(0)\Big| \le C. \tag{4.28}$$

Since

$$4\sum_{j=1}^{n-1}\frac{\partial^2 u}{\partial z_j \partial \overline{z}_j} + \frac{\partial^2 u}{\partial y_n^2} + \frac{\partial^2 u}{\partial x_n^2} = \Delta u \ge 0,$$

the proof of (4,28) can be reduced to prove

$$\frac{\partial^2 u}{\partial x_n^2}(0) \le C.$$

In fact, we only need to prove

$$\frac{\partial^2 u}{\partial z_n \partial \overline{z}_n}(0) \le C. \tag{4.29}$$

Since

$$F^{i\bar{j}}(H(u))\partial_{i\bar{j}}(\underline{u} - u) \ge F(H(\underline{u})) - F(H(u)) \ge \epsilon > 0, \quad z \in D$$

we have that $\underline{u} - u$ attains its maximum over $\overline{D}$ at some point in ∂D, but $\underline{u} - u = 0$ on ∂D, this implies that $D_\nu(\underline{u} - u) > 0$ on ∂D. In particular, $-\frac{\partial}{\partial x_n}(\underline{u} - u)(0) > 0$.

Let $\Gamma' = \{(\lambda_1, \cdots, \lambda_{n-1}) : \lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma\}$ be the projection of Γ in $\mathbb{R}^{n-1}$. Since $\lambda(H(\underline{u})) \in \Gamma$, we have $\lambda(H(\underline{u})') \in \Gamma'$. Let

$$b = -\frac{\partial}{\partial x_n}(u - \underline{u})(0) < 0. \tag{4.30}$$

Then for $1 \le \alpha, \beta \le n-1$, we have

$$\frac{\partial^2 u}{\partial z_\alpha \partial \overline{z}_\beta}(0) = \frac{\partial^2 \underline{u}}{\partial z_\alpha \partial \overline{z}_\beta}(0) - \frac{\partial (u-\underline{u})}{\partial x_n}(0))\rho_{\alpha\overline{\beta}}(0) = \underline{u}_{\alpha\overline{\beta}}(0) + b\rho_{\alpha\overline{\beta}}(0). \tag{4.31}$$

It clear that for $t \ge 1$, $\lambda(H(t\underline{u}+b\rho)') \in \Gamma'$, $\lambda(H(t\underline{u}+b\rho)') \notin \Gamma'$ when t is very negative, where $H' = [h_{i\overline{j}}]_{1 \le i,j \le n-1}$ when $H = [h_{i\overline{j}}]_{n\times n}$. Let t_0 be the first $t < 1$ so that

$$\lambda(t_0 H(\underline{u})(0)' + bH(\rho)(0)') \in \partial\Gamma' \iff t_0 \lambda(H(\underline{u})(0)' + \frac{b}{t_0} H(\rho)(0)') \in \partial\Gamma'. \tag{4.32}$$

Since Γ' is a cone, if $t_0 > 0$ then (4.32) implies that $(\lambda(H(\underline{u})(0)' + \frac{b}{t_0}H(\rho)(0)') \in \Gamma'$. Without loss of generality, we may assume that $t_0 > 1/2$, otherwise,

$$\frac{\partial^2 u}{\partial z_\alpha \partial \overline{z}_\beta}(0) = (1-t_0)\frac{\partial^2 \underline{u}}{\partial z_\alpha \partial \overline{z}_\beta}(0) + t_0 \frac{\partial^2 \underline{u}}{\partial z_\alpha \partial \overline{z}_\beta}(0) + b\rho_{\alpha\overline{\beta}}(0) \tag{4.33}$$

with $t_0 \le 1/2$. From later argument, one can see that this assumption will implies that $u_{n\overline{n}}(0) \le C$.

Since $b = -\frac{\partial(u-\underline{u})}{\partial x_n}(0) < 0$, we will show that there is $\eta > 0$ independent of $u_{n\overline{n}}(z_0)$ and $z_0 \in \partial D$ so that

$$b \ge \frac{b}{t_0} + \eta \iff (1-t_0) \ge t_0\eta/|b| = t_0\eta/(-b) \ge t_0\eta/(\|u\|_1 + \|\underline{u}\|_1) \tag{4.34}$$

where $\|u\|_1 = \|u\|_{C^1(\overline{D})}$. Without loss of generality, we may assume that $[\underline{u}_{\alpha\overline{\beta}}(0) + (b/t_0)\rho_{\alpha\overline{\beta}}(0)]$ is an $(n-1)\times(n-1)$ diagonal matrix $D_{n-1}(\tilde\lambda_1, \cdots, \tilde\lambda_{n-1})$ with $\tilde\lambda_1 \le \cdots \le \tilde\lambda_{n-1}$.

Since Γ' is a convex cone in $\mathbb{R}^{n-1}$. If $\Gamma' = \mathbb{R}^{n-1}$ then it is easy to handle with. We may assume that $\Gamma' \ne \mathbb{R}^{n-1}$. Since $\tilde\lambda \in \partial\Gamma'$, there is a supporting plane for Γ'. In other words,

$$\Gamma' \subset \{\lambda \in \mathbb{R}^{n-1} : \sum_{j=1}^{n-1} \mu_j(\lambda'_j - \tilde\lambda_j) > 0\}, \quad \sum_{j=1}^{n-1} \mu_j = 1. \tag{4.35}$$

Since Γ' is symmetric in λ'_j and $\tilde\lambda_1 \le \cdots \le \tilde\lambda_{n-1}$, it was proved in [8] that one can choose μ_j with $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_{n-1} \ge 0$ so that (4.35) holds and $\sum_{j=1}^{n-1} \mu_j\tilde\lambda_j = 0$. Therefore, we have $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_{n-1} \ge 0$ and

$$\Gamma' \subset \{\lambda' \in \mathbb{R}^{n-1} : \sum_{j=1}^{n-1} \mu_j\lambda'_j > 0\}, \quad \sum_{j=1}^{n-1} \mu_j = 1, \quad \sum_{j=1}^{n-1} \mu_j\tilde\lambda_j = 0. \tag{4.36}$$

This implies that $\Gamma \subset \{\lambda \in \mathbb{R}^n : \sum_{j=1}^n \mu_j\lambda_j > 0\}$ with $\mu_n = 0$. Since $f(\lambda(H(\underline{u})(z)) \ge \psi(z) + \epsilon > 0$ on $\overline{D}$ and $\underline{u} \in C^2(\overline{D})$, we have $\{\lambda(H(\underline{u})(z)) : z \in \overline{D}\}$ is a compact subset of Γ. Thus

$$\sum_{k=1}^n \mu_k \underline{u}_{k\overline{k}}(z) \ge \sum_{k=1}^n \mu_k\lambda_k(z) \ge \min\Big\{\sum_{k=1}^n \mu_k\lambda_k(z) : z \in \overline{D}\Big\} = a > 0 \tag{4.37}$$

where $\lambda_k(z)$ are eigenvalues of $H(\underline{u})(z)$ with $\lambda_1(z) \le \cdots \le \lambda_n(z)$.

Let $t_1 = b/t_0$. Then

$$0 = \sum_{k=1}^{n-1} \mu_k \tilde{\lambda}_k = \mathrm{tr}\Big(D(\mu_1, \cdots, \mu_{n-1})(\underline{u}_{\alpha\overline{\beta}}(0) + t_1\rho_{\alpha\overline{\beta}}(0))\Big) = \sum_{k=1}^{n-1} \mu_k(\underline{u}_{k\overline{k}} + t_1\rho_{k\overline{k}})$$

Thus
(3.38)

$$\sum_{k=1}^{n-1} \mu_k \rho_{k\overline{k}}(0) = -\frac{1}{t_1}\sum_{k=1}^{n-1} \mu_k \underline{u}_{k\overline{k}}(0) = \frac{t_0}{-b}\sum_{k=1}^{n-1} \mu_k \underline{u}_{k\overline{k}}(0) \geq \frac{a}{2(\|u\|_1 + \|\underline{u}\|_1)} = a_1 > 0.$$

Let

$$d(z) = -\rho(z) + \tau|z|^2, \quad z \in D \tag{4.39}$$

where τ is a positive constant with $\tau < a_1/4$; and let

$$w(z) = \underline{u}(z) + t_1\rho(z) + \ell(z)\rho(z) + Md(z)^2. \tag{4.40}$$

Let

$$D_\delta = D \cap B(0, \delta)$$

Then, on ∂D, we have

$$u(z) - w(z) = -M\tau^2|z|^4 \leq 0 \tag{4.41}$$

and, on $D \cap \partial B(0, \delta)$, since $t_1 < 0$ and $\rho(z) \leq 0$, we have

$$u - w(z) \leq C_0(\|u - \underline{u}|_{C(\overline{D})} + \|\ell\|_\infty\delta) - M\tau^2\delta^4 \leq -\frac{M\tau^2}{2}\delta^4$$

when $M \geq C_0(\|u - \underline{u}\|_{C(\overline{D})} + \|\ell\|_{C(\overline{D})}\delta)/(\delta^4\tau^2)$. As a summary, we have that if $M \geq C_0(\|u - \underline{u}\|_{C(\overline{D})} + \|\ell\|_{C(\overline{D})}\delta)/(\delta^4\tau^2)$ with small $\delta > 0$ fixed, then

$$u \leq w, \quad \text{on } \partial D_\delta. \tag{4.42}$$

Let $(T_1(z), \cdots, T_{n-1})$ be an orthonormal basis for holomorphic tangent space of level hypersurface $d(w) = d(z)$ at z, we choose T_j are C^1 and $T_j(0, z_n) = e_j$, where $e_j = (0, \cdots, 0, 1, 0, \cdots, 0)$. Let $\mu_n = 0$, and $T_n = \partial d/|\partial d|$. Then we define

$$\Lambda = \sum_{k=1}^{n} \mu_k \overline{T}_{ki} T_{kj} \frac{\partial^2}{\partial z_i \partial \overline{z}_j}. \tag{4.43}$$

Thus,

$$\Lambda d(z)^2 = 2d\Lambda d(z) + 2\sum_{k=1}^{n-1} \mu_k \overline{T}_{ki} T_{kj} \partial_i d(z)\, \partial_{\overline{j}} d(z) = 2d(z)\Lambda d(z) \leq -\frac{a_1}{2} d(z) \tag{4.44}$$

since

$$\Lambda d(z) = \Lambda(0, z_n)d(z) + (\Lambda - \Lambda(0, z_n))d(z) \leq -a_1 + \tau + C|\rho(z)| \leq -\frac{a_1}{4}.$$

where $|\rho(z)| \le a_1/(4C)$ with $C = \|\Lambda(z)d(z) - \Lambda(0, z_n)d(z)\|_\infty + 1$.

Now we claim that $\lambda(H(w)(z)') \notin \Gamma'$ for $z \in \overline{D}_\delta$. Since

$$H(w) = H(\underline{u} + t_1\rho) + \ell H(\rho) + \partial\ell \otimes \overline{\partial}\rho + \partial\rho \otimes \overline{\partial}\ell + MH(d^2)$$

and $\Lambda(0)(\underline{u} + t_1\rho)(0) = 0$ we have

$$\Lambda(\underline{u} + t_1\rho)(z) = \sum_{j=1}^n (A_j z_j + \overline{A}_j z_j) + O_1(|z|^2) = L_1(z) + O_1(|z|^2)$$

$$\ell(z)\Lambda\rho(z) = \ell(z)\Lambda\rho(0) + O_2(|z|^2), \quad (\Lambda\rho)(0) \in (-C, -a_1]$$

and

$$\tau \sum_{k=1}^n \mu_k \overline{T}_{ki} T_{kj} (\ell_i z_j + \ell_{\overline{j}} \overline{z}_i) = \tau \sum_{k=1}^n \mu_k (\ell_i z_i + \ell_{\overline{i}} \overline{z}_i) + O_3(|z|^2)$$

Therefore,

$$\begin{aligned}
\Lambda w &= \Lambda(\underline{u} + t_1\rho) + \ell(z)\Lambda\rho(z) + \sum_{k=1}^n \mu_k \overline{T}_{ki} T_{kj} (\ell_i \overline{\partial}_j \rho + \ell_{\overline{j}} \partial_i \rho) + M\Lambda d^2 \\
&\le L_1(z) + O_1(|z|^2) + (\Lambda\rho)(0)\,\ell(z) + O_2(|z|^2) \\
&\quad + \tau \sum_{k=1}^n \mu_k \overline{T}_{ki} T_{kj} (\ell_i z_j + \ell_{\overline{j}} \overline{z}_i) - \frac{Ma_1}{2} d(z) \\
&= L_1(z) + O_1(|z|^2) + (\Lambda\rho)(0)\,\ell(z) + O_2(|z|^2) + \tau O_3(|z|^2) \\
&\quad + \tau \sum_{k=1}^n \mu_k (\ell_k z_k + \ell_{\overline{k}} \overline{z}_k) - \frac{Ma_1}{2} d(z) \\
&= \sum_{k=1}^n (A_k + \ell_k((\Lambda\rho)(0) - \tau\mu_k)) z_k + (\overline{A}_k + \ell_{\overline{k}}((\Lambda\rho)(0) - \tau\mu_k)) \overline{z}_k \\
&\quad + O(|z|^2) - \frac{Ma_1}{2} d(z) \\
&= O(|z|^2) - \frac{Ma}{2} d(z) \\
&\le -\frac{Ma}{4} d(z)
\end{aligned}$$

by choosing ℓ_k and $M > 0$ so that

$$\ell_k = -A_k/((\Lambda\rho)(0) - \tau\mu_k), \quad \text{and} \quad O(|z|^2) \le \frac{Ma_1}{4}|z|^2. \tag{4.45}$$

Let $\lambda_k(w)$ are all eigenvalues of $H(w)$ at z with $\lambda_1(w) \le \cdots \le \lambda_n(w)$. Then by Proposition 2.4 with $B = \text{Diag}(\mu_1, \cdots, \mu_{n-1}, 0)$

$$\sum_{k=1}^{n-1} \mu_k \lambda_k(w) \le \text{tr}(H(w)(z)B) = \Lambda w(z) < 0.$$

Thus $\lambda(H(w)') \notin \Gamma'$.

Let

$$\tilde{\Gamma} = \Big\{\lambda \in \Gamma : f(\lambda) \geq \min\{\psi(z) : z \in \overline{D}\} > 0\Big\} \tag{4.46}$$

and let

$$X = \Big\{\lambda \in \mathbb{R}^n : \lambda' \in \mathbb{R}^{n-1} \setminus \Gamma' : |\lambda| \leq \|w\|_{C^2(\overline{D})} + 1\Big\}. \tag{4.47}$$

Then for any $\lambda \in \partial\Gamma$ by Assumption (1.10), we have $f(\lambda) \leq \psi^0 < \min\{\psi(z) : z \in \overline{D}\}$. Since f is continuous on compact set X, there is $\eta > 0$ such that if $\lambda \in X$ and $(\eta, \cdots, \eta) + \lambda \in \Gamma$ then

$$f((\eta, \cdots, \eta) + \lambda) < \psi_0 = \min\{\psi(z) : z \in \overline{D}\}. \tag{4.48}$$

Let

$$v(z) = w(z) + \eta(|z|^2 - \frac{1}{C_0} x_n) \tag{4.49}$$

where $C_0 > 0$ be chosen so that $|z|^2 - \frac{1}{C_0} x_n \geq 0$ on ∂D_δ. Then $u(z) \leq v(z)$ on ∂D_δ. Since $\lambda(H(v)) = (\lambda(H(w) + \eta I_n) \notin \tilde{\Gamma}$ by (4.48), (4.46), (4.47) and $\lambda(H(\underline{w})') \notin \Gamma'$. By Maximum Principle, i.e., Lemma 2.1, we have $u(z) - v(z) \leq 0$ on $\overline{D}_\delta$. Since $u(0) - v(0) = 0$, we have

$$-\frac{\partial}{\partial x_n}(u - v)(0) \geq 0 \iff -\frac{\partial(u - \underline{u})}{\partial x_n}(0) \geq \frac{\eta}{C_0} + t_1 \iff b \geq \frac{\eta}{C_0} + \frac{b}{t_0} \iff$$
$$\iff (1 - t_0) \geq \frac{\eta t_0}{C_0 |b|}$$

Therefore

$$\begin{aligned} \frac{\partial^2 u}{\partial z_\alpha \partial \overline{z}_\beta}(0) &= \frac{\partial^2 \underline{u}}{\partial z_\alpha \partial \overline{z}_\beta}(0) - \frac{\partial(u - \underline{u})}{\partial x_n}(0)\rho_{\alpha\overline{\beta}}(0) \\ &= (1 - t_0)\frac{\partial^2 \underline{u}}{\partial z_\alpha \partial \overline{z}_\beta}(0) + t_0[\frac{\partial^2 \underline{u}}{\partial z_\alpha \partial \overline{z}_\beta}(0) + t_1 \rho_{\alpha\overline{\beta}}(0)] \end{aligned}$$

Thus

$$H(u)(0) = E_0 + \begin{bmatrix} (1 - t_0)\underline{u}_{\alpha\overline{\beta}}(0) & O(1) \\ O(1)^* & u_{n\overline{n}}(0) - M_1 \end{bmatrix}$$

where $\lambda(E_0) \in \overline{\Gamma}$ for some fixed M_1. Therefore, since $\lambda(H(\underline{u})(0)) \in \Gamma$ and

$$\begin{aligned} \psi(0) = F(H(u)(0)) &\geq t_0 F(E_0/t_0) + (1 - t_0)F\Big(\Big(\begin{bmatrix} \underline{u}_{\alpha\overline{\beta}} & O(1) \\ O(1)^* & (u_{n\overline{n}}(0) - M_1 0/(1 - t_0) \end{bmatrix}\Big) \\ &\geq (1 - t_0)F\Big(\Big(\begin{bmatrix} \underline{u}_{\alpha\overline{\beta}} & O(1) \\ O(1)^* & (u_{n\overline{n}}(0) - M_1)/(1 - t_0) \end{bmatrix}\Big) \end{aligned}$$

Notices that $(1 - t_0) \geq \frac{\eta}{2C_0}/(\|u\|_1 + \|\underline{u}\|_1) > 0$ and Lemma 2.2 and Condition (1.11), we have

$$u_{n\overline{n}}(0) \leq C. \tag{4.50}$$

where C is depending only on $\|\underline{u}\|_{C^3(\overline{D})}, \|\psi\|_{C^2(\overline{D})}, \epsilon, \|\phi\|_{C^4(\partial D)}$ and Γ. Thus, the proof of Lemma 4.4 is complete. □

We now are ready to prove Theorems 1.1 –1.3.

Proof of Theorem 1.1. Under the assumption of Theorem 1.1, we have that the all assumptions of Lemmas 4.1–4.4 hold. Combining the results of Lemmas 4.1–4.4, we have *a priori* estimate on the admissible solution u of (1.14) satisfying the estimate:

$$\|u\|_{C^2(\overline{D})} \leq C(\epsilon, f, \|\phi\|_{C^4(\partial D)}, \|\psi\|_{C^2(\overline{D})}) < \infty. \tag{4.51}$$

The results in [6] and [23] and arguments in [8] and [7] implies that the Dirichlet problem (1.14) has a unique solution $u \in C^\infty(\overline{D})$ with $\lambda(H(u)) \in \Gamma$ on D. Thus, the proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. The assumptions of Theorem 1.2 imply that the all assumptions of Theorems 3.1 and 3.2 hold. For any $\phi \in C^\infty(\overline{D})$, Theorem 3.2 implies that there is $\underline{u} \in C^\infty(\overline{D})$ so that $\lambda(H(\underline{u})) \in \Gamma$ for all $z \in \overline{D}$ and

$$f(\lambda(H(\underline{u})(z)) \geq \psi(z) + 1, \quad z \in \overline{D}. \tag{4.52}$$

Theorem 1.1 gives the result of Theorem 1.2. □

Proof of Theorem 1.3. The proof of the sufficient condition of Theorem 1.3 is a consequence of Proposition 3.3 and Theorem 1.2. The proof of the necessary condition can be followed directly from an argument in [8], we omit the detail here. □

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