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Abstract

In this note, we extend the idea of startpoint to a quasi-uniform space. We present two main results, first for single-valued maps and second for multi-valued maps.

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1 Introduction

In this section, we give the basic concepts and corresponding notations attached to the idea of quasi-uniform space. The idea of startpoint for multivalued maps defined on a quasi-pseudometric space has been introduced and discussed by Gaba [6]. The startpoint theory naturally generalizes, in the asymmetric settings, well-known results of the fixed point theory. In this article, we extend and improve the results in [6, 7, 8] by using quasi-uniform spaces. Quasi-uniform spaces have been extensively discussed in the literature and key results in the theory can be read in [2, 4, 5, 10, 11] and more recently the work by Andrikopoulos [1], Coghetto [3] and Künzi [9].

Definition 1.1. A **quasi-uniformity** $\mathcal{U}$ on a set X is a filter on $X\times X$ such that

i) each member U of $\mathcal{U}$ contains the diagonal $\Delta_X = \{(x,x) : x \in X\}$ of X,

ii) for each member U of $\mathcal{U}$ there exists a $V \in \mathcal{U}$ such that $V^2 \subseteq U$, where
$V^2 := V \circ V = \{(x,z) \in X\times X : \exists y \in X : (x,y) \in V \text{ and } (y,z) \in V\}$.

The members U of $\mathcal{U}$ are called **entourages** of $\mathcal{U}$ and the pair $(X,\mathcal{U})$ is called a **quasi-uniform space**.

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To any quasi-uniformity $\mathcal{U}$ corresponds a dual quasi-uniformity $\mathcal{U}^{-1}$ defined as

$$\mathcal{U}^{-1} = \{U : U^{-1} \in \mathcal{U}\}$$

where

$$U^{-1} = \{(x,y) : (y,x) \in U\}.$$

A quasi-uniformity is a uniformity if $V \in \mathcal{U}$ implies $V^{-1} \in \mathcal{U}$.

Example 1.2. (Compare [9]) Let d be a quasi-pseudometric on a set X. The filter on $X \times X$ generated by the base $\{U_\epsilon : \epsilon > 0\}$ where

$$U_\epsilon = \{(x,y) \in X \times X : d(x,y) < \epsilon\},$$

is a quasi-uniformity called **quasi-pseudometric quasi-uniformity** and denoted $\mathcal{U}_d$. It is the quasi-uniformity induced by d on X. Indeed, just observe that for each $\epsilon > 0$, $U^2_{\epsilon/2} \subseteq U_\epsilon$.

The dual (conjugate) quasi-uniformity $\mathcal{U}_d^{-1}$ is generated by the entourages

$$U_\epsilon^{-1} = \{(x,y) \in X \times X : d(y,x) < \epsilon\}.$$

Moreover the intersection $\bigcap \mathcal{U} := \bigcap_{U \in \mathcal{U}} U$ is the graph of a preorder on X (see [10]). Hence for any quasi-uniform space $(X,\mathcal{U})$, we shall then call $(X, \bigcap \mathcal{U})$ the preordered space associated with $(X,\mathcal{U})$.

Definition 1.3. A quasi-uniformity $\mathcal{U}$ is called T_0 if $\bigcap \mathcal{U}$ is a partial order on X.

Definition 1.4. Let $(X,\mathcal{U})$ be a quasi-uniform space and $T : X \to X$ be a single-valued map. A point $x_0 \in X$ is said to be a $\mathcal{U}$-fixed point for T if

$$(x_0, Tx_0) \in \bigcap_{U \in \mathcal{U}} U.$$

Definition 1.5. Let $(X,\mathcal{U})$ be a quasi-uniform space and $T : X \to 2^X$ be a multi-valued map. A point $x_0 \in X$ is said to be a $\mathcal{U}$-startpoint (resp. endpoint) for T if there exists $z \in Tx_0$ such that

$$(x_0, z) \in \bigcap_{U \in \mathcal{U}} U \quad \left(\text{resp. } (z, x_0) \in \bigcap_{U \in \mathcal{U}} U\right).$$

Let $(X,\mathcal{U})$ be a quasi-uniform space. For each $x \in X$ and $U \in \mathcal{U}$, set

$$U(x) = \{y \in X : (x,y) \in U\}.$$

2 Main results

Theorem 2.1. *Let $(X,\mathcal{U})$ be a quasi-uniform space and $T : X \to X$ be a single-valued map. If there exists $x^* \in X$ such that $U(x^*) \cap U^{-1}(Tx^*) \neq \emptyset$ for any $U \in \mathcal{U}$, then T has a $\mathcal{U}$-fixed point.*

Proof.

We shall proceed by the way of contradiction. So assume that for any $z \in X$, there is $U_z \in \mathcal{U}$ such that $(z, Tz) \notin U_z$. So by Definition1.1 (*ii*), we know that there exists $V_z \in \mathcal{U}$ such that $V_z \circ V_z \subseteq U_z$. Therefore

$$\begin{aligned}(z,Tz) \notin U_z &\Longrightarrow (z,Tz) \notin V_z \circ V_z,\\ &\Longrightarrow (z,y) \notin V_z \text{ or } (y,Tz) \notin V_z \text{ for any } y \in X,\\ &\Longrightarrow (z,y) \notin V_z(z) \text{ or } (y,Tz) \notin V_z^{-1}(Tz).\end{aligned}$$

Therefore we obtain $V_z(z) \cap V_z^{-1}(Tz) = \emptyset$ –a contradiction. Hence there exists $x_0 \in X$ such that

$$(x_0, Tx_0) \in \bigcap_{U \in \mathcal{U}} U,$$

i.e. x_0 is a $\mathcal{U}$-fixed point for T. □

We now give the following lemma in view of a fixed point result. The proof is straightforward.

Lemma 2.2. *Let $(X,\mathcal{U})$ be a T_0-quasi-uniform space and $T : X \to X$ be a single-valued map. If T is an involution (i.e. $T(Tx) = x$ for any $x \in X$) and order preserving (i.e. $(x,y) \in \bigcap \mathcal{U} \Longrightarrow (Tx,Ty) \in \bigcap \mathcal{U}$), then any $\mathcal{U}$-fixed point for T is a fixed point for T.*

Theorem 2.3. *Let $(X,\mathcal{U})$ be a T_0-quasi-uniform space and $T : X \to X$ be an order preserving involution. If there exists $x^* \in X$ such that $U(x^*) \cap U^{-1}(Tx^*) \neq \emptyset$ for any $U \in \mathcal{U}$, then T has a fixed point.*

Proof. The result is an immediate consequence of Theorem 2.1 and Lemma 2.2.

Theorem 2.4. *Let $(X,\mathcal{U})$ be a quasi-uniform space and $T : X \to 2^X$ be a multi-valued map. If there exists $x^* \in X$ such that $U(x^*) \cap U^{-1}(z) \neq \emptyset$ for any $U \in \mathcal{U}$, for any $z \in Tx^*$, then T has a $\mathcal{U}$-startpoint.*

Proof.

Again we shall proceed by the way of contradiction. So assume that for any $x^* \in X$, and for any $z \in Tx^*$, there is $U_z \in \mathcal{U}$ such that $(x^*, z) \notin U_z$. Hence there exists $V_z \in \mathcal{U}$ such that $V_z \circ V_z \subseteq U_z$ and hence $(x^*, z) \notin V_z \circ V_z$. Therefore for any $y \in X$, $(x^*, y) \notin V_z$ or $(y, z) \notin V_z$. So we obtain $V_z(x^*) \cap V_z^{-1}(z) = \emptyset$ –a contradiction. Then T has a $\mathcal{U}$-startpoint.

□

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