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On Symplectic Dynamics

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Abstract

This paper continues to carry out a foundational study of Banyaga's topologies of a closed symplectic manifold (M,ω) [4]. Our intention in writing this paper is to work out several "symplectic analogues" of some results found in the study of Hamiltonian dynamics. By symplectic analogue, we mean if the first de Rham's group (with real coefficients) of the manifold is trivial, then the results of this paper reduce to some results found in the study of Hamiltonian dynamics. Especially, without appealing to the positivity of the symplectic displacement energy, we point out an impact of the $L^\infty-$version of Hofer-like length in the investigation of the symplectic nature of the C^0-limit of a sequence of symplectic maps. This yields a symplectic analogue of a result that was proved by Hofer-Zehnder [10] (for compactly supported Hamiltonian diffeomorphisms on $\mathbb{R}^{2n}$); then reformulated by Oh-Müller [14] for Hamiltonian diffeomorphisms in general. Furthermore, we show that Polterovich's regularization process for Hamiltonian paths extends over the whole group of symplectic isotopies, and then use it to prove the equality between the two versions of Hofer-like norms. This yields the symplectic analogue of the uniqueness result of Hofer's geometry proved by Polterovich [13]. Our results also include the symplectic analogues of some approximation lemmas found by Oh-Müller [14]. As a consequence of a result of this paper, we prove by other method a result found by McDuff-Salamon [12].

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1 Introduction

The Hofer geometry originated with the remarkable paper of Hofer [9] that introduced the Hofer topologies on the group of Hamiltonian diffeomorphisms of a symplectic manifold (so-called Hofer metrics, [9]). In particular, Hofer-Zehnder [10] elaborated almost all the basic formulae and some perspectives for the subsequent development of Hamiltonian dynamics based on Hofer's metrics.
Recently, Banyaga [4] showed that on a closed symplectic manifold (M,ω), each Hofer's

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metric generalizes over the whole group of time-one maps of all symplectic isotopies (so-called Hofer-like metric, [4]). By generalization we mean, if the first de Rham cohomology group (with real coefficients) of a closed symplectic manifold is trivial, then the Hofer-like metrics reduce to Hofer's metrics [4].
Furthermore, when the above de Rham's group is non-trivial, then the restriction of the $L^{(1,\infty)}-$version of Banyaga's norm to the group of all Hamiltonian diffeomorphisms is equivalent to Hofer's norm (see [15] and [17]).
However, we have some thorough discussions based on Hofer's topologies whose symplectic analogues with respect to Banyaga's topologies are still unknown (see [10], [11], [14], and [13]). These facts seem to attest that to better understand the Hofer-like geometry, it is judicious to do further investigations based on Banyaga's topologies. This is the main goal of the present paper.

We organize this paper as follows. In Section 2, we recall some fundamental notions concerning symplectic diffeomorphisms and isotopies: Subsection 2.4 deals with the description of symplectic isotopies introduced in [5]; Subsection 2.5 illustrates some implications of Hopf-Rinow's theorem from Riemannian geometry in the study of Hofer geometry, while in Subsection 2.10, we use Hodge's theory to prove that Polterovich's regularization method for Hamiltonian isotopies admits a natural symplectic analogue.
The main results are presented in Section 3: The first main result Theorem 3.3 shows that without appealing to the positivity of the symplectic displacement energy [6] one can use the $L^\infty-$version of Hofer-like length to investigate the symplectic nature of the C^0-limit of a sequence of symplectic diffeomorphisms.
A key ingredient in the proof of Theorem 3.3 is the fact that: if a sequence $\mathcal{H}_i$ of smooth families of smooth harmonic $1-$forms converges uniformly to a smooth family $\mathcal{H}$ of harmonic $1-$forms, then the sequence of symplectic paths generated by $\mathcal{H}_i$ converges in both C^0-metric and $L^\infty-$metric to the symplectic path generated by $\mathcal{H}$ (Lemma 3.4). This follows as a simultaneous application of Hodge's theory, ODE's continuity theorem together with a result found in Subsection 2.5.
The second main result Theorem 3.8 shows that the Hofer like-geometry is independent to the choice of the Hofer-like norm. The main idea behind the proof of Theorem 3.8 is the deformation of any non-trivial symplectic isotopy Φ into a regular symplectic isotopy Ψ so that the $L^\infty-$Hofer-like length of Ψ is bounded from above by the $L^{(1,\infty)}-$Hofer-like length of Φ up to an additive positive ϵ, and both paths Φ and Ψ have the same extremities (Lemma 3.9). This is based on the general regularization method of symplectic paths introduced in Subsections 2.10 and Lemma 3.10.
Section 4 deals with the symplectic analogues of some approximation lemmas found by Oh-Müller [14]. Finally, in Section 5 we give an alternative prove of a result from flux geometry found by McDuff-Salamon [12].

2 Preliminaries

Let M be a smooth closed manifold of dimension $2n$. In brief, a $2-$form ω on M is called a symplectic form if it is closed and non-degenerate. Then, a symplectic manifold is a manifold which can be equipped with a symplectic form. In particular, note that any symplectic

manifold is oriented, and not all even dimensional manifold can be equipped with a symplectic form.
In the rest of this paper, we shall always assume that M is a closed manifold that admits a symplectic form ω, and we shall fix a Riemannian metric g on M (any differentiable manifold M can be equipped with a Riemannian metric). Furthermore, we shall write d to denote the distance induced on M by the Riemannian metric g. The metric topology induced by d on M coincides with the underlying topological structure of M.

2.1 Symplectic vector fields

The symplectic structure ω on M being non-degenerate, it induces an isomorphism between vector fields and 1−forms on M. This isomorphism is given by: to each vector field Y on M, one assigns the 1−form $\iota(Y)\omega := \omega(Y, .)$, where ι is the usual interior product. A vector field Y on M is symplectic if the 1−form $\iota(Y)\omega$ is closed, and in particular, a symplectic vector field Y is said to be a Hamiltonian vector field if the 1−form $\iota(Y)\omega$ is exact. For instance, any harmonic 1−form α on M determines a unique symplectic vector field Y such that $\iota(Y)\omega = \alpha$ (so-called harmonic vector field, [4]). It follows from the definition of symplectic vector fields that if the first de Rham's cohomology group of M is trivial, then any symplectic vector field on M is Hamiltonian.

According to Hodge's theory, a sufficient condition that guarantees the existence of non-trivial harmonic vector fields on a symplectic manifold is the non-triviality of its first de Rham cohomology group (with real coefficients). Note that the first de Rham's group is a topological invariant, i.e. it does not depend on the differentiable structure on M and only depends on the underlying topological structure of M [8].

2.2 Symplectic diffeomorphisms and symplectic isotopies

A diffeomorphism $\phi : M \to M$, is called symplectic if it preserves the symplectic form ω, i.e. $\phi^*(\omega) = \omega$. We denote by $Symp(M, \omega)$, the group of all symplectic diffeomorphisms of (M, ω).
An isotopy $\{\phi_t\}$ of a symplectic manifold (M, ω) is said to be symplectic if $\phi_t \in Symp(M, \omega)$ for each t, or equivalently, the vector field $\dot{\phi}_t := \frac{d\phi_t}{dt} \circ \phi_t^{-1}$ is symplectic for each t. In particular, a symplectic isotopy $\{\psi_t\}$ is a Hamiltonian isotopy if for each t, the vector field $\dot{\psi}_t := \frac{d\psi_t}{dt} \circ \psi_t^{-1}$ is Hamiltonian, i.e. there exists a smooth function $F : [0,1] \times M \to \mathbb{R}$, called generating Hamiltonian such that $\iota(\dot{\psi}_t)\omega = dF_t$, for each t. Any Hamiltonian isotopy determines its generating Hamiltonian up to an additive constant. Throughout this paper we assume that every generating Hamiltonian $F : [0,1] \times M \to \mathbb{R}$ is normalized, i.e. we require that $\int_M F_t \omega^n = 0$, for all t. Let $\mathcal{N}([0,1] \times M, \mathbb{R})$ denote the vector space of all smooth normalized Hamiltonians. In addition, note that a symplectic isotopy $\{\theta_t\}$ is said to be harmonic if for each t, the vector field $\dot{\theta}_t$ is harmonic.
We let $Iso(M, \omega)$ denote the group of all symplectic isotopies of (M, ω), and let $Symp_0(M, \omega)$ denote the group of time-one maps of all symplectic isotopies.

2.3 Harmonics $1-$forms

Let $H^1(M,\mathbb{R})$ denote the first de Rham cohomology group (with real coefficients) of M, and let $\mathcal{H}^1(M,g)$ denote the space of harmonic $1-$forms on M with respect to the Riemannian metric g. The set $\mathcal{H}^1(M,g)$ forms a finite dimensional vector space over $\mathbb{R}$ which is isomorphic to $H^1(M,\mathbb{R})$, and whose dimension is denoted $b_1(M)$, and called the first Betti number of the manifold M [8]. Taking $(h_i)_{1\leq i\leq b_1(M)}$ as a basis of the vector space $\mathcal{H}^1(M,g)$, we equip $\mathcal{H}^1(M,g)$ with the norm $|.|$ defined as follows: for all $H\in\mathcal{H}^1(M,g)$ with $H=\sum_{i=1}^{b_1(M)}\lambda_i h_i$, its norm is defined as

$$|H| := \sum_{i=1}^{b_1(M)} |\lambda_i|. \tag{2.1}$$

We denote by $\mathcal{PH}^1(M,g)$ the space of all smooth mappings $\mathcal{H}:[0,1]\to\mathcal{H}^1(M,g)$.

2.3.1 Comparison of norms

To avoid some heavy computations throughout the paper, we have found necessary to compare the above norm $|.|$ with the well-known uniform sup norm of differential $1-$forms. For this purpose, let us recall the definition of the uniform sup norm of differential $1-$forms. Consider α to be a differential $1-$form on M: for each $x\in M$, we know that α induces a linear map $\alpha_x : T_xM\to\mathbb{R}$, whose norm is given by

$$\|\alpha_x\|^g = \sup\{|\alpha_x(X)| : X\in T_xM, \|X\|_g = 1\} \tag{2.2}$$

where $\|.\|_g$ is the norm induced on each tangent space T_xM (at the point x) by the Riemannian metric g. Therefore, the uniform sup norm of α, say $|.|_0$ is defined as

$$|\alpha|_0 = \sup_{x\in M}\|\alpha_x\|^g. \tag{2.3}$$

In particular, if α is a harmonic $1-$form, i.e.

$$\alpha = \sum_{i=1}^{b_1(M)}\lambda_i h_i,$$

then we obtain

$$|\alpha|_0 \leq \sum_{i=1}^{b_1(M)}|\lambda_i||h_i|_0 \leq E|\alpha| \tag{2.4}$$

where

$$E := \max_{1\leq i\leq b_1(M)}|h_i|_0. \tag{2.5}$$

If the basis $(h_i)_{1\leq i\leq b_1(M)}$ is such that $E>1$, then one can always normalize such a basis so that E equals 1. So, without loss of generality, instead of (2.4), we shall often use the following inequality,

$$|\alpha|_0 \leq |\alpha|. \tag{2.6}$$

2.4 A description of symplectic isotopies [5]

We now recall the description of symplectic isotopies introduced in [5]. Given any symplectic isotopy $\Phi = \{\phi_t\}$, one derives from Hodge's theory that the closed $1-$form $\iota(\dot{\phi}_t)\omega$ decomposes in a unique way as the sum of an exact $1-$form dU_t^Φ and a harmonic $1-$form $\mathcal{H}_t^\Phi$ [8]. Denote by U the normalized Hamiltonian of $U^\Phi = (U_t^\Phi)$, and by $\mathcal{H}$ the smooth family of harmonic $1-$forms $\mathcal{H}^\Phi = (\mathcal{H}_t^\Phi)$. In [5], the Cartesian product $\mathcal{N}([0,1]\times M,\mathbb{R})\times\mathcal{PH}^1(M,g)$ is denoted $\mathfrak{T}(M,\omega,g)$, and equipped with a group structure which makes the bijection

$$\mathfrak{A} : Iso(M,\omega) \to \mathfrak{T}(M,\omega,g), \Phi \mapsto (U,\mathcal{H}) \tag{2.7}$$

a group isomorphism. Under this identification, any symplectic isotopy Φ is denoted by $\phi_{(U,\mathcal{H})}$ to mean that $\mathfrak{A}$ maps Φ onto $(U,\mathcal{H})$, and the pair $(U,\mathcal{H})$ is called the "generator" of the symplectic isotopy Φ. For instance, a symplectic isotopy $\phi_{(0,\mathcal{H})}$, is a harmonic isotopy, and a symplectic isotopy $\phi_{(U,0)}$, is a Hamiltonian isotopy.

2.4.1 Group structure on $\mathfrak{T}(M,\omega,g)$ [5]

The product rule in $\mathfrak{T}(M,\omega,g)$ is given by,

$$(U,\mathcal{H}) \bowtie (V,\mathcal{K}) = (U + V\circ\phi^{-1}_{(U,\mathcal{H})} + \widetilde{\Delta}(\mathcal{K},\phi^{-1}_{(U,\mathcal{H})}), \mathcal{H}+\mathcal{K}). \tag{2.8}$$

The inverse of $(U,\mathcal{H})$, say $\overline{(U,\mathcal{H})}$ is given by

$$\overline{(U,\mathcal{H})} = (-U\circ\phi_{(U,\mathcal{H})} - \widetilde{\Delta}(\mathcal{H},\phi_{(U,\mathcal{H})}), -\mathcal{H}). \tag{2.9}$$

In (2.8) and (2.9) the quantity $\widetilde{\Delta}$ is defined as follows: for any symplectic isotopy $\Psi = \{\psi^t\}$, and for any smooth family of closed $1-$forms $\alpha = (\alpha_t)$, we have

$$\widetilde{\Delta}_t(\alpha,\Psi) = \Delta_t(\alpha,\Psi) - \frac{\int_M \Delta_t(\alpha,\Psi)\omega^n}{\int_M \omega^n},$$

where $\Delta_t(\alpha,\Psi) := \int_0^t \alpha_t(\dot{\psi}^s)\circ\psi^s ds$, for all t (see [5]).

Here is a consequence of (2.7) and (2.8).

Corollary 2.1. *Every symplectic isotopy decomposes into the composition of a harmonic isotopy and a Hamiltonian isotopy, and this decomposition is unique, or equivalently, every generator $(U,\mathcal{H})$ decomposes in a unique way as:*

$$(U,\mathcal{H}) = (0,\mathcal{H}) \bowtie (U\circ\phi_{(0,\mathcal{H})}, 0). \tag{2.10}$$

Since the proof of the above corollary is easy, we leave it to readers and refer them to [4, 5] for more convenience.

2.5 Scholium

So far, we have introduced the function $\Delta(\mathfrak{H},\Phi)$ whenever $\mathfrak{H}$ is a smooth family of closed $1-$forms, and $\Phi = \{\phi_t\}$ is an isotopy of a compact manifold M with $\phi_0 = id_M$, but have not produced any of its algebraic, geometric or analytic properties, so one might be wondering if they are plentiful or rare.

2.5.1 Boundedness properties of $\Delta(\mathfrak{H}, \Phi)$

Here, we use Hopf-Rinow's theorem to point out some boundedness properties of Hofer's norms of the functions $\Delta(\mathfrak{H}, \Phi)$. Firstly, recall that for any smooth family of closed $p-$forms $\{\Omega_t\}$, and for any isotopy $\Phi = \{\phi_t\}$ of a compact manifold, it is well-known that

$$\phi_t^*(\Omega_t) - \Omega_t = d\{\int_0^t \phi_s^*\left(\iota(\dot{\phi}_s)\Omega_t\right)ds\} \tag{2.11}$$

for each t, where d stands for the usual differential operator (see [4] for a quick proof). In particular, when $\{\Omega_t\}$ is a smooth family of closed $1-$forms $\mathfrak{H} = (\mathfrak{H}_t)$, it follows from (2.11) that

$$\phi_t^*(\mathfrak{H}_t) - \mathfrak{H}_t = d\{\Delta_t(\mathfrak{H}, \Phi)\} \tag{2.12}$$

for each t. Now fix a point $x_0 \in M$, and for all $x \in M$ pick a curve γ_x from x_0 to x, then define a smooth function by setting

$$\bar{u}_t(x) := \int_{\gamma_x} (\phi_t^*(\mathfrak{H}_t) - \mathfrak{H}_t) \tag{2.13}$$

for each t. The function $\bar{u}$ defined in (2.13) does not depend on the choice of the curve γ_x from x_0 to x, and it follows from (2.12) that for each (t, x), we have

$$\bar{u}_t(x) = (\Delta_t(\mathfrak{H}, \Phi))(x) - (\Delta_t(\mathfrak{H}, \Phi))(x_0),$$

i.e. both functions $\bar{u}$ and $\Delta(\mathfrak{H}, \Phi)$ have the same Hofer norms. By Hopf-Rinow's theorem, for each $x \in M$, one can choose the path γ_x to be a geodesic, and its length $l(\gamma_x) := \int_0^1 \|\dot{\gamma}_x(s)\|_g ds$ satisfies

$$l(\gamma_x) \leq diam(M) \tag{2.14}$$

where $diam(M)$ stands for the diameter of M with respect to the Riemannian metric g. For instance, for each t, consider y_0 to be any point of M that realizes the supremum of the function $x \mapsto |\bar{u}_t(x)|$, and derive from the triangle inequality that,

$$\sup_x |\bar{u}_t(x)| \leq |\int_{\gamma_{y_0}} \mathfrak{H}_t| + |\int_{\gamma_{y_0}} \phi_t^*(\mathfrak{H}_t)|$$

$$\leq |\mathfrak{H}_t|_0 \int_0^1 \|\dot{\gamma}_{y_0}(s)\|_g ds + \sup_{v,s} |D\phi_v(\gamma_{y_0}(s))||\mathfrak{H}_t|_0 \int_0^1 \|\dot{\gamma}_{y_0}(s)\|_g ds,$$

$$\leq diam(M)(1 + \sup_{s,v} |D\phi_v(\gamma_{y_0}(s))|)|\mathfrak{H}_t|_0,$$

for each t, and where $D\phi_v$ is the tangent map of ϕ_v. So, we obtain

$$\sup_x |\bar{u}_t(x)| \leq diam(M)(1 + \sup_{s,v} |D\phi_v(\gamma_{y_0}(s))|)|\mathfrak{H}_t|_0 \tag{2.15}$$

for each t, and since we always have,

$$osc(\Delta_t(\mathfrak{H}, \Phi)) = osc(\bar{u}_t) \leq 2 \sup_x |\bar{u}_t(x)| \tag{2.16}$$

for each t, then we deduce from (2.15) together with (2.16) that

$$\int_0^1 osc(\Delta_t(\mathfrak{H},\Phi))dt \leq 2diam(M)(1+\sup_{s,v}|D\phi_v(\gamma_{y_0}(s))|)\int_0^1 |\mathfrak{H}_t|_0 dt, \qquad (2.17)$$

and

$$\max_{t\in[0,1]} osc(\Delta_t(\mathfrak{H},\Phi)) \leq 2diam(M)(1+\sup_{s,v}|D\phi_v(\gamma_{y_0}(s))|)\max_{t\in[0,1]}|\mathfrak{H}_t|_0. \blacksquare \qquad (2.18)$$

Notice that (2.17) and (2.18) are useful in the rest of this paper.

2.5.2 Algebraic properties of $\Delta(\mathfrak{H},\Phi)$

Proposition 2.2. *Let $\Phi = \{\phi^t\}$ and $\Psi = \{\psi^t\}$ be two isotopies. Let $\mathcal{H} = (\mathcal{H}_t)$ be a smooth family of closed $1-$forms. Then, for each t, there exists a constant η which depends only on Φ, Ψ, and $\mathcal{H}$ such that,*

$$\Delta_t(\mathcal{H},\Phi\circ\Psi)(x) = \Delta_t(\mathcal{H},\Psi)(x) + \Delta_t(\mathcal{H},\Phi)\circ\psi^t(x) + \eta,$$

for all $x \in M$.

Proof. Using (2.11) we derive that,

$$\begin{aligned}
d\Delta_t(\mathcal{H},\Phi\circ\Psi) &= (\phi_t\circ\psi_t)^*\mathcal{H}_t - \mathcal{H}_t \\
&= \psi_t^*(\phi_t^*(\mathcal{H}_t)) - \mathcal{H}_t \\
&= \psi_t^*(\mathcal{H}_t + d\Delta_t(\mathcal{H},\Phi)) - \mathcal{H}_t \\
&= \psi_t^*(\mathcal{H}_t) + d(\Delta_t(\mathcal{H},\Phi)\circ\psi_t) - \mathcal{H}_t \\
&= \mathcal{H}_t + d\Delta_t(\mathcal{H},\Psi) + d(\Delta_t(\mathcal{H},\Phi)\circ\psi_t) - \mathcal{H}_t \\
&= d\Delta_t(\mathcal{H},\Psi) + d(\Delta_t(\mathcal{H},\Phi)\circ\psi_t),
\end{aligned}$$

for each $t \in [0,1]$. Since M is connected, it follows from the above equalities that there exists a constant η such that:

$$\Delta_t(\mathcal{H},\Phi\circ\Psi)(x) = \Delta_t(\mathcal{H},\Psi)(x) + \Delta_t(\mathcal{H},\Phi)\circ\psi_t(x) + \eta,$$

for all $x \in M$. Note that η depends only on Φ, t, Ψ, and $\mathcal{H}$, but not on x. This achieves the proof. ∎

For further survey of the class of functions $\Delta(\mathcal{H},\Phi)$, let us consider the following Poincaré's scalar product :

$$\langle,\rangle_P : H^1(M,\mathbb{R})\times H^{2n-1}(M,\mathbb{R}) \to \mathbb{R},$$

$$([\alpha],[\beta]) \mapsto \int_M \alpha\wedge\beta,$$

where $H^*(M,\mathbb{R})$ represents the $*-$th de Rham cohomology group with real coefficients, and $[\alpha]$ stands for the de Rham cohomology class of a closed differential form α.

We have the following fact.

Proposition 2.3. *Let $\mathcal{H} = (\mathcal{H}_t)$ be a smooth family of closed $1-$forms, and let $\Phi = \{\phi_t\}$ be a symplectic isotopy. Then for each $t \in [0,1]$, we have*

$$\int_M \Delta_t(\mathcal{H},\Phi)\omega^n = n\langle Flux(\bar{\Phi}_t), [\mathcal{H}_t \wedge \omega^{(n-1)}]\rangle_P \tag{2.19}$$

where $\bar{\Phi}_t$ is the isotopy $s \mapsto \phi_{st}$, and Flux stands for the flux homomorphism. In particular, if Φ is a Hamiltonian isotopy then the right hand side in (2.19) vanishes.

Proof. For each fixed $t \in [0,1]$, since the differential form $\mathcal{H}_t \wedge \omega^n$ is of degree $(2n+1)$ over a $2n-$dimensional manifold, we derive that $\mathcal{H}_t \wedge \omega^n = 0$. This implies that

$$\mathcal{H}_t(\dot{\phi}_s)\omega^n - n\iota(\dot{\phi}_s)\omega \wedge \mathcal{H}_t \wedge \omega^{(n-1)} = 0,$$

for each $s \in [0,t]$. Composing the above equality in both sides by ϕ_s^* yields,

$$\phi_s^*\Big(\mathcal{H}_t(\dot{\phi}_s)\Big)\omega^n - n\phi_s^*\Big(\iota(\dot{\phi}_s)\omega\Big) \wedge \phi_s^*(\mathcal{H}_t) \wedge \omega^{(n-1)} = 0 \tag{2.20}$$

for each $s \in [0,t]$. Using (2.12), we derive that for each $s \in [0,t]$, we have

$$\phi_s^*(\mathcal{H}_t) = \mathcal{H}_t + df^s_{\{\phi_t\},\mathcal{H}_t} \tag{2.21}$$

where

$$f^s_{\{\phi_t\},\mathcal{H}_t} := \int_0^s \mathcal{H}_t(\dot{\phi}_u) \circ \phi_u du.$$

Relations (2.20) and (2.21) immediately imply that,

$$\phi_s^*\Big(\mathcal{H}_t(\dot{\phi}_s)\Big)\omega^n = n\phi_s^*\Big(\iota(\dot{\phi}_s)\omega\Big) \wedge \mathcal{H}_t \wedge \omega^{(n-1)} + n\phi_s^*\Big(\iota(\dot{\phi}_s)\omega\Big) \wedge df^s_{\{\phi_t\},\mathcal{H}_t} \wedge \omega^{(n-1)} \tag{2.22}$$

for each $s \in [0,t]$. That is,

$$\int_M \left(\int_0^t \phi_s^*\Big(\mathcal{H}_t(\dot{\phi}_s)\Big)ds\right)\omega^n = n\int_M \left(\int_0^t \phi_s^*\Big(\iota(\dot{\phi}_s)\omega\Big)ds\right) \wedge \mathcal{H}_t \wedge \omega^{(n-1)} \tag{2.23}$$

$$+n\int_M \left(\int_0^t \Big(\phi_s^*\Big(\iota(\dot{\phi}_s)\omega\Big) \wedge df^s_{\{\phi_t\},\mathcal{H}_t}\Big)ds\right) \wedge \omega^{(n-1)}.$$

On the other hand, since the differential forms $\phi_s^*\Big(\iota(\dot{\phi}_s)\omega\Big)$, and $\omega^{(n-1)}$ are closed, we compute

$$\int_M \int_0^t \phi_s^*\Big(\iota(\dot{\phi}_s)\omega\Big) \wedge df^s_{\{\phi_t\},\mathcal{H}_t} \wedge \omega^{(n-1)}ds = \int_M d[\int_0^t f^s_{\{\phi_t\},\mathcal{H}_t}\phi_s^*\Big(\iota(\dot{\phi}_s)\omega\Big) \wedge \omega^{(n-1)}ds] \tag{2.24}$$

and since the manifold M is without boundary, we derive from Stokes' theorem that

$$\int_M d[\int_0^t f^s_{\{\phi_t\},\mathcal{H}_t}\phi_s^*\Big(\iota(\dot{\phi}_s)\omega\Big) \wedge \omega^{(n-1)}ds] = \int_{\partial M}\int_0^t f^s_{\{\phi_t\},\mathcal{H}_t}\phi_s^*\Big(\iota(\dot{\phi}_s)\omega\Big) \wedge \omega^{(n-1)}ds = 0 \tag{2.25}$$

for each t. We combine (2.23), (2.24), and (2.25) to obtain

$$\int_M \int_0^t \phi_s^*\Big(\mathcal{H}_t(\dot{\phi}_s)\Big) ds\,\omega^n = n\int_M \left(\int_0^t \phi_s^*\Big(\iota(\dot{\phi}_s)\omega\Big) ds\right) \wedge \mathcal{H}_t \wedge \omega^{(n-1)}. \tag{2.26}$$

Observe that for each t, the de Rham cohomology class of the 1−form $\int_0^t \phi_s^*\Big(\iota(\dot{\phi}_s)\omega\Big) ds$ is exactly the flux of the isotopy $\bar{\Phi}_t$, i.e. (2.26) can be written as

$$\int_M \Delta_t(\mathcal{H}, \Phi)\omega^n = n\langle Flux(\bar{\Phi}_t), [\mathcal{H}_t \wedge \omega^{(n-1)}]\rangle_P. \tag{2.27}$$

In particular, if the isotopy Φ is Hamiltonian, then so is $\bar{\Phi}_t$ for each t, and in this case, a result found in [2] implies that $Flux(\bar{\Phi}_t) = 0$, for each t, i.e. for each time t, the right hand side in (2.27) is zero. This completes the proof. ■

2.6 A geometric interpretation of the functions $\Delta(\alpha, \Phi)$

Given a smooth family of diffeomorphisms $\Phi = \{\phi_t\}$, by setting $X_t(x) = \dfrac{d}{ds}(\phi_s(x))_{|s=t}$, for each t and for all $x \in M$, one defines a family of tangent vector fields (X_t) on M: $X_t(x)$ is the tangent vector to the curve $s \mapsto \phi_s(x)$ at the time t and we have $\dot{\phi}_t = X_t \circ \phi_t^{-1}$ for each t (see page 5 in [3] for more convenience). In particular, for each fixed $(t, x) \in [0, 1] \times M$, if we consider the curve $\gamma_{x,t} : s \mapsto \phi_{st}(x)$, then

$$\dot{\gamma}_{x,t}(s) = \frac{d}{ds}(\phi_{st}(x)) = t\frac{d}{du}(\phi_u(x))_{|u=st} = tX_u(x)_{|u=st} \tag{2.28}$$

for all s. Now, assume Φ is a symplectic isotopy, and α is a closed 1−form. We have

$$\int_{\gamma_{x,t}} \alpha = \int_0^1 \alpha_{\gamma_{x,t}(s)}(\dot{\gamma}_{x,t}(s))ds = \int_0^1 \phi_{st}^*\Big(\alpha(t\dot{\phi}_{st})\Big)(x)ds = \int_0^t \phi_u^*\Big(\alpha(\dot{\phi}_u)\Big)(x)du. \tag{2.29}$$

On the other hand, if $\{\theta^t\}$ is the symplectic flow generated by α, then we derive as in [12] that

$$\int_{\gamma_{x,t}} \alpha = \int_{\gamma_{x,t}} \left(\int_0^1 \iota(\dot{\theta}^u)\omega du\right) = \int_0^1 \int_0^1 \omega_{\gamma_{x,t}(s)}(\dot{\theta}^u(\gamma_{x,t}(s)), \dot{\gamma}_{x,t}(s))duds \tag{2.30}$$

i.e.

$$\int_{\gamma_{x,t}} \alpha = \int_{[0,1]\times[0,1]} (\Theta_{\gamma_{x,t}})^*\omega \tag{2.31}$$

where

$$\Theta_{\gamma_{x,t}} : [0, 1] \times [0, 1] \to M, (u, s) \mapsto \theta^u(\phi^{st}(x)).$$

Geometrically, the equalities (2.29) and (2.31) tell us that for each t, the real number $\Delta_t(\alpha, \Phi)(x)$ can be interpreted as the algebraic value of the symplectic area of the 2−chain $\{\theta^u(\phi^{st}(x))|0 \leq u, s \leq 1\}$. In particular, each zero of the function $\Delta_t(\alpha, \Phi)$ gives rise to a 2−chain with vanishing symplectic area. The following consequence of Proposition 2.3 gives a sufficient condition that guarantees the existence of at least one zero for such a function.

Lemma 2.4. *If α is a smooth family of closed $1-$forms on M and Φ is a Hamiltonian isotopy, then for each t, the function $x \mapsto \Delta_t(\alpha, \Phi)(x)$ has at least one vanishing point on M.*

Proof. Assume α is a smooth family of closed $1-$forms and Φ is a Hamiltonian isotopy. For each fixed $t \in [0,1]$, we have a smooth function $x \mapsto \Delta_t(\alpha, \Phi)(x)$ from the compact M into the set of real numbers. Thus, the latter function achieves its bounds; and this implies that

$$\min_{x\in M} \Delta_t(\alpha, \Phi)(x) \int_M \omega^n \leq \int_M \Delta_t(\alpha, \Phi)\omega^n \leq \max_{x\in M} \Delta_t(\alpha, \Phi)(x) \int_M \omega^n,$$

i.e. $\min_{x\in M} \Delta_t(\alpha, \Phi)(x) \leq 0$, and $0 \leq \max_{x\in M} \Delta_t(\alpha, \Phi)(x)$ since by Proposition 2.3 we have $\int_M \Delta_t(\alpha, \Phi)\omega^n = 0$. ■

2.7 Reparameterization of symplectic isotopies [5, 14]

We shall need the following basic formula for the generators of reparameterized symplectic isotopies. If $\Phi = \{\phi_t\}$ is a symplectic isotopy generated by $(U, \mathcal{H})$, and $\xi : [0,1] \to [0,1]$ is a smooth function, then the reparameterized path $t \mapsto \phi_{\xi(t)}$, denoted Φ^ξ, is generated by the element $(U, \mathcal{H})^\xi$ defined as

$$(U, \mathcal{H})^\xi := (U^\xi, \mathcal{H}^\xi) \tag{2.32}$$

where $\mathcal{H}^\xi$ is the smooth family of harmonic $1-$forms $t \mapsto \dot{\xi}(t)\mathcal{H}_{\xi(t)}$; U^ξ is the smooth family of smooth functions $(x, t) \mapsto \dot{\xi}(t)U_{\xi(t)}(x)$; while $\dot{\xi}(t)$ is the derivative of the function ξ at the point t. The inverse element of $(U, \mathcal{H})^\xi$ is given by

$$\overline{(U, \mathcal{H})^\xi} = (-U^\xi \circ \Phi^\xi - \widetilde{\Delta}(\mathcal{H}^\xi, \Phi^\xi), -\mathcal{H}^\xi) \tag{2.33}$$

with

$$\Delta_t(\mathcal{H}^\xi, \Phi^\xi) = \int_0^{\xi(t)} \dot{\xi}(t)\mathcal{H}_{\xi(t)}(\dot{\phi}_u) \circ \phi_u du = \dot{\xi}(t)\Delta_{\xi(t)}(\mathcal{H}, \Phi) \tag{2.34}$$

for each t, i.e.

$$\Delta(\mathcal{H}^\xi, \Phi^\xi) = \Delta^\xi(\mathcal{H}, \Phi). \tag{2.35}$$

Definition 2.5. ([14]). Given a smooth function $\xi : [0,1] \to \mathbb{R}$, its norm $\|\xi\|_{ham}$ is defined by

$$\|\xi\|_{ham} = \|\xi\|_{C^0} + \|\dot{\xi}\|_{L_1},$$

with $\|\dot{\xi}\|_{L_1} = \int_0^1 |\dot{\xi}(t)|dt$, and $\|\xi\|_{C^0} = \sup_t |\xi(t)|$.

2.8 Boundary flat symplectic isotopies [5, 14]

Definition 2.6. ([5]). Given $(U, \mathcal{H}) \in \mathfrak{T}(M, \omega, g)$, we say that $(U, \mathcal{H})$ is boundary flat if there exists $\delta \in]0,1[$ such that $(U_t, \mathcal{H}_t) = (0,0)$, for all t in $[0, \delta[\cup]1-\delta, 1]$.

In other words, a symplectic path $\{\phi_t\}$ is boundary flat if there exists a constant $0<\delta<1$ such that $\phi_t = id_M$ for all $0 \leq t<\delta$, and $\phi_t = \phi_1$ for all $1-\delta<t \leq 1$.

2.9 The C^0−topology

Let $Homeo(M)$ be the group of all homeomorphisms of M equipped with the C^0- compact-open topology [1]. This is the metric topology induced by the following distance

$$d_0(f,h) = \max(d_{C^0}(f,h), d_{C^0}(f^{-1},h^{-1})) \tag{2.36}$$

where

$$d_{C^0}(f,h) = \sup_{x\in M} d(h(x), f(x)). \tag{2.37}$$

On the space of all continuous paths $\lambda : [0,1] \to Homeo(M)$ such that $\lambda(0) = id_M$, we consider the C^0−topology as the metric topology induced by the following metric

$$\bar{d}(\lambda,\mu) = \max_{t\in[0,1]} d_0(\lambda(t),\mu(t)). \tag{2.38}$$

2.10 Regularization of symplectic isotopies

Definition 2.7. A symplectic path $\{\phi_t\}$ is said to be regular if for every t, the vector field $\dot{\phi}_t$ does not vanish.

First of all, note that a regularization method (in the sense of Definition 2.7) for Hamiltonian paths is due to Polterovich [13]. As far as I know, when $H^1(M,\mathbb{R}) \neq \{0\}$, a general regularization method (in the sense of Definition 2.7) for the whole group of symplectic isotopies is unknown. The goal of this subsection is to provide such a result for non-Hamiltonian symplectic paths.
Given a symplectic isotopy Φ generated by $(U,\mathcal{H})$; in view of Proposition 5.2.A found in [13], for the above Hamiltonian U, there exists a Hamiltonian loop $\phi_{(r,0)}$ which is close to the constant loop identity (in the C^∞−sense), and in particular, its generating function r is arbitrarily small in the $L^{(1,\infty)}$−version of Hofer's norm so that

$$osc(-r_t + U_t) \neq 0 \tag{2.39}$$

for all t. Now consider $(V,\mathcal{K})$ to be the product $\overline{(r,0)} \bowtie (U,\mathcal{H})$, which can be written immediately as

$$(V,\mathcal{K}) = (-r\circ\phi_{(r,0)} + U\circ\phi_{(r,0)} + \widetilde{\Delta}(\mathcal{H},\phi_{(r,0)}),\mathcal{H}). \tag{2.40}$$

We claim that the isotopy generated by the element $(V,\mathcal{K})$ defined in (2.40) is regular in the sense of Definition 2.7. The proof of this claim relies on Hodge's theory and Proposition 5.2.A found in [13]. Arguing indirectly, we assume that there exists a time s for which the vector field $X_s = \dot{\phi}^s_{(V,\mathcal{K})}$ vanishes identically, i.e.

$$\iota(X_s)\omega = dV_s + \mathcal{K}_s = 0. \tag{2.41}$$

Inserting (2.40) in (2.41) gives

$$d(-r_s\circ\phi^s_{(r,0)} + U_s\circ\phi^s_{(r,0)} + \Delta_s(\mathcal{H},\phi_{(r,0)})) + \mathcal{H}_s = 0. \tag{2.42}$$

From (2.42), we see that the harmonic 1−form $\mathcal{H}_s$ is exact, and the latter 1−form must be trivial because of Hodge's theory: any exact harmonic form of a compact oriented Riemannian manifold is trivial [8], i.e. $\mathcal{H}_s = 0$. This implies that

$$\Delta_s(\mathcal{H}, \phi_{(r,0)}) = \int_0^s \mathcal{H}_s(\dot{\phi}^u_{(r,0)}) \circ \phi^u_{(r,0)} du = 0,$$

i.e. the function $-r_s \circ \phi^s_{(r,0)} + U_s \circ \phi^s_{(r,0)}$ is constant since M is connected. The latter argument contradicts (2.39), and the claim follows. ■

As a consequence of the above regularization method, we derive as in [13] that using any regular symplectic path $\phi_{(V,\mathcal{K})}$, we can define a function $\zeta : [0,1] \to [0,1]$ to be the inverse of the map,

$$s \mapsto \frac{\int_0^s (osc(V_t) + |\mathcal{K}_t|)dt}{\int_0^1 (osc(V_t) + |\mathcal{K}_t|)dt}. \tag{2.43}$$

The derivative of ζ is given explicitly by :

$$\zeta'(s) = \frac{\int_0^1 (osc(V_t) + |\mathcal{K}_t|)dt}{osc(V_{\zeta(s)}) + |\mathcal{K}_{\zeta(s)}|} \tag{2.44}$$

for each s. If ζ is only C^1, then we can approximate ζ in the C^1−topology by a smooth diffeomorphism $\kappa : [0,1] \to [0,1]$, that fixes 0 and 1 [1].
Later in the next section, we will understand the role that plays the above regularization method in the comparison of the infima of Banyaga's lengths relatively to each fixed $\phi \in Symp_0(M,\omega)$.

3 Main results

Throughout this section, we introduce the main results of this paper. We shall start by recalling the notions of lengths for symplectic isotopies introduced by [4].

According to [4], given a symplectic isotopy Φ generated by $(U,\mathcal{H})$, the $L^{(1,\infty)}$−version and the L^∞−version of Hofer-like lengths of a Φ are defined respectively by

$$l^{(1,\infty)}(\Phi) = \int_0^1 (osc(U_t) + |\mathcal{H}_t|)\,dt, \tag{3.1}$$

$$l^\infty(\Phi) = \max_{t\in[0,1]} (osc(U_t) + |\mathcal{H}_t|). \tag{3.2}$$

In the case that $H^1(M,\mathbb{R})$ vanishes, the above lengths are called Hofer's lengths. It is clear that if Φ is a Hamiltonian isotopy generated by $(U,0)$, then its inverse Φ^{-1} is generated by $(-U \circ \Phi, 0)$; and we have $l^{(1,\infty)}(\Phi) = l^{(1,\infty)}(\Phi^{-1})$, and $l^\infty(\Phi) = l^\infty(\Phi^{-1})$ (symmetry). On the other hand, for non-Hamiltonian symplectic isotopies we do not know whether the above Hofer-like lengths are symmetric or not. Indeed, for such a symplectic isotopy Φ generated by $(U, \mathcal{H})$, the formula for inversion of generators tells us that Φ^{-1} is generated by $(-U \circ \Phi - \widetilde{\Delta}(\mathcal{H}, \Phi), -\mathcal{H})$, and the gap of symmetry in each of the above lengths seems to come from the appearance of the function $\widetilde{\Delta}(\mathcal{H}, \Phi)$ in the generator of Φ^{-1}. This does not mean that the appearance of $\widetilde{\Delta}(\mathcal{H}, \Phi)$ in the generator of Φ^{-1} justifies the non-symmetry of the lengths of Φ. In fact, the following example exhibits the existence of a larger number of non-Hamiltonian symplectic isotopies whose Hofer-like lengths are symmetric.

Example 3.1. (Non-Hamiltonian symplectic isotopies with symmetric lengths). Let α be a non-trivial harmonic $1-$form. It is clear that the symplectic flow $\{\rho_t\}$ generated by $(0, \alpha)$ is non-Hamiltonian since its flux is non-trivial. Next, consider Y to be the smooth autonomic vector field such that $\iota(Y)\omega = \alpha$, and for each t compute

$$\Delta_t(\alpha, \{\rho_t\})(x) = \int_0^t \rho_s^*(\alpha(\dot{\rho}_s))(x)ds = \int_0^t \alpha(Y) \circ \rho_s(x)ds = \int_0^t \omega(Y, Y) \circ \rho_s(x)ds = 0 \quad (3.3)$$

for all $x \in M$ because $\omega(Y, Y) = 0$. On the other hand, since $\{\rho_t\}^{-1}$ is generated by $(-\widetilde{\Delta}(\alpha, \{\rho_t\}), -\alpha)$, one derives from (3.3) that

$$l^{(1,\infty)}(\{\rho_t\}^{-1}) = \int_0^1 \left[osc\left(-\widetilde{\Delta}_t(\alpha, \{\rho_t\})\right) + |\alpha|\right] dt = |\alpha| = l^{(1,\infty)}(\{\rho_t\}), \quad (3.4)$$

and

$$l^{\infty}(\{\rho_t\}^{-1}) = \max_{t \in [0,1]} \left[osc\left(-\widetilde{\Delta}_t(\alpha, \{\rho_t\})\right) + |\alpha|\right] = |\alpha| = l^{\infty}(\{\rho_t\}). \quad (3.5)$$

Remark 3.2. Given a symplectic isotopy Φ and a smooth function $\xi : [0, 1] \to [0, 1]$ that fixes 0 and 1, one can derive Subsection 2.7 that

$$l^{(1,\infty)}(\Phi^{\xi}) = l^{(1,\infty)}(\Phi) \quad (3.6)$$

i.e. the $L^{(1,\infty)}-$length is invariant under reparameterization via smooth curves ξ that fix 0 and 1.

To put the above lengths into further perspective, note that Hofer-Zehnder [10] showed that one can use the Hofer lengths to investigate the Hamiltonian nature of the C^0-limit of a sequence of Hamiltonian diffeomorphisms (see Theorem 6, [10]). Since the Hofer-like lengths generalize the Hofer lengths, it is natural to investigate whether one can use the Hofer-like lengths to elaborate the symplectic analogues of Theorem 6−[10] or not. Of course, by analogy with the Hamiltonian case, in presence of a positive symplectic displacement energy, such symplectic analogues can be provided (see [6]). The worst imaginable scenario is to think of the above symplectic analogues of Theorem 6−[10] in the lack of a positive symplectic displacement energy. However, the following result shows that using exclusively the $L^{\infty}-$Hofer-like length, we can provide a symplectic analogue of Theorem 6−[10] without appealing to the positivity of the symplectic displacement energy.

Theorem 3.3. *Let (M,ω) be a closed symplectic manifold. Let $\{\phi_i^t\}$ be a sequence of symplectic isotopies, let $\{\psi^t\}$ be another symplectic isotopy, and let $\phi : M \to M$ be a map such that*

- *(ϕ_i^1) converges uniformly to ϕ, and*
- *$l^\infty(\{\psi^t\}^{-1} \circ \{\phi_i^t\}) \to 0, i \to \infty$.*

Then we must have $\phi = \psi^1$.

The choice of the L^∞−length in the statement of Theorem 3.3 is technically supported by the following facts.

Lemma 3.4. (Naturality of the uniform sup norm) *Let $\{\rho_i^t\}$ be a sequence of harmonic isotopies generated by $(0,\mathcal{H}^i)$ and let $\{\rho^t\}$ be another harmonic isotopy generated by $(0,\mathcal{H})$ such that $\max_{t\in[0,1]}|\mathcal{H}_t^i - \mathcal{H}_t| \to 0, i \to \infty$. Then, the following properties hold*

1. *$l^\infty(\{\rho^t\}^{-1} \circ \{\rho_i^t\}) \to 0, i \to \infty$, and*
2. *$\{\rho_i^t\}$ converges uniformly to $\{\rho^t\}$.*

Proof. For (2), we define a sequence (Z_t^i) of smooth families of harmonic vector fields such that $\iota(Z_t^i)\omega = \mathcal{H}_t^i$, for each i and for all t. Likewise, we define a smooth family (Z_t) of harmonic vector fields such that $\iota(Z_t)\omega = \mathcal{H}_t$, for all t. Since by assumption the sequence $(\mathcal{H}^i)$ converges uniformly to $\mathcal{H}$, it turns out that the sequence (Z_t^i) converges uniformly to (Z_t). Therefore, it follows from the standard continuity theorem of ODE for Lipschitz vector fields that the sequence of paths generated by (Z_t^i) converges uniformly to the path generated by (Z_t), i.e. $\{\rho_i^t\}$ converges uniformly to $\{\rho^t\}$. For (1), compute

$$\overline{(0,\mathcal{H})} \bowtie (0,\mathcal{H}^i) = (\widetilde{\Delta}(\mathcal{H}^i - \mathcal{H},\{\rho^t\}),\mathcal{H}^i - \mathcal{H}),$$

for each i, and derive from (2.18) that

$$\max_{t\in[0,1]} osc(\Delta_t(\mathcal{H}^i - \mathcal{H},\{\rho^t\})) \leq 2diam(M)(1 + \sup_{t,s}|D\rho^t(\gamma_{y_0}(s))|) \max_{t\in[0,1]}|\mathcal{H}_t - \mathcal{H}_t^i| \tag{3.7}$$

where $D\rho^t$ stands for the tangent map of ρ^t, $y_0 \in M$ and γ_{y_0} is a geodesic (see Subsection 2.5). The right hand side in (3.7) tends to zero when i goes at infinity since the quantity $(1 + \sup_{t,s}|D\rho^t(\gamma_{y_0}(s))|)$ is bounded, and by assumption we have $\max_{t\in[0,1]}|\mathcal{H}_t^i - \mathcal{H}_t| \to 0, i \to \infty$. This completes the proof. ■

Remark 3.5. It is clear from Lemma 3.4 that if $\{\rho_i^t\}$ is a sequence of harmonic isotopies generated by $(0,\mathcal{H}^i)$ and $\{\rho^t\}$ is another harmonic isotopy generated by $(0,\mathcal{H})$, then the convergence $l^\infty(\{\rho^t\}^{-1} \circ \{\rho_i^t\}) \to 0, i \to \infty$, is equivalent to the convergence $\max_{t\in[0,1]}|\mathcal{H}_t^i - \mathcal{H}_t| \to 0, i \to \infty$.

The following fact is a consequence of Lemma 3.4.

Corollary 3.6. *Let Φ_i be a sequence of symplectic isotopies and let Φ be another symplectic isotopy such that $l^\infty(\Phi^{-1}\circ\Phi_i)\to 0, i\to\infty$. If for each i, μ_i is the Hamiltonian isotopy in the Hodge decomposition of Φ_i, and μ is the Hamiltonian isotopy in the Hodge decomposition of Φ, then we have $l^\infty(\mu^{-1}\circ\mu_i)\to 0, i\to\infty$.*

Proof. Let Φ be a symplectic isotopy generated by $(U,\mathcal{H})$, and Φ_i be a sequence of symplectic isotopies generated by $(U^i,\mathcal{H}^i)$. Then, by Corollary 2.1, the Hamiltonian part μ in the Hodge decomposition of Ψ is generated by $(U\circ\phi_{(0,\mathcal{H})},0)$, and for each i, the Hamiltonian part μ_i in the Hodge decomposition of Φ_i is generated by $(U^i\circ\phi_{(0,\mathcal{H}^i)},0)$. For each i, compute

$$osc(U^i_t\circ\phi^t_{(0,\mathcal{H}^i)}-U_t\circ\phi^t_{(0,\mathcal{H})})\leq osc(U^i_t-U_t)+osc(U_t\circ\phi^t_{(0,\mathcal{H}^i)}-U_t\circ\phi^t_{(0,\mathcal{H})}) \tag{3.8}$$

for all t, and derive from the uniform continuity of the map $(t,x)\mapsto U_t(x)$ together with Lemma 3.4 that $\max_t(osc(U_t\circ\phi^t_{(0,\mathcal{H}^i)}-U_t\circ\phi^t_{(0,\mathcal{H})}))\to 0, i\to\infty$, while by assumption we have $\max_t(osc(U^i_t-U_t))\to 0, i\to\infty$. Hence,

$$\max_t(osc(U^i_t\circ\phi^t_{(0,\mathcal{H}^i)}-U_t\circ\phi^t_{(0,\mathcal{H})}))\to 0, i\to\infty. \tag{3.9}$$

This completes the proof. ■

Proof of Theorem 3.3. Firstly, note that we do not use the positivity of the symplectic displacement energy; and this renders the proof rather delicate. So, we shall proceed in several steps.

- Step (1). **(Convergence of symplectic isotopies)**. For each i, let $\{\rho^t_i\}$ (resp. $\{\rho^t\}$) be the harmonic isotopy arising in the Hodge decomposition of the isotopy $\{\phi^t_i\}$ (resp. $\{\psi^t\}$). From the convergence $l^\infty(\{\psi^t\}^{-1}\circ\{\phi^t_i\})\to 0, i\to\infty$, we derive by the mean of Lemma 3.4 that
    1. $l^\infty(\{\rho^t\}^{-1}\circ\{\rho^t_i\})\to 0, i\to\infty$, and
    2. $\{\rho^t_i\}$ converges uniformly to $\{\rho^t\}$.
- Step (2). **(Decomposition of the map $\phi=\lim_{C^0}(\phi^1_i)$)**. For each i, let $\{\mu^t_i\}$ (resp. $\{\mu^t\}$) denote the Hamiltonian isotopy arising in the Hodge decomposition of the isotopy $\{\phi^t_i\}$ (resp. $\{\psi^t\}$). One derives from the assumption that the sequence of time-one maps (ϕ^1_i) converges uniformly to ϕ, and according to step (1) the sequence of time-one maps (ρ^1_i) converges uniformly to the time-one map ρ^1. The preceding arguments imply that the sequence of time-one maps (μ^1_i) converges uniformly to a continuous map $\sigma: M\to M$ because $\mu^1_i=(\rho^1_i)^{-1}\circ\phi^1_i$ for each i, and each factor of the composition converges in the C^0-metric. Note that at this level, we have no guarantee that the maps σ and ϕ are invertible. Since the composition of maps is continuous with respect to the C^0-metric, it follows from the above C^0-convergences that the map ϕ decomposes as follows:

$$\phi=\rho^1\circ\sigma. \tag{3.10}$$

- Step (3). **(The Hamiltonian nature of the map σ)**. To achieve the proof, all we have to show is that $\sigma = \mu^1$, where μ^1 is the time-one map of the Hamiltonian isotopy $\{\mu^t\}$. Arguing indirectly, i.e. assuming that $\sigma \neq \mu^1$, we derive that there exists a small non-empty open ball $B \subset M$ such that $\overline{B}$ is completely displaced by $(\mu^1)^{-1} \circ \sigma$, i.e. $\overline{B} \cap [(\mu^1)^{-1} \circ \sigma](\overline{B}) = \emptyset$. Since $\overline{B}$ is compact and the convergence $\mu_i^1 \to \sigma$ is uniform, we must have $\overline{B} \cap [(\mu^1)^{-1} \circ \mu_i^1](\overline{B}) = \emptyset$, for i large enough. The above arguments tell us that we can apply the energy-capacity inequality theorem [11] to obtain

$$0<C(\overline{B})/2 \leq l^\infty(\{\mu^t\}^{-1} \circ \{\mu_i^t\}) \tag{3.11}$$

for i large enough, where $C(\overline{B})$ represents the Gromov area of the ball $\overline{B}$. But, Corollary 3.6 implies that the right hand side in (3.11) tends to zero for i large enough, and this contradicts the positivity of the Gromov area $C(\overline{B})$. Hence, we have proved that

$$\sigma = \mu^1. \tag{3.12}$$

- Step (4). Finally, (3.10) together with (3.12) implies that,

$$\phi = \rho^1 \circ \sigma = \rho^1 \circ \mu^1 = \psi^1. \tag{3.13}$$

This completes the proof. ■

The following result is an immediate consequence of Theorem 3.3, and it can justify the definition of strong symplectic isotopies in the L^∞−context [7, 16].

Corollary 3.7. *Let $\Phi_i = \{\phi_i^t\}$ be a sequence of symplectic isotopies, $\Psi = \{\psi_t\}$ be another symplectic isotopy, and let $\eta : t \mapsto \eta_t$ be a family of maps $\eta_t : M \to M$, such that the sequence Φ_i converges uniformly to η and $l^\infty(\Psi^{-1} \circ \Phi_i) \to 0, i \to \infty$. Then $\eta = \Psi$.*

Proof. Assume the contrary i.e. assume that $\Psi \neq \eta$. This is equivalent to say that there exists $t \in]0, 1]$ such that $\eta_t \neq \psi_t$. Therefore, the sequence of symplectic paths $\Xi_i : s \mapsto \phi_i^{st}$ contradicts Theorem 3.3. This completes the proof. ■

In order to introduce the second main result of this paper, we need to recall the following definitions.

3.1 Banyaga's Hofer-like norms

Let $\phi \in Symp_0(M, \omega)$. Using the lengths introduced in (3.1) and (3.2), Banyaga [4] defined respectively the $L^{(1,\infty)}$−energy and the L^∞−energy of ϕ by

$$e_0(\phi) = \inf(l^{(1,\infty)}(\Phi)), \tag{3.14}$$

and

$$e_0^\infty(\phi) = \inf(l^\infty(\Phi)) \tag{3.15}$$

where each infimum is taken over the set of all symplectic isotopies Φ with time-one map equal to ϕ.

Therefore, the $L^{(1,\infty)}-$version and the $L^{\infty}-$version of the Hofer-like norms of ϕ are respectively defined by

$$||\phi||_{HL}^{(1,\infty)} = (e_0(\phi) + e_0(\phi^{-1}))/2, \tag{3.16}$$

and

$$||\phi||_{HL}^{\infty} = (e_0^{\infty}(\phi) + e_0^{\infty}(\phi^{-1}))/2. \tag{3.17}$$

The norms $||.||_{HL}^{\infty}$ and $||.||_{HL}^{(1,\infty)}$ are the symplectic analogues of the Hofer norms for Hamiltonian diffeomorphisms in the following sense: if $H^1(M,\mathbb{R})$ vanishes, then the norm $||.||_{HL}^{\infty}$ is called the $L^{\infty}-$version of Hofer's norm, and the norm $||.||_{HL}^{(1,\infty)}$ is called the $L^{(1,\infty)}-$version of Hofer's norm [9]. In [13] it is proved that the two versions of Hofer's norms are equal, i.e. the two norms $||.||_{HL}^{\infty}$ and $||.||_{HL}^{(1,\infty)}$ are equal whenever $H^1(M,\mathbb{R})$ vanishes (see Lemma 5.1.C found in [13]). However, the following main result shows that the two norms $||.||_{HL}^{\infty}$ and $||.||_{HL}^{(1,\infty)}$ continue to coincide regardless of whether $H^1(M,\mathbb{R})$ is trivial or not, i.e. the Hofer-like geometry is independent to the choice of the Hofer-like norm.

Theorem 3.8. *Let (M,ω) be a closed symplectic manifold. For every $\phi \in Symp_0(M,\omega)$, we have*

$$||\phi||_{HL}^{\infty} = ||\phi||_{HL}^{(1,\infty)}.$$

Theorem 3.8 was announced in [5], it yields the symplectic analogue of Lemma 5.1.C found in [13]. Its proof appeals to the following refined version of a result found in [5].

Lemma 3.9. (Fundamental Lemma of Hofer-like Geometry) *Let Φ be a symplectic isotopy. For any positive real number ϵ, there exists a regular symplectic isotopy Ψ with the same extremities than Φ such that*

$$l^{\infty}(\Psi) < l^{(1,\infty)}(\Phi) + \epsilon. \tag{3.18}$$

Proof of Theorem 3.8. It is clear from the definition of the Hofer-like energies that $||.||_{HL}^{(1,\infty)} \leq ||.||_{HL}^{\infty}$. For the converse, consider $\phi \in Symp_0(M,\omega)$, and derive from the characterization of the infimum that for each positive real number ϵ, there exists a symplectic isotopy Φ_ϵ with time-one map equals to ϕ such that $l^{(1,\infty)}(\Phi_\epsilon) \leq e_0(\phi) + \epsilon$. By Lemma 3.9, there exists another symplectic isotopy Ψ_ϵ with the same extremities than Φ_ϵ such that $l^{\infty}(\Psi_\epsilon) < l^{(1,\infty)}(\Phi_\epsilon) + \epsilon$. This yields, $e_0^{\infty}(\phi) \leq l^{\infty}(\Psi_\epsilon) < e_0(\phi) + 2\epsilon$, i.e.

$$e_0^{\infty}(\phi) < e_0(\phi) + 2\epsilon. \tag{3.19}$$

In a similar way we use once more the Lemma 3.9 to deduce that for each positive real number ϵ, we have

$$e_0^{\infty}(\phi^{-1}) < e_0(\phi^{-1}) + 2\epsilon. \tag{3.20}$$

Therefore, adding (3.19) and (3.20) member to member leads to

$$||\phi||_{HL}^{\infty} = (e_0^{\infty}(\phi^{-1}) + e_0^{\infty}(\phi))/2 < (e_0(\phi) + e_0(\phi^{-1}) + 4\epsilon)/2 = ||\phi||_{HL}^{(1,\infty)} + 2\epsilon. \tag{3.21}$$

Since (3.21) holds for any arbitrary positive ϵ, we conclude that $||\phi||_{HL}^{\infty} \leq ||\phi||_{HL}^{(1,\infty)}$. This completes the proof. ∎

3.2 Proof of Lemma 3.9

In this subsection, we will always denote by $r(g)$ the injectivity radius of the Riemannian metric g. We shall need the following fact.

Lemma 3.10. *Let (M,g) be a closed oriented Riemannian manifold, and let $\mathcal{H} = (\mathcal{H}_t)$ be a smooth family of closed $1-$forms. The following facts hold:*

1. *If $\Psi = \{\psi_t\}$ is an isotopy, and $\xi_j : [0,1] \to [0,1]$, $j = 1,2$ are two smooth monotonic functions that fix 0, then there exists a constant B_2 which depends only on $\mathcal{H}$ and Ψ such that*
$$\int_0^1 osc(\Delta_t(\mathcal{H},\Psi^{\xi_1}) - \Delta_t(\mathcal{H},\Psi^{\xi_2}))dt \leq B_2\|\xi_1 - \xi_2\|_{ham}. \tag{3.22}$$

2. *If $\Phi = \{\phi_t\}$ and $\Psi = \{\psi_t\}$ are two isotopies such that $\bar{d}(\Phi,\Psi) \leq r(g)/2$, then*
$$\max_{t\in[0,1]} osc(\Delta_t(\mathcal{H},\Phi) - \Delta_t(\mathcal{H},\Psi)) \leq 4 \max_{t\in[0,1]} |\mathcal{H}_t|_0\bar{d}(\Phi,\Psi). \tag{3.23}$$

Proof. For each $j = 1,2$, differentiating the reparameterized path Ψ^{ξ_j} in the variable t yields $\dot{\Psi}^{\xi_j}(t) = \dot{\xi}_j(t)\dot{\psi}_{\xi_j(t)}$, for all $t \in [0,1]$. Now, compute

$$\Delta_t(\mathcal{H},\Psi^{\xi_j}) = \int_0^t \mathcal{H}_t(\dot{\Psi}^{\xi_j}(s)) \circ \Psi^{\xi_j}(s)ds = \int_0^t \dot{\xi}_j(s)\mathcal{H}_t(\dot{\psi}_{\xi_j(s)}) \circ \psi_{\xi_j(s)}ds \tag{3.24}$$

for each t, and we use a suitable variable change to see that the right hand side in (3.24) can be written as
$$\int_0^t \dot{\xi}_j(s)\mathcal{H}_t(\dot{\psi}_{\xi_j(s)}) \circ \psi_{\xi_j(s)}ds = \int_0^{\xi_j(t)} \mathcal{H}_t(\dot{\psi}_u) \circ \psi_u du. \tag{3.25}$$
Relation (3.24) together with (3.25) yields

$$\int_0^1 osc(\Delta_t(\mathcal{H},\Psi^{\xi_1}) - \Delta_t(\mathcal{H},\Psi^{\xi_2}))dt = \int_0^1 osc\left(\int_{\min\{\xi_1(t),\xi_2(t)\}}^{\max\{\xi_1(t),\xi_2(t)\}} \mathcal{H}_t(\dot{\psi}_u) \circ \psi_u du\right)dt, \tag{3.26}$$
$$\leq 2\sup_{s,t,x} |\mathcal{H}_t(\dot{\psi}_s)(x)|\|\xi_1 - \xi_2\|_{C^0},$$
$$\leq 2\sup_{s,t,x} |\mathcal{H}_t(\dot{\psi}_s)(x)|\|\xi_1 - \xi_2\|_{ham}.$$

Therefore, the last inequality in (3.26) suggests that the desired B_2 can be chosen as $B_2 := 2\sup_{s,t,x} |\mathcal{H}_t(\dot{\psi}_s)(x)| < \infty$. For (2), as in Subsection 2.5, fix a point $m \in M$, and for all x in M, pick any curve γ_x from m to x, and therefore define a smooth function $\bar{\mu}_t$ by

$$\bar{\mu}_t(x) := \int_{\gamma_x} (\phi_t^*(\mathcal{H}_t) - \psi_t^*(\mathcal{H}_t)) \tag{3.27}$$

for each t. Now, let y_0 be any point of M that realizes the supremum of the function $x \mapsto |\bar{\mu}_t(x)|$, i.e.

$$\sup_x |\bar{\mu}_t(x)| = |\int_{\gamma_{y_0}} (\phi_t^*(\mathcal{H}_t) - \psi_t^*(\mathcal{H}_t))| = |\int_{\phi_t\circ\gamma_{y_0}} \mathcal{H}_t - \int_{\psi_t\circ\gamma_{y_0}} \mathcal{H}_t|. \tag{3.28}$$

In the what follows, for each fixed t, we shall express the quantity $(\int_{\phi_t\circ\gamma_{y_0}}\mathcal{H}_t-\int_{\psi_t\circ\gamma_{y_0}}\mathcal{H}_t)$ as a difference between two integrals of the closed 1−form $\mathcal{H}_t$ over two minimizing geodesics, and next we shall combine it with a well-known result from Riemannian geometry to achieve the proof. For this purpose, using the topological assumption $\bar{d}(\Phi,\Psi)\leq r(g)/2$, we derive that for each fixed $t\in[0,1]$, and for all $z\in M$, the points $\phi_t(z)$ and $\psi_t(z)$ can be connected through a minimizing geodesic χ_z. This induces a homotopy $H^t:[0,1]\times[0,1]\to M$ between the curves $s\mapsto(\phi_t\circ\gamma_{y_0})(s)$ and $s\mapsto(\psi_t\circ\gamma_{y_0})(s)$, i.e. we have $H^t(0,s)=\phi_t(\gamma_{y_0}(s))$ and $H^t(1,s)=\psi_t(\gamma_{y_0}(s))$ for all $s\in[0,1]$. We may define $H^t(u,s)$ to be the unique minimizing geodesic $\chi_{(\gamma_{y_0}(s))}$ that connects $\phi_t(\gamma_{y_0}(s))$ to $\psi_t(\gamma_{y_0}(s))$ for all $s\in[0,1]$. On the other hand, put

$$\boxplus^t:=\{H^t(u,s)\mid(u,s)\in[0,1]\times[0,1]\}\tag{3.29}$$

and derive from Stokes' theorem that $\int_{\partial\boxplus^t}\mathcal{H}_t=0$, where $\partial\boxplus^t$ represents the boundary of the set $\boxplus^t$, i.e.

$$\int_{\phi_t\circ\gamma_{y_0}}\mathcal{H}_t-\int_{\psi_t\circ\gamma_{y_0}}\mathcal{H}_t=\int_{\chi_{y_0}}\mathcal{H}_t-\int_{\chi_m}\mathcal{H}_t\tag{3.30}$$

for each t. In (3.30), each of the integrals $\int_{\chi_{y_0}}\mathcal{H}_t$ and $\int_{\chi_m}\mathcal{H}_t$ is bounded from above by $|\mathcal{H}_t|_0\bar{d}(\Phi,\Psi)$ because the speed is constant and equals to the distance between the end points. Finally, we see that inserting (3.28) in (3.30) implies

$$\max_{t\in[0,1]}osc(\Delta_t(\mathcal{H},\Phi)-\Delta_t(\mathcal{H},\Psi))\leq 4\max_{t\in[0,1]}|\mathcal{H}_t|_0\bar{d}(\Phi,\Psi).$$

This completes the proof. ■

Proof of Lemma 3.9. Let Φ be a symplectic isotopy generated by $(U,\mathcal{H})$, and let ϵ be a positive real number. Consider Ξ to be the isotopy obtained by regularizing the isotopy Φ as explained in Subsection 2.10. It follows from Subsection 2.10 that the isotopy Ξ is generated by the element $(V,\mathcal{K})$ defined in (2.40) so that

$$l^{(1,\infty)}(\Xi)=\int_0^1(osc(V_t)+|\mathcal{K}_t|)dt\leq l^{(1,\infty)}(\Phi)+\int_0^1 osc(r_t)dt+\int_0^1 osc(\Delta_t(\mathcal{H},\phi_{(r,0)}))dt,$$

where $\phi_{(r,0)}$ is a Hamiltonian loop such that $\int_0^1 osc(r_t)dt<\epsilon/2$ (see Subsection 2.10). Since Polterovich's arguments provided in Subsection 2.10 state that the path $\phi_{(r,0)}$ is arbitrarily close to the constant path identity (in the C^∞−topology), we derive from Lemma 3.10 that one can assume $\int_0^1 osc(\Delta_t(\mathcal{H},\phi_{(r,0)}))dt<\epsilon/2$. Hence,

$$l^{(1,\infty)}(\Xi)<l^{(1,\infty)}(\Phi)+\epsilon.\tag{3.31}$$

Now we use the isotopy Ξ to define a curve ζ as in Subsection 2.10 (see (2.43)), and we let Ξ^ζ to be the path obtained by a reparameterization of Ξ via curve ζ. Next, consider

$$\Omega_s:=\zeta'(s)(osc(V_{\zeta(s)})+|\mathcal{K}_{\zeta(s)}|),$$

for each s, and derive from (2.44) that $\Omega_s=l^{(1,\infty)}(\Xi)$, i.e.

$$l^\infty(\Xi^\zeta)=\max_s\Omega_s=l^{(1,\infty)}(\Xi).\tag{3.32}$$

Relations (3.31) and (3.32) imply that

$$l^{\infty}(\Xi^{\zeta})<l^{(1,\infty)}(\Phi)+\epsilon.$$

Therefore, to complete the proof, it suffices to take $\Psi=\Xi^{\zeta}$. ∎

Remark 3.11. As in [14], we have $l^{(1,\infty)}(.)\leq l^{\infty}(.)$ in general, where the former is invariant under reparameterization, while the latter is far from being invariant. Actually, it is clear from the proof of Lemma 3.9 that any regular symplectic path Φ can be reparameterized to obtain another path Ψ with the same extremities than Φ so that $l^{(1,\infty)}(\Psi)=l^{\infty}(\Psi)$.

4 Some auxiliary results

The goal of this section is to elaborate the symplectic analogues of some approximation lemmas found in [14].

Definition 4.1. ([4]) The $L^{(1,\infty)}$–version of Hofer-like topology on the group $Iso(M,\omega)$ is the topology induced by the following metric:

$$D^1((U,\mathcal{H}),(V,\mathcal{K}))=\frac{D_0((U,\mathcal{H}),(V,\mathcal{K}))+D_0(\overline{(U,\mathcal{H})},\overline{(V,\mathcal{K})})}{2} \tag{4.1}$$

where

$$D_0((U,\mathcal{H}),(V,\mathcal{K}))=\int_0^1 osc(U_t-V_t)+|\mathcal{H}_t-\mathcal{K}_t|dt. \tag{4.2}$$

We will need the following lemma.

Lemma 4.2. (Reparameterization Lemma) *Let (M,g) be a closed oriented Riemannian manifold. Let $\mathcal{H}\in\mathcal{PH}^1(M,g)$, and let $\Phi=\{\phi_t\}\in Iso(M,\omega)$. If $\xi_j:[0,1]\to[0,1]$, $j=1,2$ are two smooth functions such that ξ_1 is monotonic, then there exists a constant B_1 which depends on $\mathcal{H}$ and Φ such that*

$$\int_0^1 osc(\Delta_t(\mathcal{H}^{\xi_1},\Phi)-\Delta_t(\mathcal{H}^{\xi_2},\Phi))dt\leq B_1\|\xi_1-\xi_2\|_{ham}.$$

Proof. From the equality,

$$\Delta(\mathcal{H}^{\xi_2},\Phi)-\Delta(\mathcal{H}^{\xi_1},\Phi)=\Delta_t(\mathcal{H}^{\xi_2}-\mathcal{H}^{\xi_1},\Phi),$$

we derive by the mean of (2.17) that

$$\int_0^1 osc(\Delta_t(\mathcal{H}^{\xi_2},\Phi)-\Delta_t(\mathcal{H}^{\xi_1},\Phi))dt\leq 2diam(M)(1+\sup_{t,s}|D\phi_t(\gamma_{y_0}(s))|)\int_0^1|\mathcal{H}_t^{\xi_1}-\mathcal{H}_t^{\xi_2}|dt \tag{4.3}$$

where $D\phi_t$ is the tangent map of ϕ_t, and γ_{y_0} is a geodesic (see Subsection 2.5). Since we always have

$$|\mathcal{H}_t^{\xi_1}-\mathcal{H}_t^{\xi_2}|\leq|\dot{\xi}_1(t)\mathcal{H}_{\xi_1(t)}-\dot{\xi}_1(t)\mathcal{H}_{\xi_2(t)}|+|\dot{\xi}_1(t)\mathcal{H}_{\xi_2(t)}-\dot{\xi}_2(t)\mathcal{H}_{\xi_2(t)}|,$$

we use the Lipschitz nature of the map $t \mapsto \mathcal{H}_t$ to derive the existence of a constant $c_0>0$ (which depends on $\mathcal{H}$) such that,

$$|\mathcal{H}_t^{\xi_1} - \mathcal{H}_t^{\xi_2}| \leq \max_t |\mathcal{H}_t||\dot{\xi}_1(t) - \dot{\xi}_2(t)| + c_0\|\xi_1 - \xi_2\|_{C^0}|\dot{\xi}_1(t)| \tag{4.4}$$

for each t. Therefore, integrating (4.4) between 0 and 1 with respect to t gives

$$\begin{aligned} \int_0^1 |\mathcal{H}_t^{\xi_1} - \mathcal{H}_t^{\xi_2}|dt &\leq \max_t |\mathcal{H}_t| \int_0^1 |\dot{\xi}_1(t) - \dot{\xi}_2(t)|dt + c_0\|\xi_1 - \xi_2\|_{C^0} \\ &\leq 2\max(c_0, \max_t |\mathcal{H}_t|)\|\xi_1 - \xi_2\|_{ham}. \end{aligned} \tag{4.5}$$

Finally, (4.3) together with (4.5) yields

$$\int_0^1 osc(\Delta_t(\mathcal{H}^{\xi_2}, \Phi) - \Delta_t(\mathcal{H}^{\xi_1}, \Phi))dt \leq B_1\|\xi_1 - \xi_2\|_{ham},$$

where $B_1 = 4diam(M)\max(c_0, \max_t |\mathcal{H}_t|)(1 + \sup_{t,s} |D\phi_t(\gamma_{y_0}(s))|)<\infty$. This completes the proof.■

Lemma 4.3. *If* $(U, \mathcal{H}) \in \mathfrak{T}(M, \omega, g)$, *and* $\xi_j : [0,1] \to [0,1]$, $j = 1,2$ *are two smooth monotonic functions, then there exists a constant C which depends on* $(U, \mathcal{H})$ *such that,*

$$D^1((U, \mathcal{H})^{\xi_1}, (U, \mathcal{H})^{\xi_2}) \leq C\|\xi_1 - \xi_2\|_{ham}.$$

We shall give a complete proof of Lemma 4.3 later on. The following result is an immediate consequence of Lemma 4.3.

Lemma 4.4. *Let* $(U^i, \mathcal{H}^i)$ *be a Cauchy sequence in* D^1, *and* $\xi_l : [0,1] \to [0,1]$, $l = 1,2$ *be two monotonic smooth functions. Given* $\epsilon>0$, *there exists a positive constant* $\delta = \delta((U^i, \mathcal{H}^i))$, *and a larger positive integer* $j_0 = j_0((U^i, \mathcal{H}^i))$ *such that if the inequality* $\|\xi_1 - \xi_2\|_{ham}<\delta$ *holds, then*

$$D^1((U^i, \mathcal{H}^i)^{\xi_1}, (U^i, \mathcal{H}^i)^{\xi_2})<\epsilon,$$

for all $i \geq j_0$.

Proof. Since $(U^i, \mathcal{H}^i)$ is Cauchy in D^1, one can choose an integer j_0 large enough such that $D^1((U^i, \mathcal{H}^i)^{\xi_1}, (U^{j_0}, \mathcal{H}^{j_0})^{\xi_1})<\epsilon/3$ for all $i \geq j_0$. Assume this is done. Now we apply Lemma 4.3 with $(U^{j_0}, \mathcal{H}^{j_0})$, ξ_1, and ξ_2 to derive that there exists a constant C which depends on $(U^{j_0}, \mathcal{H}^{j_0})$ such that,

$$D^1((U^{j_0}, \mathcal{H}^{j_0})^{\xi_1}, (U^{j_0}, \mathcal{H}^{j_0})^{\xi_2}) \leq C\|\xi_1 - \xi_2\|_{ham}. \tag{4.6}$$

Taking $\delta = \epsilon/3C$, we compute

$$\begin{aligned} D^1((U^i, \mathcal{H}^i)^{\xi_1}, (U^i, \mathcal{H}^i)^{\xi_2}) &\leq D^1((U^i, \mathcal{H}^i)^{\xi_1}, (U^{j_0}, \mathcal{H}^{j_0})^{\xi_1}) \\ +D^1((U^{j_0}, \mathcal{H}^{j_0})^{\xi_1}, (U^{j_0}, \mathcal{H}^{j_0})^{\xi_2}) &+ D^1((U^{j_0}, \mathcal{H}^{j_0})^{\xi_2}, (U^i, \mathcal{H}^i)^{\xi_2}) \\ &\leq \epsilon/3 + \epsilon/3 + \epsilon/3, \end{aligned}$$

as long as $\|\xi_1 - \xi_2\|_{ham}<\delta$, and $i \geq j_0$. This completes the proof. ■

The following result is the symplectic analogue of a slight variation of the $L^{(1,\infty)}-$ approximation lemma found in [14]. It implies that a symplectic isotopy Φ can be approximated in each of the metrics D^1 and $\bar{d}$ by a boundary flat symplectic path with the same extremities than Φ.

Lemma 4.5. *Let Φ be a symplectic isotopy generated by $(U,\mathcal{H})$, and let ϵ be a positive real number. Then, there exists a boundary flat symplectic isotopy $\Psi = \psi_{(V,\mathcal{K})}$ with the same extremities than Φ such that $D^1((U,\mathcal{H}),(V,\mathcal{K}))<\epsilon$, and $\bar{d}(\Psi,\Phi)<\epsilon$.*

Proof. Let ϵ be a positive real number, and consider $\xi : [0,1] \to [0,1]$ to be any smooth and increasing function which is constant on the intervals $[0,\delta]$ and $[1-\delta,1]$ where $0<\delta<1/13$. Therefore, define $(V,\mathcal{K})$ to be the element $(U,\mathcal{H})^\xi$ obtained by a reparameterization of $(U,\mathcal{H})$ via the curve ξ as explained in Subsection 2.7. It follows from the definition of the curve ξ that the symplectic isotopy Ψ generated by $(V,\mathcal{K})$ is boundary flat and it has the same extremities than Φ. Applying Lemma 4.3 with $\xi_1 = id_{[0,1]}$ and $\xi_2 = \xi$, we derive from the above arguments that

$$D^1((U,\mathcal{H}),(V,\mathcal{K})) \leq C\|\xi - id_{[0,1]}\|_{ham},$$

where C is the constant in Lemma 4.3 which only depends on $(U,\mathcal{H})$. On the other hand, since the maps $(t,x) \mapsto \phi^t_{(U,\mathcal{H})}(x)$ and $(t,x) \mapsto \phi^{-t}_{(U,\mathcal{H})}(x)$ are Lipschitz continuous, it turns out that there exists a constant $l_0>0$ which depends only on $(U,\mathcal{H})$ such that

$$\bar{d}(\phi_{(U,\mathcal{H})},\psi_{(V,\mathcal{K})}) \leq l_0\|\xi - id_{[0,1]}\|_{C^0}<l_0\|\xi - id_{[0,1]}\|_{ham}.$$

Finally, to conclude, it suffices to choose the function ξ so that
$\|\xi - id_{[0,1]}\|_{ham} \leq \min\{\epsilon/C; \epsilon/l_0; \epsilon\}$. This completes the proof. ∎

Remark 4.6. Notice that the proof of Lemma 4.5 will still hold for any choice of δ such that $[0,\delta]\cap[1-\delta,1] = \emptyset$. This means that in Lemma 4.5 the choice of δ less than $1/13$ has no particular meaning than to satisfy the condition $[0,\delta]\cap[1-\delta,1] = \emptyset$.

Proof of Lemma 4.3. Let Φ to be a symplectic isotopy generated by $(U,\mathcal{H})$.

- Step (1). Consider the normalized function $V = U^{\xi_1} - U^{\xi_2}$, and compute

$$|V_t| = |\dot{\xi}_1(t)U_{\xi_1(t)} - \dot{\xi}_2(t)U_{\xi_2(t)}| \leq |\dot{\xi}_1(t)||U_{\xi_1(t)} - U_{\xi_2(t)}| + |\dot{\xi}_1(t) - \dot{\xi}_2(t)||U_{\xi_2(t)}| \tag{4.7}$$

for each t. Since the map $(t,x) \mapsto U_t(x)$ is smooth on a compact set, it is Lipschitz, i.e. there exists a positive constant k_0 which depends on U such that $\max_{x\in M}|U_t(x) - U_s(x)| \leq k_0|t-s|$ for all $t,s \in [0,1]$. This together with (4.7) yields

$$0 \leq \max_{x\in M} V_t(x) \leq k_0|\dot{\xi}_1(t)||\xi_1(t) - \xi_2(t)| + \max_x(U_t(x))|\dot{\xi}_1(t) - \dot{\xi}_2(t)|. \tag{4.8}$$

Similarly, one can check that

$$0 \leq -\min_{x\in M} V_t(x) \leq k_0|\dot{\xi}_1(t)||\xi_1(t) - \xi_2(t)| - \min_x(U_t(x))|\dot{\xi}_1(t) - \dot{\xi}_2(t)|. \tag{4.9}$$

Adding (4.8) and (4.9) member to member, and integrating the resulting inequality in the variable t gives

$$\int_0^1 osc(V_t)dt \leq 2k_0 \max_t|\xi_1(t) - \xi_2(t)| + \max_t(osc(U_t))\int_0^1 |\dot{\xi}_1(t) - \dot{\xi}_2(t)|dt. \tag{4.10}$$

- Step (2). Put $\mathcal{K} = \mathcal{H}^{\xi_1} - \mathcal{H}^{\xi_2}$, and compute

$$|\mathcal{K}_t| = |\dot{\xi}_1(t)\mathcal{H}_{\xi_1(t)} - \dot{\xi}_2(t)\mathcal{H}_{\xi_2(t)}| \leq |\mathcal{H}_{\xi_1(t)} - \mathcal{H}_{\xi_2(t)}||\dot{\xi}_1(t)| + |\dot{\xi}_1(t) - \dot{\xi}_2(t)||\mathcal{H}_{\xi_2(t)}| \tag{4.11}$$

for each t. The Lipschitz nature of the smooth map $t \mapsto \mathcal{H}_t$ implies that there exists a positive constant c_0 such that $|\mathcal{H}_t - \mathcal{H}_s| \leq c_0|t-s|$ for all $s,t \in [0,1]$. This tells us that (4.11) implies

$$|\mathcal{K}_t| \leq c_0|\xi_1(t) - \xi_2(t)||\dot{\xi}_1(t)| + |\dot{\xi}_1(t) - \dot{\xi}_2(t)||\mathcal{H}_{\xi_2(t)}|. \tag{4.12}$$

Integrating (4.12) in the variable t yields,

$$\int_0^1 |\mathcal{K}_t| dt \leq c_0 \max_t |\xi_1(t) - \xi_2(t)| + \max_t |\mathcal{H}_t| \int_0^1 |\dot{\xi}_1(t) - \dot{\xi}_2(t)| dt. \tag{4.13}$$

Adding (4.10) and (4.13) member to member gives

$$D_0((U,\mathcal{H})^{\xi_1}, (U,\mathcal{H})^{\xi_2}) \leq B_3 \|\xi_1 - \xi_2\|_{ham} \tag{4.14}$$

where

$$B_3 = 2\max\{2k_0 + c_0, \max_t |\mathcal{H}_t| + \max_t osc(U_t)\}.$$

- Step (3). On the other hand, for each $j = 1,2$, we compute

$$\overline{(U,\mathcal{H})^{\xi_j}} = (-U^{\xi_j} \circ \Phi^{\xi_j} - \widetilde{\Delta}(\mathcal{H}^{\xi_j}, \Phi^{\xi_j}), -\mathcal{H}^{\xi_j}),$$

with

$$\widetilde{\Delta}_t(\mathcal{H}^{\xi_j}, \Phi^{\xi_j}) = \dot{\xi}_j(t)\widetilde{\Delta}_{\xi_j(t)}(\mathcal{H}, \Phi),$$

for all t, and derive from the definition of D_0 that

$$D_0(\overline{(U,\mathcal{H})^{\xi_1}}, \overline{(U,\mathcal{H})^{\xi_2}}) \leq \int_0^1 osc\Big(\widetilde{\Delta}_t(\mathcal{H}^{\xi_1}, \Phi^{\xi_1}) - \widetilde{\Delta}_t(\mathcal{H}^{\xi_2}, \Phi^{\xi_2})\Big) dt \tag{4.15}$$

$$+ \int_0^1 osc(U_t^{\xi_2} - U_t^{\xi_1}) + |\mathcal{H}_t^{\xi_1} - \mathcal{H}_t^{\xi_2}| dt + \int_0^1 osc\Big(U_t^{\xi_1} \circ \Phi^{\xi_2}(t) - U_t^{\xi_1} \circ \Phi^{\xi_1}(t)\Big) dt$$

$$\leq \int_0^1 osc\Big(\widetilde{\Delta}_t(\mathcal{H}^{\xi_1}, \Phi^{\xi_1}) - \widetilde{\Delta}_t(\mathcal{H}^{\xi_2}, \Phi^{\xi_2})\Big) dt + D_0((U,\mathcal{H})^{\xi_1}, (U,\mathcal{H})^{\xi_2}) + k_1\|\xi_1 - \xi_2\|_{ham}.$$

Note that in (4.15), to obtain the quantity $k_1\|\xi_1 - \xi_2\|_{ham}$ we have used the Lipschitz natures of the maps $(x,t) \mapsto U_t(x)$, $(x,t) \mapsto \Phi^{-1}(t)(x)$ and $(x,t) \mapsto \Phi(t)(x)$ to derive the existence of a positive constant k_1 which depends on Φ such that

$$\int_0^1 osc\Big(\dot{\xi}_1(t) U_{\xi_1(t)} \circ \Phi^{\xi_1}(t) - \dot{\xi}_1(t) U_{\xi_1(t)} \circ \Phi^{\xi_2}(t)\Big) dt \leq k_1 \|\xi_1 - \xi_2\|_{ham}.$$

Since the map $(x,t) \mapsto \Delta_t(\mathcal{H}, \Phi)(x)$ is smooth and M is compact, we derive as in step (1) that there exists a Lipschitz constant k_2 which depends on Φ such that

$$\int_0^1 osc(\dot{\xi}_1(t)\widetilde{\Delta}_{\xi_1(t)}(\mathcal{H},\Phi) - \dot{\xi}_2(t)\widetilde{\Delta}_{\xi_2(t)}(\mathcal{H},\Phi))dt \leq 2k_2 \max_t |\xi_1(t) - \xi_2(t)| \tag{4.16}$$

$$+ \max_t(osc(\Delta_t(\mathcal{H},\Phi))) \int_0^1 |\dot{\xi}_1(t) - \dot{\xi}_2(t)|dt.$$

Observe that $\max_t(osc(\Delta_t(\mathcal{H},\Phi))) \leq B_2(\mathcal{H},\Phi)$, where B_2 is the constant in Lemma 3.10, and derive from (4.16) that

$$\int_0^1 osc(\dot{\xi}_1(t)\widetilde{\Delta}_{\xi_1(t)}(\mathcal{H},\Phi) - \dot{\xi}_2(t)\widetilde{\Delta}_{\xi_2(t)}(\mathcal{H},\Phi))dt \leq 2\max\{2k_2, B_2\}\|\xi_1 - \xi_2\|_{ham}. \tag{4.17}$$

- Step (4). Since,

$$D^1((U,\mathcal{H})^{\xi_1},(U,\mathcal{H})^{\xi_2}) = \left(D_0(\overline{(U,\mathcal{H})^{\xi_1}},\overline{(U,\mathcal{H})^{\xi_2}}) + D_0((U,\mathcal{H})^{\xi_1},(U,\mathcal{H})^{\xi_2})\right)/2,$$

it follows from step (2) and step (3) that

$$D^1((U,\mathcal{H})^{\xi_1},(U,\mathcal{H})^{\xi_2}) \leq C\|\xi_1 - \xi_2\|_{ham},$$

where

$$C = B_3 + \frac{k_1}{2} + \max\{2k_2, B_2\} < \infty.$$

This completes the proof. ∎

5 Final remark

McDuff-Salamon [12] proved that the orbits of Hamiltonian loops are null-homologous (see Lemma 10.31, [12]). This is equivalent to the following result.

Proposition 5.1. *([12]) Let Ψ be any symplectic isotopy whose flux is non-trivial. Then any loop γ in the homotopy class of an orbit of any Hamiltonian loop (relatively to a fixed base point) trivializes the flux of Ψ, i.e. $Flux(\Psi).[\gamma] = 0$.*

Here, we provide the following alternative proof of Proposition 5.1 based on Lemma 2.4.

Proof. Let Ψ be a symplectic isotopy with a non-trivial flux. Let $\mathcal{H}^\Psi$ denote the harmonic representative in the de Rham cohomology class $Flux(\Psi)$. Consider a Hamiltonian loop $\Phi = \{\phi_t\}$ in $Symp_0(M,\omega)$, and derive from (2.12) that $d\Delta_1(\mathcal{H}^\Psi,\Phi) = 0$, i.e. the function $\Delta_1(\mathcal{H}^\Psi,\Phi)$ is constant. In fact, we have $\Delta_1(\mathcal{H}^\Psi,\Phi)(x) = 0$ for all $x \in M$ because Lemma 2.4 implies that the function $x \mapsto \Delta_1(\mathcal{H}^\Psi,\Phi)(x)$ has a vanishing point on M, and M is connected. On the other hand, for each $x \in M$, consider the loop $\gamma_{x,\Phi} : t \mapsto \phi_t(x)$, and check by the mean of (2.29) that

$$Flux(\Psi).[\gamma_{x,\Phi}] = \int_{\gamma_{x,\Phi}} \mathcal{H}^\Psi = \Delta_1(\mathcal{H}^\Psi,\Phi)(x) = 0.$$

In addition, if β is any representative in the homotopic class $[\gamma_{x,\Phi}]$ (relatively to fix base point), then Stokes' theorem implies that $\int_{\gamma_{x,\Phi}} \mathcal{H}^{\Psi} = \int_{\beta} \mathcal{H}^{\Psi}$. This completes the proof. ■

Proposition 5.1 tells us that on the 2−dimensional revolution torus $\mathbb{T}^2$ equipped with its natural symplectic form ω, no meridian circle of $\mathbb{T}^2$ is an orbit of a Hamiltonian loop in $Symp_0(\mathbb{T}^2, \omega)$ since a meridian of $\mathbb{T}^2$ yields a non-trivial homology class. This tells us how does an orbit of a Hamiltonian loop wind on $\mathbb{T}^2$.

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