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Semigroup and Blow-Up Dynamics of Attraction Keller-Segel Equations in Scale of Banach Spaces

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Abstract

In this paper, we study the asymptotic and blow-up dynamics of the attraction Keller-Segel chemotaxis system of equations in scale of Banach spaces $E_q^\alpha = H^{2\alpha,q}(\Omega)$, $-1 \le \alpha \le 1, 1 < q < \infty$, where $\Omega \subset \mathbb{R}^N$ is a bounded spatial domain. We show that the system of equations is well-posed for a perturbed analytic semigroup, whenever $2\chi + a < \left(\frac{Ne\pi}{2}\right)^{\beta+\frac{\gamma}{2}-\frac{1}{2}}$, where χ is the chemical attractivity coefficient, a is the rate of production of chemical, and q, β, γ are of the scale spaces. Thus, as $t \nearrow \infty$, the asymptotic dynamics are captured in the limit set $\mathcal{M} \cup \{0\}$, where $\mathcal{M} = |\Omega| L^1-$ spatial average solutions. The constants for the sharp space embedding $E_q^\alpha \subset L^\Theta(\Omega)$ $(1 < \Theta \le \infty)$ indicate that for either the application of Banach fixed point theorem, or the global existence of solutions, no need of either the time for a contraction mapping, nor the initial data of the system of equations, to be small, respectively. In blow-up dynamics, we prove that the solutions blow-up at the borderline scale spaces $E_q^\alpha, \alpha = \frac{N}{2q}$, independent of time $t > 0$, if the chemo-attractivity coefficient dominates the Moser-Trudinger threshold value. An analysis of the finite time bounds for blow-up of solutions in norm of $L^{2p}(\Omega), 1 \le p \le 6$ and $\Omega \subset \mathbb{R}^N, N = 2, 3$, is also furnished.

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1 Introduction

In this paper, we study the asymptotic dynamics of the following classical Keller-Segel chemotaxis[1] system of equations.

$$\left\{\begin{array}{rcll} v_t &=& \Delta v-\lambda v+aw & \text{in } \Omega\times(0,T),\\ w_t &=& \Delta w-\nabla\cdot(w\chi\nabla v) & \text{in } \Omega\times(0,T),\\ \partial_{\vec{n}}v &=& \partial_{\vec{n}}w=0 & \text{on } \partial\Omega=\Gamma,\\ v(0) &=& v_0,\quad w(0)=w_0 & \text{in } \Omega, \end{array}\right. \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ is an open bounded domain with smooth boundary Γ,

$$\begin{array}{rcl} v &:=& \text{chemical concentration},\\ w &:=& \text{cell density},\\ \lambda &:=& \text{rate of decay of chemical},\\ a &:=& \text{rate of production of chemical},\\ \chi &:=& \text{chemical attractivity coefficient},\\ \nabla\cdot &=& \text{div},\qquad \vec{n}:=\text{unit normal vector pointing outwards of } \Gamma. \end{array}$$

In what follows, we will often let $I=[0,T)$, $\dot{I}=(0,T)$.

We will study the system of equations (1.1) in nested scale of Banach spaces

$$E_q^\alpha := H^{2\alpha,q}(\Omega)=(I-\mathcal{A}_0)^{-\alpha}L^q(\Omega),\quad \text{for}\quad -1\le\alpha\le 1, 1<q<\infty, \tag{1.2}$$

with dual spaces $[E_{q'}^\alpha]^*$, for $\alpha\ge 0$, $\frac{1}{q}+\frac{1}{q'}=1$, endowed with the dual spaces product $\langle\cdot,\cdot\rangle_{q,q'}$ of the $L^q(\Omega)-$spaces. By nested, we mean that if $\alpha\ge\beta$, then the identity mapping $i: E_q^\alpha\mapsto E_q^\beta$ is such that $i\in\mathcal{L}(E_q^\alpha,E_q^\beta)$, and in operator norm, $\|i\|_{\alpha,\beta}\le 1$. The defined scale of Banach spaces (1.2) are a special case of inhomogeneous Sobolev spaces $H^{s,q}(\Omega)$, $s\in\mathbb{R}$, $1<q<\infty$, or Bessel potential spaces, which coincide with the standard Soblolev spaces when $s=\alpha\in\mathbb{Z}$ [2, 9].

To achieve the above-mentioned objective, we first make precise the notations to be used for the function spaces. To this end, we state that in the absence of any danger of causing uncertainties or confusions, we will use the same space notation

$$H^{s,q}(\Omega)=H^{s,q}(\Omega;\mathbb{R}^2) \tag{1.3}$$

for both Sobolev spaces of either single or vector real-valued functions. More prominently, if the components of the vector-valued functions are in different spaces ($s_1\neq s_2$), then we will use the notation

$$H^{s,q}(\Omega):=H^{s_1+s_2,q}(\Omega)=H^{s_1,q}(\Omega)\times H^{s_2,q}(\Omega):=Z_q^{s_1+s_2}.$$

In a similar fashion, we will use the notation

$$Z_q^s:=E_q^{s_1}\oplus E_q^{s_2}:=E_q^s,\quad \text{if } s_1=s_2. \tag{1.4}$$

[1] Chemotaxis is the migration and organisation of cells induced by changes in the concentration of chemical substances secreted by the cells themselves.

Now, consider in (1.1) the uncoupled elliptic equation in $L^q(\Omega)$ and define the operator

$$\mathcal{A}_0 = \begin{pmatrix} -\Delta+\lambda & 0 \\ 0 & -\Delta \end{pmatrix} : D(\mathcal{A}_0) \subset L^q(\Omega) \to L^q(\Omega), \tag{1.5}$$

with domain

$$D(\mathcal{A}_0) = \left\{ u = (v,w)^\top \in H^{2,q}(\Omega) : \partial_{\vec{n}} u = \vec{0} \text{ on } \Gamma,\ \ 1 < q < \infty \right\}. \tag{1.6}$$

It is well known, following [3, 11, 29, 31], that (1.5) is a sectorial (or C^+ for short) operator in $L^q(\Omega)$. Therefore, by the complex interpolation- extrapolation method (see [3, 32]), the product scale spaces of (1.2)

$$\begin{aligned} E_q^\alpha := H^{2\alpha,q}(\Omega) &= (I-\mathcal{A}_0)^{-\alpha} L^q(\Omega) \\ &= \begin{pmatrix} (1+\Delta-\lambda)^{-\alpha} & 0 \\ 0 & (1+\Delta)^{-\alpha} \end{pmatrix} L^q(\Omega;\mathbb{R}^2) \\ &= E_q^\alpha \times E_q^\alpha = E_q^\alpha, \quad \text{for } -1 \le \alpha \le 1, 1 < q < \infty, \end{aligned} \tag{1.7}$$

incorporating the boundary conditions, are well defined, and we can identify

$$E_q^1 \cong D(\mathcal{A}_0), \quad E_q^{\frac{1}{2}} \cong H^{1,q}(\Omega), \quad E_q^0 \cong L^q(\Omega), \quad E_q^{-\frac{1}{2}} \cong H^{-1,q}(\Omega).$$

In particular, if we denote the complex interpolation by $[\cdot,\cdot]_\theta$, then the fractional order scale spaces associated with the operator (1.5)-(1.6) satisfy [3, 11]

$$X_q^\alpha = [L^q(\Omega), E_q^1 \cong D(\mathcal{A}_0)]_\alpha \hookrightarrow H^{2\alpha,q}(\Omega) := E_q^\alpha, \quad 0 \le \alpha \le 1,\ 1 < q < \infty, \tag{1.8}$$

where the inclusions are strictly continuous. In general, the embeddings (1.8) are not known, except in the case where Ω is of class C^∞, or the boundary conditions are of the Dirichlet type. We refer the reader to [3, 11] for more details.

For discussion convenience, we formulate the system of equations (1.1) in an abstract evolutionary equations framework. To this end, we let $u = (v,w)^\top \in E_q^\alpha$ for some $\alpha \in \mathbb{R}$, and observe that the system of equations (1.1) takes the form

$$\begin{cases} u_t + \mathcal{A}u &= h(u,\nabla u), \\ u(0) &= u_0 \in Z_q^{\beta+\gamma} := E_q^\beta \times E_q^\gamma, \quad \beta \ge \gamma, \end{cases} \tag{1.9}$$

where $\mathcal{A} = \mathcal{A}_\alpha \in \mathcal{L}(E_q^{\alpha+1}, E_q^\alpha)$ denotes the realisation of the sectorial operator (1.5)-(1.6) in $L^q(\Omega)$, such that $\mathcal{A}_{-\alpha} \in \mathcal{L}(E_q^{\alpha+1}, E_q^{-\alpha})$ is defined by

$$\langle \mathcal{A}_{-\alpha} u, z \rangle_{q,q'} = \int_\Omega \nabla v \nabla \varphi + \lambda \int_\Omega v\varphi + \int_\Omega \nabla w \nabla \psi, \quad \forall z = (\varphi,\psi) \in E_{q'}^\alpha, \tag{1.10}$$

in the weak variational form if $\alpha = \frac{1}{2}$, that is as well very week if $\alpha = 1$, since the distributional derivatives of the variables in system of equations are passed onto the test functions. The non-linear term

$$h(u,\nabla u) = aw + P(w\chi\nabla v) \in E_q^\beta \oplus E_{q'}^{-\alpha},$$

for $\alpha \geq \beta \geq 0$ such that $E_{q'}^{-\alpha} \supseteq E_{q'}^{-\beta}$, is defined by

$$\begin{aligned}\langle h(u,\nabla u), z\rangle &:= \langle (aw, -\nabla\cdot(w\chi\nabla v))^\top, (\varphi,\psi)^\top\rangle_{q,q'}\\ &:= a\int_\Omega w\varphi + \chi\int_\Omega w\nabla v\nabla\psi\\ &:= \langle aw,\varphi\rangle_{q,q'} + \langle P(w\chi\nabla v),\psi\rangle_{q,q'} \quad \forall z = (\varphi,\psi)^\top \in E_{q'}^{\alpha}. \end{aligned} \tag{1.11}$$

Thus, (1.9) is understood in the context of the identity

$$\langle u_t + \mathcal{A}u, z\rangle_{q,q'} = \langle h(u,\nabla u), z\rangle_{q,q'}, \quad \forall z = (\varphi,\psi)^\top \in E_{q'}^{\alpha}.$$

It is important to observe that the variational equivalent formulation (1.9) of the system of equations (1.1) gives an easier configuration to treat the non-linear term

$$-\nabla\cdot(w\chi\nabla v) = -\chi\nabla w\cdot\nabla v - \chi w\Delta v$$

in solving the equations for a solution compared to working directly with the original system of equations, since the action of the semigroup $\{T(t) := e^{\Delta t} :\ t > 0\}$ on the operator $\nabla\cdot$ do not commute [16]. This can however be resolved to some extent (see [16], and/or independently using Hardy-Littlewood-Sobolev inequality [9]). Nevertheless, the results in this paper are much finer as far as the well-definition of (1.11), and the treatment of the question are concerned.

In this regard, it follows, using [3, 11, 30, 31], that the realisation of the operator (1.5) is an infinitesimal generator of an analytic semigroup

$$\{S(t) = e^{-\mathcal{A}t} :\ t \in \mathbb{R}^+\} : E_q^{\alpha} \longmapsto E_q^{\beta} \quad \text{for any} \quad \alpha,\beta \in \mathbb{R} \tag{1.12}$$

in the scale spaces. Thus we can solve the homogeneous equations corresponding to (1.1). Moreover, Duhamel's principle provides the integral equation

$$u(t) = e^{-\mathcal{A}t}u_0 + \int_0^t e^{-\mathcal{A}(t-s)}h(u(s))ds \tag{1.13}$$

as the solution to the evolutionary equation (1.9) in adequate function spaces by means of a contraction mapping, and vice-versa.

If we take, in (1.1), the dual spaces product between the spaces $L^q(\Omega)$ and $L^{q'}(\Omega)$, where $\frac{1}{q} + \frac{1}{q'} = 1$, using $z = (1,1)^\top$ as the test function, then we deduce that

$$\begin{aligned} &\frac{d}{dt}\int_\Omega v = -\lambda\int_\Omega v + a\int_\Omega w \Rightarrow v_\Omega(t) = e^{-\lambda t}v_\Omega^0 + a\int_0^t e^{-\lambda(t-s)}w_\Omega(s)ds\\ &\Longleftrightarrow v_\Omega(t) = e^{-\lambda t}v_\Omega^0 + \frac{aw_\Omega^0}{\lambda}(1-e^{-\lambda t}), \quad \forall t\in(0,T),\\ &\quad\text{since}\quad \frac{d}{dt}\int_\Omega w = 0 \Rightarrow w_\Omega(t) = \int_\Omega w_0 = w_\Omega^0, \quad \forall t\in(0,T), \end{aligned}$$

in which we have set $\varphi_\Omega = \int_\Omega \varphi$.

Thus, if $T = \infty$, then we have the limit set

$$\mathcal{M} = \left\{ A \in \mathbb{R}^2 : A = \left(\frac{aw_\Omega^0}{\lambda}, w_\Omega^0 \right)^\top \right\}, \tag{1.14}$$

of L^1- spatially integrable solutions, in the distributions sense.

On the other hand, considering the corresponding stationary equations to the system of equations (1.1), if we take $\varphi = \ln w - \chi v \in H^1(\Omega) = E_2^{\frac{1}{2}}$, $w \in L^\infty(\Omega)$, $w \neq 0$ as a test function in the cell density equation, and then integrate by parts over Ω, taking into account the boundary conditions, then we find, by using Green's formula, that

$$\begin{aligned} 0 &= \int_\Omega \nabla \cdot (\nabla w - w\chi \nabla v)\varphi = \int_\Omega \nabla \cdot w \left(\frac{\nabla w}{w} - \chi \nabla v \right) \varphi \\ &= -\int_\Omega w |\nabla (\ln w - \chi v)|^2 \\ \Leftrightarrow & \inf_\Omega w \int_\Omega |\nabla(\ln w - \chi v)|^2 \leq \int_\Omega w |\nabla(\ln w - \chi v)|^2 = 0 \\ \Rightarrow & \ln w - \chi v = \text{cte}, \quad \text{and} \quad w = Ke^{\chi v} \end{aligned} \tag{1.15}$$

where $K = e^{\frac{aw_\Omega^0}{\lambda}}$, using (1.14). Otherwise if we suppose that (1.15) is false, then we would have that

$$\varphi = \ln w - \chi v \in V = \left\{ \phi \in H^1(\Omega) : \int_\Omega \phi = 0 \right\}$$

which is the set of functions orthogonal to constant functions in $H^1(\Omega)$. Thus, since $\varphi \neq 0$, $\int_\Omega |\nabla \varphi|^2 = 0$ would leads us to conclude, with the aid of Poincaré inequality, that $\|\varphi\|_{H^1(\Omega)} = 0$ is valid, leading us to a contradiction. Ergo, φ must be a constant, and the implied in (1.15) must be valid.

It henceforth follows, by substituting $w = Ke^{\chi v}$ into the stationary $v-$ equation, that we obtain the semi-linear elliptic problem

$$\begin{cases} 0 = \Delta v - \lambda v + aKe^{\chi v} & \text{in } \Omega, \\ \partial_{\vec{n}} v = 0 & \text{on } \Gamma. \end{cases} \tag{1.16}$$

Due to the exponential nature of its non-linear term, its well-posedness in the scale of Banach spaces $E_q^\beta, \beta = \frac{N}{2q}$ depends strongly on the following Moser-Trudinger lemma.

Lemma 1.1 ([1, 21, 37])**.** *Let $\beta \in (0, N)$ be a positive real number and $1 < q = \frac{N}{2\beta} < \infty$. Then,*

$$\sup_{\substack{f \in E_q^\beta \\ \|(I-\Delta)^\beta f\|_q \leq 1}} \int_\Omega e^{\chi |f|} dx \begin{cases} \leq & C_{q,N}|\Omega| \quad \text{if } \chi < \chi_{N,\beta} \\ = & +\infty \quad \text{if } \chi \geq \chi_{N,\beta} \end{cases} \tag{1.17}$$

where,

$$\chi_{N,\beta} = \left(\frac{N}{\omega_{N-1}} \right)^{\frac{1}{q'}} \left[\frac{\pi^{\frac{N}{2}} 2^{2\beta} \Gamma(\beta)}{\Gamma(\frac{N-2\beta}{2})} \right] \tag{1.18}$$

is the Moser-Trudinger threshold value, and $\omega_{N-1} = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}$ *is the measure of a unit sphere in* $\mathbb{R}^N$.

Furthermore, (1.16) can be viewed either as a non-linear eigenvalue problem, or a non-local elliptic problem (see [8, 14, 15]), under adequate transformations of the equations variables in (1.1) of the nature

$$\psi(t,x) = \frac{|\Omega| w(t,x)}{\int_\Omega w_0(x)dx}, \quad \varphi(t,x) = \chi\left(v(t,x) - \frac{1}{|\Omega|}\int_\Omega v(t,x)dx\right), \tag{1.19}$$

leading to a transformed version (1.31) of the original Keller-Segel chemotaxis equations in (1.1), which we will discuss more towards the end of this section.

It is important to note that the study of (1.34) can be significantly involving, in the context of establishing the Palais-Smale condition, in view of the Trudinger-Moser inequality and Pohozaev's identity for non-existence of solutions. It is in this regard, that we will only confine ourselves to its impact on the blow-up dynamics of the system of equations (1.1) on the basis of (1.18).

Among the questions to be answered in this paper, before the above-mentioned blow-up analysis, is the question of whether the complete system elliptic differential operator

$$\begin{aligned} \mathcal{A}(w) &= \mathcal{A}_0 - \tilde{P}(w) \\ &= \begin{pmatrix} -\Delta + \lambda & -a \\ \nabla\cdot(w\chi\nabla\cdot) & -\Delta \end{pmatrix} : Z_q^{\beta+\gamma} := E_q^\beta \times E_q^\gamma \mapsto E_q^{-\beta} \times E_q^{-\gamma}, \end{aligned} \tag{1.20}$$

is an infinitesimal generator of an analytic perturbed semigroup to the semigroup (1.12), defined by the uncoupled system elliptic differential operator, where $\gamma \le \beta \le \alpha < \gamma + 1$ and $w \in E_q^\gamma$ fixed . In this direction, we first have the following lemma.

Lemma 1.2. *Consider the system* (1.1)*, and assume that* $u = (v, w) \in Z_q^{\alpha+\gamma} := E_q^\alpha \times E_q^\gamma$*, with the scales satisfying* $\alpha \ge \frac{1}{2}$, $0 \le \alpha - \gamma < 1$,

$$\frac{1}{2} + \frac{N}{2q} \le \alpha + \gamma, \quad \textit{and} \quad 1 + \frac{N}{2q} \le 2\alpha + \gamma. \tag{1.21}$$

Then the product $w\chi\nabla v \in E_q^\alpha$*, with weak form* $P(w\chi\nabla v) \in E_{q'}^{-\beta}$*, defined by*

$$\langle P(w\chi\nabla v), \psi\rangle_{q,q'} = \langle w\chi\nabla v, \nabla\psi\rangle_{q,q'} = \chi\int_\Omega w\nabla v\nabla\psi \in \mathbb{R}, \quad \forall\psi \in E_{q'}^\alpha, \tag{1.22}$$

is well defined. Moreover,

$$||P||_{\mathcal{L}(E_q^\alpha, E_{q'}^{-\beta})} := \sup_{||\psi||_{\alpha,q'}\le 1} \frac{\langle P(w\chi\nabla v), \psi\rangle_{E_q^\alpha, E_{q'}^\gamma}}{||w\chi\nabla v||_{\alpha,q}} \le \left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\gamma}{2}-\frac{1}{2}}. \tag{1.23}$$

In particular, the weak form $P \in \mathcal{L}_{lip}(E_q^\alpha, E_{q'}^\beta)$ *is a linear, continuous and Lipschitz operator between the scale spaces.*

It is worthwhile to remark the following about Lemma 1.2.

Remark 1.3. **a)** The bounding estimate from above in (1.23) changes with the space embeddings $E_q^{\frac{N}{2q}} \subset L^{\Theta}(\Omega), 1 < \Theta \leq \infty$, in the critical situation, see (2.2) in the next section.

b) The first yielding condition in (1.21) is consistent with studying the drift function only with the div-operator (see [29]). The conditions in (1.21) are special cases of the following

$$\frac{1}{2}+\frac{N}{2}\left(\frac{1}{q}+\frac{1}{p}-\frac{1}{\rho}\right) \leq \beta+\gamma \quad \text{and} \quad 1+\frac{N}{2}\left(\frac{1}{q}+\frac{1}{p}-\frac{1}{\rho}\right) \leq 2\beta+\gamma, \tag{1.24}$$

relating to initial data spaces with all different exponents, implying $P : E_\rho^\gamma \longmapsto E_{\rho'}^{-\beta}$, $P \in \mathcal{L}_{lip}(E_\rho^\gamma, E_{\rho'}^\beta)$, and (1.21) is obtained when $\rho = q$. Since $\frac{1}{2} \leq \gamma, \beta \leq 1$, the generalised condition yields Young's inequality for convolutions, and $\rho \geq q, p$. Moreover, the sharp optimal scale of Banach spaces embedding (2.1) are verified with $\rho = \Theta$.

c) The estimate (1.23) simplifies the action of the semigroup (1.12) in controlling (1.13), which does not commute with the div-operator $\nabla\cdot$ in (1.1), but behaves in a similar fashion as it does for $t \geq 1$. See [17, 36] for alternatives to this issue in dealing with the $w-$integral equation solution directly from (1.1) and not from (1.13).

Thanks to Lemma 1.2, we have, on existence and uniqueness of solutions, the following theorem.

Theorem 1.4. *Assume in the $v-$equation of* (1.1) *that $w \in L^\sigma(\dot{I}; E_q^\beta)$, with $1 \leq \sigma \leq \infty$ and $0 \leq \alpha - \beta < \frac{1}{\sigma'}$. Then,*

(i)

$$v \in C(I; E_q^\beta) \cap C(\dot{I}; E_q^\alpha) \cap C(\dot{I}; E_q^{\beta+1}) \cap C^1(\dot{I}; E_q^{\gamma'}), \tag{1.25}$$

for any $\gamma' < \beta + \frac{1}{\sigma'}$, $\beta \leq \alpha < \beta + 1$. Moreover, since (1.5) *is a sectorial operator, $w \in L^\sigma(0,\infty; E_q^\beta)$, $v \in L^\sigma(0,\infty; E_q^\alpha)$, $\beta \leq \alpha < \beta+1$, for any $\beta \in \mathbb{R}$,, and*

$$\limsup_{t \nearrow \infty} \|w\|_\alpha = 0 \quad \text{and} \quad \limsup_{t \nearrow \infty} \|\nabla v\|_{\alpha - \frac{1}{2}} = 0. \tag{1.26}$$

(ii) *Assume Lemma 1.2 is verified. Then, the $w-$solution of* (1.1) *satisfies*

$$w \in C(I; E_q^\gamma) \cap C(\dot{I}; E_q^\alpha) \cap C(\dot{I}; E_q^{\gamma+1}) \cap C^1(\dot{I}; E_q^{\gamma'}) \tag{1.27}$$

for any $\gamma' < \gamma + 1$.

(iii) *The system* (1.1) *admits a unique globally defined strong solution given by* (1.13) *and conversely. Furthermore, if*

$$1 > \frac{2\chi + a}{q}\left(\frac{2}{Ne\pi}\right)^{\beta+\frac{\gamma}{2}-\frac{1}{2}} \tag{1.28}$$

holds, then the complete system differential operator (1.20) *generates a perturbed analytic semigroup in* $Z_q^{\beta+\gamma} = E_q^{\beta} \times E_q^{\gamma}$*, and*

$$\limsup_{t \nearrow \infty} \|(v(t), w(t))^{\top}\|_{\beta+\gamma} = A^* \in \mathcal{M} \cup \{0\}, \tag{1.29}$$

where the limit set $\mathcal{M}$*, is as defined in* (1.14).

(iv) *If the first condition in* (1.21) *is satisfied with a strict inequality, then the unique globally defined strong solution in (iii) is a classical solution.*

In particular, using (1.21) with $\gamma = 0$, $q = 2$, and since $\alpha - \frac{1}{2} < 1$, Theorem 1.4 implies, by a density argument, the following corollary, which is important in the analysis for much finer blow-up results and local controllability of the system for a control function $f \in L^2(\omega \times (0,T))$, where $\omega \subset \Omega$.

Corollary 1.5. *Consider the system of equations* (1.1) *with initial data* $(v_0, w_0) \in L^2(\Omega) \times L^2(\Omega)$ *and* $\Omega \subset \mathbb{R}^N$, $N = 1, 2, 3$. *Then the solution* (2.5) *satisfies*

$$u,\, w \in L^{\infty}\Big(0, T;\, L^2(\Omega)\Big) \cap L^2\Big(0, T;\, H^1(\Omega)\Big); \quad v_t,\, w_t \in L^2\Big(0, T;\, L^2(\Omega)\Big). \tag{1.30}$$

Moreover, the system (1.1) *is given, for a.e.* $t \in (0,T)$*, by*

$$\begin{aligned} \int_{\Omega} v_t \varphi &= -\int_{\Omega} \nabla v \nabla \varphi - \lambda \int_{\Omega} v\varphi + a \int_{\Omega} u\varphi, \quad \forall \varphi \in H^1(\Omega), \\ \int_{\Omega} w_t \psi &= -\int_{\Omega} \nabla w \nabla \psi + \chi \int_{\Omega} w \nabla v \nabla \psi, \quad \forall \psi \in H^1(\Omega), \end{aligned}$$

in distribution sense of functions in $H^{-1}(\Omega)$.

The system of equations (1.1) is the simplest description of a cell population, which produces a chemical signal and responds to it by performing chemotaxic movements. It was originally developed in [18] by Evelyn Fox Keller and Lee A. Segel (1970), in the context of investigating the aggregation of the cellular slime mold *Dictyostelium discoideum*. The system of equations is a macroscopic model for chemotactic cell migration, where besides diffusing randomly, the cells partly orient their movement towards increasing concentrations of a chemical signal substance. Eversince its discovery, the system of equations has attracted the attention of many scientists from varied perspectives. See [7, 13, 22, 33, 34] for other natural phenomena that describe generalized models from the system equations (1.1).

Now we give a brief review of some of the contributions to the topic. For some literature, we cite, among others, [5, 8, 12, 13, 16, 14, 19, 23, 24]. In [5], Corrias and Perthame (2006) studied the system of equations (1.1) in the entire space $\mathbb{R}^N, N \geq 3$, with initial data

$$(v_0, w_0) \in W^{1,N}(\mathbb{R}^N) \times L^q(\mathbb{R}^N) \cap L^1(\mathbb{R}^N), \quad q > \frac{N}{2}, \quad \|\nabla v_0\|_N \ll 1,$$

near optimum critical spaces. They proved that the system of equations is well-posed for a solution satisfying

$$\begin{aligned} (v, w) \in L^{\infty}(\dot{I}; W^{1,N}(\mathbb{R}^N)) \quad &\times \\ \times \quad & L^{\infty}(\dot{I}; L^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)) \cap L^{\infty}_{loc}(\dot{I}; L^p(\mathbb{R}^N)), \end{aligned}$$

for all $p > q$, and

$$t^{\frac{1}{2}-\frac{N}{2r}}\|\nabla v(t)\|_r \le C(T; \|u_0\|_q, \|\nabla v_0\|_N),$$

for $r > N$. In order to obtain global in time existence of solutions to the system of equations, they had to impose an extra requirement on the cell initial data, that $\|w_0\|_q \ll 1$ be sufficiently small, owing to a need to establish $L^p(\mathbb{R}^N)$ energy estimates for the solution. An important role in their proofs is played by $L^p(\Omega) - L^q(\Omega), 1 < p \le q \le \infty$, heat kernel estimates, to show that the solution to the complete system of equations behaves in a similar way as does the semigroup defined by the uncoupled elliptic differential operator, with much higher contractivity in the smoothening or regularization effect estimates of the semigroup. Similar results, to a great extent, were derived in [24] by Nagai and Yamada (2007), but with initial data in much stronger spaces, i.e.

$$(v_0, w_0) \in W^{1,\infty}(\mathbb{R}^N) \times L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \quad N \ge 1,$$

and without the need of the initial data to be small.

The objective of this paper, is to study the system of equations (1.1) in nested scale of Banach spaces $E_q^\alpha, -1 \le \alpha \le 1, 1 < q < \infty$. It turns out that our results will agree, to a great degree, with those of [5, 24], but without the need of the initial data to the equations to either be small, nor to immediately belong in scale spaces embedded into the space of uniformly bounded functions in Ω. Special thanks are due to the Professors W. Kryszewski and M. Clapp for referring the authors to the research paper [16] by D. Horstmann and M. Winkler, whose approach is somehow related to that of this paper, but its study is technically different in view of (1.8), taken in conjunction with that the system of equations are not entirely the same. In [16], the system of equations (1.1) is of a semi-linear chemotactic sensitivity function, to which a precise exponent threshold value for blowing-up of solutions was established.

The question of blow-up of solutions to the system of equations (1.1) has been previously investigated by many other authors. See [8, 12, 16, 14, 15], to cite just a few. The study insofar has been either via the radically symmetric method, or the Lyapunov function approach associated to the equations. In relation to (1.19), the corresponding transformed version of the system of equation (1.1) is the following

$$\begin{cases} \varphi_t &= \Delta\varphi - \lambda\varphi + a\chi(\psi - 1) & \text{in } \Omega \times (0,T), \\ \psi_t &= \Delta\psi - \nabla\cdot(\psi\nabla\varphi) & \text{in } \Omega \times (0,T), \\ \partial_{\vec{n}}\varphi &= \partial_{\vec{n}}\psi = 0 & \text{on } \partial\Omega = \Gamma, \\ \varphi(0) &= \varphi_0, \quad \psi(0) = \psi_0 & \text{in } \Omega. \end{cases} \tag{1.31}$$

The well-posedness of (1.31) for a local weak solution satisfying (1.30) was established in [8] by H. Gajewski and K. Zacharias. The transformed system of equations (1.31) was well shown to accept the Lyapunov function

$$J(\varphi,\psi)(t) = \frac{1}{2a\chi}\int_\Omega \left(|\nabla\varphi|^2 + \lambda\varphi^2\right) + \int_\Omega (\psi(\log\psi - 1) + 1) - \int_\Omega (\psi - 1)\varphi, \tag{1.32}$$

with lower estimate

$$J(\varphi,\psi)(t) \ge \mathcal{T}(\varphi) = \frac{1}{2a\chi}\int_\Omega \left(|\nabla\varphi|^2 + \lambda\varphi^2\right) - |\Omega|\log\left(\frac{1}{\Omega}\int_\Omega e^\varphi\right) \tag{1.33}$$

for $t \geq 0$, and is bounded from below in a smooth domain $\Omega \subset \mathbb{R}^2$, provided that $\frac{a\chi|\Omega|}{4\pi} < 1$. It follows again, by [8, 14, 15], that the solutions to (1.31), in time subsequence $(t_k)_{k\in\mathbb{N}}$ such that $t_k \nearrow \infty$, converge to the stationary solution of the system of equations. In fact, this was validated to hold for $t \nearrow \infty$, and

$$\varphi(t) \rightharpoonup \varphi^* \quad \text{weakly in } H^1(\Omega), \quad \psi(t) \to \psi^* \quad \text{strongly in } L^2(\Omega),$$

as $t \nearrow \infty$, where $\psi^* = \frac{|\Omega|e^{\varphi^*}}{\int_\Omega e^{\varphi^*}}$ and φ^* solves the non-local elliptic boundary problem (1.16) of the form

$$\begin{cases} 0 = \Delta\varphi - \lambda\varphi + a\chi\left(\frac{|\Omega|e^{\varphi}}{\int_\Omega e^{\varphi}dx} - 1\right) & \text{in } \Omega, \\ \partial_{\vec{n}}\varphi = 0 & \text{on } \Gamma. \end{cases} \tag{1.34}$$

A complete treatise of this non-local elliptic problem and blow-up of solutions to (1.31), in either finite or infinite time, was furnished in [14, 15], including the case without symmetry assumptions in [15].

This paper is organized as follows. In section 2, we give some preliminaries. Section 3 is concerned with the proof of Theorem 1.4. Note that in essence, it implies the following; parting from the natural yielding condition (1.21) relating the initial data spaces of the cell density and chemical concentration variable, the system of equations is globally well-posed within the large time asymptotic dynamics, orthogonal to constant solutions decaying to the null states. This in fact takes place at the same rate of the semigroup, by virtue of an *a priori* approach to zero of the drift attracting chemical cue from that of the cell density. Furthermore, using the best constant of the spaces embedding into $L^\Theta(\Omega)$-spaces, $1 < \Theta \leq \infty$, it shows that for either application of Banach fixed point theorem, or global existence of solutions, no need of either the time for a contraction mapping, or initial data of the equations, to be necessarily small respectively. In particular, if (1.28) holds, the system coupled differential operator is an infinitesimal generator of a perturbed analytic semigroup in product scale of Banach spaces.

In Section 4, we give an alternative proof to Theorem 1.4 (iv), that if the first condition in (1.21) is strictly attained, the solution of the system of equation (1.1) is a classical solution. Section 5 gives some highlights on the blow-up of solution to the system of equations at the borderline space $E_q^\alpha, \alpha = \frac{N}{2q}$, independent of time $t > 0$. We comment that the question of finite time blow-up of solution of the system of equations (1.1) has been previously investigated by many other authors, see [12, 23, 16] among others. Section 6, studies finite time upper and lower bounds for blow-up of solution of the system of equations (1.1) in norm of $L^{2p}(\Omega), 1 \leq p \leq 6$, and $\Omega \subset \mathbb{R}^N, N = 2, 3$. It complements the elegant work initiated by L.E.Payne and J.C. Song (2010) in [26] pertinent to the parabolic-elliptic equations of the minimal chemotaxis model. The importance of both finite time bounds for blow-up of solutions is in indicating the time for initial stage of aggregation and also when the final stage of aggregation is reached

2 Preliminaries

Throughout this paper, generic constants will be denoted by $C \in \mathbb{R}^+$. In the sequel, relating to the particular spaces $E_q^\alpha, -1 \leq \alpha \leq 1$ associated with the operator (1.5), we will use $\|\cdot\|_\alpha$

as the norm notation, while in the case of the space $L^{\sigma}(\dot{I}; E_q^{\alpha}), 1 \leq \sigma \leq \infty$, the norm notation to be used will be $\|\cdot\|_{\alpha,\sigma}$. Important for our analysis are the following Sobolev type spaces embeddings [2, 3, 11, 30, 32];

$$E_q^{\alpha} \subset L^{\Theta}(\Omega) \iff \Theta \begin{cases} \leq & \frac{qN}{N-2\alpha q} & \text{if } 2\alpha - \frac{N}{q} < 0, \\ < & \infty & \text{if } 2\alpha - \frac{N}{q} = 0, \\ \leq & \infty & \text{if } 2\alpha - \frac{N}{q} > 0, \end{cases} \tag{2.1}$$

with the best constants of the inclusions ([10, 37]) given by

$$C_{\alpha} = \begin{cases} \left(\frac{2}{\pi^{\frac{1}{2}}}\right)^{\frac{2\alpha}{2}} \frac{\Gamma(\frac{N-2\alpha}{2})}{\Gamma(\frac{N+2\alpha}{2})} \left(\frac{\Gamma(N)}{\Gamma(\frac{N}{2})}\right)^{\frac{2\alpha}{N}} \simeq (2(Ne\pi)^{-1})^{\alpha} & \text{if } 1 < q < \infty, \\ & 0 < 2\alpha < \frac{N}{q}, \\ \dfrac{2\Gamma\left(\frac{N}{2q'}\right)|\Omega|^{\frac{1}{p}}}{2^{\frac{1}{q}+\frac{N}{q}} \pi^{\frac{N}{2q}} N^{\frac{1}{q'}} \left(1-\frac{1}{q'}\left(\frac{1}{q'}+\frac{1}{p}\right)\right)^{\frac{1}{q'}+\frac{1}{p}} [\Gamma(\frac{N}{2})]^{\frac{1}{q'}} \Gamma\left(\frac{N}{2q}\right)} & \text{if } 2\alpha = \frac{N}{q}, \\ & q \leq p < \infty, \end{cases} \tag{2.2}$$

respectively, using Stirling's formula in the case $1 < q < \infty$ and $0 < 2\alpha < \frac{N}{q}$ of (2.1). The inverse type embeddings to (2.1);

$$L^{\Theta}(\Omega) \subset E_q^{-\alpha} \iff \infty \geq \Theta \begin{cases} \geq & \frac{qN}{N+2\alpha q} & \text{if } 2\alpha < \frac{N}{q'}, \\ > & 1 & \text{if } 2\alpha = \frac{N}{q'}, \end{cases} \tag{2.3}$$
$$\text{and } \mathcal{M}(\Omega) \subset E_q^{-\alpha} \quad \text{if } 2\alpha > \frac{N}{q'}$$

are as well verified.

Thanks to [3, 11, 30, 31], the semigroup (1.12) smoothening effect estimates

$$\|S(t)\varphi_0\|_{\alpha} \leq \frac{Me^{-\omega t}}{t^{\alpha-\beta}} \|\varphi_0\|_{\beta}, \quad t > 0, \quad \varphi_0 \in E_q^{\beta}, \tag{2.4}$$

whenever $\alpha \geq \beta$, for $M \geq 1$, $\omega > 0$, are satisfied.

As for the notion of a strong solution to (1.9) (as given in (1.1)) to be used, we observe that in equivalence to (1.13), we have the integral equations

$$\begin{aligned} v(t) &= e^{(\Delta-\lambda)t} v_0 + a \int_0^t e^{(\Delta-\lambda)(t-s)} w(s) ds, \\ w(t) &= e^{\Delta t} w_0 - \int_0^t e^{\Delta(t-s)} P(u(s)) ds, \end{aligned} \tag{2.5}$$

using the notation of the weak non-linear form $P(u) \in E_{q'}^{-\beta}$ in scale of Banach spaces introduced in Lemma 1.2. Accordingly, as per integral formula, we have the following definition;

Definition 2.1.

(i) If $w \in L^{\sigma}(\dot{I}; E_q^{\beta})$, $1 \le \sigma \le \infty$, a function v satisfying (1.25), (2.5) and equation in (1.1) in distribution sense as an identity in E_q^{β}, is a strong solution.

(ii) A function $w \in E_q^{\alpha}$ for which (1.22) is well defined in E_q^{γ}, with $0 \le \alpha - \gamma < 1$, and (1.27) holds, as well as the equation in (1.1) in distribution sense as an identity in E_q^{γ}, is a strong solution.

(iii) If (i)-(ii) above are satisfied, then $u = (v, w)^{\top}$ is a strong solution to (1.9) and the equation is verified in distribution sense as an identity in $Z_q^{\beta+\gamma} = E_q^{\beta} \times E_q^{\gamma}$.

We now proceed to the main sections of the paper. Note that in most of the proofs, we use the subcritical case of the spaces embedding (2.1), with obvious changes (although not necessarily trivial) in the critical and supercritical cases, which we might not highlight to shorten an already too lengthy paper.

3 Proof of Theorem 1.4

We carry out the proof of Theorem 1.4 in a sequence of lemmas, starting with the proof of Lemma 1.2.

Proof. of Lemma 1.2. It suffices to note that the proof follows by space embeddings (2.1) and Hölder's inequality. In fact, the mappings

$$\begin{aligned} E_q^{\alpha} \times E_q^{\gamma} \times E_{q'}^{\alpha} &\ni (v, w, \psi) \mapsto (w\chi\nabla v, \psi) \in E_q^{\alpha} \times E_{q'}^{\alpha} \quad \text{and} \\ E_q^{\alpha} \times E_{q'}^{\alpha} &\ni (w\chi\nabla v, \psi) \mapsto \chi \int_{\Omega} w\nabla v \nabla\psi \in \mathbb{R} \end{aligned}$$

are well defined and continuous, since $\nabla v \in E_q^{\alpha-\frac{1}{2}} \subset E_q^0 = L^q(\Omega)$ if $\alpha \ge \frac{1}{2}$. Thus, $w\nabla v \in E_q^0$, using (2.1) of E_q^{α}, provided that $\gamma \ge \frac{N}{2q}$, as one needs that $\frac{1}{q} - \frac{2\gamma}{N} + \frac{1}{q} \le \frac{1}{q}$.

Furthermore, relaxing the embedding into space for $E_q^{\alpha-\frac{1}{2}}$ yields that $\frac{1}{q} - \frac{2\gamma}{N} + \frac{1}{q} - \frac{2\alpha}{N} + \frac{1}{N} \le \frac{1}{q}$, giving our conclusion, as long as the first condition in (1.21) is satisfied. Thanks again to the space embeddings (2.1) and Hölder's inequality in more general setting, we require

$$1 \ge \frac{N - 2(\alpha - \frac{1}{2})q}{qN} + \frac{N - 2\gamma q}{qN} + \frac{N - 2(\alpha - \frac{1}{2})q'}{q'N}$$

must hold. This implies that the second hypothesis of (1.21) has to be satisfied. Consequently,

$$
\begin{aligned}
|\int_{\Omega}\chi w\nabla v\nabla\psi| &\leq \|w\chi\nabla v\|_{\Theta}\|\nabla\psi\|_{\Theta'} \\
&\leq \chi\left(\frac{2}{Ne\pi}\right)^{\alpha-\frac{1}{4}}\|w\nabla v\|_{\alpha,q}\|\nabla\psi\|_{\alpha-\frac{1}{2},q'} \\
&\leq \chi\left(\frac{2}{Ne\pi}\right)^{\alpha-\frac{1}{2}}\|w\|_{\Theta_0}\|\nabla v\|_{\alpha-\frac{1}{2},q}\|\nabla\psi\|_{\alpha-\frac{1}{2},q'} \\
&\leq \chi\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\gamma}{2}-\frac{1}{2}}\|w\|_{\gamma,q}\|v\|_{\alpha,q}\|\psi\|_{\alpha,q'}, \qquad (3.1)
\end{aligned}
$$

using (2.1) and (2.2), taking $\frac{1}{\Theta}=\frac{1}{\Theta_0}+\frac{1}{\Theta_1}$. Lastly, we recognize that the linearity of the mapping implies that it is Lipschitz continuous. The proof of the lemma is complete. □

Now in proceedings to prove (i) and (ii) of Theorem 1.4, we observe that in (1.25)-(1.27), the initial smoothness of solutions are due to the fact that an analytic semigroup (1.12) is as well a C^0- semigroup, hence [30, 31] yields the assertions, using (2.5).

It as well follows, with either $\alpha=\beta$ if $\sigma=1$, or $0\leq\alpha-\beta<\frac{1}{\sigma'}$ if $1<\sigma<\infty$, that

$$
\begin{aligned}
\|v(t)\|_{\alpha} &\leq \|e^{(\Delta-\lambda)t}v_0\|_{\alpha}+a\int_0^t\|e^{(\Delta-\lambda)(t-s)}w(s)\|_{\alpha}\,ds \\
&\leq Mt^{-(\alpha-\beta)}\|v_0\|_{\beta}+aM\int_0^t(t-s)^{-(\alpha-\beta)}\|w(s)\|_{\beta}\,ds \\
&\leq Mt^{-(\alpha-\beta)}\|v_0\|_{\beta}+aM\left(\int_0^t(t-s)^{-\sigma'(\alpha-\beta)}\,ds\right)^{\frac{1}{\sigma'}}\left(\int_0^t\|w(s)\|_{\beta}^{\sigma}\,ds\right)^{\frac{1}{\sigma}} \\
&\leq Mt^{-(\alpha-\beta)}\|v_0\|_{\beta}+aM\left(\frac{1}{1-\sigma'(\alpha-\beta)}\right)^{\frac{1}{\sigma'}}t^{\frac{1}{\sigma'}-(\alpha-\beta)}\|w(t)\|_{\sigma,\beta}, \qquad (3.2)
\end{aligned}
$$

which imply boundedness of the $v-$ solution on finite time intervals, away from $t=0$. In particular, $v\in\mathcal{L}^{\infty}_{\alpha-\beta}(\dot{I};E^{\alpha}_{q})$.

To prove the continuity, fix $t>0$ (or even $t=0$ if $v_0\in E^{\alpha}_{q}$), $h>0$, then compute using (2.5), that

$$v(t+h)-v(t)=e^{(\Delta-\lambda)h}v(t)-v(t)+a\int_t^{t+h}e^{(\Delta-\lambda)(t+h-s)}w(s)ds.$$

Taking the norm, we get that

$$
\begin{aligned}
\|v(t+h)-v(t)\|_{\alpha} \leq \|(e^{(\Delta-\lambda)h}-I)v(t)\|_{\alpha} &+ \\
+ \quad aM\int_t^{t+h}(t+h-s)^{-(\alpha-\beta)}\|w(s)\|_{\beta}ds& \\
\leq \quad \|(e^{(\Delta-\lambda)h}-I)v(t)\|_{\alpha}+aM\left(\int_t^{t+\tau}(t+h-s)^{-\sigma'(\alpha-\beta)}ds\right)^{\frac{1}{\sigma'}} &\times \\
\times \quad \left(\int_t^{t+\tau}\|w(s)\|_{\beta}^{\sigma}ds\right)^{\frac{1}{\sigma}}& \\
\leq \quad \|(e^{(\Delta-\lambda)h}-I)v(t)\|_{\alpha}+M_{1-\sigma'(\alpha-\beta)}\|w\|_{\sigma,\beta}h^{\frac{1}{\sigma'}-(\alpha-\beta)} &\searrow 0,
\end{aligned}
$$

as $h \searrow 0$, concluding the desired continuity of the $v-$ solution.

Next, for any $\beta, \alpha \in \mathbb{R}$ such that $\beta \leq \alpha \leq \beta + \frac{1}{\sigma'}$, if we let

$$c_{\beta,\alpha}(t) = e^{(\Delta-\lambda)t} \in L^1(0,\infty),$$

then $c_{\beta,\alpha}(t)$ is not bounded at $t = 0$, unless $\alpha = \beta$. Coupled with this, if $\sigma = 1$, then we let

$$\varphi(t) = \int_0^t e^{(\Delta-\lambda)(t-s)} w(s)ds,$$

so that, since

$$e^{(\Delta-\lambda)(t-s)} v_0 \in L^1(0,\infty; E_q^\alpha)$$

whenever $v_0 \in E_q^\alpha$, we only need to prove that $\varphi(t) \in L^1(0,\infty; E_q^\alpha)$. Thus, let $s = t\rho$ for $\rho \in [0,1]$ fixed. Then we get that

$$\begin{aligned} \|\varphi(t)\|_{1,\alpha} &\leq \int_0^1 \|\varphi(t)\|_{1,\alpha} d\rho = \int_0^1 \int_0^\infty \|e^{(\Delta-\lambda)t(1-\rho)} w(t\rho)\|_\alpha dt \\ &\leq \int_0^1 \int_0^\infty \frac{r}{\rho^2} c_{\beta,\alpha}(r(1-\rho)\rho^{-1}) \|w(r)\|_\beta dr d\rho \\ &\leq \left(\int_0^\infty c_{\beta,\alpha}(s) ds \right) \left(\int_0^\infty \|w(r)\|_\beta dr \right), \end{aligned}$$

in which the last inequality follows from changes to the time variables $r = t\rho$, $s = r(\frac{1-\rho}{\rho})$, then integrating with respect to ρ. Consequently,

$$\|v(t)\|_{1,\alpha} \leq \|c_{\alpha,\alpha}(t)\|_1 \|v_0\|_\alpha + \|c_{\beta,\alpha}(s)\|_1 \|w(r)\|_{1,\beta}.$$

The case $\sigma = \infty$ is proven as in the first lines of the proof to (i) of the theorem. The rest is by using interpolation. Thus, the second-last result in (1.26) is proven.

Furthermore, owing to (2.5), if we apply ∇ to the $v-$ formula and taking the norm in $\alpha - \frac{1}{2}$, we obtain that

$$\begin{aligned} \|\nabla v\|_{\alpha-\frac{1}{2}} &\leq \|\nabla\left(e^{\Delta-\lambda)t} v_0\right)\|_{\alpha-\frac{1}{2}} + a\int_0^t \|\nabla\left(e^{\Delta-\lambda)(t-s)} w(s)\right)\|_{\alpha-\frac{1}{2}} ds \\ &\leq Mt^{-(\alpha-\beta)}\|v_0\|_\beta + aM\int_0^t (t-s)^{-(\alpha-\beta)}\|w(s)\|_\beta ds \\ &\leq Mt^{-(\alpha-\beta)}\|v_0\|_\beta + aM\left(\int_0^t (t-s)^{-\sigma'(\alpha-\beta)}\right)^{\frac{1}{\sigma'}} \|w\|_{\sigma,\beta} \qquad (3.3) \\ &\leq Mt^{-(\alpha-\beta)}\|v_0\|_\beta + aM_{1-\sigma(\alpha-\beta)} t^{\frac{1}{\sigma'}-(\alpha-\beta)}\|w\|_{\sigma,\beta} \end{aligned}$$

holds. This is because (1.21) is assumed to hold, and thus $\alpha - \frac{1}{2} \geq \beta$ provided that $\beta \geq \frac{N}{2q}$ and $t \in (0,T)$ is large. Therefore, $\nabla v \in \mathcal{L}^\infty_{\alpha-\beta}(0,\infty; E_q^{\alpha-\frac{1}{2}})$. Hence, if we set $f(t) = t^{\alpha-\beta}\|\nabla v\|_{\alpha-\frac{1}{2}}$

and consider the $w-$equation of (2.5), then we get that

$$\begin{aligned}
\|w(t)\|_\alpha &\leq Mt^{-(\alpha-\gamma)}\|w_0\|_\gamma + \int_0^t \|\nabla e^{\Delta(t-s)}(w\chi\nabla v)(s)\|_\alpha ds \\
&\leq Mt^{-(\alpha-\gamma)}\|w_0\|_\gamma + \chi M\int_0^t (t-s)^{-\left(\frac{1}{2}+\alpha-\beta\right)}\|(w\nabla v)(s)\|_\beta ds \\
&\leq Mt^{-(\alpha-\gamma)}\|w_0\|_\gamma + \chi M\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\gamma}{2}-\frac{1}{2}} \times \\
&\quad \times \int_0^t (t-s)^{-\left(\frac{1}{2}+\alpha-\beta\right)}\|w(s)\|_\gamma\|\nabla v(s)\|_{\alpha-\frac{1}{2}} ds \\
&\leq Mt^{-(\alpha-\gamma)}\|w_0\|_\gamma + \chi M\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\gamma}{2}-\frac{1}{2}} \times \\
&\quad \times \int_0^t (t-s)^{-\left(\frac{1}{2}+\alpha-\beta\right)} f(s) s^{-(\alpha-\beta)}\|w(s)\|_\gamma ds \\
&= Mt^{-(\alpha-\gamma)}\|w_0\|_\gamma + \chi M\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\gamma}{2}-\frac{1}{2}} J,
\end{aligned}$$

where $J = \int_0^t (t-s)^{-\left(\frac{1}{2}+\alpha-\beta\right)} f(s) s^{-(\alpha-\beta)}\|w(s)\|_\gamma ds$. Making a change of time variable $s=\rho t$ leads to

$$\begin{aligned}
J &\leq \\
&\sup_{t>0} f(t)\left(\int_0^1 t^{-\sigma'\left(\frac{1}{2}+\alpha-\beta\right)} t^{-\sigma'(\alpha-\beta)}\left(\frac{1}{(1-\rho)^{\sigma'\left(\frac{1}{2}+\alpha-\beta\right)}\rho^{\sigma'(\alpha-\beta)}}\right) d\rho\right)^{\frac{1}{\sigma'}} \|w\|_{\sigma,\beta} \\
&\leq t^{-\left(\frac{1}{2}+2(\alpha-\beta)\right)} \sup_{t>0} f(t)\left(\int_0^1 \frac{1}{(1-\rho)^{\sigma'\left(\frac{1}{2}+\alpha-\beta\right)}\rho^{\sigma'(\alpha-\beta)}} d\rho\right)^{\frac{1}{\sigma'}} \|w\|_{\sigma,\beta} \\
&\leq t^{-\left(\frac{1}{2}+(\alpha-\beta)\right)} \sup_{t>0}\|\nabla v\|_{\alpha-\frac{1}{2}} \times \\
&\quad \times \left(\int_0^1 \frac{1}{(1-\rho)^{\sigma'\left(\frac{1}{2}+\alpha-\beta\right)}\rho^{\sigma'(\alpha-\beta)}} d\rho\right)^{\frac{1}{\sigma'}} \|w\|_{\sigma,\beta}. \qquad (3.4)
\end{aligned}$$

A backward combination with (3.4) yields $\limsup_{t\nearrow\infty}\|w\|_\alpha = 0$. If we allow the exponential decay effect of the semigroup (2.4) in the norm estimates of (3.3) in $\sigma=\infty$, then we conclude that the last statement in (1.26) is true. In fact,

$$\limsup_{t\nearrow+\infty}\|\nabla v\|_{\alpha-\frac{1}{2}} \leq aM\left(\int_0^\infty \frac{e^{-\omega t}}{t^{\alpha-\beta}} dt\right)\limsup_{t\nearrow+\infty}\|w\|_\beta = 0,$$

from which our result follows.

To complete the proof of (i), we need extra results on (ii).

Lemma 3.1. *Let $w \in E_q^\gamma$ be as given in* (2.5) *and $\Xi = \alpha-\beta \in (0,1)$. Then, $w \in C_{loc}^\Xi((0,T);E_q^\beta)$. That is, w is Hölder continuous in time.*

Proof. Indeed, if $0 < t < t+h < T$, then

$$\begin{aligned} w(t+h) - w(t) &= (e^{\Delta h} - I)e^{\Delta t} w_0 + \int_0^t (e^{\Delta h} - I)e^{\Delta(t-s)} \nabla(w(s)\chi \nabla v(s))ds \\ &+ \int_t^{t+h} e^{\Delta(t+h-s)} \nabla(w(s)\chi\nabla v(s))ds. \end{aligned}$$

Thus, by virtue of Lemma 1.2, taking $\gamma = \beta$, we get that

$$\begin{aligned} ||w(t+h) - w(t)||_\beta &\leq ||(e^{\Delta h} - I)e^{\Delta t} w_0||_\beta \quad + \\ &+ \int_0^t ||(e^{\Delta h} - I)e^{\Delta(t-s)} \nabla(w(s)\chi\nabla v(s))||_\beta ds \quad + \\ &+ \int_t^{t+h} ||e^{\Delta(t+h-s)} \nabla(w(s)\chi\nabla v(s))||_\beta ds \\ \leq & \quad M_{\alpha-\beta} h^{\alpha-\beta} ||e^{\Delta t} w_0||_\alpha + M_{\alpha-\beta} h^{\alpha-\beta} \int_0^t ||\nabla e^{\Delta(t-s)} (w(s)\chi\nabla v(s))||_\alpha ds \\ &+ \int_t^{t+h} ||\nabla e^{\Delta(t+h-s)} (w(s)\chi\nabla v(s))||_\alpha ds \\ \leq & \quad M_{\alpha-\beta} M h^{\alpha-\beta} t^{-(\alpha-\beta)} ||w_0||_\beta + \chi M_{\alpha-\beta} M h^{\alpha-\beta} \int_0^t (t-s)^{-\frac{1}{2}-(\alpha-\beta)} \times \\ \times & \quad ||w(s)\nabla v(s)||_\beta ds + \chi M \int_t^{t+h} (t+h-s)^{-\frac{1}{2}-(\alpha-\beta)} ||w(s)\nabla v(s)||_\beta ds \\ \leq & \quad M_{\alpha-\beta} M h^{\alpha-\beta} t^{-(\alpha-\beta)} ||w_0||_\beta + \chi M \left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\beta}{2}-\frac{1}{2}} M_{\alpha-\beta} h^{\alpha-\beta} \quad \times \\ \times & \quad \int_0^t (t-s)^{-\frac{1}{2}-(\alpha-\beta)} ||w(s)||_\beta ||\nabla v(s)||_{\alpha-\frac{1}{2}} ds \quad + \\ + & \quad \chi M \left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\beta}{2}-\frac{1}{2}} \int_t^{t+h} (t+h-s)^{-\frac{1}{2}-(\alpha-\beta)} ||w(s)||_\beta ||\nabla v(s)||_{\alpha-\frac{1}{2}} ds \\ \leq & \quad \Bigg(M_{\alpha-\beta} M t^{-(\alpha-\beta)} ||w_0||_\beta + \chi \left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\beta}{2}-\frac{1}{2}} M_{\alpha-\beta} M_{1-(\alpha-\beta)} t^{1-(\alpha-\beta)} + \\ + & \quad \chi \left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\beta}{2}-\frac{1}{2}} M_{1-(\alpha-\beta)} \sup_{t\in(0,T)} \left\{ ||w||_\beta ||\nabla v||_{\alpha-\frac{1}{2}} \right\} \Bigg) h^{\alpha-\beta}, \end{aligned}$$

which gives the desired Hölder continuity of the $w-$ integral solution in (2.5), and the proof of the lemma is complete. □

Lemma 3.2. *Consider the subset*

$$W := \left\{ \psi \in C(I; E_q^\alpha);\ \sup_{t\in(0,T)} ||\psi(t)||_\alpha \leq C||\psi_0||_\gamma \right\}, \tag{3.5}$$

and let in (2.5)

$$\mathcal{F}(w)(t) = e^{\Delta t}w_0 - \int_0^t e^{\Delta(t-s)}P(u(s))ds.$$

Then,

(i) *$\mathcal{F}W \subset W$, i.e. it maps W to itself.*

(ii) *The mapping $\mathcal{F} : E_q^\beta \to E_q^\alpha$ is a contraction.*

(iii) *There exists a unique $w \in W$ such that $\mathcal{F}(w)(t) = w(t)$ is a solution to* (1.1) *up to maximal time $T^*(\|w_0\|_\gamma)$ of existence of solutions of* (1.9).

Proof. We first note that we can read the right hand side of (2.5), in taking the norm of $E_q^\alpha = E_q^\gamma \times E_q^{\alpha-\gamma}$, as in the scale spaces product, whereas, thanks to Lemma 1.2, $w\chi\nabla v$ is well defined in $E_q^0 \cong L^q(\Omega)$. Therefore, if $w \in W$, then we find that

$$\begin{aligned}
\|\mathcal{F}(w)(t)\|_\alpha &\leq M\|w_0\|_\gamma + M\int_0^t (t-s)^{-\frac{1}{2}-(\alpha-\gamma)}\|w\chi\nabla v\|_0 ds \\
&\leq M\|w_0\|_\gamma + \chi\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\gamma}{2}-\frac{1}{2}} M\int_0^t (t-s)^{-\frac{1}{2}-(\alpha-\gamma)}\|w\|_\gamma\|\nabla v\|_{\alpha-\frac{1}{2}} ds \\
&\leq M\|w_0\|_\gamma + \chi MC\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\gamma}{2}-\frac{1}{2}} \sup_{t\in(0,T)}\|\nabla v\|_{\alpha-\frac{1}{2}}\|w_0\|_\gamma \quad \times \\
&\quad \times \int_0^t (t-s)^{-\frac{1}{2}-(\alpha-\gamma)} ds \\
&\leq M\|w_0\|_\gamma + \chi MC\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\gamma}{2}-\frac{1}{2}} \sup_{t\in(0,T)}\|\nabla v\|_{\alpha-\frac{1}{2}}\|w_0\|_\gamma T^{\frac{1}{2}-(\alpha-\gamma)}.
\end{aligned}$$

Thus, with

$$T = \left(\left(\frac{1}{M} - \frac{1}{C}\right)\frac{1}{\chi\sup_{t\in(0,T)}\|\nabla v\|_{\alpha-\frac{1}{2}}}\left(\frac{2}{Ne\pi}\right)^{\frac{1-2\alpha-\gamma}{2}}\right)^{\frac{2}{1-2(\alpha-\gamma)}},$$

we get that (i) is satisfied.

To prove (ii), we evaluate $\mathcal{F}$ at $w_1, w_2 \in W$, using the same initial data, to obtain that

$$\begin{aligned}
\|\mathcal{F}(w_1)(t) - \mathcal{F}(w_2)(t)\|_\alpha &\leq \int_0^t \|\nabla e^{\Delta(t-s)}((w_1-w_2)\chi\nabla v)(s)\|_\alpha ds \\
&\leq M\int_0^t (t-s)^{-\frac{1}{2}-(\alpha-\gamma)}\|(w_1-w_2)\chi\nabla v\|_\gamma ds \\
&\leq \chi M\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\gamma}{2}-\frac{1}{2}}\int_0^t (t-s)^{-\frac{1}{2}-(\alpha-\gamma)}\|w_1-w_2\|_\gamma\|\nabla v\|_{\alpha-\frac{1}{2}} \\
&\leq \chi M\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\gamma}{2}-\frac{1}{2}} T^{\frac{1}{2}-(\alpha-\gamma)}\sup_{t\in(0,T)}\|\nabla v\|_{\alpha-\frac{1}{2}}\sup_{t\in(0,T)}\|w_1-w_2\|_\gamma.
\end{aligned}$$

Whence, (ii) is proven by taking

$$T < \left(\frac{1}{\chi M \sup_{t\in(0,T)} \|\nabla v\|_{\alpha-\frac{1}{2}}} \left(\frac{2}{Ne\pi} \right)^{\frac{1-2\alpha-\gamma}{2}} \right)^{\frac{2}{1-2(\alpha-\gamma)}},$$

and viewed together with (i) of the lemma.

Applying the Banach contraction mapping theorem followed by the use of the Picard's method, or classical continuation for extension of the finite existence time to maximal time $T^* = T(\|w_0\|_\gamma)$, yields the last assertion of the lemma. □

To complete the proof of the theorem, we prove (iii). But before we do this we make the following observation on the smoothness of the solution, as given in the theorem. The solution regularity in (1.25) holds, using [11, 30, 31]. Since (1.5) is a C^+ operator, Lemma 3.1 is valid, and $w(t) \in L^\sigma(\dot{I}; E_q^\beta)$ is Hölder continuous. Consequently, the conclusion is due to linear non-homogeneous evolutionary equations theory, which imply the time regularity of the solution component, even with T at ∞. Analogously, writing the weak form in Lemma 1.2-(1.22) as

$$g(t) = \langle (w\chi\nabla v)(t), \nabla\varphi \rangle_{q,q'}, \quad \text{for any} \quad \varphi \in E_{q'}^\alpha, \tag{3.6}$$

we conclude that (1.27) holds, because $\nabla v \in E_q^{\alpha-\frac{1}{2}}$ is bounded, and by Lemma 3.1, $w \in C^\Xi(\dot{I}, E_q^\beta)$ for $0 \le \Xi = \alpha - \beta < 1$, yielding that (3.6) is Hölder continuous in time. Furthermore, from [11, 30, 31] (in particular [11]; Lemma 3.2.1 and Theorem 3.2.2.) we get the existence and uniqueness of solutions to (1.9)-(1.1). The fact that the solution is given by (2.5) follows from Definition 2.1.

Now, to prove the generation of a perturbed analytic semigroup, we have the following lemma.

Lemma 3.3. *Assume in* (1.20) *that* (1.28) *holds. Then,* (1.20) *is an infinitesimal generator of a perturbed analytic semigroup in scale spaces* $Z_q^{\beta+\gamma}$*, and the strong solution coincides with the one generated by the operator* (1.20).

Proof. First, we observe from what has been proved up to now that (3.2) implies $v \in \mathcal{L}^\infty_{\alpha-\beta}(0,T;E_q^\alpha)$, and $\limsup_{t\nearrow+\infty} t^{\alpha-\beta}\|v(t)\|_\alpha \le M\|v_0\|_\beta$, using (3.4) with $\sigma = \infty$, while still with (3.4) we obtain $\limsup_{t\nearrow+\infty} t^{(\beta-\gamma)}\|w(t)\|_\beta \le M\|w_0\|_\gamma$, and the assertion should follow. More precisely, to complete ideas, we prove that (1.20) is well defined, continuous, coercive, strictly monotone and is a sectorial operator in $E_q^0 \cong L^q(\Omega)$.

To this end, we define $b : Z_q^{\beta+\gamma} \times Z_q^{\beta+\gamma} \longmapsto \mathbb{R}$ by

$$b(u,z) = \int_\Omega \nabla v \nabla\varphi + \lambda \int_\Omega v\varphi + \int_\Omega \nabla w \nabla\psi - \chi \int_\Omega w \nabla v \nabla\psi - a \int_\Omega w\varphi, \tag{3.7}$$

where $z = (\varphi,\psi)^\top$. Note that, since Lemma 1.2-(1.21) is assumed, continuity of the mapping (3.7) is clear. We therefore only need to prove the coercivity (since if we apply Browder-Minty theorem, strict monotonicity can be easily deduced).

Thus, taking $z = u$, we find that

$$
\begin{aligned}
b(u,u) &\geq \|\nabla v\|^q_{\beta-\frac{1}{2}} + \\
&+ \|\nabla w\|^q_{\gamma-\frac{1}{2}} - \chi\left(\frac{2}{Ne\pi}\right)^{\beta+\frac{\gamma}{2}-\frac{1}{2}} \|w\|_\gamma \|\nabla v\|_{\beta-\frac{1}{2}} \|\nabla w\|_{\gamma-\frac{1}{2}} \\
&- \frac{a}{q}\left(\frac{2}{Ne\pi}\right)^{\beta+\frac{\gamma}{2}-\frac{1}{2}} \|\nabla w\|^q_{\gamma-\frac{1}{2}} - \frac{a}{q}\left(\frac{2}{Ne\pi}\right)^{\beta+\frac{\gamma}{2}-\frac{1}{2}} \|\nabla v\|^q_{\beta-\frac{1}{2}} \\
&\geq \left(1 - \frac{2\chi + a}{q}\left(\frac{2}{Ne\pi}\right)^{\beta+\frac{\gamma}{2}-\frac{1}{2}}\right) \|u\|^q_{\beta+\gamma},
\end{aligned}
\tag{3.8}
$$

implying the coercivity of (3.7), using (1.28). Thus, (1.20) is uniquely invertible by using Browder-Minty's theorem, and is a sectorial operator in $E^0_q \cong L^q(\Omega)$, since

$$
\begin{aligned}
\|(\mathcal{A}+\mu)^{-\alpha}\tilde{P}\|_0 &= \sup_{\|u\|_0\leq 1} \leq \left\{\frac{\|(\mathcal{A}+\mu)^{-\alpha}\tilde{P}(u)\|_0}{\|u\|_0}\right\} \\
&\leq \frac{C}{\mu^\alpha}(a + \|P\|_{\alpha,0}) \leq \frac{C}{\mu^\alpha}\left(a + \left(\frac{2}{Ne\pi}\right)^{\alpha-\frac{1}{2}}\right)
\end{aligned}
$$

for any $0 \leq \alpha < 1$ satisfying Lemma 1.2-(1.21), for some $C \in \mathbb{R}^+$, $|\pi - \arg\mu| \geq \vartheta$, $\vartheta < \frac{\pi}{2}$. The conclusion of the lemma is obtained by using Corollary 1.4.5 in [11]. Clearly, (1.14) and (1.26) imply that (1.29) is true. The proof of the lemma is complete. □

To complete the proof of Theorem 1.4-(iv), it suffices to note that, since $\alpha - \frac{1}{2} > \frac{N}{2q}$, we have $E_q^{\alpha-\frac{1}{2}} \subset L^\infty(\Omega)$ by virtue of (2.1), and Theorem 1.4-(1.26) implies that $\nabla v \in L^\infty(\Omega)$ is bounded for all $t > 0$. Since $w \in E^0_q \cong L^q(\Omega)$, $q > \frac{N}{2}$ because $1 \geq \alpha - \frac{1}{2} > \frac{N}{2q}$. Viewing the weak form (1.22) in $L^q(\Omega)$, as well as the equation in elliptic form by passing w_t to the right hand side, and using [30], we get that $w \in L^\infty(\Omega)$ is bounded for all $t > 0$. The rest is trivial or immediate. The proof of Theorem 1.4 is complete.

4 Uniform boundedness

In this section, we give an alternative proof for Theorem 1.4-(iv) without using the space embeddings. More precisely, we have the following theorem.

Theorem 4.1. *Assume that the minimal condition of* (1.21) *is attained strictly. If* $\gamma = 0$, *then* $u = (v,w) \in L^\infty(0,\infty; H^{1,\infty}(\Omega) \times L^\infty(\Omega))$,

$$
\sup_{t>0} \|u\|_{\frac{1}{2},\infty;\infty} \leq M\left(t^{-(\alpha-\beta)}\|v_0\|_\beta + t^{-\beta}\|w_0\|_0\right) + C,
\tag{4.1}
$$

and the solution semigroup to (1.9) *is a classical solution semigroup.*

Proof. Assume $w_0 = 0$, and let $\frac{N}{2} < q \leq N$, $|w|^{q-2}w \in E_2^{\frac{1}{2}} \cong H^1(\Omega)$. Then we find, using the second line from above of (3.1) and Gagliardo-Nirenberg's inequality [11], that

$$\frac{1}{q}\frac{d}{dt}\int_\Omega |w|^q + \frac{4(q-1)}{q^2}\int_\Omega |\nabla |w|^{\frac{q}{2}}|^2 = \chi\int_\Omega w\nabla v\nabla(|w|^{q-2}w)$$
$$\leq \quad (q-1)\chi\int_\Omega |\nabla v|w|^{q-1}\nabla w| = \frac{2\chi(q-1)}{q}\int_\Omega |w|^{\frac{q}{2}}\nabla|w|^{\frac{q}{2}}|\nabla v|$$
$$\leq \quad \frac{2\chi(q-1)}{q}\left(\frac{2}{Ne\pi}\right)^{\frac{1}{4}}||\nabla v||_{\infty,\infty}\left(||\nabla|w|^{\frac{q}{2}}||_2|||w|^{\frac{q}{2}}||_2\right)$$
$$\leq \quad \frac{2\chi(q-1)}{q}\left(\frac{2}{Ne\pi}\right)^{\frac{1}{4}}||\nabla v||_{\infty,\infty}||\nabla|w|^{\frac{q}{2}}||_2\left(\left(\frac{2}{Ne\pi}\right)^{\frac{1}{4}}||\nabla|w|^{\frac{q}{2}}||_2^{\frac{N}{N+2}}\times\right.$$
$$\left.\times|||w|^{\frac{q}{2}}||_1^{1-\frac{N}{N+2}} + \int_\Omega |w|^{\frac{q}{2}}\right)$$
$$\leq \quad \frac{2\chi(q-1)}{q}\left(\frac{2}{Ne\pi}\right)^{\frac{1}{2}}||\nabla v||_{\infty,\infty}||\nabla|w|^{\frac{q}{2}}||_2^{1+\frac{N}{N+2}}|||w|^{\frac{q}{2}}||_1^{1-\frac{N}{N+2}} \quad +$$
$$+ \quad \frac{2\chi(q-1)}{q}\left(\frac{2}{Ne\pi}\right)^{\frac{1}{4}}||\nabla v||_{\infty,\infty}||\nabla|w|^{\frac{q}{2}}||_2\int_\Omega |w|^{\frac{q}{2}}.$$

This yields, after multiplying throughout by q and using Young's inequality, that

$$\frac{d}{dt}\int_\Omega |w|^q + \frac{4}{q'}\int_\Omega |\nabla|w|^{\frac{q}{2}}|^2$$
$$\leq \quad \left(\frac{2}{Ne\pi}\right)^{\frac{1}{2}}\left(1+\frac{N}{N+2}\right)\int_\Omega |\nabla|w|^{\frac{q}{2}}|^2 \quad +$$
$$+ \quad \left((2q\chi||\nabla v||_{\infty,\infty})^{N+2} + (2q\chi||\nabla v||_{\infty,\infty})^2\right)\left(\int_\Omega |w|^{\frac{q}{2}}\right)^2$$
$$\leq \quad \left(\frac{2}{Ne\pi}\right)^{\frac{1}{2}}\left(1+\frac{N}{N+2}\right)\int_\Omega |\nabla|w|^{\frac{q}{2}}|^2 + (2\chi N||\nabla v||_{\infty,\infty})^2 \quad \times$$
$$\times\left(1+q^N\right)\left(\int_\Omega |w|^{\frac{q}{2}}\right)^2,$$

where we have used the fact that for $T \gg 1$ sufficiently large, $||\nabla v||_{\infty,\infty} \ll 1$ is adequately small.

Thus, if we set

$$\omega = \frac{4}{q'} - \left(\frac{2}{Ne\pi}\right)^{\frac{1}{2}}\left(1+\frac{N}{N+2}\right) > 0, \quad \text{and} \quad C_\Omega = (2\chi N)^2,$$

then we get that

$$\frac{d}{dt}\int_\Omega |w|^q + \omega\int_\Omega |w|^q \leq C_\Omega(1+q)^N\left(\int_\Omega |w|^{\frac{q}{2}}\right)^2$$
$$\implies \int_\Omega |w|^q \leq C_\Omega(1+q)^N \sup_{t>0}\left(\int_\Omega |w|^{\frac{q}{2}}\right)^2.$$

Now, let $\Lambda(q) = \int_\Omega |w|^q$, so that

$$\Lambda(q) \le [C_\Omega(1+q)^N]^{\frac{1}{q}} \Lambda\left(\frac{q}{2}\right), \quad \forall q \ge 2.$$

Consequently, if we let $q_i = 2^i, i \in \mathbb{N}^*$, then we conclude that

$$\begin{aligned}
\Lambda(2^i) &\le C_\Omega^{2^{-i}} (1+2^i)^{\frac{N}{2^i}} \Lambda(2^{i-1}) \\
&\le \ldots \le C_\Omega^{\sum_{k=1}^{i} 2^{-k}} (1+2^i)^{2^{-i}N} \ldots (1+2)^{2^{-1}N} \Lambda(1) \\
&\le C_\Omega \left[2^{i2^{-i}N}(2^{-i})^{2^{-i}N}\right] \ldots\ldots \left[2^{2^{-1}N}(2^{-1})^{2^{-1}N}\right] \Lambda(1) \\
&\le C_\Omega 2^{N\sum_{k=1}^{i} k2^{-k}} \times 2^{N\sum_{k=1}^{i} 2^{-k}} \Lambda(1) \le C_\Omega 2^{3N} \Lambda(1).
\end{aligned}$$

Therefore, taking the limit as $i \to \infty$ gives

$$\|w(t)\|_\infty \le C_\Omega 2^{3N} \Lambda(1) \le C_\Omega 2^{3N} \|w_0\|_1 < \infty. \tag{4.2}$$

Now, let's write $w(t) = \psi_1(t) + \psi_2(t)$, where $\psi_1(t)$ satisfies the homogeneous equation in (1.1) with $w(0) = w_0$, and $\psi_2(t)$ the non-homogeneous equation with, $w_0 = 0$. It follows, by (2.1) and (2.4), that $\|\psi_1(t)\|_\infty \le Mt^{-\frac{N}{2q}} \|w_0\|_0$ for all $t > 0$, while (4.2) implies $\|\psi_2(t)\|_\infty \le C$. Thus, we obtain

$$\|w(t)\|_\infty \le Mt^{-\beta} \|w_0\|_0 + C,$$

where $\beta = \frac{N}{2q}$, so that combining with the v-solution gives (4.1). The proof of the theorem is complete. □

5 Blow-up dynamics independent of time

In this section, we give some highlights on the blow-up dynamics of the system of equations (1.1) at the borderline spaces $E_q^\alpha, \alpha = \frac{N}{2p}$, and independent of the condition (1.28) yielding that the complete system coupled differential operator (1.20) is an infinitesimal generator of a perturbed analytic semigroup to the semigroup (1.12) defined by the uncoupled system differential operator (1.5). To this end, we first notice that the stationary equations to the system can be derived as a limit process at time ∞, to the following Lyapunov function

$$\mathcal{J}(t) = \int_\Omega w \ln w - \chi \int_\Omega wv + \frac{\chi}{aq} \int_\Omega (|\nabla v|^q + \lambda |v|^q), \tag{5.1}$$

using La-Salle- Hale-Henry invariance principle [11].

Theorem 5.1. *The dynamical system defined by the equations* (1.1) *admits* (5.1) *as a Lyapunov function, and the elliptic equation* (1.16) *is verified at* $T = \infty$ *with, if the initial data is in spaces* $E_q^\beta, \beta = \frac{N}{2q}$ *such that* $\chi > \chi_{N,\beta} = (1.18)$*, then*

$$\|(v, w)^\top\|_{\beta+\gamma} = \infty \quad \textit{for any } t \in (0, \infty).$$

Moreover, $\infty > \mathcal{J}(0) \ge \mathcal{J}(t) > -\infty$*, and Proposition 3.2 in [28] holds for blow-up of solutions for any finite time* $t^* \in (0, \infty)$.

Proof. To show that (5.1) is a Lyapunov function, we take the dual spaces product in (1.1) with $\ln w - \chi v \in E^{\gamma}_{q'}$ as a test function, in the $w-$ equation, and let $v_t \in E^{\beta}_{q'}$. Then we find that

$$
\begin{aligned}
\frac{d\mathcal{J}(t)}{dt} &= \int_\Omega w_t \ln w + \int_\Omega w_t - \chi \int_\Omega w_t v - \kappa \int_\Omega v_t w \quad + \\
+ \quad \frac{\chi}{a}\left(\int_\Omega |\nabla v|^{q-2}\nabla v \nabla v_t + \lambda \int_\Omega |v|^{q-2} v v_t\right) &= \int_\Omega w_t(\ln w - \chi v) - \frac{\chi}{a}\int_\Omega |v_t|^q \\
&= \int_\Omega \nabla(\nabla w - \chi w \nabla v)(\ln w - \chi v) - \frac{\chi}{a}\int_\Omega |v_t|^q \\
&= -\int_\Omega (\nabla w - \chi w \nabla v)\nabla(\ln w - \chi v) - \frac{\chi}{a}\int_\Omega |v_t|^q \\
&= -\int_\Omega w|\nabla(\ln w - \chi v)|^q - \frac{\chi}{a}\int_\Omega |v_t|^q \leq 0, \qquad (5.2)
\end{aligned}
$$

using the dual space function characterization for functions in $L^{q'}$, and the fact that

$$\int_\Omega w_t = 0, \quad w\nabla(\ln w - \chi v) = w\left(\frac{\nabla w}{w} - \chi \nabla v\right),$$

to yield that (5.1) is a Lyapunov function for the system of equations (1.1). The proof asserts that the Lyapunov function decreases along trajectories of orthogonal to constant solutions of the system of equations as time increases to infinity. Thanks to La-Salle-Hale-Henry invariance principle [11], at time $T = \infty$, we have that the stationary equations corresponding to (1.9) are verified, and consequently, so is the elliptic problem (1.16), since (1.28) is not assumed to be verified.

To prove the blow-up of solutions, we note that (3.8) holds, using the best constant of the inclusion $E^{\beta}_q, \beta = \frac{N}{2q}$ in (2.2), while, associated to (1.16) is the energy functional

$$\mathcal{E}(t) = \frac{1}{q}\|\nabla v\|^q_{\beta - \frac{1}{2}} + \frac{\lambda}{q}\|v\|^q_\beta - \frac{aK}{\chi}\int_\Omega (e^{\chi v} - 1). \qquad (5.3)$$

This results in (3.8) yielding

$$b(u,u) \geq \omega \|\nabla w\|^q_{\gamma - \frac{1}{2}} + \frac{aK}{\chi\omega}\int_\Omega (e^{\chi v} - 1),$$

using the second embedding condition in (2.2), If we take $u \in E^{\beta}_q \times E^{\gamma}_q$ as a test function in the complete system equations (1.9), and integrate in time $t \in (0,T)$, using a reduction to absurd argument, then we get our conclusion.

In fact, if we suppose that the conclusion was false, then we would get that

$$
\begin{aligned}
0 \quad &= \quad \frac{d}{dt}\|u\|^\rho_{\beta+\gamma} + b(u,u) \\
&\geq \quad \frac{d}{dt}\|u\|^\rho_{\beta+\gamma} + \omega\|\nabla w\|^q_{\gamma-\frac{1}{2}} + \frac{aK}{\chi\omega}\int_\Omega e^{\chi v} - \frac{aK|\Omega|}{\chi\omega} \\
&\Longleftrightarrow \quad \frac{aK|\Omega|T}{\chi\omega} + \|u_0\|^\rho_{\beta+\gamma} \geq \|u\|^\rho_{\beta+\gamma} + \frac{aK}{\chi\omega}\int_0^t \int_\Omega e^{\gamma v(s)} ds \\
&\geq \quad \frac{aK}{\chi\omega}\int_0^t \int_\Omega e^{\chi v(s)} ds = \infty,
\end{aligned}
$$

using (1.17) at $\chi = (1.18)$. The contrary to the premises is true, since the norm $\|u_0\|^{\rho}_{\beta+\gamma} = \|v_0\|^q_\beta + \|w_0\|^q_\gamma$, and $t \in (0,T)$ are finite. Therefore, the conclusion of the theorem is valid.

To prove the second half of the theorem, we first note that

$$
\begin{aligned}
\chi \int_\Omega wv &\leq \chi\|w\|_{q'}\|v\|_q \leq \chi|\Omega|^{\frac{1}{q}-\frac{1}{\Theta}}\|w\|_{q'}\|v\|_\Theta \\
&\leq \chi|\Omega|^{\frac{1}{q}-\frac{1}{\Theta}}\left(\frac{2}{Ne\pi}\right)^{\frac{1}{2}}\|w\|_{q'}\|v\|_{\frac{1}{2}} \\
&\leq \frac{1}{q'}\left(|\Omega|^{\frac{1}{q}-\frac{1}{\Theta}}\right)^q\|w\|^{q'}_{q'} + \frac{1}{q}\chi^q\int_\Omega(|\nabla v|^q + \lambda|v|^q),
\end{aligned}
$$

where we have used (2.1)-(2.2), Hölder's and Young inequalities. Now, since $\inf w\ln w = -\frac{1}{e}$, we have, from (5.1), that

$$
\begin{aligned}
\mathcal{J}(t) &\geq -\frac{|\Omega|}{e} - \frac{1}{q'}\left(|\Omega|^{\frac{1}{q}-\frac{1}{\Theta}}\right)^q\|w\|^{q'}_{q'} - \frac{1}{q}\left(\chi^q - \frac{\chi}{a}\right)\int_\Omega(|\nabla v|^q + \lambda|v|^q) \\
&> -\frac{|\Omega|}{e} - \frac{1}{q'}\left(|\Omega|^{\frac{1}{q}-\frac{1}{\Theta}}\right)^q\|w\|^{q'}_{q'} - \left(\chi^q - \frac{\chi}{a}\right)\frac{aK}{\chi}\int_\Omega e^{\chi v} = -\infty,
\end{aligned}
$$

following from (5.3) at $\chi = (1.18)$. On the other hand, thanks to (5.2), we have that $\mathcal{J}(t) \leq \mathcal{J}(0) < \infty$ for any $\chi \in (0,\infty)$ and for all $t > 0$. The conclusions of Proposition 3.2 of [28] are satisfied for any $q' > 1$.

Alternative proofs can be found in [8, 14] for the case E^α_q, $\alpha = \frac{1}{2}, q = 2$, using the Lyapunov function (5.1), embedding into Orlicz spaces [6, 20] and properties. The proof of the theorem is complete. □

6 Upper and lower finite time bounds for blow-up dynamics

In this section, we study finite time bounds for blow-up of solutions in $E^{\frac{1}{2}}_2 \cong H^1(\Omega)$ to the system of equations (1.1) in norms of $L^{2p}(\Omega)$-spaces. The following theorem yields existence of a finite time upper bound for blow-up of solutions using the concavity method in [4] .

Theorem 6.1. *Assume that Theorem 4.1 holds and that the classical solutions are bounded on $\Omega \times I$ with $T = t^*$. Then, there exists a finite time upper bound to the maximal time of existence of solutions*

$$
\begin{aligned}
t^* &\leq \frac{1}{\nu M(\mathcal{E}(0))^\nu}, \quad \textit{where} \quad \mathcal{E}(0) = \frac{\kappa}{4}\int_\Omega(v_0^2 + w_0^2) \\
M &= \frac{\mathcal{J}(0)}{(\mathcal{E}(0))^{1+\nu}} \quad \textit{with} \\
\mathcal{J}(0) &= -\frac{1}{2}\left(\int_\Omega|\nabla v_0|^2 + \lambda\int_\Omega v_0^2 - a\int_\Omega w_0v_0\right) \\
&- \frac{1}{2}\left(\int_\Omega|\nabla w_0|^2 - \chi\int_\Omega w_0\nabla v_0\nabla w_0\right)
\end{aligned}
$$

for some $\nu > 0$ *such that* $\liminf_{t\nearrow t^*} \|(v,w)^\top\|_0^2 = \infty$. *That is, solutions to system of equations* (1.1) *blow-up in finite time in norm of* $E_2^0 \cong L^2(\Omega)$, *with* $\Omega \subset \mathbb{R}^N$, $N = 2,3$.

Note that the restriction $\Omega \subset \mathbb{R}^N$, $N = 2,3$ follows as in the proof of Theorem 1.4-(iv). Since, if (1.28) is attained strictly, and $\gamma = 0$, then we have $1 \geq \alpha - \frac{1}{2} > \frac{N}{2q}$. This, combined with $q = 2$, yields that $4 > N$, and remains valid in Theorem 4.1.

Proof. Let $\kappa = \frac{1}{1+\nu}$ for some $\nu > 0$ and $\mathcal{E}(t) = \frac{\kappa}{4}\int_\Omega (v^2 + w^2)$. Then

$$
\begin{aligned}
\mathcal{E}'(t) &= \frac{\kappa}{2}\int_\Omega (v_t v + w_t w) \\
&= -\frac{\kappa}{2}\left(\int_\Omega |\nabla v|^2 + \lambda\int_\Omega v^2 - a\int_\Omega wv\right) \\
&\qquad - \frac{\kappa}{2}\left(\int_\Omega |\nabla w|^2 - \chi\int_\Omega w\nabla v\nabla w\right) \\
&\geq -\frac{1}{2}\left(\int_\Omega |\nabla v|^2 + \lambda\int_\Omega v^2 - a\int_\Omega wv\right) \\
&\qquad - \frac{1}{2}\left(\int_\Omega |\nabla w|^2 - \chi\int_\Omega w\nabla v\nabla w\right) := \mathcal{J}(t). \qquad (6.1)
\end{aligned}
$$

Next, we observe that if we multiply through (1.1) by $(v_t, w_t)^\top$ and then integrate by parts over Ω using the boundary conditions, then we have

$$
\begin{aligned}
\mathcal{J}'(t) &= -\left(\int_\Omega \nabla v\nabla v_t + \lambda\int_\Omega vv_t - a\int_\Omega wv_t - a\int_\Omega w_t v\right) \\
&\quad - \int_\Omega \nabla w\nabla w_t + \chi\int_\Omega w_t\nabla v\nabla w + \chi\int_\Omega w\nabla v_t\nabla w + \chi\int_\Omega w\nabla v\nabla w_t \\
&\geq \int_\Omega (v_t^2 + w_t^2).
\end{aligned}
$$

This leads us to conclude that

$$
\begin{aligned}
&a\int_\Omega w_t v = 0, \\
&\chi\int_\Omega w_t\nabla v\nabla w + \chi\int_\Omega w\nabla v_t\nabla w = 0. \qquad (6.2)
\end{aligned}
$$

Another way to justify the above is as follows; independent of the sign of the integrands, using the fact that solutions to (1.1) are classical solutions, one can find bounds above and below in terms of $\int_\Omega w_t = 0$, with, in the last conclusion, an argument leading to (6.2), to imply that $\chi\int_\Omega w\nabla v_t\nabla w = 0$.

Next, note that $\mathcal{J}(0) \geq 0$ and $\mathcal{J}'(t) \geq 0$ for $t \geq 0$, imply that $\mathcal{J}(t) \geq 0$ for $t \geq 0$. Now, from (6.1) we obtain, using Cauchy-Schwartz inequality, that

$$\begin{aligned}
\mathcal{E}'(t)\mathcal{E}'(t) &= (\mathcal{E}'(t))^2 \\
&= \frac{\kappa^2}{4}\left(\left(\int_\Omega vv_t\right)^2 + \left(\int_\Omega ww_t\right)^2 + 2\left(\int_\Omega vv_t\right)\left(\int_\Omega ww_t\right)\right) \\
&\le \frac{\kappa^2}{4}\left(\int_\Omega v^2 \int_\Omega v_t^2 + \int_\Omega w^2 \int_\Omega w_t^2 + 2\left(\int_\Omega v^2 \int_\Omega v_t^2 \int_\Omega w^2 \int_\Omega w_t^2\right)^{\frac{1}{2}}\right) \\
&\le \frac{\kappa^2}{4}\left(\int_\Omega (v^2+w^2)\right)\left(\int_\Omega (v_t^2+w_t^2)\right) \\
&= \kappa\mathcal{E}(t)\mathcal{J}'(t) = \frac{1}{1+\nu}\mathcal{E}(t)\mathcal{J}'(t),
\end{aligned}$$

where the second from last follows using the elementary inequality $\sqrt{ab} \le \frac{a+b}{2}, a,b \ge 0$. Thus, using (6.1) we obtain that

$$(1+\nu)\frac{\mathcal{E}'(t)}{\mathcal{E}(t)} \le \frac{\mathcal{J}'(t)}{\mathcal{J}(t)} \Rightarrow \frac{1}{(\mathcal{E}(t))^\nu} \le \frac{1}{(\mathcal{E}(0))^\nu} - \nu Mt,$$

using [25]. Since the implied inequality cannot be true for all $t \ge 0$, we infer that at least one of either v or w must blow-up in norm of $L^2(\Omega)$ in finite time. The last assertion follows from the yielding condition of Theorem 4.1, working in the function space $E_2^{1/2} \cong H^1(\Omega)$. □

In continuation, to find the lower finite time bound for blow-up of solutions to (1.1) via the differential inequality technique due to P.E. Payne, *et al.*[25, 27], the following Sobolev type inequality is required.

Lemma 6.2. *Let $\varphi \in C^1(\Omega) \cap C^+(\Omega)$, where the plus sign imply non-negative. Then,*

$$\int_\Omega \varphi^{\frac{3}{2}n} \le \left[\frac{3}{2\rho}\int_\Omega \varphi^n + \frac{n}{2}\left(1+\frac{d}{\rho}\right)\int_\Omega \varphi^{n-1}|\nabla\varphi|\right]^{\frac{3}{2}}, \tag{6.3}$$

where $\Omega \subset \mathbb{R}^N, N = 2,3$ is such that $\vec{0} \in \Omega$ and is a star-shaped, convex domain in two orthogonal directions , $n \ge 1$, $\rho = \min_\Gamma(x\cdot\vec{n})$, $d = \max_{\overline{\Omega}}|x|$.

The following Theorem provides the lower finite time bound for blow-up of solutions to (1.1).

Theorem 6.3. *Consider the system of equations* (1.1) *in $\Omega \subset \mathbb{R}^N, N = 2,3$ such that Lemma 6.2 and Theorem 4.1 hold, or equivalently, just $w \in L^\infty(\Omega)$ is bounded. Then, there exists a lower finite time bound for blow-up of solutions*

$$t^* \ge \int_{\mathcal{E}(0)}^\infty \frac{d\eta}{\eta(K_1\eta^2 + K_2\eta^{\frac{1}{2}} + K_3)}, \quad \textit{with} \quad \mathcal{E}(0) = \int_\Omega (v_0^{2p} + w_0^{2p}), \tag{6.4}$$

and on setting $\vartheta > 0$, $\nu > 0$, the computable constants $K_j, j = 1,2,3$ are

$$\begin{aligned}
K_1 &= \max\left\{\frac{|\Omega|^{\frac{1}{2}}}{3^{\frac{3}{4}}\times 4\vartheta^3}, \frac{a|\Omega|^{\frac{1}{3}}}{2\nu^3}\right\}\left(\frac{d}{\rho}+1\right)^{\frac{3}{2}}, \\
K_2 &= \max\left\{\frac{|\Omega|^{\frac{1}{2}}\sqrt{2}}{3^{\frac{3}{4}}}, \frac{2\sqrt{2}a|\Omega|^{\frac{1}{3}}}{3^{\frac{3}{4}}}\right\}\left(\frac{3}{2\rho}\right)^{\frac{3}{2}}, K_3 = \max\{21 + \frac{1}{p'}, 2p\lambda\}
\end{aligned}$$

such that solutions to the system of equations (1.1) *in* $E_2^{\frac{1}{2}} \cong H^1(\Omega)$ *blow-up in norm of* $L^{2p}(\Omega), 1 \le p \le 6$, *whenever* $t \nearrow t^*$, *i.e.* $\inf_{t\nearrow t^*} \|(v,w)^\top\|_{2p} = \infty$.

Proof. We carry out the proof of the theorem in $\Omega \subset \mathbb{R}^N, N = 3$. Consider the energy function

$$\mathcal{E}(t) = \int_\Omega (w^{2p} + v^{2p}).$$

Then, compute the time derivative through the system of equations, to find that

$$\begin{aligned}
\mathcal{E}'(t) &= 2p\int_\Omega (w^{2p-1}w_t + v^{2p-1}v_t) \\
&= 2p\int_\Omega w^{2p-1}\,(\nabla\cdot(\nabla w - w\chi\nabla v)) \quad + \\
&\quad + \quad 2p\int_\Omega v^{2p-1}\,(\Delta v - \lambda v + aw) = \sum_{j=1}^{2} I_j.
\end{aligned}$$

Next, we note that the alternative requirement of the hypotheses $w \in L^\infty(\Omega)$ implies that

$$v \in C^2(\Omega)\cap C^1(\Omega), \text{ and } \nabla\varphi^{2p-1} = (2p-1)\varphi^{2p-2}|\nabla\varphi|^2 = \frac{2p-1}{p^2}|\nabla\varphi^p|^2.$$

Therefore, we can write

$$I_1 = -\frac{2(2p-1)}{p}\int_\Omega |\nabla w^p|^2 + 2p(2p-1)\chi\int_\Omega w^{2p-1}\nabla w\nabla v, \tag{6.5}$$

from which, observing that $w^{2p-1}\nabla w = \frac{1}{2p}\nabla w^{2p} = \frac{1}{p}w^p\nabla w^p$, and since $v > 0$ is classical solution, there is

$$\delta = \sqrt{\inf_\Omega v^{p-1}} > 0 \quad \text{such that} \quad \frac{1}{(p\delta)^2}|\nabla v^p|^2 \ge \|\nabla v|^2.$$

This, in conjunction with the fact that $w \in L^\infty(\Omega)$, implies that

$$\begin{aligned}
\frac{1}{p}\int_\Omega w^p\nabla w^p\nabla v &\le \frac{1}{p}\left(\int_\Omega |w^p\nabla w^p|^2\right)^{\frac{\theta_0}{2}}\left(\int_\Omega |\nabla v|^2\right)^{\frac{\theta_3}{2}} \\
&\le \frac{1}{p(p\delta)^{\theta_3}}\left(\int_\Omega w^{2p}\right)^{\frac{\theta_1}{2}}\left(\int_\Omega |\nabla w^p|^2\right)^{\frac{\theta_1}{2}}\left(\int_\Omega |\nabla v^p|^2\right)^{\frac{\theta_3}{2}},
\end{aligned}$$

for some $0 < \sum_{j=1}^3 \frac{\theta_j}{2} = 1$ to be determined.

Thus, getting back to (6.5), adequately associating multiplying coefficients of the integrand involving v, and using the generalized Young's inequality [35], we obtain

$$\begin{aligned}
I_1 &\le -(2p-1)\left(\frac{2}{p} - \theta_2\right)\int_\Omega |\nabla w^p|^2 + \theta_1(2p-1)\int_\Omega w^{2p} \quad + \\
&\quad + \quad \frac{\theta_3\chi}{2p\delta}\int_\Omega |\nabla v^p|^2.
\end{aligned} \tag{6.6}$$

But $p \leq 6$, $\theta_1 < 2$, so we have

$$\theta_1(2p-1)\int_\Omega w^{2p} \leq 22\int_\Omega w^{2p} = 21\int_\Omega w^{2p} + \int_\Omega w^{2p}.$$

From this, controlling only the last added integrand term, using the fact that $L^{3p}(\Omega) \subset L^{2p}(\Omega)$, we get the following, with additional consequences due to (6.3) of Lemma 6.2;

$$\begin{aligned}
\int_\Omega w^{2p} &\leq \left(\int_\Omega w^{2p}\right)^{\frac{3}{2}} \leq |\Omega|^{\frac{1}{2}}\int_\Omega w^{3p} \\
&\leq |\Omega|^{\frac{1}{2}}\frac{1}{3^{\frac{3}{4}}}\left\{\frac{3}{2\rho}\int_\Omega w^{2p} + \left(\frac{d}{\rho}+1\right)\left(\int_\Omega w^{2p}\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla w^p|^2\right)^{\frac{1}{2}}\right\}^{\frac{3}{2}} \\
&\leq |\Omega|^{\frac{1}{2}}\frac{1}{3^{\frac{3}{4}}}\left\{\sqrt{2}\left(\frac{3}{2\rho}\right)^{\frac{3}{2}}\left(\int_\Omega w^{2p}\right)^{\frac{3}{2}} + \left(\frac{d}{\rho}+1\right)^{\frac{3}{2}}\left(\int_\Omega u^{2p}\right)^{\frac{3}{4}}\left(\int_\Omega |\nabla w^p|^2\right)^{\frac{3}{4}}\right\} \\
&\leq |\Omega|^{\frac{1}{2}}\frac{1}{3^{\frac{3}{4}}}\left\{\sqrt{2}\left(\frac{3}{2\rho}\right)^{\frac{3}{2}}\left(\int_\Omega w^{2p}\right)^{\frac{3}{2}} + \frac{1}{4\vartheta^3}\left(\frac{d}{\rho}+1\right)^{\frac{3}{2}}\left(\int_\Omega w^{2p}\right)^{3} + \right. \\
&\quad \left. + \frac{3\vartheta}{4}\left(\frac{d}{\rho}+1\right)^{\frac{3}{2}}\int_\Omega |\nabla w^p|^2\right\},
\end{aligned}$$

by virtue of the elementary inequalities $(a+b)^{\frac{3}{2}} \leq 2^{\frac{1}{2}}(a^{\frac{3}{2}}+b^{\frac{3}{2}})$, and $a^{\frac{1}{4}}b^{\frac{3}{4}} \leq \frac{1}{4}a+\frac{3}{4}b$, for numbers $a,b \in \mathbb{R}^+$, with weight $\vartheta > 0$ to be found. Therefore, (6.6) is extended in a manner of the sense that

$$\begin{aligned}
I_1 \leq\; & -\left((2p-1)\left(\frac{2}{p}-\theta_2\right) - \frac{3^{\frac{1}{4}}\vartheta|\Omega|^{\frac{1}{2}}}{4}\left(\frac{d}{\rho}+1\right)^{\frac{3}{2}}\right)\int_\Omega |\nabla w^p|^2 + 21\int_\Omega w^{2p} + \\
& + \frac{|\Omega|^{\frac{1}{2}}\sqrt{2}}{3^{\frac{3}{4}}}\left(\frac{3}{2\rho}\right)^{\frac{3}{2}}\left(\int_\Omega u^{2p}\right)^{\frac{3}{2}} + \frac{|\Omega|^{\frac{1}{2}}}{3^{\frac{3}{4}}\times 4\vartheta^3}\left(\frac{d}{\rho}+1\right)^{\frac{3}{2}}\left(\int_\Omega w^{2p}\right)^{3} + \\
& + \frac{\theta_3\chi}{2p\delta}\int_\Omega |\nabla v^p|^2,
\end{aligned} \tag{6.7}$$

so that with properly chosen $\theta_2, \vartheta > 0$, we must have

$$-\left((2p-1)\left(\frac{2}{p}-\theta_2\right) - \frac{3^{\frac{1}{4}}\vartheta|\Omega|^{\frac{1}{2}}}{4}\left(\frac{d}{\rho}+1\right)^{\frac{3}{2}}\right) \leq 0.$$

Now we pay attention to I_2, for which it holds that

$$\begin{aligned}
I_2 &= -2(2p-1)\int_\Omega |\nabla v^p|^2 - 2p\lambda\int_\Omega v^{2p} + 2pa\int_\Omega v^{2p-1}w \\
&\le -2(2p-1)\int_\Omega |\nabla v^p|^2 + 2p\lambda\int_\Omega v^{2p} + 2pa\left(\int_\Omega v^{3p}\right)^{\frac{2p-1}{3p}}\left(\int_\Omega w^{\frac{3p}{p+1}}\right)^{\frac{p+1}{3p}} \\
&\le -2(2p-1)\int_\Omega |\nabla v^p|^2 + 2p\lambda\int_\Omega v^{2p} + 2pa|\Omega|^{\frac{1}{3}}\left(\int_\Omega v^{3p}\right)^{\frac{2p-1}{3p}}\left(\int_\Omega w^{2p}\right)^{\frac{1}{2p}} \\
&\le -2(2p-1)\int_\Omega |\nabla v^p|^2 - 2p\lambda\int_\Omega v^{2p} + 2pa|\Omega|^{\frac{1}{3}-\frac{1}{6p}}\left(\int_\Omega v^{3p}\right)^{\frac{1}{p}}\left(\int_\Omega w^{2p}\right)^{\frac{1}{p'}} \\
&\le -2(2p-1)\int_\Omega |\nabla v^p|^2 + 2p\lambda\int_\Omega v^{2p} + 2a|\Omega|^{\frac{1}{3}}\int_\Omega v^{3p} + \frac{1}{p'}\int_\Omega w^{2p}.
\end{aligned}$$

Using similar arguments leading to (6.7) we find that

$$\begin{aligned}
&I_2 \le -2(2p-1)\int_\Omega |\nabla v^p|^2 + 2p\lambda\int_\Omega v^{2p} + 2a|\Omega|^{\frac{1}{3}}\times \\
\times\ & \frac{1}{3^{\frac{3}{4}}}\left\{\sqrt{2}\left(\frac{3}{2\rho}\right)^{\frac{3}{2}}\left(\int_\Omega v^{2p}\right)^{\frac{3}{2}} + \left(\frac{d}{\rho}+1\right)^{\frac{3}{2}}\left(\int_\Omega v^{2p}\right)^{\frac{3}{4}}\left(\int_\Omega |\nabla v^p|^2\right)^{\frac{3}{4}}\right\} \\
&\quad + \frac{1}{p'}\int_\Omega w^{2p} \\
\le\ & -\left(2(2p-1) - \frac{3a|\Omega|^{\frac{1}{3}}\nu}{2}\left(\frac{d}{\rho}+1\right)^{\frac{3}{2}}\right)\int_\Omega |\nabla v^p|^2 + 2p\lambda\int_\Omega v^{2p} \quad + \\
+\ & \frac{2\sqrt{2}a|\Omega|^{\frac{1}{3}}}{3^{\frac{3}{4}}}\left(\frac{3}{2\rho}\right)^{\frac{3}{2}}\left(\int_\Omega v^{2p}\right)^{\frac{3}{2}} + \frac{a|\Omega|^{\frac{1}{3}}}{2\nu^3}\left(\frac{d}{\rho}+1\right)^{\frac{3}{2}}\left(\int_\Omega v^{2p}\right)^3 \\
&\quad + \frac{1}{p'}\int_\Omega w^{2p},
\end{aligned}$$

with $\theta_3 > 0$ of (6.7), and $\nu > 0$ such that

$$-\left(2(2p-1) - \frac{3a|\Omega|^{\frac{1}{3}}\nu}{2}\left(\frac{d}{\rho}+1\right)^{\frac{3}{2}} - \frac{\theta_3\chi}{2p\delta}\right) \le 0.$$

Combining all the above, we get that

$$\begin{aligned}
\mathcal{E}'(t) &\le (21 + \frac{1}{p'})\int_\Omega w^{2p} + \frac{|\Omega|^{\frac{1}{2}}\sqrt{2}}{3^{\frac{3}{4}}}\left(\frac{3}{2\rho}\right)^{\frac{3}{2}}\left(\int_\Omega w^{2p}\right)^{\frac{3}{2}} \quad + \\
&\quad + \frac{|\Omega|^{\frac{1}{2}}}{3^{\frac{3}{4}}\times 4\vartheta^3}\left(\frac{d}{\rho}+1\right)^{\frac{3}{2}}\left(\int_\Omega w^{2p}\right)^3 + 2p\lambda\int_\Omega v^{2p} \\
&\quad + \frac{2\sqrt{2}a|\Omega|^{\frac{1}{3}}}{3^{\frac{3}{4}}}\left(\frac{3}{2\rho}\right)^{\frac{3}{2}}\left(\int_\Omega v^{2p}\right)^{\frac{3}{2}} + \frac{a|\Omega|^{\frac{1}{3}}}{2\nu^3}\left(\frac{d}{\rho}+1\right)^{\frac{3}{2}}\left(\int_\Omega v^{2p}\right)^3,
\end{aligned}$$

yielding Payne *et al.* [25, 27] type differential inequality

$$\mathcal{E}'(t) \leq K_1\mathcal{E}^3(t) + K_2\mathcal{E}^{\frac{3}{2}}(t) + K_1\mathcal{E}(t), \quad \text{or} \quad t \geq \int_{\mathcal{E}(0)}^{\mathcal{E}(t)} \frac{d\eta}{K_1\eta^3 + K_2\eta^{\frac{3}{2}} + K_3\eta},$$

and if $\mathcal{E}(t)$ blows-up at time t^*, then (6.4) must be verified.

This completes the proof of the Theorem in $\mathbb{R}^3$ and complements the beautiful work initiated in [26], pertinent to the parabolic-elliptic equations of the minimal chemotaxis model. Lastly, we note that the conclusion of the theorem holds in dimension $N = 2$, as a result of the yielding condition from Theorem 4.1, working in product spaces of $E_2^{\frac{1}{2}} \cong H^1(\Omega)$. □

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