African Diaspora Journal of Mathematics
Volume 20, Number 2, pp. 39–44 (2017)
ISSN 1539-854X
www.math-res-pub.org/adjm

On Commutativity of Prime $\Gamma-$Rings with $\theta-$Derivations

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Abstract

Let M be a prime $\Gamma-$ring, I a nonzero ideal, θ an automorphism and d a $\theta-$derivation of M. In this article we have proved the following result: (1) If $d([x,y]_\alpha) = \pm([x,y]_\alpha)$ or $d((x\circ y)_\alpha) = \pm((x\circ y)_\alpha)$ for all $x,y\in I;\alpha\in\Gamma$, then M is commutative. (2) Under the hypothesis $d\theta = \theta d$ and $CharM \neq 2$, if $(d(x)\circ d(y))_\alpha = 0$ or $[d(x),d(y)]_\alpha = 0$ for all $x,y\in I;\alpha\in\Gamma$, then M is commutative. (3) If d acts as a homomorphism or an anti-homomorphism on I, then $d = 0$ or M is commutative. Moreover, an example is given to demonstrate that the primeness imposed on the hypothesis of the various results is essential.

AMS Subject Classification: 16N60, 16U80, 16W25.

Keywords: commutativity, prime $\Gamma-$rings, $\theta-$derivations.

1 Introduction

In 1964, Nobusawa [11] introduced the notion of a Γ-ring, an object more general than a ring. Barnes [4] slightly weakened the conditions in the definition of a Γ-ring in the sense of Nobusawa. Since then, many researchers have done a lot of work on Γ-rings and have obtained some generalizations of the corresponding results in ring theory (see [10] for references). If M and Γ are additive Abelian groups and there exists a mapping $(.,.,.): M\times\Gamma\times M\to M$ which satisfies the following conditions:
(i) $(a,\beta,b)\in M$;
$(ii)(a+b)\alpha c = a\alpha c+b\alpha c$, $a(\alpha+\beta)b = a\alpha b+a\beta b, a\alpha(b+c) = a\alpha b+a\alpha c$;
(iii) $(a\alpha b)\beta c = a\alpha(b\beta c)$, for $a,b,c\in M$ and $\alpha,\beta\in\Gamma$;
then M is called a Γ- ring.

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Obviously every associative ring is a Γ-ring with $M = \Gamma$, but the converse is in general not true. Recall that a Γ-ring M is prime if $a\Gamma M\Gamma b = 0$ implies that $a = 0$ or $b = 0$. A Γ-ring M is said to be a commutative if $x\alpha y = y\alpha x$ for all $x,y \in M$ and $\alpha,\beta \in \Gamma$. A Γ-ring M is said to be 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in M$. Moreover, the set $Z(M) = \{x \in M | x\alpha y = y\alpha x \ \forall\ \alpha \in \Gamma, y \in M\}$ is called the center of the Γ-ring M. We shall write $[x,y]_\alpha = x\alpha y - y\alpha x$ and $(x \circ y)_\alpha = x\alpha y + y\alpha x$ for all $x,y \in M; \alpha \in \Gamma$. Throughout the paper, we shall assume that $x\alpha y\beta z = x\beta y\alpha z$ for all $x,y,z \in M, \alpha,\beta \in \Gamma$ and in this case we have some basic identities: $[x\beta y,z]_\alpha = [x,z]_\alpha\beta y + x\beta[y,z]_\alpha$; $[x,y\beta z]_\alpha = [x,y]_\alpha\beta z + y\beta[x,z]_\alpha$ and $(x \circ (y\beta z))_\alpha = (x \circ y)_\alpha\beta z - y\beta[x,z]_\alpha = y\beta(x \circ z)_\alpha + [x,y]_\alpha\beta z$ for all $x,y,z \in M$ and $\alpha,\beta \in \Gamma$.

An additive subgroup U of a Γ-ring M is called a left (resp. right) ideal of M if $M\Gamma U \subseteq U$ (resp. $U\Gamma M \subseteq U$). If U is both a left ideal and a right ideal, then we say that U is an ideal of M. An additive mapping $d : M \to M$ is called a derivation on M if $d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$ for all $x,y \in M$ and $\alpha \in \Gamma$. Following [7], an additive mapping $d : M \to M$ is called a θ-derivation on M if $d(x\alpha y) = d(x)\alpha y + \theta(x)\alpha d(y)$ for all $x,y \in M$ and $\alpha \in \Gamma$, where θ is an automorphism on M. Let S be a nonempty subset of M and d a $\theta-$derivation of M. If $d(x\alpha y) = d(x)\alpha d(y)$ or $d(x\alpha y) = d(y)\alpha d(x)$ for all $x,y \in S; \alpha \in \Gamma$, then d is said to be a $\theta-$derivation which acts as a homomorphism or an anti-homomorphism on S, respectively.

During the past few decades, there has been an ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain special types of derivations (see [2, 5, 6, 13], where further references can be found). The first result in this direction is due to Posner [14] who proved that if a prime ring R admits a nonzero derivation d such that $[d(x),x] \in Z(R)$ for all $x \in R$, then R is commutative. Recently some authors have obtained commutativity of prime and semiprime rings with derivations, generalized derivations et al., satisfying certain polynomial constraints (see [3, 9, 15], where further references can be found). In the year 2014, Ashraf and Jamal [1] investigated the commutativity of prime Γ-rings satisfying certain differential identities. In this paper, we shall attempt to extend some known commutativity results of rings to Γ-rings involving $\theta-$derivations on some appropriate subset of the Γ-ring M.

2 Main results

Theorem 2.1. *Let M be a prime $\Gamma-$ring, θ an automorphism of M and I a nonzero ideal of M. If M admits a $\theta-$derivation d such that $d([x,y]_\alpha) = [x,y]_\alpha$ for all $x,y \in I$ and $\alpha \in \Gamma$, then M is commutative.*

Proof. By the given hypothesis we have

$$d([x,y]_\alpha) = [x,y]_\alpha \text{ for all } x,y \in I; \alpha \in \Gamma. \tag{2.1}$$

If $d = 0$, then $[x,y]_\alpha = 0$ for all $x,y \in I$. Thus, I is commutative and so is M by [8, Lemma 2.3]. Hence, in the sequel we assume that $d \neq 0$. Replacing y by $y\beta x$ in (2.1) we get $d([x,y\beta x]_\alpha) = [x,y\beta x]_\alpha$, which reduces to $d([x,y]_\alpha\beta x) = [x,y]_\alpha\beta x$ for all $x,y \in I; \alpha,\beta \in \Gamma$. Since d is a $\theta-$derivation, we deduce that

$$d([x,y]_\alpha)\beta x + \theta([x,y]_\alpha)\beta d(x) = [x,y]_\alpha\beta x \text{ for all } x,y \in I; \alpha,\beta \in \Gamma. \tag{2.2}$$

Combining (2.1) and (2.2), we obtain that

$$\theta([x,y]_\alpha)\beta d(x) = 0 \text{ for all } x,y \in I; \alpha,\beta \in \Gamma. \tag{2.3}$$

Replacing y by $z\gamma y$ in (2.3) and using (2.3), we get $\theta([x,z]_\alpha)\gamma\theta(y)\beta d(x) = 0$ for all $x,y,z \in I;\alpha,\beta,\gamma \in \Gamma$. Since θ is an automorphism of M, the above relation implies that

$$[x,z]_\alpha \Gamma I \Gamma \theta^{-1} d(x) = 0 \text{ for all } x,z \in I; \alpha \in \Gamma. \tag{2.4}$$

The primeness of I [12, Lemma 2] forces that for each fixed $x \in I$, either $[x,z]_\alpha = 0$ for all $z \in I$ or $\theta^{-1}d(x) = 0$. Let $K = \{x \in I \mid [x,z]_\alpha = 0\}$ and $L = \{x \in I \mid \theta^{-1}d(x) = 0\}$. Then, K and L are both additive subgroups of I such that $I = K \cup L$. Since a group can't be a union of its two proper subgroups, we have either $I = K$ or $I = L$. If $I = K$, then $[I,I]_\alpha = 0$ and we are done in this case. If $I = L$, then $\theta^{-1}d(I) = 0$. In this case, $d(I) = 0$ and so $0 = d(I\Gamma M) = d(I)\Gamma M + \theta(I)\Gamma d(M) = \theta(I)\Gamma d(M)$. Now, $\theta(I)\Gamma d(M) = 0$ implies $\theta(I)\Gamma\theta(M)\Gamma d(M) = 0$, the primeness of M forces that $\theta(I) = 0$ or $d(M) = 0$. Hence, $I = 0$ or $d = 0$, a contradiction.

Theorem 2.2. *Let M be a prime $\Gamma-$ring, θ an automorphism of M and I a nonzero ideal of M. If M admits a $\theta-$derivation d such that $d([x,y]_\alpha) + [x,y]_\alpha = 0$ for all $x,y \in I$ and $\alpha \in \Gamma$, then M is commutative.*

Proof. If d is $\theta-$derivation such that $d([x,y]_\alpha) + [x,y]_\alpha = 0$ for all $x,y \in I$, then $-d$ is also a $\theta-$derivation and satisfies $(-d)([x,y]_\alpha) = [x,y]_\alpha$ for all $x,y \in I$. It follows from Theorem 2.1 that M is commutative.

Theorem 2.3. *Let M be a prime $\Gamma-$ring, θ an automorphism of M and I a nonzero ideal of M. If M admits a $\theta-$derivation d such that $d((x \circ y)_\alpha) = (x \circ y)_\alpha$ for all $x,y \in I$ and $\alpha \in \Gamma$, then M is commutative.*

Proof. If $d = 0$, then $(x \circ y)_\alpha = 0$ for all $x,y \in I$. Replacing y by $y\beta z$ in above relation and using the identity $(x \circ (y\beta z))_\alpha = (x \circ y)_\alpha \beta z - y\beta[x,z]_\alpha$, we conclude that $y\alpha[x,z]_\beta = 0$ for all $x,y,z \in I; \alpha,\beta \in \Gamma$. This implies that $I\Gamma[I,I]_\beta = 0$ and hence $[I,I]_\beta = 0$. Thus, we get the required result.

Suppose that $d \neq 0$ and we have

$$d((x \circ y)_\alpha) = (x \circ y)_\alpha \text{ for all } x,y \in I; \alpha \in \Gamma. \tag{2.5}$$

Replacing y by $y\beta x$ in (2.5) and using (2.5) we arrive at

$$\theta((x \circ y)_\alpha)\beta d(x) = 0 \text{ for all } x,y \in I; \alpha,\beta \in \Gamma. \tag{2.6}$$

Again replacing y by $y\gamma z$ in (2.6) and using (2.6), we obtain $\theta([x,y]_\alpha)\gamma\theta(z)\beta d(x) = 0$ for all $x,y,z \in I; \alpha,\beta,\gamma \in \Gamma$. This implies that

$$[x,y]_\alpha \Gamma I \Gamma \theta^{-1} d(x) = 0 \text{ for all } x,y \in I. \tag{2.7}$$

This expression is similar to the equation (2.4) and hence repeat the same process to get the required result.

Similarly, we can prove the following:

Theorem 2.4. *Let M be a prime $\Gamma-$ring, θ an automorphism of M and I a nonzero ideal of M. If M admits a $\theta-$derivation d such that $d((x\circ y)_\alpha)+(x\circ y)_\alpha=0$ for all $x,y\in I$ and $\alpha\in\Gamma$, then M is commutative.*

Theorem 2.5. *Let M be a 2-torsion free prime $\Gamma-$ring, θ an automorphism of M and I a nonzero ideal of M. If M admits a nonzero $\theta-$derivation d commuting with θ such that $(d(x)\circ d(y))_\alpha=0$ for all $x,y\in I$ and $\alpha\in\Gamma$, then M is commutative.*

Proof. We are given that

$$(d(x)\circ d(y))_\alpha=0 \text{ for all } x,y\in I;\alpha\in\Gamma. \tag{2.8}$$

Replacing y by $y\beta z$ in (2.8) and using (2.8) we find that

$$[d(x),\theta(y)]_\alpha\beta d(z)-d(y)\beta[d(x),z]_\alpha=0 \text{ for all } x,y,z\in I;\alpha,\beta\in\Gamma. \tag{2.9}$$

Again replacing y by $y\gamma d(x)$ in (2.9) and using (2.9), we obtain

$$[d(x),\theta(y)]_\alpha\beta\theta(z)\gamma d^2(x)=0 \text{ for all } x,y\in I;z\in d(I);\alpha,\beta,\gamma\in\Gamma. \tag{2.10}$$

The above equation implies that $[\theta^{-1}d(x),y]_\alpha\Gamma I\Gamma d^2(x)=0$ for all $x,y\in I$ and $\alpha\in\Gamma$. For each fixed $x\in I$, either $[\theta^{-1}d(x),y]_\alpha=0$ for all $y\in I$ or $d^2(x)=0$. Using similar arguments as in the proof of given in the proof of Theorem 2.1, we have $[\theta^{-1}d(I),I]_\alpha=0$ or $d^2(I)=0$. If $[\theta^{-1}d(I),I]_\alpha=0$, then $\theta^{-1}d(I)\subseteq Z(I)$. By [10, Lemma 1.2.2], [12, Lemma 4], M is commutative. If $d^2(I)=0$, we have $0=d^2(u\alpha v)=d^2(u)\alpha v+2\theta d(u)\alpha d(v)+\theta^2(u)d^2(v)=2\theta d(u)\alpha d(v)$ for all $u,v\in I$. Since M is 2-torsion free, we get $\theta d(u)\Gamma d(I)=0$. In view of [12, Lemma 3], either $\theta d(I)=0$ or $d=0$. The former case implies that $d(I)=0$ and so $d=0$. This is a contradiction and the proof is complete.

Using the same techniques with necessary variations we get the following:

Theorem 2.6. *Let M be a 2-torsion free prime $\Gamma-$ring, θ an automorphism of M and I a nonzero ideal of M. If M admits a nonzero $\theta-$derivation d commuting with θ such that $[d(x),d(y)]_\alpha=0$ for all $x,y\in I$ and $\alpha\in\Gamma$, then M is commutative.*

Theorem 2.7. *Let M be a 2-torsion free prime $\Gamma-$ring, θ an automorphism of M and I a nonzero ideal of M. If M admits a $\theta-$derivation d acting as a homomorphism on I, then $d=0$ or M is commutative.*

Proof. If M is commutative, then we are done. Assume that d acts as a homomorphism on I. By our hypothesis, we have $d(x\alpha y)=d(x)\alpha d(y)$, which can be rewritten as

$$d(x)\alpha y+\theta(x)\alpha d(y)=d(x)\alpha d(y) \text{ for all } x,y\in I;\alpha\in\Gamma. \tag{2.11}$$

Replacing y by $y\beta z$ in (2.11) and using (2.11), we get $(\theta(x)-d(x))\alpha\theta(y)\beta d(z)=0$, which implies that $(x-\theta^{-1}d(x))\Gamma I\Gamma\theta^{-1}d(z)=0$ for all $x,z\in I$. By the primeness of I, either $\theta(x)=d(x)$ for all $x\in I$ or $d(I)=0$. In the former case, $\theta(x)\alpha\theta(y)=d(x)\alpha d(y)=d(x\alpha y)=d(x)\alpha y+\theta(x)\alpha d(y)=d(x)\alpha y+\theta(x)\alpha\theta(y)$ for all $x,y\in I$. Thus, $d(x)\alpha y=0$ and hence $d(I)\Gamma I=0$. In light of [12, Lemma 3], $d(I)=0$. In both cases, we conclude $d(I)=0$ and so $d=0$.

Theorem 2.8. *Let M be a 2-torsion free prime $\Gamma-$ring, θ an automorphism of M and I a nonzero ideal of M. If M admits a $\theta-$derivation d acting as an anti-homomorphism on I, then $d = 0$ or M is commutative.*

Proof. Assume that d acts as an anti-homomorphism on I, then

$$d(x)\alpha y + \theta(x)\alpha d(y) = d(x\alpha y) = d(y)\alpha d(x) \text{ for all } x, y \in I; \alpha \in \Gamma. \tag{2.12}$$

Replacing x by $x\beta y$ in (2.12) and using (2.12), we get

$$d(y)\alpha\theta(x)\beta d(y) = \theta(x)\beta\theta(y)\alpha d(y) \text{ for all } x, y \in I; \alpha, \beta \in \Gamma. \tag{2.13}$$

Replacing x by $z\gamma x$ in (2.13) and using (2.13), we have

$$d(y)\alpha\theta(z)\gamma\theta(x)\beta d(y) = \theta(z)\gamma\theta(x)\beta\theta(y)\alpha d(y) = \theta(z)\gamma d(y)\alpha\theta(x)\beta d(y) \tag{2.14}$$

for all $x, y, z \in I$ and $\alpha, \beta, \gamma \in \Gamma$. This implies that $[d(y), \theta(z)]_\gamma\alpha\theta(x)\beta d(y) = 0$ and hence $[\theta^{-1}d(y), z]_\gamma\Gamma I\Gamma\theta^{-1}d(y) = 0$ for all $y, z \in I; \gamma \in \Gamma$. For each fixed $y \in I$, either $[\theta^{-1}d(y), z]_\gamma = 0$ for all $z \in I$ or $\theta^{-1}d(y) = 0$. Repeating similar arguments as given in in the proof of Theorem 2.1, we obtain $[\theta^{-1}d(I), I]_\gamma = 0$ or $\theta^{-1}d(I) = 0$. If $[\theta^{-1}d(I), I]_\gamma = 0$, then the same arguments as in the proof of Theorem 2.5 forces M to be commutative. In the latter case, $\theta^{-1}d(I) = 0$ implies that $d(I) = 0$ and we deduce that $d = 0$.

The following example shows that the primeness in the above theorems can not be omitted.

Example 2.9. Let Q be rational number field and $M = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in Q\right\}$. Then it is easy to check that $I = \left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in Q\right\}$ is a nonzero ideal of M. The fact that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\Gamma M\Gamma\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ proves that M is not prime. Define maps $d\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ and $\theta\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & 0 \end{pmatrix}$. Then d is a nonzero $\theta-$derivation on M. It can be easily checked that (i) $d([x, y]_\alpha) = \pm([x, y]_\alpha)$ (ii) $d((x \circ y)_\alpha) = \pm((x \circ y)_\alpha)$ (iii) $(d(x) \circ d(y))_\alpha = 0$ (iv) $[d(x), d(y)]_\alpha = 0$ (v) $d(x\alpha y) = d(x)\alpha d(y)$ (vi) $d(x\alpha y) = d(y)\alpha d(x)$ for all $x, y \in I; \alpha \in \Gamma$. However, M is not commutative.

Acknowledgments. The authors are greatly indebted to the referee for her/his useful suggestions. The first author was supported by the Anhui Provincial Natural Science Foundation (1408085QA08) and the key University Science Research Project of Anhui Province (KJ2014A183) and also Training Program of Chuzhou University (2014PY06) of China.

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