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Hammerstein Equations with Lipschitz and Strongly Monotone Mappings in Classical Banach spaces

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Abstract

Let E be a Banach space either l_p or L_p or $W^{m,p}$, $1 < p < \infty$ with dual E^*, and let $F : E \to E^*$, $K : E^* \to E$ be Lipschitz and strongly monotone mappings with $D(K) = R(F) = E^*$. Assume that the Hammerstein equation $u + KFu = 0$ has a unique solution $\bar{u}$. For given $u_1 \in E$ and $v_1 \in E^*$, let $\{u_n\}$ and $\{v_n\}$ be sequences generated iteratively by: $u_{n+1} = J^{-1}(Ju_n - \lambda(Fu_n - v_n))$, $n \geq 1$ and $v_{n+1} = J(J^{-1}v_n - \lambda(Kv_n + u_n))$, $n \geq 1$, where J is the duality mapping from E into E^* and λ is a positive real number in $(0,1)$ satisfying suitable conditions. Then it is proved that the sequence $\{u_n\}$ converges strongly to $\bar{u}$, the sequence $\{v_n\}$ converges strongly to $\bar{v}$, with $\bar{v} = F\bar{u}$. Furthermore, our technique of proof is of independent interest.

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1 Introduction

Let H be a real Hilbert space. A mapping $A : D(A) \subset H \to H$ is called *monotone* if for all $x, y \in D(A)$, the following inequality holds:

$$\langle Ax - Ay, x - y\rangle \quad \geq \quad 0. \tag{1.1}$$

The notion of monotonicity has been extended to real normed spaces E in two ways.

The *first* involves mappings from E to E^*. A mapping $A : D(A) \subset E \to E^*$ is called *monotone* if for all $x, y \in D(A)$,

$$\langle x - y, Ax - Ay\rangle \quad \geq \quad 0, \tag{1.2}$$

where $\langle , \rangle$ denotes the duality pairing between elements of E and elements of E^*. It is said to be *strongly monotone* if there exists a positive constant k such that for all $x, y \in D(A)$,

$$\langle x - y, Ax - Ay\rangle \quad \geq \quad k\|x - y\|^2. \tag{1.3}$$

Note that if E is a real Hilbert space H, then $H = H^*$ and (1.2) coincides with (1.1).

The *second* extension of the notion of monotonicity to real normed spaces involves mappings E *into itself*. A mapping $A : D(A) \subset E \to E$ is called *accretive* if for every $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that the following inequality holds:

$$\langle Ax - Ay, j(x - y)\rangle \geq 0, \tag{1.4}$$

where $J : E \to 2^{E^*}$ is the normalized duality mapping of E defined by:

$$J(x) := \left\{ f \in E^* : \langle x, f\rangle = \|x\|^2 \text{ and } \|f\| = \|x\| \right\}.$$

Here, if E is a real Hilbert space, J becomes the identity map and condition (1.4) reduces to (1.1). Hence, in real Hilbert spaces, *accretive* operators become *monotone*. Consequently, accretive operators can be regarded as extension of Hilbert space monotonicity condition to real normed spaces.

A mapping $A : D(A) \subset E \to E$ is called *strongly accretive* if there exists a constant $k > 0$ such that for every $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq k\|x - y\|^2.$$

An integral *equation of Hammerstein type* (see, e.g., Hammerstein [29]) is an equation of the form:

$$u(x) + \int_{\Omega} \kappa(x, y) f(y, u(y))dy = h(x), \tag{1.5}$$

where dy is a σ-finite measure on the measure space Ω; the real kernel κ is defined on $\Omega \times \Omega$, f is a real-valued function defined on $\Omega \times \mathbb{R}$ and is, in general, nonlinear and h is a given function on Ω. If we now define an operator $K : \mathcal{F}(\Omega, \mathbb{R}) \to \mathcal{F}(\Omega, \mathbb{R})$ by

$$Kv(x) := \int_{\Omega} \kappa(x, y)v(y)dy, \ a.e. \ x \in \Omega$$

and the so-called *superposition* or *Nemytskii* operator $F : \mathcal{F}(\Omega,\mathbb{R}) \to \mathcal{F}(\Omega,\mathbb{R})$ by

$$Fu(y) := f(y,u(y)) \; a.e. \; y \in \Omega,$$

where $\mathcal{F}(\Omega,\mathbb{R})$ denotes a space of functions mapping Ω into $\mathbb{R}$, then, the integral equation (1.5) can be put in operator theoretic form as follows:

$$u + KFu = 0, \tag{1.6}$$

where, without loss of generality, we have taken $h \equiv 0$.

Interest in equation (1.6) stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear parts possess Green's functions can, as a rule, be transformed into the form (1.6). Among these, we mention the problem of the forced oscillations of finite amplitude of a pendulum (see, e.g., Pascali and Sburlan [34], Chapter IV).

Equations of Hammerstein type play a crucial role in the theory of optimal control systems and in automation and network theory (see, e.g., Dolezale [28]).

Several existence and uniqueness theorems have been proved for equations of Hammerstein type (see, e.g., Brézis and Browder ([9],[7],[8]), Browder [10], Browder and De Figueiredo [11], Bowder and Gupta [12], Chepanovich [13], De Figueiredo [25]).

In general, equations of Hammerstein type (1.6) are nonlinear and there is no known method to find closed form solutions for them. Consequently, methods of approximating solutions of such equations are of interest.

In the special case in which the operator F is angle bounded (defined below) and weakly compact, Brézis and Browder [9, 8] proved the strong convergence of a suitably defined *Galerkin approximation* to a solution of (1.6). Before we state this theorem, we need the following definitions.

Let H be a real Hilbert space. A *nonlinear* operator $A : H \to H$ is said to be *angle-bounded* with angle $\beta > 0$ if

$$\langle Ax - Az, z - y\rangle \le \beta\langle Ax - Ay, x - y\rangle \tag{1.7}$$

for any triple of elements $x, y, z \in H$. For $y = z$, inequality (1.7) implies the monotonicity of A.

A *monotone linear* operator $A : H \to H$ is said to be *angle-bounded* with angle $\alpha > 0$ if

$$|\langle Ax, y\rangle - \langle Ay, x\rangle| \le 2\alpha\langle Ax, x\rangle^{\frac{1}{2}}\langle Ay, y\rangle^{\frac{1}{2}} \tag{1.8}$$

for all $x, y \in H$. It is known that the two definitions of angle boundedness are equivalent (see Pascali and Sburlan, [34], Ch. IV, p.189).

We now state the theorem of Brézis and Browder referring to above.

Theorem BB (Brézis and Browder, [8]). *Let H be a separable real Hilbert space and C be a closed subspace of H. Let $K : H \to C$ be a bounded continuous monotone operator and $F : C \to H$ be an angle-bounded and weakly compact mapping. For a given $f \in C$, consider the Hammerstein equation*

$$(I + KF)u = f \tag{1.9}$$

and its n-th Galerkin approximation given by

$$(I + K_n F_n)u_n = P^* f, \tag{1.10}$$

where $K_n = P_n^* K P_n : H \to C_n$ and $F_n = P_n F P_n^* : C_n \to H$, where the symbols have their usual meanings (see, [34]). Then, for each $n \in \mathbb{N}$, the Galerkin approximation (1.10) *admits a unique solution u_n in C_n and $\{u_n\}$ converges strongly in H to the unique solution $u \in C$ of the equation* (1.9).

It is obvious that if an *iterative algorithm* can be developed for the approximation of solutions of equations of Hammerstein type (1.9), this will certainly be preferred.

The first satisfactory results on *iterative methods* for approximating solutions of Hammerstein equations, as far as we know, were obtained by Chidume and Zegeye [24, 23, 22]. Under the setting of a real Hilbert space H, for $F, K : H \to H$, they defined an auxiliary map on the Cartesian product $E := H \times H$, $T : E \to E$ by

$$T[u, v] = [Fu - v, Kv + u].$$

We note that

$$T[u, v] = 0 \Leftrightarrow u \quad \text{solves} \quad (1.6) \ \text{ and } \ v = Fu.$$

With this, they were able to obtain *strong convergence* of an *iterative scheme* defined in the Cartesian product space E to a solution of Hammerstein equation (1.6). Extensions to a real Banach space setting were also obtained.

Let X be a real Banach space and $K, F : X \to X$ be *accretive type* mappings. Let $E := X \times X$. The same authors (see, [24, 23]) defined $T : E \to E$ by

$$T[u, v] = [Fu - v, Kv + u]$$

and obtained strong convergence theorems for solutions of Hammerstein equations under various continuity conditions in the Cartesian product space E.

The method of proof used by Chidume and Zegeye provided the clue to the establishment of the following coupled explicit algorithm for computing a solution of the equation $u + KFu = 0$ *in the original space* X. With initial vectors $u_0, v_0 \in X$, sequences $\{u_n\}$ and $\{v_n\}$ in X are defined iteratively as follows:

$$u_{n+1} = u_n - \alpha_n (Fu_n - v_n), n \geq 0, \tag{$*$}$$

$$v_{n+1} = v_n - \alpha_n(Kv_n + u_n), n \geq 0. \qquad (**)$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying appropriate conditions. The recursion formulas (*) and (**) have been used successfully to approximate solutions of Hammerstein equations involving nonlinear accretive-type operators. Following this, Chidume and Djitte studied this explicit coupled iterative algorithms and proved several strong convergence theorems (see, Chidume and Djitte [16], [19]). For recent results using these recursion formulas or their modifications, the reader may consult any of the following references: Amata et. al. [6], Chidume and Djitte [15], [17], [18], Djitte and Sene [26], [27], Chidume and Ofeodu [20], Chidume and Shehu [21], Irina [30], Javadi [31], Ofoedu and Onyi [33], Shehu [36], Sweilam et. al. [38], and also Chapter 13 of [14].

For Hammerstein equations involving *monotone mappings* from E to E^*, very little has been achieved. Part of the difficulty is that inequalities involving vectors in E do not generally hold in E^*. For instance, if $E = L_p \;\; (p > 2)$, then $E^* = L_q$ with $(1 < q < 2)$ and generally an inequality that holds in $L_p \;\; (p > 2)$ is reversed in $L_p \;\; (1 < p < 2)$, (see, e.g., Chidume [14], Chapter 5.) Interestingly enough, *almost all the existence theorems* proved for Hammerstein equations involve *monotone mappings* (see, e.g., Brézis and Browder [9, 7, 8], Browder [10], Browder et al. [11], and Browder and Gupta [12]). We note that it has been remarked that in dealing with the Nemytskii operator, which is intimately connected with the Hammertsein integral equation, its properties are distinguished, in applications, according to two important cases: $L_p(\Omega)$ spaces, $\;1 < p < \infty$, and $L_1(\Omega)$, (see Pascali and Sburlan [34], Chapter IV, pp. 165, 172). Thus, developing iterative methods for approximating solutions of nonlinear Hammerstein integral equations in these cases is of paramount importance.

Recently, in [37] Sow et. al. proposed an algorithm of Mann type for approximating solutions of Hammerstein equations with bounded and strongly monotone mappings.

It is our purpose in this paper to construct a coupled iterative process of Krasnoselskii type and prove its strong convergence to the unique solution of the Hammerstein equation $u + KFu = 0$ with *Lipschitz and strongly monotone mappings* in certain Banach spaces including all Hilbert spaces and all l_p, L_p or $W^{m,p}$-spaces, $1 < p < \infty$. Furthermore, our thechnique of proof is of independent interest.

2 Preliminaries

Let E be a normed linear space. The modulus of convexity of E, $\delta_E : (0,2] \to [0,1]$ is defined by:

$$\delta_E(\epsilon) := \inf\Big\{1 - \frac{1}{2}\|x+y\| \;:\; \|x\| = \|y\| = 1, \; \|x-y\| \geq \epsilon\Big\}.$$

For $p > 1$, E is said to be p-uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in (0,2]$.

Let E be a real normed space and let $S := \{x \in E : \|x\| = 1\}$. E is said to be smooth if the limit

$$\lim_{t\to 0^+} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in S$. E is said to be uniformly smooth if it is smooth and the limit is attained uniformly for each $x, y \in S$.

The modulus of smoothness of E, ρ_E, is defined by :

$$\rho_E(\tau) := \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\right\}; \quad \tau > 0.$$

E is said to be q-uniformly smooth, if there exists a constant $c > 0$ and a real number $q > 1$ such that $\rho_E(\tau) \leq c\tau^q$.

Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$ where,

$$L_p \ (or\ l_p)\ or\ W_p^m \text{ is } \begin{cases} 2-\text{uniformly smooth and } p-\text{uniformly convex} & \text{if } 2 \leq p < \infty; \\ 2-\text{uniformly convex and } p-\text{uniformly smooth} & \text{if } 1 < p < 2. \end{cases}$$

Let J_q denote the *generalized duality mapping* from E to 2^{E^*} defined by

$$J_q(x) := \left\{f \in E^* : \langle x, f\rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1}\right\}$$

where $\langle ., . \rangle$ denotes the generalized duality pairing. J_2 is called the *normalized duality mapping* and is denoted by J.

It is well known that E is smooth if and only if J is single valued. Moreover, if E is a reflexive smooth and strictly convex Banach space, then J^{-1} is single valued, one-to-one, surjective and it is the duality mapping from E^* into E.

Remark 2.1. Note also that a duality mapping exists in each Banach space. We recall from [2] some of the examples of this mapping in $l_p, L_p, W^{m,p}$-spaces, $1 < p < \infty$.

(i) $l_p :\ Jx = \|x\|_{l_p}^{2-p} y \in l_q,\ x = (x_1, x_2, \cdots, x_n, \cdots),\ y = (x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, \cdots, x_n|x_n|^{p-2}, \cdots)$,

(ii) $L_p :\ Ju = \|u\|_{L_p}^{2-p} |u|^{p-2} u \in L_q$,

(iii) $W^{m,p} :\ Ju = \|u\|_{W^{m,p}}^{2-p} \sum_{|\alpha \leq m|} (-1)^{|\alpha|} D^\alpha\left(|D^\alpha u|^{p-2} D^\alpha u\right) \in W^{-m,q}$,

where $1 < q < \infty$ is such that $1/p + 1/q = 1$.

Theorem 2.2 (H. K. Xu [39])**.** *Let $p > 1$ be a given real number. Then the following are equivalent in a Banach space:*

(i) E is p-uniformly convex.

(ii) There is a constant $c_1 > 0$ such that for every $x, y \in E$ and $j_x \in J_p(x)$, The following inequality holds:

$$||x+y||^p \geq ||x||^p + p\langle y, j_x\rangle + c_1||y||^p.$$

(iii) There is a constant $c_2 > 0$ such that for every $x, y \in E$ and $j_x \in J_p(x), j_y \in J_p(y)$, the following inequality holds:

$$\langle x-y, j_x - j_y\rangle \geq c_2||x-y||^p.$$

Lemma 2.3. *Let E be a 2-uniformly convex and smooth real Banach space. Then J^{-1} is Lipschitz from E^* into E, i.e., there exists a constant $L > 0$ such that for all $u, v \in E^*$,*

$$||J^{-1}u - J^{-1}v|| \leq L||u-v||. \tag{2.1}$$

Proof. The proof follows from inequality (iii) of Theorem 2.2 with $p = 2$. □

Lemma 2.4 ([5])**.** *Let $p \geq 2$ and $q > 1$, and let E be a p-uniformly convex and q-uniformly smooth real Banach space. Then, the duality mapping $J : E \to E^*$ is Lipschitz on bounded subsets of E; that is, for all $R > 0$, there exists a positive constant m_2 such that*

$$||Jx - Jy|| \leq m_2||x-y||,$$

for all $x, y \in E$ with $||x|| \leq R$, $||y|| \leq R$.

Let E be a smooth real Banach space with dual E^*. The Lyapunov function $\phi : E \times E \to \mathbb{R}$, defined by

$$\phi(x,y) = ||x||^2 - 2\langle x, Jy\rangle + ||y||^2,\ x, y \in E, \tag{2.2}$$

introduced by Alber, has been studied by Alber [3], Alber and Guerre-Delabriere [4], Kamimura and Takahashi[32], Reich[35] and a host of other authors. This functional ϕ will play a central role in what follows. If $E = H$, a real Hilbert space, then equation (2.2) reduces to $\phi(x,y) = ||x-y||^2$ for $x, y \in H$. It is obvious from the definition of the function ϕ that

$$(||x|| - ||y||)^2 \leq \phi(x,y) \leq (||x|| + ||y||)^2\ \forall\, x, y \in E. \tag{2.3}$$

Let $V : E \times E^* \to \mathbb{R}$ be the functional defined by:

$$V(x, x^*) = ||x||^2 - 2\langle x, x^*\rangle + ||x^*||^2,\ \forall\, x \in E, x^* \in E^*. \tag{2.4}$$

Then, it is easy to see that

$$V(x, x^*) = \phi(x, J^{-1}x^*)\ \forall\, x \in E,\ x^* \in E^*. \tag{2.5}$$

Lemma 2.5 (Alber, [3])**.** *Let X be a reflexive striclty convex and smooth real Banach space with X^* as its dual. Then,*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*) \tag{2.6}$$

for all $x \in X$ and $x^*, y^* \in X^*$.

Similarly, if E is a reflexive smooth and strictly convex real Banach space, we introduce the functional $\phi_* : E^* \times E^* \to \mathbb{R}$, defined by:

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^*\rangle + \|y^*\|^2,\ x^*, y^* \in E^*, \tag{2.7}$$

and the functional $V_* : E^* \times E \to \mathbb{R}$ defined from $E^* \times E$ to $\mathbb{R}$ by:

$$V_*(x^*, x) = \|x^*\|^2 - 2\langle x, x^*\rangle + \|x\|^2,\ x \in E, x^* \in E^*. \tag{2.8}$$

It is easy to see that

$$V_*(x^*, x) = \phi_*(x^*, Jx)\ \forall\, x \in E,\ x^* \in E^*. \tag{2.9}$$

In what follows, the product space $E \times E^*$ is equiped with the following norm:

$$\|w\| = \Big(\|x\|^2 + \|x^*\|^2\Big)^{\frac{1}{2}}\ \ \forall\, w = (x, x^*) \in E \times E^*.$$

Finally, we introduce the functional $\psi : (E \times E^*) \times (E \times E^*) \to \mathbb{R}$ defined by:

$$\psi(w_1, w_2) := \phi(x, y) + \phi_*(x^*, y^*)\ \ \forall\, w_1 = (x, x^*) \in E \times E^*,\ w_2 = (y, y^*) \in E \times E^*.$$

We now prove the following results.

3 Convergence in l_p, L_p and $W^{m,p}$-spaces, $1 < p \le 2$

Theorem 3.1. *For $q > 1$, let E be a 2-uniformly convex and q-uniformly smooth real Banach space. Let $F : E \to E^*$, $K : E^* \to E$ be Lipschitz and strongly monotone mappings with $D(K) = R(F) = E^*$. For given $u_1 \in E$ and $v_1 \in E^*$, let $\{u_n\}$ and $\{v_n\}$ be generated iteratively by:*

$$\begin{aligned} u_{n+1} &= J^{-1}(Ju_n - \lambda(Fu_n - v_n)),\ n \ge 1,\\ v_{n+1} &= J(J^{-1}v_n - \lambda(Kv_n + u_n)),\ n \ge 1, \end{aligned} \tag{3.1}$$

where J is normalized duality mapping from E into E^* and $\lambda \in (0,1)$. Suppose that the equation $u + KFu = 0$ has a unique solution $\bar{u}$. Then, there exists some $\delta > 0$ such that if $\lambda \in (0, \delta)$, the sequence $\{u_n\}$ converges strongly to $\bar{u}$, the sequence $\{v_n\}$ converges strongly to $\bar{v}$, with $\bar{v} = F\bar{u}$.

Proof. Let $X = E \times E^*$ with the norm $\|z\| = (\|u\|^2 + \|v\|^2)^{\frac{1}{2}}$, where $z = (u, v) \in E \times E^*$. Define the sequence $\{w_n\}$ in X by: $w_n = (u_n, v_n)$. Let $\bar{u} \in E$ be the unique solution of $u + KFu = 0$ and set $\bar{v} := F\bar{u}$ and $\bar{w} := (\bar{u}, \bar{v})$. We observe that $\bar{u} = -K\bar{v}$.

In what follows, k_1 and k_2 denote the strongly monotonicity constants of F and K, L_1 the Lipschitz constant of J^{-1} and L_F and L_K the Lipschitz constants of F and K.

The proof is in two steps.

Step 1: We first prove that $\{w_n\}$ is bounded. There exists $r > 0$ such that: $\psi(\bar{w}, w_1) \le r$. We show that $\psi(\bar{w}, w_n) \le r$ for all $n \ge 1$. The proof is by induction. We have $\psi(\bar{w}, w_1) \le r$.

Assume that $\psi(\bar{w}, w_n) \le r$ for some $n \ge 1$. We show that $\psi(\bar{w}, w_{n+1}) \le r$. Using the fact that K is bounded (because Lipschitz) and Lemma 2.4, there exists a positive constant m_2 such that

$$||J(J^{-1}v - \lambda(Kv+u)) - J(J^{-1}v)|| \le \lambda m_2 ||Kv+u|| \;\; \forall \lambda \in (0,1),\; (u,v) \in E \times E^* : \psi(\bar{w},(u,v)) \le r. \tag{3.2}$$

Set

$$k := \min\{k_1, k_2\},\; L = \max\{1, L_F, L_K\},\text{ and } \delta = \min\Big\{1, \frac{k}{4L^2(L_1+m_2)}\Big\}. \tag{3.3}$$

Using the definition of u_{n+1}, we compute as follows:

$$\begin{aligned}\phi(\bar{u}, u_{n+1}) &= \phi(\bar{u}, J^{-1}(Ju_n - \lambda(Fu_n - v_n))) \\ &= V(\bar{u}, Ju_n - \lambda(Fu_n - v_n)).\end{aligned}$$

Using Lemma 2.5, with $y^* = \lambda(Fu_n - v_n)$, we have:

$$\begin{aligned}\phi(\bar{u}, u_{n+1}) &= V(\bar{u}, Ju_n - \lambda(Fu_n - v_n)) \\ &\le V(\bar{u}, Ju_n) - 2\lambda\langle J^{-1}(Ju_n - \lambda(Fu_n - v_n)) - \bar{u}, Fu_n - v_n\rangle \\ &= \phi(\bar{u}, u_n) - 2\lambda\langle u_n - \bar{u}, Fu_n - v_n\rangle - 2\lambda\langle J^{-1}(Ju_n - \lambda(Fu_n - v_n)) - u_n, Fu_n - v_n\rangle \\ &= \phi(\bar{u}, u_n) - 2\lambda\langle u_n - \bar{u}, Fu_n - F\bar{u}\rangle \\ &\quad -2\lambda\langle J^{-1}(Ju_n - \lambda(Fu_n - v_n)) - J^{-1}(Ju_n), Fu_n - v_n\rangle - 2\lambda\langle u_n - \bar{u}, \bar{v} - v_n\rangle.\end{aligned}$$

Using the strong monotonocity of F, Schwartz inequality and the Lipschitz property of J^{-1}, we obtain

$$\begin{aligned}\phi(\bar{u}, u_{n+1}) &\le \phi(\bar{u}, u_n) - 2\lambda k_1||u_n - \bar{u}||^2 + \\ &\quad 2\lambda||J^{-1}(Ju_n - \lambda(Fu_n - v_n)) - J^{-1}(Ju_n)||||Fu_n - v_n|| + 2\lambda\langle u_n - \bar{u}, v_n - \bar{v}\rangle \\ &\le \phi(\bar{u}, u_n) - 2\lambda k_1||u_n - \bar{u}||^2 + 2\lambda^2 L_1||Fu_n - v_n||^2 + 2\lambda\langle u_n - \bar{u}, v_n - \bar{v}\rangle.\end{aligned}$$

Using the fact that F is Lipschitz and $\bar{v} = F\bar{u}$, we have:

$$\begin{aligned}\phi(\bar{u}, u_{n+1}) &\le \phi(\bar{u}, u_n) - 2\lambda k_1||u_n - \bar{u}||^2 + 2\lambda^2 L_1(||Fu_n - F\bar{u} + \bar{v} - v_n||)^2 + 2\lambda\langle u_n - \bar{u}, v_n - \bar{v}\rangle \\ &\le \phi(\bar{u}, u_n) - 2\lambda k_1||u_n - \bar{u}||^2 + 2\lambda^2 L_1(||Fu_n - F\bar{u}|| + ||v_n - \bar{v}||)^2 + 2\lambda\langle u_n - \bar{u}, v_n - \bar{v}\rangle \\ &\le \phi(\bar{u}, u_n) - 2\lambda k_1||u_n - \bar{u}||^2 + 2\lambda^2 L_1(2||Fu_n - F\bar{u}||^2 + 2||v_n - \bar{v}||^2) + 2\lambda\langle u_n - \bar{u}, v_n - \bar{v}\rangle \\ &\le \phi(\bar{u}, u_n) - 2\lambda k_1||u_n - \bar{u}||^2 + 4\lambda^2 L_1(L_F^2||u_n - \bar{u}||^2 + ||v_n - \bar{v}||^2) + 2\lambda\langle u_n - \bar{u}, v_n - \bar{v}\rangle.\end{aligned}$$

Finally, we obtain:

$$\phi(\bar{u}, u_{n+1}) \le \phi(\bar{u}, u_n) - 2\lambda k_1||u_n - \bar{u}||^2 + 4\lambda^2 L_1 L^2||w_n - \bar{w}||^2 + 2\lambda\langle u_n - \bar{u}, v_n - \bar{v}\rangle \tag{3.4}$$

Similarly, using the definition of v_{n+1}, we compute as follows:

$$\begin{aligned}\phi_*(\bar{v}, v_{n+1}) &= \phi_*(\bar{v}, J(J^{-1}v_n - \lambda(Kv_n + u_n))) \\ &= V_*(\bar{v}, J^{-1}v_n - \lambda(Kv_n + u_n)).\end{aligned}$$

Using Lemma 2.5, with $y^* = \lambda(Kv_n + u_n)$, we have:

$$\begin{aligned}\phi_*(\bar{v}, v_{n+1}) &= V_*(\bar{v}, J^{-1}v_n - \lambda(Kv_n + v_n))\\ &\leq V_*(\bar{v}, J^{-1}v_n) - 2\lambda\langle Kv_n + u_n, J(J^{-1}v_n - \lambda(Kv_n + u_n)) - \bar{v}\rangle\\ &= \phi_*(\bar{v}, v_n) - 2\lambda\langle Kv_n + u_n, v_n - \bar{v}\rangle - 2\lambda\langle Kv_n + u_n, J(J^{-1}v_n - \lambda(Kv_n + u_n)) - v_n\rangle\\ &= \phi_*(\bar{v}, v_n) - 2\lambda\langle Kv_n - K\bar{v}, v_n - \bar{v}\rangle\\ &\quad -2\lambda\langle Kv_n + u_n, J(J^{-1}v_n - \lambda(Kv_n + u_n)) - J(J^{-1}v_n)\rangle - 2\lambda\langle u_n - \bar{u}, v_n - \bar{v}\rangle.\end{aligned}$$

Using the strong monotonocity of K, the Lipschitz property of K, the fact that $\bar{u} = -K\bar{v}$, Schwartz inequality and inequality (3.2), it follows that

$$\begin{aligned}\phi_*(\bar{v}, v_{n+1}) &\leq \phi_*(\bar{v}, v_n) - 2\lambda k_2\|v_n - \bar{v}\|^2 +\\ &\quad 2\lambda\|J(J^{-1}v_n - \lambda(Kv_n + u_n)) - J(J^{-1}v_n)\|\|Kv_n + u_n\| - 2\lambda\langle u_n - \bar{u}, v_n - \bar{v}\rangle\\ &\leq \phi_*(\bar{v}, v_n) - 2\lambda k_2\|v_n - \bar{v}\|^2 + 2\lambda^2 m_2\|Kv_n - K\bar{v} - \bar{u} + u_n\|^2\\ &\quad -2\lambda\langle v_n - \bar{v}, u_n - \bar{u}\rangle.\end{aligned}$$

Using the fact that K is Lipschitz , $\bar{u} = -K\bar{v}$ we have:

$$\begin{aligned}\phi_*(\bar{v}, v_{n+1}) &\leq \phi_*(\bar{v}, v_n) - 2\lambda k_2\|v_n - \bar{v}\|^2 + 2\lambda^2 m_2(2\|Kv_n - K\bar{v}\|^2 + 2\| - \bar{u} + u_n\|^2)\\ &\quad -2\lambda\langle u_n - \bar{u}, v_n - \bar{v}\rangle.\\ &\leq \phi_*(\bar{v}, v_n) - 2\lambda k_2\|v_n - \bar{v}\|^2 + 4\lambda^2 m_2 L^2(\|v_n - \bar{v}\|^2 + \|u_n - \bar{u}\|^2)\\ &\quad -2\lambda\langle u_n - \bar{u}, v_n - \bar{v}\rangle.\end{aligned}$$

Therefore,

$$\phi_*(\bar{v}, v_{n+1}) \leq \phi_*(\bar{v}, v_n) - 2\lambda k_2\|v_n - \bar{v}\|^2 + 4\lambda^2 m_2 L^2\|w_n - \bar{w}\|^2 - 2\lambda\langle u_n - \bar{u}, v_n - \bar{v}\rangle. \tag{3.5}$$

Adding up (3.4) and (3.5), using the fact that $\lambda \in (0, \delta)$ and the definition of δ in (3.3), we have:

$$\begin{aligned}\psi(\bar{w}, w_{n+1}) &\leq \psi(\bar{w}, w_n) - 2\lambda k\|w_n - \bar{w}\|^2 + 4\lambda^2 L^2(L_1 + m_2)\|w_n - \bar{w}\|^2\\ &\leq \psi(\bar{w}, w_n) - \lambda(2k - 4\lambda L^2(L_1 + m_2))\|w_n - \bar{w}\|^2\\ &\leq \psi(\bar{w}, w_n) - \lambda k\|w_n - \bar{w}\|^2 \leq r.\end{aligned}$$

Hence, by induction, $\{w_n\}$ is bounded.

Step 2: We prove that $\{w_n\}$ converges to $\bar{w} = (\bar{u}, \bar{v})$. We have:

$$\psi(\bar{w}, w_{n+1}) \leq \psi(\bar{w}, w_n) - \lambda k\|w_n - \bar{w}\|^2.$$

Therefore, the sequence $\{\psi(\bar{w}, w_{n+1})\}$ converges, since it is monotone decreasing and bounded below by zero. Consequently,

$$\lambda k\|w_n - \bar{w}\|^2 \leq \psi(\bar{w}, w_n) - \psi(\bar{w}, w_{n+1}) \to 0, \text{ as } n \to +\infty.$$

This yields $w_n \to \bar{w}$ as $n \to +\infty$. Hence, $u_n \to \bar{u}$ and $v_n \to \bar{v}$. This completes the proof. □

Corollary 3.2. *Let E be a real Banach space either l_p, or L_p or $W^{m,p}$, $1 < p \le 2$ with dual E^* and let $F : E \to E^*$, $K : E^* \to E$ be Lipschitz and strongly monotone mappings with $D(K) = R(F) = E^*$. For given $u_1 \in E$ and $v_1 \in E^*$, let $\{u_n\}$ and $\{v_n\}$ be generated iteratively by:*

$$\begin{aligned} u_{n+1} &= J^{-1}(Ju_n - \lambda(Fu_n - v_n)),\ n \ge 1, \\ v_{n+1} &= J(J^{-1}v_n - \lambda(Kv_n + u_n)),\ n \ge 1, \end{aligned} \tag{3.6}$$

where J is normalized duality mapping from E into E^* and $\lambda \in (0,1)$. Suppose that the equation $u + KFu = 0$ has a unique solution $\bar{u}$. Then, there exists some $\delta > 0$ such that if $\lambda \in (0,\delta)$, the sequence $\{u_n\}$ converges strongly to $\bar{u}$, the sequence $\{v_n\}$ converges strongly to $\bar{v}$, with $\bar{v} = F\bar{u}$.

Proof. Since l_p, L_p or $W^{m,p}$-spaces, $1 < p \le 2$ are 2-uniformly convex and p-uniformly smooth Banach spaces, then the proof follows from Theorem 3.1. □

4 Convergence in l_p, L_p and $W^{m,p}$-spaces, $2 \le p < \infty$

Note that for p, $1 < p < \infty$, a Banach space E with dual E^* is p-uniformly convex if and only if E^* is q-uniformly smooth, where $q > 1$ is the conjugate of p. Therefore, for E, a 2-uniformly smooth and s-uniformly convex real Banach space, the following hold:

(i) The dual space E^* is $2-$uniformly convex and q-uniformly smooth, where q is the conjugate of s;

(ii) The duality mapping J_* of E^* coincides with J^{-1}, the inverse of the duality mapping of E.

Using these observations and Lemmas 2.3 and 2.4, we have:

Lemma 4.1. *Let E be a 2-uniformly smooth and p-uniformly convex real Banach space. Then J is Lipschitz from E into E^*, i.e., there exists a constant $L > 0$ such that for all $x, y \in E$,*

$$||Jx - Jy|| \le L||x - y||. \tag{4.1}$$

Lemma 4.2. *Let E be a 2-uniformly smooth and p-uniformly convex real Banach space. Then, $J^{-1} : E^* \to E$, the inverse of the duality mapping of E , J is Lipschitz on bounded subsets of E^*; that is, for all $R > 0$, there exist a positive constant m_2 such that*

$$||J^{-1}u - J^{-1}v|| \le m_2||u - v||,$$

for all $u, v \in E^*$ with $||u|| \le R$, $||v|| \le R$.

From (i) and (ii), using the same method of proof as in Theorem 3.1 and using Lemmas 4.1 and 4.2 we have the following result.

Theorem 4.3. *For $s > 1$, let E be a 2-uniformly smooth and s-uniformly convex real Banach space with dual E^*. Let $F : E \to E^*$, $K : E^* \to E$ be Lipschitz and strongly monotone mappings with $D(K) = R(F) = E^*$. For given $u_1 \in E$ and $v_1 \in E^*$, let $\{u_n\}$ and $\{v_n\}$ be generated iteratively by:*

$$\begin{aligned} u_{n+1} &= J^{-1}(Ju_n - \lambda(Fu_n - v_n)),\ n \geq 1, \\ v_{n+1} &= J(J^{-1}v_n - \lambda(Kv_n + u_n)),\ n \geq 1, \end{aligned} \tag{4.2}$$

where J is normalized duality mapping from E into E^* and $\lambda \in (0,1)$. Suppose that the equation $u + KFu = 0$ has a unique solution $\bar{u}$. Then, there exists some $\delta > 0$ such that if $\lambda \in (0,\delta)$, the sequence $\{u_n\}$ converges strongly to $\bar{u}$, the sequence $\{v_n\}$ converges strongly to $\bar{v}$, with $\bar{v} = F\bar{u}$.

Corollary 4.4. *Let E be a real Banach space either l_p, or L_p, or $W^{m,p}$, $2 \leq p < \infty$ with dual E^* and let $F : E \to E^*$, $K : E^* \to E$ be Lipschitz and strongly monotone mappings with $D(K) = R(F) = E^*$. For given $u_1 \in E$ and $v_1 \in E^*$, let $\{u_n\}$ and $\{v_n\}$ be generated iteratively by:*

$$\begin{aligned} u_{n+1} &= J^{-1}(Ju_n - \lambda(Fu_n - v_n)),\ n \geq 1, \\ v_{n+1} &= J(J^{-1}v_n - \lambda(Kv_n + u_n)),\ n \geq 1, \end{aligned} \tag{4.3}$$

where J is normalized duality mapping from E into E^* and $\lambda \in (0,1)$. Suppose that the equation $u + KFu = 0$ has a unique solution $\bar{u}$. Then, there exists some $\delta > 0$ such that if $\lambda \in (0,\delta)$, the sequence $\{u_n\}$ converges strongly to $\bar{u}$, the sequence $\{v_n\}$ converges strongly to $\bar{v}$, with $\bar{v} = F\bar{u}$.

Proof. Since l_p, L_p or $W^{m,p}$-spaces, $2 \leq p < \infty$ are 2-uniformly smooth and p-uniformly convex Banach spaces, then the proof follows from Theorem 4.3. □

Remark 4.5. In [37], the main theorems (Theorem 3.3 and Theorem 3.4) are valid for the class of *bounded strongly monotone* mappings and the algorithm used is of Mann Type. Here, the algorithm used is of Krasnoselskii type and strong convergence are proved for the class of *Lipschitz strongly monotone mappings*.

The class of mappings used in [37] is larger than the one used here but the arguments used in the proofs in [37] do not work when the algorithm used in [37] is replaced by the algorithm (4.3) proposed in the present work. The results obtained in both papers are complementary but not comparable in the sense that the class of mappings used are different and the algorithms used also are different.

For the class of *Lipschitz strongly monotone mappings*, class considered in the present work, it is kown that the algorithm (4.3) is preferable than the algorithm used in [37]. In fact, the recursion formula (4.3) used here is known to be superior to the recursion formula in [37] in the following sense: (i) The recursion formula (4.3) requires less computation time than the recursion formula of the Mann algorithm used in [37] because the parameter λ is fixed in $(0,1)$ whereas in the algorithm used in [37], λ is replaced by a sequence $\{\alpha_n\}$ in $(0,1)$ satisfying some conditions. In addition, The α_n must be choosen at each step of the iteration process. (ii) The Krasnoselskii-type algorithm usually yields rate of convergence as fast as that of a geometric progression whereas the Mann algorithm usually has order of convergence of the form $o(1/n)$.

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