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Generalized Weighted Pseudo-Almost Periodic Solutions and Generalized Weighted Pseudo-Almost Automorphic Solutions of Abstract Volterra Integro-Differential Inclusions

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Abstract

In this paper, we analyze the existence and uniqueness of generalized weighted pseudo-almost automorphic solutions of abstract Volterra integro-differential inclusions in Banach spaces. The main results are devoted to the study of various types of weighted pseudo-almost periodic (automorphic) properties of convolution products. We illustrate our abstract results with some examples and possible applications.

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1 Introduction and Preliminaries

The class of weighted pseudo-almost periodic functions with values in Banach spaces was introduced by T. Diagana in [15] (2006), while the class of weighted pseudo-almost automorphic functions with values in Banach spaces was introduced by J. Blot, G. M. Mophou, G. M. N'Guérékata and D. Pennequin in [8] (2009). In this paper, we analyze the existence and uniqueness of generalized weighted pseudo-almost periodic solutions and generalized weighted pseudo-almost automorphic solutions of abstract Volterra integro-differential inclusions in Banach spaces.

Concerning already made applications to abstract Volterra integro-differential equations, we will mention here only one illustrative. In [14, Chapter 10], T. Diagana has examined the existence and uniqueness of weighted pseudo almost-periodic solutions of the autonomous abstract differential equation

$$\frac{d}{dt}\Big[u''(t)+f(t,Bu(t))\Big]=w(t)Au(t)+g(t,Cu(t)),\quad t\in\mathbb{R}, \tag{1.1}$$

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where A is a sectorial operator on X, B and C are closed linear operators acting on X, and $f : \mathbb{R}\times X \to X$, $g : \mathbb{R}\times X \to X$ are pseudo almost-periodic functions in $t \in \mathbb{R}$ uniformly in $x \in X$. Motivated by his ideas, a great number of authors has considered the qualitative properties of solutions for various types of the Sobolev-type differential equations (see e.g. [2], [10]-[11], [17], [45] and references cited therein).

Fractional calculus and fractional differential equations have received much attention recently due to its wide range of applications in various fields of science, such as mathematical physics, engineering, aerodynamics, biology, chemistry, economics etc (see e.g. [4], [22], [29]-[30] and [41]-[42]). In this paper, we essentially use only the Weyl-Liouville fractional derivatives (for more details, we refer the reader to the paper [39] by J. Mu, Y. Zhoa and L. Peng). The Weyl-Liouville fractional derivative $D_{t,+}^{\gamma}u(t)$ of order $\gamma \in (0,1)$ is defined for those continuous functions $u : \mathbb{R} \to X$ such that $t \mapsto \int_{-\infty}^{t} g_{1-\gamma}(t-s)u(s)\,ds$, $t \in \mathbb{R}$ is a well-defined continuously differentiable mapping, by $D_{t,+}^{\gamma}u(t) := \frac{d}{dt}\int_{-\infty}^{t} g_{1-\gamma}(t-s)u(s)\,ds$, $t \in \mathbb{R}$; here and hereafter, $g_{\zeta}(t) := t^{\zeta-1}/\Gamma(\zeta)$, $t > 0$, where $\Gamma(\cdot)$ denotes the Gamma function ($\zeta > 0$).

The organization of paper can be simply described as follows. At the end of introductory part, we give a brief overview of definitions and results about multivalued linear operators in Banach spaces. The main aim of Section 2 is to analyze various generalizations of almost periodic functions and almost automorphic functions; besides some new notions introduced, like Besicovitch pseudo-almost automorphy, the only original contribution here is Proposition 2.2, which states that the class of Weyl-p-vanishing functions (extended by zero outside $[0,\infty)$; see [36] for the notion) is contained in the Weyl ergodic type space $W_pPAA_0(\mathbb{R} : X)$, introduced by S. Abbas in [1]; we paid special attention to generalized two-parameter almost automorphic functions. In Section 3, we remind ourselves of the basic facts about weighted pseudo-almost periodic functions, weighted pseudo-almost automorphic functions, their generalizations and composition principles for these classes (see especially Subsection 3.1). We introduce the notions of a weighted Weyl p-pseudo almost automorphic function and a weighted Besicovitch p-pseudo almost automorphic function, investigating also the convolution invariance of space $B^1WPAA_0(\mathbb{R},X,\rho_1,\rho_2)$ and translation invariance of space $PAP_0(\mathbb{R},X,\rho_1,\rho_2)$. Our main results are stated in Section 4, where we investigate the generalized weighted pseudo-almost periodic (automorphic) properties of convolution products; in this paper, we are primarily concerned with the infinite convolution product $\int_{-\infty}^{\cdot} R(\cdot - s)f(s)\,ds$, where $(R(t))_{t>0}$ is a strongly continuous operator family. Weighted pseudo-almost automorphic solutions of semilinear (fractional) Cauchy inclusions are analyzed in Section 5, where we also provide some applications in the analysis of the following abstract Cauchy inclusion of first order

$$u'(t) \in \mathcal{A}u(t) + f(t),\ t \in \mathbb{R},$$

and its fractional relaxation analogue

$$D_{t,+}^{\gamma}u(t) \in \mathcal{A}u(t) + f(t),\ t \in \mathbb{R},$$

where $D_{t,+}^{\gamma}$ denotes the Riemann-Liouville fractional derivative of order $\gamma \in (0,1)$, and $f : \mathbb{R} \to X$ satisfies certain properties, as well as their semilinear analogues

$$u'(t) \in \mathcal{A}u(t) + f(t,u(t)),\ t \in \mathbb{R}, \tag{1.2}$$

and

$$D_{t,+}^{\gamma}u(t)\in \mathcal{A}u(t)+f(t,u(t)),\ t\in \mathbb{R}, \tag{1.3}$$

where $f:\mathbb{R}\times X\to X$ satisfies certain applications and $\mathcal{A}$ is a multivalued linear operator in X. It should be emphasized that our results seem to be new even for a class of almost sectorial operators [40].

Let X and Y be two Banach spaces over the field of complex numbers. A multivalued map $\mathcal{A}:X\to P(Y)$ is said to be a multivalued linear operator (MLO) iff $D(\mathcal{A}):=\{x\in X : \mathcal{A}x\neq\emptyset\}$ is a linear submanifold of X, $\mathcal{A}x+\mathcal{A}y\subseteq \mathcal{A}(x+y)$, $x,\ y\in D(\mathcal{A})$ and $\lambda\mathcal{A}x\subseteq\mathcal{A}(\lambda x)$, $\lambda\in\mathbb{C}$, $x\in D(\mathcal{A})$. If $X=Y$, then we say that $\mathcal{A}$ is an MLO in X. We know that the equality $\lambda\mathcal{A}x+\eta\mathcal{A}y=\mathcal{A}(\lambda x+\eta y)$ holds for every $x,\ y\in D(\mathcal{A})$ and for every $\lambda,\ \eta\in\mathbb{C}$ with $|\lambda|+|\eta|\neq 0$. Set $R(\mathcal{A}):=\{\mathcal{A}x : x\in D(\mathcal{A})\}$. Then the set $N(\mathcal{A}):=\mathcal{A}^{-1}0=\{x\in D(\mathcal{A}) : 0\in\mathcal{A}x\}$ is called the kernel space of $\mathcal{A}$. The inverse $\mathcal{A}^{-1}$ of an MLO is defined through $D(\mathcal{A}^{-1}):=R(\mathcal{A})$ and $\mathcal{A}^{-1}y:=\{x\in D(\mathcal{A}) : y\in\mathcal{A}x\}$. It can be easily verified that $\mathcal{A}^{-1}$ is an MLO in X, as well as that $N(\mathcal{A}^{-1})=\mathcal{A}0$ and $(\mathcal{A}^{-1})^{-1}=\mathcal{A}$. If $N(\mathcal{A})=\{0\}$, i.e., if $\mathcal{A}^{-1}$ is single-valued, then $\mathcal{A}$ is said to be injective. Assuming $\mathcal{A},\ \mathcal{B}:X\to P(Y)$ are two MLOs, then we define its sum $\mathcal{A}+\mathcal{B}$ by $D(\mathcal{A}+\mathcal{B}):=D(\mathcal{A})\cap D(\mathcal{B})$ and $(\mathcal{A}+\mathcal{B})x:=\mathcal{A}x+\mathcal{B}x$, $x\in D(\mathcal{A}+\mathcal{B})$. It is clear that $\mathcal{A}+\mathcal{B}$ is an MLO. Products and integer powers are well-known operations for MLOs ([24]).

Suppose now that $\mathcal{A}$ is an MLO in X. The resolvent set of $\mathcal{A}$, $\rho(\mathcal{A})$ for short, is defined as the union of those complex numbers $\lambda\in\mathbb{C}$ for which $X=R(\lambda-\mathcal{A})$and $(\lambda-\mathcal{A})^{-1}$ is a single-valued linear continuous operator on X. The operator $\lambda\mapsto(\lambda-\mathcal{A})^{-1}$ is called the resolvent of $\mathcal{A}$ ($\lambda\in\rho(\mathcal{A})$). For further information about multivalued linear operators and abstract degenerate integro-differential equations, we refer the reader to the monographs [9], [13], [24], [34], [38] and [43].

2 Various Generalizations of Almost Periodic Functions and Almost Automorphic Functions

The concept of almost periodicity was introduced by Danish mathematician H. Bohr around 1924-1926 and later generalized by many other authors (cf. [14], [25], [33] and [47] for more details on the subject). Let $I=\mathbb{R}$ or $I=[0,\infty)$, and let $f:I\to X$ be continuous. Given $\epsilon>0$, we call $\tau>0$ an ϵ-period for $f(\cdot)$ iff $\|f(t+\tau)-f(t)\|\leq\epsilon$, $t\in I$. The set constituted of all ϵ-periods for $f(\cdot)$ is denoted by $\vartheta(f,\epsilon)$. It is said that $f(\cdot)$ is almost periodic, a.p. for short, iff for each $\epsilon>0$ the set $\vartheta(f,\epsilon)$ is relatively dense in I, which means that there exists $l>0$ such that any subinterval of I of length l meets $\vartheta(f,\epsilon)$.

Let $f:\mathbb{R}\to X$ be continuous. Then we say that $f(\cdot)$ is almost automorphic, a.a. for short, iff for every real sequence (b_n) there exist a subsequence (a_n) of (b_n) and a map $g:\mathbb{R}\to X$ such that

$$\lim_{n\to\infty} f(t+a_n)=g(t)\ \text{ and }\ \lim_{n\to\infty} g(t-a_n)=f(t), \tag{2.1}$$

pointwise for $t\in\mathbb{R}$. The vector space consisting of all almost automorphic functions is denoted by $AA(\mathbb{R}:X)$; see [14] and [33] for more details about the subject.

A continuous function $f : \mathbb{R} \to X$ is said to be asymptotically almost automorphic, a.a.a. for short, iff there are a function $h \in C_0([0,\infty) : X)$ and an almost automorphic function $q : \mathbb{R} \to X$ such that $f(t) = h(t) + q(t)$, $t \geq 0$. Using Bochner's criterion again, we can easily show that any (asymptotically) almost periodic function $[0,\infty) \mapsto X$ is (asymptotically) almost automorphic. The spaces of almost periodic, almost automorphic and asymptotically almost automorphic functions are closed subspaces of $C_b(\mathbb{R} : X)$ when equipped with the sup-norm.

Suppose now that $1 \leq p < \infty$, $l > 0$ and $f,\ g \in L^p_{loc}(I : X)$, where $I = \mathbb{R}$ or $I = [0,\infty)$. We say that a function $f \in L^p_{loc}(I : X)$ is Stepanov p-bounded, S^p-bounded shortly, iff $\|f\|_{S^p} := \sup_{t\in I}(\int_t^{t+1} \|f(s)\|^p\, ds)^{1/p} < \infty$. The above norm turns the space $L^p_S(I : X)$ consisting of all S^p-bounded functions into a Banach space. It is said that a function $f \in L^p_S(I : X)$ is Stepanov p-almost periodic, S^p-almost periodic or S^p-a.p. shortly, iff the function $\hat{f} : I \to L^p([0,1] : X)$, defined by $\hat{f}(t)(s) := f(t+s)$, $t \in I,\ s \in [0,1]$ is almost periodic. It is said that $f \in L^p_S([0,\infty) : X)$ is asymptotically Stepanov p-almost periodic, asymptotically S^p-a.p. shortly, iff $\hat{f} : [0,\infty) \to L^p([0,1] : X)$ is asymptotically almost periodic.

The following notion of a Stepanov p-almost automorphic function has been introduced by G. M. N'Guérékata and A. Pankov in [27]: A function $f \in L^p_{loc}(\mathbb{R} : X)$ is called Stepanov p-almost automorphic, S^p-almost automorphic or S^p-a.a. shortly, iff for every real sequence (a_n), there exists a subsequence (a_{n_k}) and a function $g \in L^p_{loc}(\mathbb{R} : X)$ such that

$$\lim_{k\to\infty} \int_t^{t+1} \left\| f(a_{n_k} + s) - g(s) \right\|^p ds = 0 \text{ and } \lim_{k\to\infty} \int_t^{t+1} \left\| g(s - a_{n_k}) - f(s) \right\|^p ds = 0$$

for each $t \in \mathbb{R}$; a function $f \in L^p_{loc}([0,\infty) : X)$ is said to be asymptotically Stepanov p-almost automorphic, asymptotically S^p-almost automorphic or asymptotically S^p-a.a. shortly, iff there exists an S^p-almost automorphic function $g(\cdot)$ and a function $q \in L^p_S([0,\infty) : X)$ such that $f(t) = g(t) + q(t)$, $t \geq 0$ and $\hat{q} \in C_0([0,\infty) : L^p([0,1] : X))$; the vector space consisting of such functions $q(\cdot)$ will be denoted by $S^p_0([0,\infty) : X)$. It is well known that any S^p-almost automorphic function $f(\cdot)$ has to be S^p-bounded ($1 \leq p < \infty$). The vector space consisting of all S^p-almost automorphic functions, resp., asymptotically S^p-almost automorphic functions, will be denoted by $AAS^p(\mathbb{R} : X)$, resp., $AAAS^p([0,\infty) : X)$.

We refer the reader to [36] for the notions of an (equi-)Weyl-p-almost periodic function and an (equi-)Weyl-p-vanishing function ($1 \leq p < \infty$); see also [3] for scalar-valued case. Denote by $W^p_0([0,\infty) : X)$ and $e - W^p_0([0,\infty) : X)$ the sets consisting of all Weyl-p-vanishing functions and equi-Weyl-p-vanishing functions, respectively. Then we know that $e - W^p_0([0,\infty) : X) \subseteq W^p_0([0,\infty) : X)$.

The concepts of Weyl almost automorphy and Weyl pseudo-almost automorphy, more general than those of Stepanov almost automorphy and Stepanov pseudo-almost automorphy, were introduced by S. Abbas [1] in 2012. The vector space consisted of all Weyl p-almost automorphic functions will be denoted by $W^pAA(\mathbb{R} : X)$. This space contains the space of Weyl-p-almost periodic functions; see [1] and [33] for more details.

The class of Stepanov pseudo-almost automorphic functions has been introduced by T. Diagana in [18] (cf. also the paper [23], where Z. Fan, J. Liang and T.-J. Xiao have analyzed composition theorems for the corresponding class):

Definition 2.1. Let $1 \leq p < \infty$. A Stepanov p-bounded function $f(\cdot)$ is said to be Stepanov p-pseudo almost periodic (automorphic) iff it admits a decomposition $f(t) = g(t) + q(t)$, $t \in \mathbb{R}$, where $g(\cdot)$ is S^p-almost periodic (automorphic) and $q(\cdot) \in L^p_{loc}(\mathbb{R} : X)$ satisfies

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \left[\int_t^{t+1} \|q(s)\|^p \, ds \right]^{1/p} dt = 0. \tag{2.2}$$

Denote by $S^pPAP(\mathbb{R} : X)$ ($S^pPAA(\mathbb{R} : X)$) the collection of such functions, and by $S^pPAA_0(\mathbb{R} : X)$ the collection of locally p-integrable X-valued functions $q(\cdot)$ such that (2.2) holds.

It is well known that $S^pPAA(\mathbb{R}, X, \rho_1, \rho_2)$, endowed with the Stepanov norm, is a Banach space which contains all pseudo-almost automorphic functions. A similar statement holds for the space $S^pPAP(\mathbb{R}, X, \rho_1, \rho_2)$. By $W_pPAA(\mathbb{R} : X)$ we denote the set consisted of all Weyl p-pseudo almost automorphic functions (see [1] for the notion); by $W_pPAA_0(\mathbb{R} : X)$ we denote the space consisting of all functions $q \in L^p_{loc}(\mathbb{R} : X)$ satisfying $\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} [\lim_{l\to+\infty} \frac{1}{2l} \int_{x-l}^{x+l} \|q(t)\|^p \, dt]$ 0.

It can be simply proved that the class of Weyl-p-vanishing functions (extended by zero outside $[0, \infty)$) is contained in $W_pPAA_0(\mathbb{R} : X)$. More precisely, we have the following result (see [33] for a proof):

Proposition 2.2. *Let $1 \leq p < \infty$, and let $q \in L^p_{loc}([0, \infty) : X)$ be a Weyl-p-vanishing function. Let $q_e \in L^p_{loc}(\mathbb{R} : X)$ be given by $q_e(t) := q(t)$, $t \geq 0$ and $q_e(t) := 0$, $t < 0$. Then $q_e \in W_pPAA_0(\mathbb{R} : X)$.*

For Besicovitch and Besicovitch-Doss generalizations of almost periodic functions in Banach spaces, we refer the reader to [37]; see also [6]. The reader may consult the paper [5] by F. Bedouhene, N. Challali, O. Mellah, P. Raynaud de Fitte and M. Smaali for further information concerning the class $B^pAA(\mathbb{R} : X)$ consisted of all Besicovitch almost automorphic functions. This class extends the class of Weyl p-almost automorphic functions. It can be easily proved that the set $B^pAA(\mathbb{R} : X)$, equipped with the usual operations, forms a vector space. In the present situation, the author does not know whether a Besicovitch p-almost periodic function is necessarily Besicovitch p-almost automorphic.

Definition 2.3. Let $p \geq 1$. Then we say that a function $f \in L^p_{loc}(\mathbb{R} : X)$ is Besicovitch p-pseudo almost automorphic iff $f(\cdot) = g(\cdot) + q(\cdot)$, where $g(\cdot)$ is Besicovitch p-almost automorphic and $q \in L^p_{loc}(\mathbb{R} : X)$ satisfies

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \left[\limsup_{l\to+\infty} \frac{1}{2l} \int_{x-l}^{x+l} \|q(t)\|^p \, dt \right]^{1/p} dx = 0.$$

The set of all such functions are denoted by $B_pPAA(\mathbb{R} : X)$.

A very simple analysis shows that $B_pPAA(\mathbb{R} : X)$ is a vector space, as well as that $W_pPAA(\mathbb{R} : X) \subseteq B_pPAA(\mathbb{R} : X)$.

In order to shorten our study, we will skip here all details concerning two-parameter almost periodic functions and their generalizations; see T. Diagana [14] and M. Kostić [33] for further information in this direction. Regarding generalized two-parameter almost automorphic functions, the following facts should be recalled: Let $(Y, \|\cdot\|_Y)$ be another pivot space over the field of complex numbers. Then we say that a continuous function $F : \mathbb{R}\times Y \to X$ is almost automorphic iff for every sequence of real numbers (s'_n) there exists a subsequence (s_n) such that $G(t,y) := \lim_{n\to\infty} F(t+s_n,y)$ is well defined for each $t \in \mathbb{R}$ and $y \in Y$, and $\lim_{n\to\infty} G(t-s_n,y) = F(t,y)$ for each $t \in \mathbb{R}$ and $y \in Y$. The vector space consisting of such functions will be denoted by $AA(\mathbb{R}\times Y : X)$.

The notion of a pseudo almost-automorphic function was introduced by T.-J. Xiao, J. Liang and J. Zhang [46] in 2008. Let us recall that the space of pseudo-almost automorphic functions, denoted shortly by $PAA(\mathbb{R} : X)$, is defined as the direct sum of spaces $AA(\mathbb{R} : X)$ and $PAP_0(\mathbb{R} : X)$, where $PAP_0(\mathbb{R} : X)$ denotes the space consisting of all bounded continuous functions $\Phi : \mathbb{R} \to X$ such that $\lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} \|\Phi(s)\|\,ds = 0$. Equipped with the sup-norm, the space $PAA(\mathbb{R} : X)$ becomes one of Banach's. A bounded continuous function $f : \mathbb{R}\times Y \to X$ is said to be pseudo-almost automorphic iff $F = G+\Phi$, where $G \in AA(\mathbb{R}\times Y : X)$ and $\Phi \in PAP_0(\mathbb{R}\times Y : X)$; here, $PAP_0(\mathbb{R}\times Y : X)$ denotes the space consisting of all continuous functions $\Phi : \mathbb{R}\times Y \to X$ such that $\{\Phi(t,y) : t \in \mathbb{R}\}$ is bounded for all $y \in Y$, and $\lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} \|\Phi(s,y)\|\,ds = 0$, uniformly in $y \in Y$. The collection of such functions will be denoted henceforth by $PAA(\mathbb{R}\times Y : X)$.

The notion of a Stepanov p-almost automorphic function $f : \mathbb{R}\times Y \to X$ will be taken in the sense of [16], while the notion of a Stepanov p-pseudo almost automorphic function $f : \mathbb{R}\times Y \to X$ will be taken in the sense of [18]. Denote by $S^pPAA(\mathbb{R}\times Y, X, \rho_1, \rho_2)$ the collection of such functions, and by $S^pPAA_0(\mathbb{R}\times Y, X, \rho_1, \rho_2)$ the collection of functions satisfying that for each $y \in Y$ one has that $q \in L^p_{loc}(\mathbb{R} : X)$ and $\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} [\int_t^{t+1} \|q(s,y)\|^p\,ds]^{1/p}\,dt = 0$. If $\rho = \rho_1 = \rho_2$, then we write $S^pPAA(\mathbb{R}\times Y, X, \rho_1, \rho_2) \equiv S^pPAA(\mathbb{R}\times Y, X, \rho)$.

We refer the reader to [35] for various composition principles for almost automorphic functions, pseudo-almost automorphic functions and Stepanov almost automorphic functions. The reader may consult S. Abbas [1] for the notion of a Weyl pseudo-almost automorphic function as well as for the formulation of a composition principle for Weyl pseudo-almost automorphic functions.

3 Weighted Pseudo-Almost Periodic Functions, Weighted Pseudo-Almost Automorphic Functions and Their Generalizations

Set $\mathbb{U} := \{\rho \in L^1_{loc}(\mathbb{R}) : \rho(t) > 0$ a.e. $t \in \mathbb{R}\}$, $\mathbb{U}_\infty := \{\rho \in \mathbb{U} : \inf_{x\in\mathbb{R}} \rho(x) < \infty$ and $\nu(T,\rho) := \lim_{T\to+\infty}\int_{-T}^{T} \rho(t)\,dt = \infty\}$ and $\mathbb{U}_b := L^\infty(\mathbb{R}) \cap \mathbb{U}_\infty$. Then it is clear that $\mathbb{U}_b \subseteq \mathbb{U}_\infty \subseteq \mathbb{U}$. We say that weights $\rho_1(\cdot)$ and $\rho_2(\cdot)$ are equivalent, $\rho_1 \sim \rho_2$ for short, iff $\rho_1/\rho_2 \in \mathbb{U}_b$. By $\mathbb{U}_T$ we denote the space consisting of all weights $\rho \in \mathbb{U}_\infty$ which are equivalent with all its translations.

Next, we introduce the spaces consisting of so-called ergodic components, depending on one or two variables. Assume that ρ_1, $\rho_2 \in \mathbb{U}_\infty$. We define the space $PAP_0(\mathbb{R}, X, \rho_1, \rho_2)$ as the space consisted of all functions $f \in C_b(\mathbb{R} : X)$ such that $\lim_{T\to+\infty} \frac{1}{2\int_{-T}^{T}\rho_1(t)\,dt}\int_{-T}^{T} \|f(t)\|\rho_2(t)\,dt = 0$ and the space $PAP_0(\mathbb{R}\times Y, X, \rho_1, \rho_2)$ as the space consisted of all functions $f \in C_b(\mathbb{R}\times Y : X)$ such that $\lim_{T\to+\infty} \frac{1}{2\int_{-T}^{T}\rho_1(t)\,dt}\int_{-T}^{T} \|f(t,y)\|\rho_2(t)\,dt = 0$, uniformly on bounded subsets of Y.

The class of weighted pseudo-almost periodic functions was introduced by T. Diagana in [15] (2006); we will slightly generalize the notion from this paper by examining two weight functions, leading to the concept of so-called double-weighted pseudo almost-periodic functions, as it has been done by T. Diagana in [19]-[20].

Definition 3.1. (i) A function $f \in C_b(\mathbb{R} : X)$ is said to be weighted pseudo-almost periodic iff it admits a decomposition $f(t) = g(t) + q(t)$, $t \in \mathbb{R}$, where $g(\cdot)$ is almost periodic and $q(\cdot) \in PAP_0(\mathbb{R}, X, \rho_1, \rho_2)$.

(ii) A function $f(\cdot,\cdot) \in C_b(\mathbb{R} \times Y : X)$ is said to be weighted pseudo-almost periodic iff it admits a decomposition $f(t,y) = g(t,y) + q(t,y)$, $t \in \mathbb{R}$, where $g(\cdot,\cdot)$ is almost periodic and $q(\cdot,\cdot) \in PAP_0(\mathbb{R} \times Y, X, \rho_1, \rho_2)$.

Denote by $WPAP(\mathbb{R}, X, \rho_1, \rho_2)$ (respectively, $WPAP(\mathbb{R} \times Y, X, \rho_1, \rho_2)$) the vector spaces of such functions. If $\rho = \rho_1 = \rho_2$, then we write $WPAP(\mathbb{R}, X, \rho_1, \rho_2) \equiv WPAP(\mathbb{R}, X, \rho)$ and $WPAP(\mathbb{R} \times Y, X, \rho_1, \rho_2) \equiv WPAP(\mathbb{R} \times Y, X, \rho)$.

The class of weighted pseudo-almost automorphic functions has been introduced by J. Blot, G. M. Mophou, G. M. N'Guérékata and D. Pennequin in [8]. The following slight generalization is introduced by T. Diagana [19]-[20] (see also T. Diagana, K. Ezzinbi, M. Miraoui [21] and S. Abbas, V. Kavitha, R. Murugesu [2]):

Definition 3.2. (i) A function $f \in C_b(\mathbb{R} : X)$ is said to be weighted pseudo-almost automorphic iff it admits a decomposition $f(t) = g(t) + q(t)$, $t \in \mathbb{R}$, where $g(\cdot)$ is almost automorphic and $q(\cdot) \in PAP_0(\mathbb{R}, X, \rho_1, \rho_2)$.

(ii) A function $f(\cdot,\cdot) \in C_b(\mathbb{R} \times Y : X)$ is said to be weighted pseudo-almost automorphic iff it admits a decomposition $f(t,y) = g(t,y) + q(t,y)$, $t \in \mathbb{R}$, where $g(\cdot,\cdot)$ is almost automorphic and $q(\cdot,\cdot) \in PAP_0(\mathbb{R} \times Y, X, \rho_1, \rho_2)$.

Denote by $WPAA(\mathbb{R}, X, \rho_1, \rho_2)$ (respectively, $WPAA(\mathbb{R} \times Y, X, \rho_1, \rho_2)$) the vector spaces of such functions. If $\rho = \rho_1 = \rho_2$, then we write $WPAA(\mathbb{R}, X, \rho_1, \rho_2) \equiv WPAA(\mathbb{R}, X, \rho)$ and $WPAA(\mathbb{R} \times Y, X, \rho_1, \rho_2) \equiv WPAA(\mathbb{R} \times Y, X, \rho)$.

It is well-known that, for every $\rho_1,\ \rho_2 \in \mathbb{U}_T$, we have that $WPAP(\mathbb{R}, X, \rho_1, \rho_2)$ and $WPAA(\mathbb{R}, X, \rho_1, \rho_2)$ are Banach spaces with the sup-norm. For the Stepanov class, we will use the following definition from [2] (see also Z. Xia, M. Fan [45] for the case that $\rho_1 = \rho_2$):

Definition 3.3. Let $1 \leq p < \infty$.

(i) A Stepanov p-bounded function $f(\cdot)$ is said to be weighted Stepanov p-pseudo almost periodic (automorphic) iff it admits a decomposition $f(t) = g(t) + q(t)$, $t \in \mathbb{R}$, where $g(\cdot)$ is S^p-almost periodic (automorphic) and $q(\cdot) \in L^p_{loc}(\mathbb{R} : X)$ satisfies

$$\lim_{T \to +\infty} \frac{1}{2\int_{-T}^{T} \rho_1(t)\, dt} \int_{-T}^{T} \left[\int_t^{t+1} \|q(s)\|^p\, ds \right]^{1/p} \rho_2(t)\, dt = 0. \tag{3.1}$$

Denote by $S^pWPAP(\mathbb{R}, X, \rho_1, \rho_2)$ ($S^pWPAA(\mathbb{R}, X, \rho_1, \rho_2)$) the vector space of such functions, and by $S^pWPAA_0(\mathbb{R}, X, \rho_1, \rho_2)$ the vector space of locally p-integrable X-valued functions $q(\cdot)$ such that (3.1) holds.

(ii) A function $f : \mathbb{R} \times Y \to X$ is said to be weighted S^p-pseudo almost periodic (automorphic) iff for each $y \in Y$ we have that $f(\cdot, y) \in L^p_{loc}(\mathbb{R} : X)$ and $f(\cdot, \cdot)$ admits a decomposition $f(t, y) = g(t, y) + q(t, y)$, $t \in \mathbb{R}$, where $g(\cdot, \cdot)$ is S^p-almost periodic (automorphic) and

$$\lim_{T\to+\infty} \frac{1}{2\int_{-T}^{T} \rho_1(t)\,dt} \int_{-T}^{T} \left[\int_t^{t+1} \|q(s,y)\|^p\,ds \right]^{1/p} \rho_2(t)\,dt = 0, \tag{3.2}$$

uniformly on bounded subsets of Y.

Denote by $S^pWPAP(\mathbb{R} \times Y, X, \rho_1, \rho_2)$ ($S^pWPAA(\mathbb{R} \times Y, X, \rho_1, \rho_2)$) the vector space of such functions, and by $S^pWPAA_0(\mathbb{R} \times Y, X, \rho_1, \rho_2)$ the vector space of functions satisfying that for each $y \in Y$ one has that $q(\cdot)$ is a locally p-integrable X-valued function and (3.2) holds. If $\rho = \rho_1 = \rho_2$, then we write $S^pWPAP(\mathbb{R} \times Y, X, \rho_1, \rho_2) \equiv S^pWPAP(\mathbb{R} \times Y, X, \rho)$ ($S^pWPAA(\mathbb{R} \times Y, X, \rho_1, \rho_2) \equiv S^pWPAA(\mathbb{R} \times Y, X, \rho)$).

Several illustrative examples of weighted S^p-pseudo almost automorphic functions can be found in [2].

Denote by $\mathbb{V}_\infty$ the collection of all weighted ρ_1, $\rho_2 \in \mathbb{U}_\infty$ such that

$$\limsup_{T\to+\infty} \frac{\rho_2(t+\tau)}{\rho_2(t)} < \infty \text{ for any } \tau \in \mathbb{R}, \text{ and } \limsup_{T\to+\infty} \frac{\int_{-T}^{T} \rho_1(t)\,dt}{\int_{-T}^{T} \rho_2(t)\,dt} < \infty.$$

Then, owing to [2, Theorem 2.1], we have that $S^pWPAA(\mathbb{R} \times Y, X, \rho_1, \rho_2)$ is a Banach space endowed with the Stepanov norm $\| \cdot \|_{S^p}$, for any $p \in [1, \infty)$. A similar statement holds for the space $S^pWPAP(\mathbb{R} \times Y, X, \rho_1, \rho_2)$.

We introduce the notions of a weighted Weyl p-pseudo almost periodic (automorphic) function and a weighted Besicovitch p-pseudo almost periodic (automorphic) function as follows:

Definition 3.4. Let $1 \leq p < \infty$.

(i) A function $f \in L^p_{loc}(\mathbb{R} : X)$ is said to be weighted Weyl p-pseudo almost periodic (automorphic) iff it admits a decomposition $f(t) = g(t) + q(t)$, $t \in \mathbb{R}$, where $g(\cdot)$ is W^p-almost periodic (automorphic) and $q(\cdot) \in L^p_{loc}(\mathbb{R} : X)$ satisfies

$$\lim_{T\to+\infty} \frac{1}{2\int_{-T}^{T} \rho_1(t)\,dt} \int_{-T}^{T} \left[\lim_{l\to+\infty} \frac{1}{2l} \int_{t-l}^{t+l} \|q(s)\|^p\,ds \right]^{1/p} \rho_2(t)\,dt = 0. \tag{3.3}$$

Denote by $W^pWPAA(\mathbb{R}, X, \rho_1, \rho_2)$ ($W^pWPAA(\mathbb{R}, X, \rho_1, \rho_2)$) the collection of such functions, and by $W^pWPAA_0(\mathbb{R}, X, \rho_1, \rho_2)$ the collection of locally p-integrable X-valued functions $q(\cdot)$ such that (3.3) holds.

(ii) A function $f \in L^p_{loc}(\mathbb{R} : X)$ is said to be weighted Besicovitch p-pseudo almost periodic (automorphic) iff it admits a decomposition $f(t) = g(t) + q(t)$, $t \in \mathbb{R}$, where $g(\cdot)$ is B^p-almost periodic (automorphic) and $q(\cdot) \in L^p_{loc}(\mathbb{R} : X)$ satisfies

$$\lim_{T\to+\infty} \frac{1}{2\int_{-T}^{T} \rho_1(t)\,dt} \int_{-T}^{T} \left[\limsup_{l\to+\infty} \frac{1}{2l} \int_{t-l}^{t+l} \|q(s)\|^p\,ds \right]^{1/p} \rho_2(t)\,dt = 0. \tag{3.4}$$

Denote by $B^pWPAP(\mathbb{R},X,\rho_1,\rho_2)$ ($B^pWPAA(\mathbb{R},X,\rho_1,\rho_2)$) the collection of such functions, and by $B^pWPAA_0(\mathbb{R},X,\rho_1,\rho_2)$ the collection of locally p-integrable X-valued functions $q(\cdot)$ such that (3.4) holds.

We will not introduce here the notions of two-parameter weighted Weyl p-pseudo almost periodic (automorphic) functions and two-parameter weighted Besicovitch p-pseudo almost periodic (automorphic) functions since these notions can be very difficultly applied in the analysis of abstract semilinear Cauchy inclusions. It is easily seen that $B^pWPAA(\mathbb{R},X,\rho_1,\rho_2)$ is a vector space, as well as that $W^pWPAA(\mathbb{R},X,\rho_1,\rho_2) \subseteq B^pWPAA(\mathbb{R},X,\rho_1,\rho_2)$. Similar statements hold in the case of consideration of weighted pseudo-almost periodicity.

Concerning the convolution invariance of the space $B^pWPAA_0(\mathbb{R},X,\rho_1,\rho_2)$, we have the following result with $p = 1$:

Proposition 3.5. *Let $q(\cdot) \in L^1_{loc}(\mathbb{R} : X)$ be S^1-bounded, let $g \in L^1(\mathbb{R})$, and let $q \in B^1WPAA_0(\mathbb{R},X,\rho_1,\rho_2)$. Then $g * q \in B^1WPAA_0(\mathbb{R},X,\rho_1,\rho_2)$.*

Proof. First of all, let us recall the well-known fact that

$$\limsup_{l\to+\infty} \frac{1}{2l}\int_{t-l}^{t+l} \|q(s)\|\,ds = \limsup_{l\to+\infty} \frac{1}{2l}\int_{t-l-r}^{t+l-r} \|q(s)\|\,ds, \quad r \in \mathbb{R}. \tag{3.5}$$

Put $f_{l,t}(r) := \frac{|g(r)|}{2l}\int_{t-l-r}^{t+l-r} \|q(s)\|\,ds$, $l \geq 1$, t, $r \in \mathbb{R}$. By the S^1-boundedness of $q(\cdot)$, we get the existence of a function $G_{l,t}(\cdot) \in L^1(\mathbb{R})$ such that $f_{l,t}(r) \leq G_{l,t}(r)$ for a.e. $r \in \mathbb{R}$. Hence, we can apply the reverse Fatou's lemma in order to see that

$$\begin{aligned}\limsup_{l\to+\infty} &\int_{-\infty}^{\infty}\left[\frac{1}{2l}\int_{t-l-r}^{t+l-r} \|q(s)\|\,ds\right]|g(r)|\,dr \\ &\leq \int_{-\infty}^{\infty} \limsup_{l\to+\infty}\left[\frac{1}{2l}\int_{t-l-r}^{t+l-r} \|q(s)\|\,ds\right]|g(r)|\,dr.\end{aligned} \tag{3.6}$$

Then the final conclusion follows by using (3.5)-(3.6), the validity of (3.4) for $q(\cdot)$, and the following integral computation with Fubini theorem:

$$
\begin{aligned}
&\frac{1}{2\int_{-T}^{T}\rho_1(t)\,dt}\int_{-T}^{T}\Biggl[\limsup_{l\to+\infty}\frac{1}{2l}\int_{t-l}^{t+l}\Biggl\|\int_{-\infty}^{\infty}g(s-r)q(r)\,dr\Biggr\|ds\Biggr]\rho_2(t)\,dt\\
&\leq \frac{1}{2\int_{-T}^{T}\rho_1(t)\,dt}\int_{-T}^{T}\Biggl[\limsup_{l\to+\infty}\frac{1}{2l}\int_{t-l}^{t+l}\int_{-\infty}^{\infty}|g(r)|\|q(s-r)\|\,dr\,ds\Biggr]\rho_2(t)\,dt\\
&= \frac{1}{2\int_{-T}^{T}\rho_1(t)\,dt}\int_{-T}^{T}\Biggl[\limsup_{l\to+\infty}\int_{-\infty}^{\infty}\frac{1}{2l}\int_{t-l}^{t+l}|g(r)|\|q(s-r)\|\,ds\,dr\Biggr]\rho_2(t)\,dt\\
&= \frac{1}{2\int_{-T}^{T}\rho_1(t)\,dt}\int_{-T}^{T}\Biggl[\limsup_{l\to+\infty}\int_{-\infty}^{\infty}\frac{1}{2l}\int_{t-l-r}^{t+l-r}\|q(s)\|\,ds\,|g(r)|\,dr\Biggr]\rho_2(t)\,dt\\
&\leq \frac{1}{2\int_{-T}^{T}\rho_1(t)\,dt}\int_{-T}^{T}\int_{-\infty}^{\infty}\limsup_{l\to+\infty}\Biggl[\frac{1}{2l}\int_{t-l-r}^{t+l-r}\|q(s)\|\,ds\Biggr]|g(r)|\,dr\,dt\\
&= \frac{1}{2\int_{-T}^{T}\rho_1(t)\,dt}\int_{-T}^{T}\int_{-\infty}^{\infty}\limsup_{l\to+\infty}\Biggl[\frac{1}{2l}\int_{t-l}^{t+l}\|q(s)\|\,ds\Biggr]|g(r)|\,dr\,dt\\
&= \frac{\int_{-\infty}^{\infty}|g(r)|\,dr}{2\int_{-T}^{T}\rho_1(t)\,dt}\int_{-T}^{T}\Biggl[\limsup_{l\to+\infty}\frac{1}{2l}\int_{t-l}^{t+l}\|q(s)\|\,ds\Biggr]\rho_2(t)\,dt.
\end{aligned}
$$

The convolution and translation invariance of (double-)weighted pseudo almost-periodic functions and some other problems for this class have been investigated by T. Diagana [19]-[20], D. Ji, Ch. Zhang [28] and A. Coronel, M. Pinto and D. Sepúlveda [12]. The translation invariance of $PAP(\mathbb{R},X,\rho)$ is ensured e.g. by the validity of condition $\sup_{r>0}\sup_{t\in\Omega_{r,s}}\frac{\rho(t+s)}{\rho(t)}<\infty$, $s\in\mathbb{R}$; cf. [12, Theorem 3.7 (b)]. In the following proposition, we provide a slightly different condition ensuring the translation invariance of the space $PAP_0(\mathbb{R},X,\rho_1,\rho_2)$ (see [33] for a proof):

Proposition 3.6. *Let* $\rho_1,\ \rho_2\in\mathbb{U}_\infty$. *Then the space* $PAP_0(\mathbb{R},X,\rho_1,\rho_2)$ *is translation invariant if* $\lim_{T\to+\infty}\frac{|\int_{-T-s}^{-T}\rho_2(t)\,dt|+|\int_{T}^{T-s}\rho_2(t)\,dt|}{\int_{-T}^{T}\rho_1(t)\,dt}=0$, $s\in\mathbb{R}$ *and there exists a function* $g:\mathbb{R}\to(0,\infty)$ *such that* $\rho_2(t-s)\leq g(s)\rho_2(t)$, $t,\ s\in\mathbb{R}$.

In order to analyze generalized weighted pseudo-almost automorphic solutions of semilinear (fractional) Cauchy inclusions, we need to repeat some known facts about composition principles for the classes of weighted pseudo-almost automorphic functions and Stepanov weighted pseudo-almost automorphic functions (weighted pseudo-almost periodic solutions can be analyzed in a similar fashion; see e.g. [20, Theorem 5.9] and [48, Theorem 3.1, Theorem 3.5].

3.1 Composition Principles for Weighted Pseudo-Almost Automorphic Solutions

In [20, Theorem 5.8], T. Diagana has proved a composition principle for the class $WPAA(\mathbb{R}\times Y,X,\rho_1,\rho_2)$. Arguing as in the proof of [45, Theorem 3.6, Theorem 3.7], we can prove the following composition principles, stated here with two generally different pivot spaces (see also [2, Theorem 2.2, Theorem 2.3, Theorem 2.4]).

Theorem 3.7. *Assume that* $\rho_1,\ \rho_2 \in \mathbb{U}_\infty$, $1 \le p < \infty$, $f : \mathbb{R}\times Y \to X$ *is weighted* S^p*-pseudo almost automorphic,* $f(t,y) = g(t,y) + q(t,y)$, $t \in \mathbb{R}$, *where* $g(\cdot,\cdot)$ *is* S^p*-almost automorphic and* $q(\cdot,\cdot)$ *satisfies* (3.2)*, uniformly on bounded subsets of Y. Let* $h : \mathbb{R} \to Y$ *be weighted* S^p*-pseudo almost automorphic,* $f(t) = g(t) + q(t)$, $t \in \mathbb{R}$, *where* $g(\cdot)$ *is* S^p*-almost automorphic with relatively compact range in Y, and* $q(\cdot) \in L^p_{loc}(\mathbb{R} : X)$ *satisfies* (3.1)*. Assume that there exist two finite constants* $L_f > 0$ *and* $L_g > 0$ *such that*

$$\|f(t,y) - f(t,z)\| \le L_f\|y - z\|_Y \text{ and } \|g(t,y) - g(t,z)\| \le L_g\|y - z\|_Y \tag{3.7}$$

for all $t \in \mathbb{R},\ y,\ z \in Y$*. Then* $f(\cdot,h(\cdot)) \in S^pWPAA(\mathbb{R}\times Y,X,\rho_1,\rho_2)$.

Theorem 3.8. *Assume that* $\rho_1,\ \rho_2 \in \mathbb{U}_\infty$, $1 < p < \infty$, $f : \mathbb{R}\times Y \to X$ *is weighted* S^p*-pseudo almost automorphic,* $f(t,y) = g(t,y) + q(t,y)$, $t \in \mathbb{R}$, *where* $g(\cdot,\cdot)$ *is* S^p*-almost automorphic and* $q(\cdot,\cdot)$ *satisfies* (3.2)*, uniformly on bounded subsets of Y. Let* $h : \mathbb{R} \to Y$ *be weighted* S^p*-pseudo almost automorphic,* $f(t) = g(t) + q(t)$, $t \in \mathbb{R}$, *where* $g(\cdot)$ *is* S^p*-almost automorphic with relatively compact range in Y, and* $q(\cdot) \in L^p_{loc}(\mathbb{R} : X)$ *satisfies* (3.1)*. Assume that* $r \ge \max(p, p/p-1)$ *and there exist two Stepanov r-almost automorphic scalar-valued functions* $L_f(\cdot)$ *and* $L_g(\cdot)$ *such that*

$$\|f(t,y) - f(t,z)\| \le L_f(t)\|y - z\|_Y \text{ and } \|g(t,y) - g(t,z)\| \le L_g(t)\|y - z\|_Y, \tag{3.8}$$

for all $t \in \mathbb{R},\ y,\ z \in Y$*. Set* $q := pr/p + r$*. Then* $q \in [1, p)$ *and* $f(\cdot,h(\cdot)) \in S^qWPAA(\mathbb{R}\times Y,X,\rho_1,\rho_2)$.

4 Generalized Weighted Almost Periodic (Automorphic) Properties of Convolution Products

We start this section by stating the following result:

Proposition 4.1. *Suppose that* $1 \le p < \infty$, $1/p + 1/q = 1$ *and* $(R(t))_{t>0} \subseteq L(X)$ *is a strongly continuous operator family satisfying that* $M := \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} < \infty$*. If the space* $PAP_0(\mathbb{R},X,\rho_1,\rho_2)$ *is translation invariant (see Proposition 3.6) and* $f : \mathbb{R} \to X$ *is weighted* S^p*-almost periodic, resp. weighted* S^p*-almost automorphic, then the function* $F(\cdot)$*, given by*

$$F(t) := \int_{-\infty}^{t} R(t-s)f(s)\,ds, \quad t \in \mathbb{R}, \tag{4.1}$$

is well-defined and belongs to the class $AP(\mathbb{R},X,\rho_1,\rho_2) + S^pWPAA_0(\mathbb{R},X,\rho_1,\rho_2)$*, resp.,* $AA(\mathbb{R},X,\rho_1,\rho_2) + S^pWPAA_0(\mathbb{R},X,\rho_1,\rho_2)$.

Proof. We will prove the theorem only for weighted S^p-almost automorphy. Let $f(\cdot) = g(\cdot) + q(\cdot)$, where $g(\cdot)$ and $q(\cdot)$ satisfy conditions from Definition 3.3(i). By [35, Proposition 5], and S^p-almost automorphy of $g(\cdot)$, we have that the function $G(\cdot)$ obtained by replacing $f(\cdot)$ in (4.2) by $g(\cdot)$, is almost automorphic. Define $Q_k(t) := \int_k^{k+1} R(s)q(t-s)\,ds$, $t \in \mathbb{R}$ ($k \in$

$\mathbb{N}$). Arguing as in the proof of afore-mentioned proposition, we can prove that $Q_k(\cdot)$ is bounded and continuous on $\mathbb{R}$ for all $k \in \mathbb{N}$, as well as that $Q_k(\cdot)$ converges uniformly to $Q(\cdot) := \int_{-\infty}^{\cdot} R(\cdot - s)q(s)\,ds$. Therefore, all we need to prove is that, for any integer $k \in \mathbb{N}$ given in advance, (3.1) holds with the function $q(\cdot)$ replaced therein by $Q_k(\cdot)$. By the Hölder inequality and an elementary change of variables in double integral, we have the existence of a positive finite constant $c_k > 0$ such that:

$$\frac{1}{2\int_{-T}^{T}\rho_1(t)\,dt}\int_{-T}^{T}\Biggl[\int_t^{t+1}\|Q_k(s)\|^p\,ds\Biggr]^{1/p}\rho_2(t)\,dt$$
$$\leq \frac{\|R(\cdot)\|_{L^q[k,k+1]}}{2\int_{-T}^{T}\rho_1(t)\,dt}\int_{-T}^{T}\Biggl[\int_t^{t+1}\int_k^{k+1}\|q(s-v)\|^p\,dv\,ds\Biggr]^{1/p}\rho_2(t)\,dt$$
$$= \frac{\|R(\cdot)\|_{L^q[k,k+1]}}{2\int_{-T}^{T}\rho_1(t)\,dt}\int_{-T}^{T}\Biggl[\int_t^{t+1}\int_{s-k}^{s-(k+1)}\|q(v)\|^p\,dv\,ds\Biggr]^{1/p}\rho_2(t)\,dt$$
$$\leq \frac{\|R(\cdot)\|_{L^q[k,k+1]}}{2\int_{-T}^{T}\rho_1(t)\,dt}\int_{-T}^{T}\Biggl[\int_{t-(k+1)}^{t-k}\int_t^{r+(k+1)}\|q(r)\|^p\,ds\,dr\Biggr]^{1/p}\rho_2(t)\,dt$$

$$+ \frac{\|R(\cdot)\|_{L^q[k,k+1]}}{2\int_{-T}^{T}\rho_1(t)\,dt}\int_{-T}^{T}\Biggl[\int_{t-(k-1)}^{t-k}\int_{t+1}^{r-k}\|q(r)\|^p\,ds\,dr\Biggr]^{1/p}\rho_2(t)\,dt$$
$$\leq \frac{\|R(\cdot)\|_{L^q[k,k+1]}}{2\int_{-T}^{T}\rho_1(t)\,dt}\int_{-T}^{T}\Biggl[\int_{t-(k+1)}^{t-k}|r+k+1-t|\|q(r)\|^p\,dr\Biggr]^{1/p}\rho_2(t)\,dt$$
$$+ \frac{\|R(\cdot)\|_{L^q[k,k+1]}}{2\int_{-T}^{T}\rho_1(t)\,dt}\int_{-T}^{T}\Biggl[\int_{t-(k-1)}^{t-k}|t+1-r+k|\|q(r)\|^p\,ds\,dr\Biggr]^{1/p}\rho_2(t)\,dt$$
$$\leq \frac{c_k\|R(\cdot)\|_{L^q[k,k+1]}}{2\int_{-T}^{T}\rho_1(t)\,dt}\int_{-T}^{T}\Biggl[\int_{t-(k+1)}^{t-k}\|q(r)\|^p\,dr\Biggr]^{1/p}\rho_2(t)\,dt$$
$$+ \frac{c_k\|R(\cdot)\|_{L^q[k,k+1]}}{2\int_{-T}^{T}\rho_1(t)\,dt}\int_{-T}^{T}\Biggl[\int_{t-(k-1)}^{t-k}\|q(r)\|^p\,ds\,dr\Biggr]^{1/p}\rho_2(t)\,dt$$
$$= \frac{c_k\|R(\cdot)\|_{L^q[k,k+1]}}{2\int_{-T}^{T}\rho_1(t)\,dt}\int_{-T}^{T}\Biggl[\int_{t+)}^{t}\|q(r-(k+1))\|^p\,dr\Biggr]^{1/p}\rho_2(t)\,dt$$
$$+ \frac{c_k\|R(\cdot)\|_{L^q[k,k+1]}}{2\int_{-T}^{T}\rho_1(t)\,dt}\int_{-T}^{T}\Biggl[\int_t^{t+1}\|q(r-(k-1))\|^p\,ds\,dr\Biggr]^{1/p}\rho_2(t)\,dt, \quad T > 0.$$

Now the final conclusion follows from the fact that (3.1) holds with the function $q(\cdot)$ replaced therein by $Q_k(\cdot)$ and the translation invariance of $PAP_0(\mathbb{R}, X, \rho_1, \rho_2)$.

Results for weighted Besicovitch almost automorphic functions are rather restrictive (cf. [33] for more details).

In the sequel, we will use the following notions. Set $\mathbb{U}_p := \{\rho \in L^1_{loc}([0,\infty)) : \rho(t) > 0 \text{ a.e. } t \geq 0\}$, $\mathbb{U}_{b,p} := \{\rho \in L^\infty([0,\infty)) : \rho(t) > 0 \text{ a.e. } t \geq 0\}$ and $\mathbb{U}_{\infty,p} := \{\rho \in \mathbb{U} : \nu(T,\rho) :=$

$\lim_{T\to+\infty}\int_0^T\rho(t)\,dt=\infty\}$. Then $\mathbb{U}_{b,p}\subseteq\mathbb{U}_{\infty,p}\subseteq\mathbb{U}_p$. If $\rho_1,\ \rho_2\in\mathbb{U}_{\infty,p}$, then we define the space $PAP_0([0,\infty),X,\rho_1,\rho_2)$ as the set of all functions $f\in C_b([0,\infty):X)$ satisfying that $\lim_{T\to+\infty}\frac{1}{\int_0^T\rho_1(t)\,dt}\int_0^T\|f(t)\|\rho_2(t)\,dt=0$ and the space $PAP_0([0,\infty)\times Y,X,\rho_1,\rho_2)$ as the set of all functions $f\in C_b([0,\infty)\times Y:X)$ satisfying that $\lim_{T\to+\infty}\frac{1}{\int_0^T\rho_1(t)\,dt}\int_0^T\|f(t,y)\|\rho_2(t)\,dt=0$, uniformly on bounded subsets of Y; see also [7] and [21]. Concerning the invariance of space $PAP_0([0,\infty),X,\rho_1,\rho_2)$ under the action of finite convolution product, we have the following result:

Proposition 4.2. *Assume that there exists a non-negative measurable function $g:[0,\infty)\to[0,\infty)$ such that $\rho_2(t)\le g(s)\rho_2(t-s)$ for $0\le s\le t<\infty$. Assume that $(R(t))_{t>0}\subseteq L(X)$ is a strongly continuous operator family satisfying $\int_0^\infty(1+g(s))\|R(s)\|\,ds<\infty$. Let $f\in PAP_0([0,\infty),X,\rho_1,\rho_2)$. Define $F(t):=\int_0^t R(t-s)f(s)\,ds$, $t\ge 0$. Then we have $F\in PAP_0([0,\infty),X,\rho_1,\rho_2)$.*

Proof. It can be easily seen that $F\in C_b([0,\infty):X)$. The claimed statement follows from the prescribed assumptions and the next computation:

$$\begin{aligned}
&\frac{1}{\int_0^T\rho_1(t)\,dt}\int_0^T\|F(t)\|\rho_2(t)\,dt\le\frac{1}{\int_0^T\rho_1(t)\,dt}\int_0^T\left[\int_0^t\|R(s)\|\|f(t-s)\|\,ds\right]\rho_2(t)\,dt\\
&=\frac{1}{\int_0^T\rho_1(t)\,dt}\int_0^T\int_s^T\|R(s)\|\|f(t-s)\|\rho_2(t)\,dt\,ds\\
&\le\frac{1}{\int_0^T\rho_1(t)\,dt}\int_0^T g(s)\|R(s)\|\left[\int_s^T\|f(t-s)\|\rho_2(t-s)\,dt\right]ds\\
&=\frac{1}{\int_0^T\rho_1(t)\,dt}\int_0^T g(s)\|R(s)\|\left[\int_0^{T-s}\|f(r)\|\rho_2(r)\,dr\right]ds\\
&\le\left[\int_0^\infty g(s)\|R(s)\|\,ds\right]\cdot\left[\frac{1}{\int_0^T\rho_1(t)\,dt}\int_0^T\|f(r)\|\rho_2(r)\,dr\right],\quad T>0.
\end{aligned}$$

Remark 4.3. Assume, in place of condition $\int_0^\infty(1+g(s))\|R(s)\|\,ds<\infty$, that $1\le p<\infty$, $1/p+1/q=1$, $M=\sum_{k=0}^\infty\|R(\cdot)\|_{L^q[k,k+1]}<\infty$ and $g(\cdot)$ additionally satisfies that it is S^p-bounded. Then we can easily shiown that $F\in PAP_0([0,\infty),X,\rho_1,\rho_2)$.

Combining Proposition 4.1, Proposition 4.2-Remark 4.3 and the argumentation contained in the proof of [31, Proposition 2.13], we can clarify the following proposition (weighted S^p-almost periodic case can be considered similarly):

Proposition 4.4. *Assume that $1\le p<\infty$, $1/p+1/q=1$, and there exists a S^p-bounded function $g:[0,\infty)\to[0,\infty)$ such that $\rho_2(t)\le g(s)\rho_2(t-s)$ for $0\le s\le t<\infty$. Assume, further, that $(R(t))_{t>0}\subseteq L(X)$ is a strongly continuous operator family satisfying that for each $s\ge 0$ we have $M_s:=\sum_{k=0}^\infty\|R(\cdot)\|_{L^q[s+k,s+k+1]}<\infty$, as well as that the space $PAA_0(\mathbb{R},X,\rho_1,\rho_2)$ is translation invariant and $g:\mathbb{R}\to X$ is weighted S^p-almost automorphic. If $q\in PAP_0([0,\infty),X,\rho_1,\rho_2)$, then the function $\mathbf{F}(\cdot)$, given by*

$$\mathbf{F}(t):=\int_0^\infty R(t-s)[g(s)+q(s)]\,ds,\quad t\ge 0,\tag{4.2}$$

is well-defined and belongs to the class

$$AA_{[0,\infty)}(\mathbb{R},X,\rho_1,\rho_2)+S^pWPAA_0^{[0,\infty)}(\mathbb{R},X,\rho_1,\rho_2) \\ +S_0^p([0,\infty):X)+PAP_0([0,\infty),X,\rho_1,\rho_2).$$

Here, $AA_{[0,\infty)}(\mathbb{R},X,\rho_1,\rho_2)$ *and* $S^pWPAA_0^{[0,\infty)}(\mathbb{R},X,\rho_1,\rho_2)$ *denote the spaces consisting of restrictions of functions belonging to* $AA(\mathbb{R},X,\rho_1,\rho_2)$ *and* $S^pWPAA_0(\mathbb{R},X,\rho_1,\rho_2)$ *to the non-negative real axis, respectively.*

5 Weighted Pseudo-Almost Automorphic Solutions of Semilinear (Fractional) Cauchy Inclusions

In this section, we will clarify a few results concerning the existence and uniqueness of weighted automorphic solutions of semilinear (fractional) Cauchy inclusions; the existence and uniqueness of weighted automorphic solutions can be analyzed similarly. Because of some obvious complications appearing in the study of existence and uniqueness of weighted pseudo-almost periodic (automorphic) solutions defined only for non-negative values of time t (see Proposition 4.4), henceforth we will restrict ourselves to the study of abstract Cauchy inclusions (1.2)-(1.3), only. The organization of section is very similar to that of [35, Section 5]; the proofs of structural results are omitted since they can be deduced by using composition principles from Subsection 3.1 and the argumentation contained in [33].

We deal with the class of multivalued linear operators $\mathcal{A}$ satisfying the condition [24, (P), p. 47] introduced by A. Favini and A. Yagi:

(P) There exist finite constants $c,\ M>0$ and $\beta\in(0,1]$ such that

$$\Psi:=\Big\{\lambda\in\mathbb{C}:\Re\lambda\geq -c(|\Im\lambda|+1)\Big\}\subseteq\rho(\mathcal{A})\text{ and }\|R(\lambda:\mathcal{A})\|\leq M(1+|\lambda|)^{-\beta},\quad \lambda\in\Psi.$$

We define the fractional power $(-\mathcal{A})^{\theta}$ for $\theta>\beta-1$ as usually. Set $Y:=[D((-\mathcal{A})^{\theta})]$ and $\|\cdot\|_Y:=\|\cdot\|_{[D((-\mathcal{A})^{\theta})]}$; then Y is a Banach space that is continuously embedded in X. Define, further,

$$T_\nu(t)x:=\frac{1}{2\pi i}\int_\Gamma(-\lambda)^{\nu}e^{\lambda t}(\lambda-\mathcal{A})^{-1}x\,d\lambda,\quad x\in X,\ t>0\ (\nu>0),$$

where Γ denotes the upwards oriented curve $\lambda=-c(|\eta|+1)+i\eta$ $(\eta\in\mathbb{R})$. Then there exists a finite constant $M>0$ such that:

(A) $\|T_\nu(t)\|\leq Me^{-ct}t^{\beta-\nu-1},\ t>0,\ \nu>0.$

Assume that $L_f(\cdot)$ is a locally bounded non-negative function, and M denotes the constant from (A), with $\nu=\theta$. Set, for every $n\in\mathbb{N}$,

$$M_n:=M^n\sup_{t\in\mathbb{R}}\int_{-\infty}^{t}\int_{-\infty}^{x_n}\cdots\int_{-\infty}^{x_2}e^{-c(t-x_n)}(t-x_n)^{\beta-\theta-1} \\ \times\prod_{i=2}^{n}e^{-c(x_i-x_{i-1})}(x_i-x_{i-1})^{\beta-\theta-1}\prod_{i=1}^{n}L_f(x_i)\,dx_1\,dx_2\cdots dx_n. \qquad (5.1)$$

Define

$$T_{\gamma,\nu}(t)x := t^{\gamma\nu}\int_0^\infty s^\nu \Phi_\gamma(s)T_0(st^\gamma)x\,ds, \quad t>0,\ x\in X,\ \nu>-\beta,$$

$$S_\gamma(t) := T_{\gamma,0}(t) \text{ and } P_\gamma(t) := \gamma T_{\gamma,1}(t)/t^\gamma, \quad t>0,$$

where $\Phi_\gamma(\cdot)$ denotes the famous Wright function, defined by

$$\Phi_\gamma(z) := \sum_{n=0}^{\infty}\frac{(-z)^n}{n!\Gamma(1-\gamma-\gamma n)}, \quad z\in\mathbb{C}.$$

Set also

$$R_\gamma(t) := t^{\gamma-1}P_\gamma(t),\ t>0 \text{ and } R_\gamma^\theta(t) := \gamma t^{\gamma-1}\int_0^\infty s\Phi_\gamma(s)T_\theta(st^\gamma)x\,ds,\ t>0,\ x\in X.$$

Consider the first inequality in (3.8) with $L_f(\cdot)$ being a measurable non-negative function. Set, for every $n\in\mathbb{N}$,

$$\begin{aligned} B_n := \sup_{t\geq 0}\int_{-\infty}^t\int_{-\infty}^{x_n}\cdots\int_{-\infty}^{x_2}\left\|R_\gamma^\theta(t-x_n)\right\| \\ \times\prod_{i=2}^n\left\|R_\gamma^\theta(x_i-x_{i-1})\right\|\prod_{i=1}^n L_f(x_i)\,dx_1\,dx_2\cdots dx_n. \end{aligned}$$

If $(Z,\|\cdot\|_Z)$ is a complex Banach space that is continuously embedded in X, then we use the following notion of a mild solution of (1.2), resp., (1.3):

Definition 5.1. Assume that $f: I\times Z\to X$. By a mild solution of (1.2), we mean any Z-continuous function $u(\cdot)$ such that $u(t)=(\Lambda u)(t)$, $t\in\mathbb{R}$, where

$$t\mapsto(\Lambda u)(t) := \int_{-\infty}^t T(t-s)f(s,u(s))\,ds,\ t\in\mathbb{R}.$$

Definition 5.2. Assume that $f: I\times Z\to X$. By a mild solution of (1.3), we mean any Z-continuous function $u(\cdot)$ such that $u(t)=(\Lambda_\gamma u)(t)$, $t\in\mathbb{R}$, where

$$t\mapsto(\Lambda_\gamma u)(t) := \int_{-\infty}^t (t-s)^{\gamma-1}P_\gamma(t-s)f(s,u(s))\,ds,\ t\in\mathbb{R}.$$

Suppose that $M>0$ denotes the constant from (A), and the sequence (M_n) is defined through (5.1). We will first state the following analogues of [33, Theorem 2.10.3-Theorem 2.10.4] and [33, Theorem 2.10.9-Theorem 2.10.10], which can be also clarified for weighted pseudo-almost periodicity.

Theorem 5.3. *Let $\rho_1,\ \rho_2\in\mathbb{U}_T$ and $1<p<\infty$. Suppose that* (P) *holds, $\beta>\theta>1-\beta$ and the following conditions hold:*

(i) $f:\mathbb{R}\times Y\to X$ is weighted S^p-pseudo almost automorphic, $f(t,y)=g(t,y)+q(t,y)$, $t\in\mathbb{R}$, where $g(\cdot,\cdot)$ is S^p-almost automorphic and $q(\cdot,\cdot)$ satisfies (3.2)*, uniformly on bounded subsets of Y.*

(ii) Assume that $r \geq \max(p, p/p-1)$, $r > p/p-1$ *and there exist two Stepanov r-almost automorphic scalar-valued functions* $L_f(\cdot)$ *and* $L_g(\cdot)$ *such that* (3.8) *holds. Set* $q := \frac{pr}{p+r}$ *and* $q' := \frac{pr}{pr-p-r}$.

If $q'(\beta-\theta-1) > -1$ *and* $M_n < 1$ *for some* $n \in \mathbb{N}$, *then there exists an weighted pseudo-almost automorphic mild solution of inclusion* (1.2). *The uniqueness of mild solutions holds in the case that* $\mathcal{A}$ *is single-valued.*

Theorem 5.4. *Let* $\rho_1,\ \rho_2 \in \mathbb{U}_T$ *and and* $1 < p < \infty$. *Suppose that* (P) *holds,* $\beta > \theta > 1-\beta$ *and the following conditions hold:*

(i) $f : \mathbb{R} \times Y \to X$ *is weighted* S^p*-pseudo almost automorphic,* $f(t,y) = g(t,y) + q(t,y)$, $t \in \mathbb{R}$, *where* $g(\cdot,\cdot)$ *is* S^p*-almost automorphic and* $q(\cdot,\cdot)$ *satisfies* (3.2), *uniformly on bounded subsets of Y.*

(ii) There exist two finite constants $L_f > 0$ *and* $L_g > 0$ *such that* (3.7) *holds.*

If $\frac{p}{p-1}(\beta-\theta-1) > -1$ *and* $M_n < 1$ *for some* $n \in \mathbb{N}$, *then there exists a weighted pseudo-almost automorphic mild solution of inclusion* (1.2). *The uniqueness of mild solutions holds provided that* $\mathcal{A}$ *is single-valued, additionally.*

Theorem 5.5. *Let* $\rho_1,\ \rho_2 \in \mathbb{U}_T$ *and* $1 < p < \infty$. *Suppose that* (P) *holds,* $\beta > \theta > 1-\beta$ *and the following conditions hold:*

(i) $f : \mathbb{R} \times Y \to X$ *is weighted* S^p*-pseudo almost automorphic,* $f(t,y) = g(t,y) + q(t,y)$, $t \in \mathbb{R}$, *where* $g(\cdot,\cdot)$ *is* S^p*-almost automorphic and* $q(\cdot,\cdot)$ *satisfies* (3.2), *uniformly on bounded subsets of Y.*

(ii) Assume that $r \geq \max(p, p/p-1)$, $r > p/p-1$ *and there exist two Stepanov r-almost automorphic scalar-valued functions* $L_f(\cdot)$ *and* $L_g(\cdot)$ *such that* (3.8) *holds. Set* $q := \frac{pr}{p+r}$ *and* $q' := \frac{pr}{pr-p-r}$.

If $q'(\gamma(\beta-\theta)-1) > -1$ *and* $B_n < 1$ *for some* $n \in \mathbb{N}$, *then there exists an weighted pseudo-almost automorphic mild solution of inclusion* (1.2). *The uniqueness of mild solutions holds in the case that* $\mathcal{A}$ *is single-valued.*

Theorem 5.6. *Let* $\rho_1,\ \rho_2 \in \mathbb{U}_T$ *and and* $1 < p < \infty$. *Suppose that* (P) *holds,* $\beta > \theta > 1-\beta$ *and the following conditions hold:*

(i) $f : \mathbb{R} \times Y \to X$ *is weighted* S^p*-pseudo almost automorphic,* $f(t,y) = g(t,y) + q(t,y)$, $t \in \mathbb{R}$, *where* $g(\cdot,\cdot)$ *is* S^p*-almost automorphic and* $q(\cdot,\cdot)$ *satisfies* (3.2), *uniformly on bounded subsets of Y.*

(ii) There exist two finite constants $L_f > 0$ *and* $L_g > 0$ *such that* (3.7) *holds.*

If $\frac{p}{p-1}(\gamma(\beta-\theta)-1) > -1$ *and* $B_n < 1$ *for some* $n \in \mathbb{N}$, *then there exists a weighted pseudo-almost automorphic mild solution of inclusion* (1.2). *The uniqueness of mild solutions holds provided that* $\mathcal{A}$ *is single-valued, additionally.*

The following theorem is an analogue of [33, Theorem 2.12.5]:

Theorem 5.7. *Assume that* ρ_1, $\rho_2 \in \mathbb{U}_T$, $1 \leq p < \infty$, *and the following conditions hold:*

(i) $f \in PAA(\mathbb{R}\times X, X, \rho_1, \rho_2)$ *is weighted pseudo-almost automorphic.*

(ii) The first inequality in (3.7) *holds with some bounded non-negative function* $L_f(\cdot)$.

(iii) $\sum_{n=1}^{\infty} M_n < \infty$.

Then there exists a unique weighted pseudo-almost automorphic solution of inclusion (1.2).

And, at the end of paper, a few words about situations in which we can apply our abstract results. As already mentioned in our previous researches, we can apply our results in the analysis of existence and uniqueness of weighted (asymptotically) almost automorphic solutions of the fractional semilinear Poisson heat equation

$$\begin{cases} D_{t,+}^{\gamma}[m(x)v(t,x)] = (\Delta - b)v(t,x) + f(t, m(x)v(t,x)), \ t \in \mathbb{R}, \ x \in \Omega; \\ v(t,x) = 0, \quad (t,x) \in [0,\infty)\times\partial\Omega. \end{cases}$$

in the space $X := L^p(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$, $b > 0$, $m(x) \geq 0$ a.e. $x \in \Omega$, $m \in L^{\infty}(\Omega)$, $\gamma \in (0,1)$ and $1 < p < \infty$. Furthermore, we can analyze the existence and uniqueness of asymptotically S^p-almost automorphic solutions of the following fractional damped Poisson-wave type equation in the spaces $X := H^{-1}(\Omega)$ or $X := L^p(\Omega)$:

$$\begin{cases} \mathbf{D}_t^{\gamma}(m(x)\mathbf{D}_t^{\gamma}u) + (2\omega m(x) - \Delta)\mathbf{D}_t^{\gamma}u + (A(x;D) - \omega\Delta + \omega^2 m(x))u(x,t) = f(x,t), \\ t \geq 0, \ x \in \Omega \ ; \ u = \mathbf{D}_t^{\gamma} = 0, \quad (x,t) \in \partial\Omega\times[0,\infty), \\ u(0,x) = u_0(x), \ m(x)[\mathbf{D}_t^{\gamma}u(x,0) + \omega u_0] = m(x)u_1(x), \quad x \in \Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded open domain with smooth boundary, $1 < p < \infty$, $m(x) \in L^{\infty}(\Omega)$, $m(x) \geq 0$ a.e. $x \in \Omega$, Δ is the Dirichlet Laplacian in $L^2(\Omega)$, acting with domain $H_0^1(\Omega) \cap H^2(\Omega)$, and $A(x;D)$ is a second order linear differential operator on Ω with coefficients continuous on $\overline{\Omega}$ and the Caputo fractional derivative $\mathbf{D}_t^{\gamma}$ is taken in a slightly weakened sense [32]; see [24, Example 6.1] and [35, Example 6.2] for more details.

As already mentioned in the introductory part, our results can be applied to almost sectorial operators so that we can examine the existence and uniqueness of weighted pseudo-almost automorphic solutions for certain classes of higher order (semilinear) elliptic differential equations in Hölder spaces (see e.g. W. von Wahl [44]). The main results of [2] can be also fomulated for inclusions.

In [26], G. M. N'Guérékata and M. Kostić have recently studied various classes of generalized almost periodic and generalized almost automorphic solutions of abstract multi-term fractional differential inclusions in Banach spaces. We close the paper with the observation that Proposition 4.4 can be applied in the qualitative analysis of solutions of certain classes of the abstract multi-term Cauchy inclusions.

References

[1] S. Abbas, A note on Weyl pseudo almost automorphic functions and their properties, *Math. Sci. (Springer)* **6:29** (2012), 5 pp, doi:10.1186/2251-7456-6-29.

[2] S. Abbas, V. Kavitha, and R. Murugesu, Stepanov-like weighted pseudo almost automorphic solutions to fractional order abstract integro-differential equations, *Proc. Indian Acad. Sci. (Math. Sci.)* **125** (2015), 323–351.

[3] J. Andres, A. M. Bersani, and R. F. Grande, Hierarchy of almost-periodic function spaces, *Rend. Mat. Appl.* **26** (2006), 121–188.

[4] E. Bazhlekova, Fractional Evolution Equations in Banach Spaces, *PhD Thesis, Eindhoven University of Technology, Eindhoven,* 2001.

[5] F. Bedouhene, N. Challali, O. Mellah, P. Raynaud de Fitte, and M. Smaali, Almost automorphy and various extensions for stochastic processes, *J. Math. Anal. Appl.* **429** (2015), 1113–1152.

[6] A. S. Besicovitch, Almost Periodic Functions, *Dover Publications Inc., New York,* 1954.

[7] J. Blot, P. Cieutat, K. Ezzinbi, New approach for weighted pseudo-almost periodic functions under the light of measure theory, basic results and applications, *Applicable Analysis* (2011), 1–34.

[8] J. Blot, G. M. Mophou, G. M. N'Guérékata, and D. Pennequin, Weighted pseudo almost automorphic functions and applications to abstract differential equations, *Nonlinear Anal.* **71** (2009), 903–909.

[9] R. W. Carroll, R. W. Showalter, Singular and Degenerate Cauchy Problems, *Academic Press, New York,* 1976.

[10] Y.-K. Chang, S. Zheng, Weighted pseudo almost automorphic solutions to functional differential equations with infinite delay, *Electron. J. Differential Equations,* **286** (2016), 1–18.

[11] Y.-K. Chang, Y.-T. Bian, Weighted asymptotic behavior of solutions to a Sobolev-type differential equation with Stepanov coeffcients in Banach spaces, *Filomat* **29** (2015), 1315–1328.

[12] A. Coronel, M. Pinto, D. Sepúlveda, Weighted pseudo almost periodic functions, convolutions and abstract integral equations, *J. Math. Anal. Appl.* **435** (2016), 1382–1399.

[13] R. Cross, Multivalued Linear Operators, *Marcel Dekker Inc., New York,* 1998.

[14] T. Diagana, Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces, *Springer, New York,* 2013.

[15] T. Diagana, Weighted pseudo almost periodic functions and applications, *C. R. Math. Acad. Sci. Paris* **343** (2006), 643–646.

[16] T. Diagana, V. Nelson, and G. M. N'Guérékata, Stepanov-like $C^{(n)}$-pseudo-almost automorphy and applications to some nonautonomous higher-order differential equations, *Opuscula Math.* **32** (2012), 455–471.

[17] T. Diagana and R. Agarwal, Existence of pseudo almost automorphic solutions for the heat equation with S^p-pseudo almost automorphic coefficients, *Boundary Value Problems,* vol. 2009, Article ID 182527, 19 pages, doi:10.1155/2009/182527.

[18] T. Diagana, Existence of pseudo-almost automorphic solutions to some abstract differential equations with S^p-pseudo-almost automorphic coefficients, *Nonlinear Anal.* **70** (2009), 3781–3790.

[19] T. Diagana, Existence of doubly-weighted pseudo almost periodic solutions to non-autonomous differential equations, *Electron. J. Differential Equations* **28** (2011), 1–15.

[20] T. Diagana, Existence of doubly-weighted pseudo almost periodic solutions to non-autonomous differential equations, *Afr. Diaspora J. Math.* **12** (2011), 121–136.

[21] T. Diagana, K. Ezzinbi, and M. Miraoui, Pseudo-almost periodic and pseudo-almost automorphic solutions to some evolution equations involving theoretical measure theory, *Cubo* **16** (2014), 01–31.

[22] K. Diethelm, The Analysis of Fractional Differential Equations, *Springer-Verlag, Berlin,* 2010.

[23] Z. Fan, J. Liang, and T.-J. Xiao, On Stepanov-like (pseudo) almost automorphic functions, *Nonlinear Anal.* **74** (2011), 2853–2861.

[24] A. Favini and A. Yagi, Degenerate Differential Equations in Banach Spaces, *Chapman and Hall/CRC Pure and Applied Mathematics, New York,* 1998.

[25] G. M. N'Guérékata, Almost Automorphic and Almost Periodic Functions in Abstract Spaces, *Kluwer Acad. Publ., Dordrecht,* 2001.

[26] G. M. N'Guérékata and M. Kostić, Generalized asymptotically almost periodic and generalized asymptotically almost automorphic solutions of abstract multi-term fractional differential inclusions, *Abstract Appl. Anal.* vol. 2018, Article ID 5947393, 17 pages, https://doi.org/10.1155/2018/5947393.

[27] G. M. N'Guérékata and A. Pankov, Stepanov-like almost automorphic functions and monotone evolution equations, *Nonlinear Anal.* **68** (2008), 2658-2667.

[28] D. Ji and Ch. Zhang, Translation invariance of weighted pseudo almost periodic functions and related problems, *J. Math. Anal. Appl.* **391** (2012), 350–362.

[29] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, *Elsevier Science B.V., Amsterdam,* 2006.

[30] M. Kostić, Abstract Volterra Integro-Differential Equations, *CRC Press, Boca Raton, Fl.,* 2015.

[31] M. Kostić, Existence of generalized almost periodic and asymptotic almost periodic solutions to abstract Volterra integro-differential equations, *Electron. J. Differential Equations* **239** (2017), 1–30.

[32] M. Kostić, Abstract degenerate fractional differential inclusions, *Appl. Anal. Discrete Math.* **11** (2017), 39–61.

[33] M. Kostić, Almost Periodic and Almost Automorphic Solutions to Integro-Differential Equations, *W. de Gruyter, Berlin,* in press.

[34] M. Kostić, Abstract Degenerate Volterra Integro-Differential Equations, *Mathematical Institute SANU, Belgrade,* in press.

[35] M. Kostić, Generalized almost automorphic and generalized asymptotically almost automorphic solutions of abstract Volterra integro-differential inclusions, *Fractional Diff. Calc.* **8** (2018), 255–284.

[36] M. Kostić, Weyl-almost periodic solutions and asymptotically Weyl-almost periodic solutions of abstract Volterra integro-differential equations, *Banach J. Math. Anal.* **13** (2019), 64–90.

[37] M. Kostić, On Besicovitch-Doss almost periodic solutions of abstract Volterra integro-differential equations, *Novi Sad J. Math.* **47** (2017), 187–200.

[38] I. V. Melnikova and A. I. Filinkov, Abstract Cauchy Problems: Three Approaches, *Chapman and Hall/CRC, Boca Raton,* 2001.

[39] J. Mu, Y. Zhoa, and L. Peng, Periodic solutions and S-asymptotically periodic solutions to fractional evolution equations, *Discrete Dyn. Nat. Soc.,* vol. 2017, Article ID 1364532, 12 pages, https://doi.org/10.1155/2017/1364532.

[40] F. Periago and B. Straub, A functional calculus for almost sectorial operators and applications to abstract evolution equations, *J. Evol. Equ.* **2** (2002), 41–68.

[41] J. Prüss, Evolutionary Integral Equations and Applications, *Birkhäuser-Verlag, Basel,* 1993.

[42] S. G. Samko, A. A. Kilbas, and O. I. Marichev, Fractional Derivatives and Integrals: Theory and Applications, *Gordon and Breach, New York,* 1993.

[43] G. A. Sviridyuk and V. E. Fedorov, Linear Sobolev Type Equations and Degenerate Semigroups of Operators, *Inverse and Ill-Posed Problems (Book* **42***), VSP, Utrecht, Boston,* 2003.

[44] W. von Wahl, Gebrochene Potenzen eines elliptischen Operators und parabolische Differentialgleichungen in Räumen hölderstetiger Funktionen *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl.* **11** (1972), 231–258.

[45] Z. Xia and M. Fan, Weighted Stepanov-like pseudo almost automorphy and applications, *Nonlinear Anal.* **75** (2012), 2378–2397.

[46] T.-J. Xiao, J. Liang, and J. Zhang, Pseudo-almost automorphic solutions to semilinear differential equations in Banach spaces, *Semigroup Forum* **76** (2006), 518–524.

[47] S. Zaidman, Almost-Periodic Functions in Abstract Spaces, *Pitman Research Notes in Math., vol. 126, Pitman, Boston,* 1985.

[48] J. Zhang, T.-J. Xiao, and J. Liang, Weighted pseudo almost-periodic functions and applications to semilinear evolution equations, *Abstract Appl. Anal.,* vol. 2012, art. ID 179525, 15 pp., doi:10.1155/2012/179525.