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On a relative Hilali conjecture

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Abstract

The well-known Hilali conjecture stated in [9] is one claiming that if X is a simply connected elliptic space, then $\dim \pi_*(X)\otimes\mathbb{Q} \le \dim H_*(X;\mathbb{Q})$. In this paper we propose that if $f: X \to Y$ is a continuous map of simply connected elliptic spaces, then $\dim \mathrm{Ker}\, \pi_*(f)_{\mathbb{Q}} \le \dim \mathrm{Ker}\, H_*(f;\mathbb{Q}) + 1$, and we prove this for certain reasonable cases. Our proposal is a *relative version* of the Hilali conjecture and it includes the Hilali conjecture as a special case.

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1 Introduction

We let $\pi_*(X)\otimes\mathbb{Q} := \bigoplus_{i\ge 1}\pi_i(X)\otimes\mathbb{Q}$ and $H_*(X;\mathbb{Q}) = \bigoplus_{i\ge 0} H_i(X;\mathbb{Q})$. A simply connected CW complex X is said to be *elliptic* if $\dim(\pi_*(X)\otimes\mathbb{Q})$ and $\dim H_*(X;\mathbb{Q})$ are both finite. In [9] M. R. Hilali conjectures that for a simply connected elliptic space X

$$\dim \pi_*(X)\otimes\mathbb{Q} \le \dim H_*(X;\mathbb{Q}). \tag{HC}$$

That is, the total sum of the Betti numbers of an elliptic space is bigger than or equal to the total sum of the homotopy ranks of it. The conjecture holds for many spaces ([1], [4], [9], [10], [11], [12], [13]).

Let $f: X \to Y$ be a continuous map between two simply connected elliptic spaces. We define

1. $\mathrm{Ker}\, \pi_*(f)_{\mathbb{Q}} := \bigoplus_{i\ge 1}\mathrm{Ker}\Big(\pi_i(f)_{\mathbb{Q}} : \pi_i(X)\otimes\mathbb{Q} \to \pi_i(Y)\otimes\mathbb{Q}\Big)$.

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2. $\text{Ker}\, H_*(f;\mathbb{Q}) := \bigoplus_{i\geq 0} \text{Ker}\Big(H_i(f;\mathbb{Q}) : H_i(X;\mathbb{Q}) \to H_i(Y;\mathbb{Q})\Big)$.

We propose the following relative Hilali conjecture (abbr. RHC):

$$\dim \text{Ker}\, \pi_*(f)_{\mathbb{Q}} \leq \dim \text{Ker}\, H_*(f;\mathbb{Q}) + 1, \tag{RHC}$$

which is in fact a generalization of the above Hilali conjecture (HC). Indeed, let us consider (RHC) for the case when $Y = *$ is a point. Then we have

$$\text{Ker}\, \pi_*(f)_{\mathbb{Q}} = \pi_*(X) \otimes \mathbb{Q} \quad \text{and} \quad \text{Ker}\, H_*(f;\mathbb{Q}) \oplus \mathbb{Q} = H_*(X;\mathbb{Q}), \tag{1.1}$$

from which we get (HC) by taking the dimension of them. Here we note that $f_* : H_0(X;\mathbb{Q}) = \mathbb{Q} \cong H_0(*;\mathbb{Q}) = \mathbb{Q}$, thus $\text{Ker}(f_* : H_0(X;\mathbb{Q}) \to H_0(*;\mathbb{Q})) = \{0\}$, hence $\oplus\mathbb{Q}$ is needed in (1.1), namely, $+1$ is needed in (RHC).

Theorem 1.1. *When $f : X \to Y$ is a spherical fibration, it satisfies (RHC).*

Example 1.2. Let us consider the Hopf fibration $S^1 \to S^3 \xrightarrow{f} S^2$. Since $\pi_1(S^3) = \pi_1(S^2) = \{0\}$, it suffices to look at $\pi_i(-)$ for $i \geq 2$. First we note

$$\pi_2(f)_{\mathbb{Q}} : \pi_2(S^3) \otimes \mathbb{Q} = \{0\} \to \pi_2(S^2) \otimes \mathbb{Q} = \mathbb{Q}.$$

Since $\pi_i(S^1) = \{0\}$ for $i \geq 2$, it follows from the homotopy long exact sequence that for $j \geq 3$ we have

$$\pi_j(f)_{\mathbb{Q}} : \pi_j(S^3) \otimes \mathbb{Q} \cong \pi_j(S^2) \otimes \mathbb{Q}.$$

Therefore $\dim \text{Ker}\, \pi_*(f)_{\mathbb{Q}} = 0$, hence it satisfies (RHC), because we have $\dim \text{Ker}\, H_*(f;\mathbb{Q}) + 1 \geq 1$.

Remark 1.3. By the above homotopy exact sequence argument we see that if $f : X \to Y$ is a fibration with a fiber F such that any homotopy group $\pi_i(F)$ is a finite group (thus $\pi_i(F) \otimes \mathbb{Q} = \{0\}$), then (RHC) holds since $\pi_i(f)_{\mathbb{Q}}$ is an isomorphism for $i \geq 1$, thus $\dim \text{Ker}\, \pi_*(f)_{\mathbb{Q}} = 0$.

A simply connected elliptic space F is said to be an F_0-space if

$$H^*(F;\mathbb{Q}) \cong \mathbb{Q}[x_1, .., x_n]/(f_1, .., f_n)$$

where $|x_i|$ are even and $f_1, .., f_n$ is a regular sequence. S. Halperin [8] conjectures that the Serre spectral sequence (E_r, d_r) of any fibration $F \to X \xrightarrow{f} Y$ with fibre F an F_0-space collapses at E_2-level. It is equivalent to saying that

$$H^*(X;\mathbb{Q}) \cong H^*(F;\mathbb{Q}) \otimes H^*(Y;\mathbb{Q})$$

as $H^*(Y;\mathbb{Q})$-modules. Such a fibration $f : X \to Y$ is said to be *totally non-cohomologous to zero* (abbr. TNCZ).

Remark 1.4. The notion of TNCZ is usually defined on an *orientable* fibration $f : X \to Y$ with fiber F, i.e., a fibration satisfying that the fundamental group $\pi_1(Y)$ acts trivially on the cohomology group $H^*(F)$ of the fiber F (e.g., see [3, Definition 4.38]). We note that since we consider a fibration of simply connected elliptic spaces, the fibration is automatically orientable.

Proposition 1.5. *Suppose that a fibration $f : X \to Y$ is TNCZ. Then, if the fibre F satisfies (HC), the fibration $f : X \to Y$ satisfies (RHC).*

Corollary 1.6. *Suppose that the Halperin conjecture is true. If the fibre of a fibration $f : X \to Y$ is an F_0-space, the fibration $f : X \to Y$ satisfies (RHC).*

Due to [14], we obtain

Example 1.7. If the fibre of a fibration $f : X \to Y$ is a homogeneous space G/H with $\mathrm{rank} G = \mathrm{rank} H$, then the fibration $f : X \to Y$ satisfies (RHC).

2 Sullivan models

We use the Sullivan minimal model $M(X)$ of a simply connected CW complex X of finite type [2]. It is a free $\mathbb{Q}$-commutative differential graded algebra $(\Lambda V, d)$ with a $\mathbb{Q}$-graded vector space $V = \bigoplus_{i\geq 2} V^i$ where $\dim V^i < \infty$ and a decomposable differential; i..e., $d(V^i) \subset (\Lambda^+ V \cdot \Lambda^+ V)^{i+1}$ and $d \circ d = 0$. Here $\Lambda^+ V$ is the ideal of ΛV generated by elements of positive degree. The degree of a homogeneous element x of a graded algebra is denoted by $|x|$. Then $xy = (-1)^{|x||y|} yx$ and $d(xy) = d(x)y + (-1)^{|x|} xd(y)$. Recall that $M(X)$ determines the rational homotopy type of X. In particular there are isomorphisms

$$V^i \cong Hom(\pi_i(X) \otimes \mathbb{Q}, \mathbb{Q}) \text{ and } H^*(\Lambda V, d) \cong H^*(X; \mathbb{Q}).$$

Thus we have

$$\dim V < \infty, \quad \dim H^*(\Lambda V, d) < \infty$$

when X is elliptic and the Hilali conjecture is equivalent to the inequality

$$\dim V \leq \dim H^*(\Lambda V, d). \tag{HC'}$$

A map $f : X \to Y$ has a minimal model, which is a DGA-map

$$M(f) : M(Y) \to M(X).$$

It is induced by a relative or Koszul-Sullivan (KS-)model

$$j \ : \ M(Y) = (\Lambda W, d_Y) \to (\Lambda W \otimes \Lambda V, D),$$

where $D|_{\Lambda W} = d_Y$, $(\Lambda V, \overline{D}) = (\Lambda V, d)$ is the minimal model of the homotopy fibre of f and there is a quasi-isomorphism

$$\rho_X : M(X) \xrightarrow{\sim} (\Lambda W \otimes \Lambda V, D)$$

such that $\rho_X \circ M(f) \simeq j$ (see [8]). Note that the differential D is not decomposable in general.

Let $j : (\Lambda W, d_Y) \to (\Lambda W \otimes \Lambda V, D)$ be the KS-model of f. The dual of $\pi_*(f)_{\mathbb{Q}} : \pi_*(X) \otimes \mathbb{Q} \to \pi_*(Y) \otimes \mathbb{Q}$ is given by

$$H^*(j, D_1) : W \to H^*(W \oplus V, D_1),$$

where D_1 is the linear part of D. Then

$$\text{Coker } H^*(j, D_1) \cong Hom(\text{Ker } \pi_*(f)_{\mathbb{Q}}, \mathbb{Q}).$$

On the other hand, we have that

$$\text{Coker } H^*(j) \cong Hom(\text{Ker } H_*(f;\mathbb{Q}), \mathbb{Q})$$

for $H^*(j) : H^*(\Lambda W, d_Y) \to H^*(\Lambda W \otimes \Lambda V, D)$. Thus (RHC) is equivalent to

$$\dim \text{Coker } H^*(j, D_1) \leq \dim \text{Coker } H^*(j) + 1. \qquad \text{(RHC')}$$

Notice that Coker $H^*(j, D_1)$ is in general identified as a subspace of V.

3 Proofs

Proof of Theorem 1.1. Let $j : M(Y) = (\Lambda W, d_Y) \to (\Lambda W \otimes \Lambda V, D)$ be the KS-model of f.

1. When the fibre is an odd-sphere S^n, $V = \mathbb{Q}\{v\}$ with $|v| = n$.
   (a) When $Dv \notin \Lambda^{>1} W$, Coker $H^*(j, D_1) = 0$. Thus (RHC') holds.
   (b) When $Dv \in \Lambda^{>1} W$, dim Coker $H^*(j, D_1) = \dim V = 1$. Thus (RHC') holds.
2. When the fibre is an even-sphere S^n,

$$V = \mathbb{Q}\{v, z\} \text{ with } dv = 0,\ dz = v^2,\ |v| = n \text{ and } |z| = 2n - 1.$$

Then $Dv = 0$ from $D \circ D = 0$, therefore Coker $H^*(j) \supset \mathbb{Q}\{v\}$. On the other hand, dim Coker $H^*(j, D_1) \leq \dim V = 2$. Thus (RHC') holds. □

Proof of Proposition 1.5. From the assumption we have

$$\text{Coker } H^*(j) \cong H^*(\Lambda W, d_Y) \otimes H^+(\Lambda V, d).$$

Thus it satisfies (RHC') if $\dim V \leq \dim H^*(\Lambda V, d)$. □

Proof of Corollary 1.6. Let $H^*(F;\mathbb{Q}) = H^*(\Lambda V, d) \cong \mathbb{Q}[x_1, .., x_n]/(f_1, .., f_n)$ for the fibre F. Since we get from [7] the following inequality

$$\dim H^*(\Lambda V, d) = \frac{|f_1| \cdots |f_n|}{|x_1| \cdots |x_n|} \geq 2^n \geq 2n = \dim V,$$

the F_0-space F satisfies (HC') as a special case of [9]. Thus this corollary follows from Proposition 1.5. □

Example 3.1. Let $SU(n)$ be the n-th special unitary group. The homogeneous space $G/H = SU(6)/SU(3)\times SU(3)$ is not an F_0-space since $\mathrm{rank}G = 5 > 4 = \mathrm{rank}H$. Let a non-trivial fibration $G/H \to X \xrightarrow{f} Y = S^3$ be given by the KS-model

$$(\Lambda w, 0) \to (\Lambda(w, v_1, v_2, v_3, v_4, v_5), D)$$

with
$|w| = 3$, $|v_1| = 4$, $|v_2| = 6$, $|v_3| = 7$, $|v_4| = 9$, $|v_5| = 11$,
$D(v_1) = 0$, $D(v_2) = wv_1$, $D(v_3) = v_1^2$, $D(v_4) = v_1v_2 + wv_3$ and $D(v_5) = v_2^2 + 2wv_4$.
Here $M(G/H) = (\Lambda(v_1, v_2, v_3, v_4, v_5), d)$ with $d(v_1) = d(v_2) = 0$, $d(v_3) = v_1^2$, $d(v_4) = v_1v_2$ and $d(v_5) = v_2^2$ (see [6]). Then since Coker $H^*(j, D_1) = \mathbb{Q}\{v_1, v_2, v_3, v_4, v_5\}$ and $\dim H^*(\Lambda(w, v_1, v_2, v_3, v_4, v_5), D) = 8$ with $\mathrm{Im}H^*(j) = \mathbb{Q}\{1, w\}$, we have

$$\dim \mathrm{Coker}\ H^*(j, D_1) = 5 < 7 = 6 + 1 = \dim \mathrm{Coker}\ H^*(j) + 1.$$

Thus the map f satisfies (RHC'), though it is not TNCZ.

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