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MULTIDIMENSIONAL BSDE WITH POISSON JUMPS IN FINITE TIME HORIZON

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Abstract

This paper is devoted to solve a multidimensional backward stochastic differential equation with jumps in finite time horizon. Under weak monotonicity condition on the generator and by means of suitable sequences, we prove existence and uniqueness of solution.

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1 Introduction

It is well known that the pioneer result on Backward stochastic differential equation (BSDE in short) was established by Pardoux and Peng [10]. Few years later the authors prove in [11] the deep connection between such equations and parabolic partial differential equations. Since then the interest in such stochastic equations has increased thanks to the many domains of applications. In order to study more general BSDEs, several authors interested in relaxing the Lipschitz condition on the generator. In this way some attempts have been done (see among others Mao [9], kobylanski [7], Lepeltier and San Martin [8]). Some authors studying parabolic integral-partial differential equation (PIDE), interested in BSDEs with Poisson Process (BSDEP in short). Among them we mention the result of Barles *et al* [1] who establish a probabilistic interpretation of a solution of a PIDE. This was done by means of a real-valued BSDEP with Lipschitzian generator.

Soon after appeared multidimensional BSDEs (MBSDE in short). Hamadène [6] proved an existence and uniqueness result of such equations with uniformly continuous generator

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and Fan *et al* [4] focused on the uniqueness of solutions of a MBSDE with linear growth generator. Recently Fan and Jiang [3] studying a MBSDE in finite time horizon prove an existence and uniqueness result under mild conditions on the generator. Their method based on four steps with suitable sequences improve subsequently the known results.

In this paper we intend to extend the result establish in [3] to MBSDEs with Poisson jumps (MBSDEP in short) introduced by a random Poisson measure independent to the underlying Brownian motion. We prove existence and uniqueness of solution under weak monotonicity and a growth condition on the generator. The paper is organized as follows. We first introduce a technical assumption and establish some preliminary results in section 2. Thanks to these statements we deal with the solvability of a MBSDEP in finite time duration in Section 3.

2 MBSDE with Poisson Jumps

2.1 Definitions and preliminary results

Let Ω be a non-empty set, $\mathcal{F}$ a $\sigma-$algebra of sets of Ω and $\mathbb{P}$ a probability measure defined on $\mathcal{F}$. The triplet $(\Omega,\mathcal{F},\ \mathbb{P})$ defines a probability space, which is assumed to be complete. We assume given two mutually independent processes:

- a $d-$dimensional Brownian motion $(B_t)_{t\geq 0}$,
- a random Poisson measure μ on $E\times\mathbb{R}_+$ with compensator $\nu(dt,de)=\lambda(de)dt$

where the space $E=\mathbb{R}-\{0\}$ is equipped with its Borel field $\mathcal{E}$ such that $\{\widetilde{\mu}([0,t]\times A)=(\mu-\nu)[0,t]\times A\}$ is a martingale for any $A\in\mathcal{E}$ satisfying $\lambda(A)<\infty$. λ is a $\sigma-$finite measure on $\mathcal{E}$ and satisfies

$$\int_E (1\wedge |e|^2)\lambda(de)<\infty.$$

We consider the filtration $(\mathcal{F}_t)_{t\geq 0}$ given by $\mathcal{F}_t=\mathcal{F}_t^B\vee\mathcal{F}_t^\mu$, where for any process $\{\eta_t\}_{t\geq 0}$, $\mathcal{F}_{s,t}^\eta=\sigma\{\eta_r-\eta_s,\ s\leq r\leq t\}\vee\mathcal{N},\ \mathcal{F}_t^\eta=\mathcal{F}_{0,t}^\eta$. Here $\mathcal{N}$ denotes the class of $\mathbb{P}-$null sets of $\mathcal{F}$.
For $Q\in\mathbb{N}^*$, $|\,.\,|$ stands for the Euclidian norm in $\mathbb{R}^Q$.
We consider the following sets (where $\mathbb{E}$ denotes the mathematical expectation with respect to the probability measure $\mathbb{P}$) and a non-random horizon time $0<T<+\infty$:

- $\mathcal{S}^2(\mathbb{R}^Q)$ the space of $\mathcal{F}_t-$adapted càdlàg processes

$$\Psi:[0,T]\times\Omega\longrightarrow\mathbb{R}^Q,\ \|\Psi\|^2_{\mathcal{S}^2(\mathbb{R}^Q)}=\mathbb{E}\left(\sup_{0\leq t\leq T}|\Psi_t|^2\right)<\infty.$$

- $\mathcal{M}^2(\mathbb{R}^Q)$ the space of $\mathcal{F}_t-$progressively measurable processes

$$\Psi:[0,T]\times\Omega\longrightarrow\mathbb{R}^{Q\times d},\ \|\Psi\|^2_{\mathcal{M}^2(\mathbb{R}^Q)}=\mathbb{E}\int_0^T|\Psi_t|^2\,dt<\infty.$$

- $\mathcal{L}^2(\widetilde{\mu},\mathbb{R}^Q)$ the space of mappings $U:\Omega\times[0,T]\times E\longrightarrow\mathbb{R}^Q$ which are $\mathcal{P}\otimes\mathcal{E}$-measurable such that

$$\|U\|^2_{\mathcal{L}^2(\mathbb{R}^Q)}=\mathbb{E}\int_0^T\|U_t\|^2_{L^2(E,\mathcal{E},\lambda,\mathbb{R})}dt<\infty,$$

where $\mathcal{P}$ denotes the $\sigma-$algebra of $\mathcal{F}_t-$predictable sets of $\Omega\times[0,T]$ and

$$\|U_t\|^2_{L^2(E,\mathcal{E},\lambda,\mathbb{R})} = \int_E |U_t(e)|^2\,\lambda(de).$$

We may often write $|\cdot|$ instead of $\|\cdot\|_{L^2(E,\mathcal{E},\lambda,\mathbb{R})}$ for a sake of simplicity.

Let $k\geq 1$ and define $\mathcal{A}=\mathbb{R}^k\times\mathbb{R}^{k\times d}\times L^2(E,\mathcal{E},\lambda,\mathbb{R}^k)$. Notice that the space $\mathcal{B}^2(\mathbb{R}^k)=\mathcal{S}^2(\mathbb{R}^k)\times\mathcal{M}^2(\mathbb{R}^{k\times d})\times\mathcal{L}^2(\widetilde{\mu},\mathbb{R}^k)$ endowed with the norm

$$\|(Y,Z,U)\|^2_{\mathcal{B}^2(\mathbb{R}^k)} = \|Y\|^2_{\mathcal{S}^2(\mathbb{R}^k)} + \|Z\|^2_{\mathcal{M}^2(\mathbb{R}^{k\times d})} + \|U\|^2_{\mathcal{L}^2(\mathbb{R}^k)}$$

is a Banach space.

Finally let $\mathbf{S}$ be the set of all nondecreasing and concave function $\varphi(\cdot):\mathbb{R}_+\to\mathbb{R}_+$ satisfying $\varphi(0)=0$, $\varphi(s)>0$ for $s>0$ and $\int_{0+}\varphi^{-1}(u)du=+\infty$.

Remark 2.1. Notice that for any $\kappa\in\mathbf{S}$, there exists a positive constant A such that $\kappa(x)\leq A(x+1)$, $x\in\mathbb{R}_+$.

Given $g:\Omega\times[0,T]\times\mathcal{A}\to\mathbb{R}^k$ a jointly measurable function and $\xi\in L^2(\mathcal{F}_T,\mathbb{R}^k)$ the set of all $\mathbb{R}^k-$valued, square integrable and $\mathcal{F}_T-$measurable random vectors, we are interested in the MBSDEP with parameters (ξ,g,T):

$$Y_t = \xi + \int_t^T g(r,\Theta_r)\,dr - \int_t^T Z_r dB_r - \int_t^T \int_E U_r(e)\widetilde{\mu}(dr,de),\ 0\leq t\leq T, \tag{2.1}$$

where Θ_r stands for the triple (Y_r,Z_r,U_r).
For instance let us precise the notion of solution to eq.(2.1).

Definition 2.2. A triplet of processes $(Y_t,Z_t,U_t)_{0\leq t\leq T}$ is called a solution to eq.(2.1), if $(Y_t,Z_t,U_t)\in\mathcal{B}^2(\mathbb{R}^k)$ and it satisfies eq.(2.1).

Now, let us introduce the following Proposition 2.3, which will play an important role in the proof of our main result. In stating it, the following assumption on the generator g is useful:

$$\begin{aligned}(\mathbf{A}):\quad & d\mathbb{P}\times dt\text{-a.e.},\quad \forall(y,z,u)\in\mathbb{R}^k\times\mathbb{R}^{k\times d}\times\mathbb{R}^k,\\ & \langle y, g(\phi,t,y,z,u)\rangle \leq \psi(|y|^2)+\alpha|y|(|z|+|u|)+|y|\varphi_t\end{aligned}$$

where $\alpha>0$, φ_t is a non-negative and $(\mathcal{F}_t)$-progressively measurable process satisfying

$$\mathbb{E}\left[\int_0^T \varphi_t^2\,dt\right] < +\infty$$

and ψ is a nondecreasing concave function from $\mathbb{R}_+$ to itself with $\psi(0)=0$.

Proposition 2.3. *Assume that g satisfies the assumption* $(\mathbf{A})$ *and let $(Y_t,Z_t,U_t)_{0\leq t\leq T}$ be a solution to the MBSDEP* (2.1). *Then for any $\theta>0$, there exists a constant $c>0$ depending only on α and θ such that, for any $0\leq u\leq t\leq T$,*

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\leq r\leq T}|Y_r|^2\,\Big|\,\mathcal{F}_u\right]+\mathbb{E}\left[\int_t^T|Z_s|^2ds\,\Big|\,\mathcal{F}_u\right]+\mathbb{E}\left[\int_t^T\int_E|U_s(e)|^2\lambda(de)ds\,\Big|\,\mathcal{F}_u\right]\\ &\leq e^{c(T-t)}\left\{c\mathbb{E}[|\xi|^2\,|\,\mathcal{F}_u]+c\int_t^T\psi(\mathbb{E}[|Y_s|^2|\mathcal{F}_u])ds+\frac{1}{\theta}\mathbb{E}\left[\int_t^T\varphi_s^2ds\,\Big|\,\mathcal{F}_u\right]\right\}\end{aligned} \tag{2.2}$$

Proof. Applying Itô's formula to $|Y_t|^2$ reads to

$$|Y_t|^2+\int_t^T|Z_s|^2ds+\int_t^T\int_E|U_s(e)|^2\lambda(de)ds+\sum_{t<s\le T}(\Delta Y_s)^2=|\xi|^2+2\int_t^T\langle Y_s,g(s,\Theta_s)\rangle ds$$
$$-2\int_t^T\langle Y_s,Z_sdB_s\rangle-2\int_t^T\int_E\langle Y_{s^-},U_s(e)\widetilde{\mu}(ds,de)\rangle,\quad 0\le t\le T. \tag{2.3}$$

By the assumption **(A)** and the inequality $2ab\le\theta a^2+b^2/\theta$ for any $\theta>0$, we have

$$\begin{aligned}2\langle Y_s,g(s,\Theta_s))\rangle &\le\psi(|Y_s|^2)+2\alpha|Y_s|(|Z_s|+|U_s|)+2|Y_s|\varphi_s\\ &\le\psi(|Y_s|^2)+(4\alpha^2+146\theta)|Y_s|^2+\frac{\varphi_s^2}{146\theta}+\frac{1}{2}(|Z_s|^2+|U_s|^2)\end{aligned} \tag{2.4}$$

Thus it follows from eq.(2.3) and (2.4) that for any $0\le u\le t\le T$,

$$\frac{1}{2}\mathbb{E}\left[\int_t^T|Z_s|^2ds+\int_t^T\int_E|U_s(e)|^2\lambda(de)ds\;\Big|\;\mathcal{F}_u\right]\le X_u^t \tag{2.5}$$

where for $0\le u\le t\le T$,

$$\begin{aligned}X_u^t&=\mathbb{E}[|\xi|^2\mathcal{F}_u]+(4\alpha^2+146\theta)\int_t^T\mathbb{E}\left[\sup_{s\le r\le T}|Y_r|^2\;\Big|\;\mathcal{F}_u\right]ds\\ &+\mathbb{E}\left[\int_t^T\left(2\psi(|Y_s|^2)+\frac{\varphi_s^2}{146\theta}\right)ds\;\Big|\;\mathcal{F}_u\right]\end{aligned}$$

From the Burkhölder-Davis-Gundy inequality, the process $\left\{M_t=\int_0^t\langle Y_s,Z_sdB_s\rangle\right\}_{0\le t\le T}$ is in fact a uniformly integrable martingale. Indeed for any $0\le u\le t\le T$, we have

$$\begin{aligned}2\mathbb{E}\left[\sup_{t\le r\le T}\left|\int_r^T\langle Y_s,Z_sdB_s\rangle\right|\;\Big|\;\mathcal{F}_u\right]&\le 6\mathbb{E}\left[\left(\sup_{t\le r\le T}|Y_r|^2\right)^{\frac{1}{2}}.\left(\int_t^T|Z_s|^2ds\right)^{\frac{1}{2}}\;\Big|\;\mathcal{F}_u\right]\\ &\le\frac{1}{4}\mathbb{E}\left[\sup_{t\le r\le T}|Y_r|^2\;\Big|\;\mathcal{F}_u\right]+36\mathbb{E}\left[\int_t^T|Z_s|^2ds\;\Big|\;\mathcal{F}_u\right]\end{aligned} \tag{2.6}$$

Similarly for the discontinuous martingale, we have

$$\begin{aligned}&2\mathbb{E}\left[\sup_{t\le r\le T}\left|\int_r^T\int_E\langle Y_{s^-},U_s(e)\widetilde{\mu}(ds,de)\rangle\right|\;\Big|\;\mathcal{F}_u\right]\\ &\le 6\mathbb{E}\left[\left(\sup_{t\le r\le T}|Y_r|^2\right)^{\frac{1}{2}}.\left(\int_t^T\int_E|U_s(e)|^2\lambda(de)ds\right)^{\frac{1}{2}}\;\Big|\;\mathcal{F}_u\right]\\ &\le\frac{1}{4}\mathbb{E}\left[\sup_{t\le r\le T}|Y_r|^2\;\Big|\;\mathcal{F}_u\right]+36\mathbb{E}\left[\int_t^T\int_E|U_s(e)|^2\lambda(de)ds\;\Big|\;\mathcal{F}_u\right]\end{aligned} \tag{2.7}$$

Taking in account (2.4) and (2.7), we deduce from eq.(2.3)

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\le r\le T}|Y_r|^2\;\Big|\;\mathcal{F}_u\right]+\mathbb{E}\left[\int_t^T|Z_s|^2ds\;\Big|\;\mathcal{F}_u\right]+\mathbb{E}\left[\int_t^T\int_E|U_s(e)|^2\lambda(de)ds\;\Big|\;\mathcal{F}_u\right]\\ &\le 2X_u^t+72\mathbb{E}\left[\int_t^T|Z_s|^2ds\;\Big|\;\mathcal{F}_u\right]+72\mathbb{E}\left[\int_t^T\int_E|U_s(e)|^2\lambda(de)ds\;\Big|\;\mathcal{F}_u\right]\end{aligned} \tag{2.8}$$

Hence combining the above inequality and (2.5), we obtain

$$\begin{aligned} f(t) &\leq 146X_u^t \\ &:= 146\mathbb{E}[|\xi|^2|\mathcal{F}_u] + 146(4\alpha^2+146\theta)\int_t^T f(s)ds + \mathbb{E}\left[\int_t^T \left(292\psi(|Y_s|^2) + \frac{\varphi_s^2}{\theta}\right)ds \,\middle|\, \mathcal{F}_u\right] \end{aligned}$$

where $f(t)$ stands for the left hand side of (2.8).

Applying Fubini's theorem and Jensen's inequality, inequality (2.2) follows from Gronwall's lemma. □

In what follows we investigate our main subject.

3 Existence and uniqueness of solution

Let us introduce the following assumptions on the generator g. We say that g satisfies assumptions (**H**) if the following hold :

• **(H1)**: g satisfies the weak monotonicity condition in y, i.e., there exists $\kappa \in \mathbf{S}$ such that $d\mathbb{P}\times dt$-a.e, $\forall y_1, y_2 \in \mathbb{R}^k, z \in \mathbb{R}^{k\times d}, u \in \mathbb{R}^k$,

$$\langle y_1 - y_2, g(\emptyset, t, y_1, z, u) - g(\emptyset, t, y_2, z, u)\rangle \leq \kappa(|y_1 - y_2|^2).$$

• **(H2)**: $d\mathbb{P}\times dt$-a.e., $\forall z \in \mathbb{R}^{k\times d}$ the function $y \longmapsto g(\emptyset, t, y, z, u)$ is continuous.

• **(H3)**: g has a general growth with respect to y, i.e., $d\mathbb{P}\times dt$-a.e,

$$\forall y \in \mathbb{R}^k, \quad |g(\emptyset, t, y, 0, 0)| \leq |g(\emptyset, 0, 0, 0)| + \phi(|y|)$$

where $\phi : \mathbb{R}^+ \longmapsto \mathbb{R}^+$ is an increasing continuous function.

• **(H4)**: g is Lipschitz continuous in (z, u) uniformly with respect to $(\emptyset, t, y)$, i.e., there exists a constant $\beta > 0$ such that $d\mathbb{P}\times dt$-a.e, $y \in \mathbb{R}^k, z_1, z_2 \in \mathbb{R}^{k\times d}, u_1, u_2 \in \mathbb{R}^k$

$$|g(\emptyset, t, y, z_1, u_1) - g(\emptyset, t, y, z_2, u_2)| \leq \beta(|z_1 - z_2| + |u_1 - u_2|).$$

• **(H5)**: The integrability condition holds a.s. $\mathbb{E}\left[\int_0^T |g(\emptyset, t, 0, 0, 0)|^2 dt\right] < +\infty.$

We recall the following result which will be useful in the proof of uniqueness. This one is a consequence of Lemma 3.6 in [9].

Lemma 3.1 (Bihari's inequality)**.** *Let $T > 0$, u, v continuous non-negative functions on $[0, T]$ and $H : \mathbb{R}_+ \to \mathbb{R}_+$ continuous and nondecreasing such that $H(r) > 0$ for $r > 0$ satisfying*

$$\int_{0^+} \frac{ds}{H(s)} = +\infty.$$

If

$$u(t) \leq \int_0^t v(s)H(u(s))ds, \quad 0 \leq t \leq T,$$

then $u(t) = 0$ for all $0 \leq t \leq T$.

We are now in position to give our main result:

Theorem 3.2. *Let* $\xi \in L^2(\mathcal{F}_T, \mathbb{R}^k)$. *If* g *satisfies the assumptions* **(H)**, *then the MBSDEP* (2.1) *with parameters* (ξ, T, g) *has a unique solution.*

Proof. **(i) Uniqueness.** Let $(Y_t^i, Z_t^i, U_t^i)_{0\le t\le T}$, $i = 1,2$ be two solutions of the MBSDEP (2.1). We consider the function $\widehat{g}$ defined by

$$\forall (y,z,u) \in \mathbb{R}^k \times \mathbb{R}^{k\times d} \times \mathbb{R}^k, \quad \widehat{g}(s,y,z,u) = g(s, y + Y_s^2, z + Z_s^2, u + U_s^2) - g(s, Y_s^2, Z_s^2, U_s^2)$$

and for $\delta \in \{Y, Z, U\}$ we define $\widehat{\delta} = \delta^1 - \delta^2$.

It is easily seen that the triple $(\widehat{Y}_t, \widehat{Z}_t, \widehat{U}_t)_{0\le t\le T}$ is a solution to the following MBSDEP with parameters $(0, T, \widehat{g})$:

$$\widehat{Y}_t = \int_t^T \widehat{g}(s, \widehat{Y}_s, \widehat{Z}_s, \widehat{U}_s)ds - \int_t^T \widehat{Z}_s dB_s - \int_t^T \int_E \widehat{U}_s(e)\widetilde{\mu}(ds,de), \quad 0 \le t \le T. \tag{3.1}$$

It follows from **(H1)** and **(H4)** that $d\mathbb{P} \times dt$-a.e.,

$$\forall (y,z,u) \in \mathbb{R}^k \times \mathbb{R}^{k\times d} \times \mathbb{R}^k, \quad \langle y, \widehat{g}(s,y,z,u)\rangle \le \kappa(|y|^2) + \beta|y|(|z| + |u|).$$

Then the generator $\widehat{g}$ of the MBSDEP (3.1) satisfies the assumption **(A)** with

$$\psi(u) = \kappa(u), \quad \alpha = \beta, \quad \text{and} \quad \phi_t \equiv 0.$$

Thus, it follows from Proposition 2.3 with $u = 0$ and $\theta = 1$ that there exists a constant $c > 0$ depending only on β and T such that, for $0 \le t \le T$,

$$\mathbb{E}\left[\sup_{t\le r\le T} |\widehat{Y}_r|^2\right] + \mathbb{E}\left[\int_t^T |\widehat{Z}_s|^2 ds\right] + \mathbb{E}\left[\int_t^T \int_E |\widehat{U}_s(e)|^2 \lambda(de) ds\right] \le c \int_t^T \kappa\left(\mathbb{E}\left[\sup_{s\le r\le T} |\widehat{Y}_r|^2\right]\right) ds.$$

Bihari's inequality implies that, for any $0 \le t \le T$,

$$\mathbb{E}\left[\sup_{t\le r\le T} |\widehat{Y}_r|^2\right] + \mathbb{E}\left[\int_t^T |\widehat{Z}_s|^2 ds\right] + \mathbb{E}\left[\int_t^T \int_E |\widehat{U}_s(e)|^2 \lambda(de) ds\right] = 0.$$

Uniqueness follows.

(ii) Existence. The proof of the existence part will be split into four steps:

Step 1: Let us consider the following condition :

(B1): for a given pair of processes $(V, W) \in \mathcal{M}^2(\mathbb{R}^{k\times d}) \times \mathcal{L}^2(\widetilde{\mu}, \mathbb{R}^k)$ there exists $K > 0$ such that

$$d\mathbb{P}\text{-a.s,} \quad |\xi|^2 \le K, \quad d\mathbb{P}\times dt\text{-a.e.,} \quad |g(t,0,0,0)| \le K \text{ and } |V_t| + |W_t| \le K. \tag{3.2}$$

We intend to prove that under the assumptions **(H)** and condition **(B1)** there exists a unique solution to the following MBSDEP:

$$Y_t = \xi + \int_t^T g(s, Y_s, V_s, W_s)ds - \int_t^T Z_s dB_s - \int_t^T U_s(e)\widetilde{\mu}(ds,de). \tag{3.3}$$

Let $\psi \in C^\infty(\mathbb{R}^k, \mathbb{R}_+)$ s.t. $\int_{\mathbb{R}^k} \psi(y) = 1$, with the closed unit ball as compact support. For an integer $n \geq 1$ and $(\emptyset, t, y) \in \Omega \times [0, T] \times \mathbb{R}^k$, we set

$$g_n(t, y, V_t, W_t) = n^k g(t, y, V_t, W_t) * \psi(ny). \tag{3.4}$$

Then, g_n is an $(\mathcal{F}_t)$-progressively mesurable process and for any $y \in \mathbb{R}^k$, we have

$$\begin{aligned} g_n(t, y, V_t, W_t) &= \int_{\mathbb{R}^k} g(t, y - \frac{u}{n}, V_t, W_t) * \psi(u) du \\ &= \int_{\{u:|u|\leq 1\}} g(t, y - \frac{u}{n}, V_t, W_t) \psi(u) du. \end{aligned} \tag{3.5}$$

It follows from assumptions **(H3)**, **(H4)** and condition **(B1)** that

$$\begin{aligned} \forall y \in \mathbb{R}^k, \quad |g(t, y, V_t, W_t)| &\leq |g(t, y, 0, 0)| + |g(t, y, V_t, W_t) - g(t, y, 0, 0)| \\ &\leq K(1 + \beta) + \phi(|y|) \end{aligned} \tag{3.6}$$

Hence from eq.(3.4), we can show that $n \geq 1$, g_n is locally Lipschitz in y. Furthermore, for any $n \geq 1$ and $y \in \mathbb{R}^k$, it follows from eq.(3.5) and (3.6) that $d\mathbb{P} \times dt$-a.e.,

$$|g_n(t, y, V_t, W_t)| \leq K(1 + \beta) + \phi(|y|) \tag{3.7}$$

Now, for some large enough integer $r > 0$ which will be chosen later, let θ_r be a smooth function such that

$$0 \leq \theta_r \leq 1, \ \theta_r(y) = 1, \text{ for } |y| \leq r \text{ and } \theta_r(y) = 0 \text{ as soon as } |y| \geq r + 1.$$

Then for any integer $n \geq 1$, the function $g_n^\theta(t, y) = \theta_r(y) g_n(t, y, V_t, W_t)$ is globally Lipschitz in y. Indeed, let us pick $(y, y') \in \mathbb{R}^k \times \mathbb{R}^k$. If $|y| > r + 1$ and $|y'| > r + 1$, then the statement is trivially satisfied and thus we reduce to the case $|y'| \leq r + 1$. Note that g_n is locally Lipschitz in y and θ_r is globally Lipschitz in y. It follows from (3.7) that there exist two positive constants C_1 and C_2 such that $d\mathbb{P} \times dt$-a.e.,

$$\begin{aligned} |g_n^\theta(t, y) - g_n^\theta(t, y')| &\leq |\theta_r(y)||g_n(t, y, V_t, W_t) - g_n(t, y', V_t, W_t)| \\ &\quad + |\theta_r(y) - \theta_r(y')||g_n(t, y', V_t, W_t)| \\ &\leq C_1|y - y'| + C_2(K + \beta K + \phi(r + 1))|y - y'| := C|y - y'|. \end{aligned}$$

From Theorem 2.1 in [1], we know that for any $n \geq 1$, the following MBSDEP

$$Y_t^n = \xi + \int_t^T g_n^\theta(s, Y_s^n) ds - \int_t^T Z_s^n dB_s - \int_t^T \int_E U_s^n(e) \widetilde{\mu}(ds, de). \tag{3.8}$$

has a unique solution $(Y_t^n, Z_t^n, U_t^n)_{0 \leq t \leq T}$.
Furthermore from (3.5) and **(H1)**, we deduce that for any integer $n \geq 1$ and $(y_1, y_2) \in \mathbb{R}^k \times \mathbb{R}^k$,

$$\begin{aligned} &\langle y_1 - y_2, g_n(t, y_1, V_t, W_t) - g_n(t, y_2, V_t, W_t) \rangle \\ &\quad \leq \int_{\mathbb{R}^k} \langle y_1 - y_2, g(t, y_1 - \frac{u}{n}, V_t, W_t) - g(t, y_2 - \frac{u}{n}, V_t, W_t) \rangle \psi(u) du \\ &\quad \leq \int_{\mathbb{R}^k} \kappa(|y_1 - y_2|^2) \psi(u) du = \kappa(|y_1 - y_2|^2). \end{aligned} \tag{3.9}$$

Moreover for any $n \geq 1$ and $y \in \mathbb{R}^k$, combining (3.9) and (3.7) we have $d\mathbb{P} \times dt$-a.e.,

$$\begin{aligned}\langle y, g_n^\theta(t,y)\rangle &= \theta_r(y)\langle y, g_n(t,y,V_t,W_t) - g_n(t,0,V_t,W_t)\rangle + \theta_r(y)\langle y, g_n(t,0,V_t,W_t)\rangle \\ &\leq \kappa(|y|^2) + |y|(K(1+\beta)+\phi(0)).\end{aligned}$$

This implies that the generator g_n^θ of the MBSDEP (3.8) satisfies the assumption **(A)** with

$$\psi(u) = \kappa(u), \quad \alpha = 0, \; \textit{and} \; \varphi_t = K(1+\beta)+\phi(0).$$

Hence applying Proposition 2.3 (with $\theta = 1$) and taking in account condition **(B1)**, we deduce that there exists a constant $c > 0$ depending only on T such that, for $n \geq 1$ and any $0 \leq u \leq t \leq T$,

$$\begin{aligned}&\mathbb{E}[|Y_t^n|^2|\mathcal{F}_u] + \mathbb{E}\left[\int_t^T |Z_s^n|^2 ds \,\middle|\, \mathcal{F}_u\right] + \mathbb{E}\left[\int_t^T \int_E |U_s^n(e)|^2 \lambda(de)ds \,\middle|\, \mathcal{F}_u\right] \\ &\qquad \leq cK^2 + c\int_t^T \kappa(\mathbb{E}[|Y_s^n|^2|\mathcal{F}_u])ds + c(K+\beta K+\phi(0))^2 T\end{aligned}$$

Furthermore, by Remark 2.1 and Gronwall's lemma, we deduce that

$$\begin{aligned}&\mathbb{E}[|Y_t^n|^2|\mathcal{F}_u] + \mathbb{E}\left[\int_t^T |Z_s^n|^2 ds \,\middle|\, \mathcal{F}_u\right] + \mathbb{E}\left[\int_t^T \int_E |U_s^n(e)|^2 \lambda(de)ds \,\middle|\, \mathcal{F}_u\right] \\ &\qquad \leq (cK^2 + cAT + c(K(1+\beta)+\phi(0))^2 T) \times e^{cAT} := r^2.\end{aligned}$$

Substituting $u = t$ in the previous inequality it follows that, for any $n \geq 1$,

$$\forall 0 \leq t \leq T, \quad |Y_t^n| \leq r \quad \text{and} \quad \mathbb{E}\left[\int_t^T |Z_s^n|^2 ds\right] + \mathbb{E}\left[\int_t^T \int_E |U_s^n(e)|^2 \lambda(de)ds\right] \leq r^2. \quad (3.10)$$

By (3.8) and (3.10), we conclude that $(Y_t^n, Z_t^n, U_t^n)_{0\leq t\leq T}$ solves the following MBSDEP:

$$Y_t^n = \xi + \int_t^T g_n(s, Y_s^n, V_s, W_s) - \int_t^T Z_s^n dB_s - \int_t^T \int_E U_s^n(e)\widetilde{\mu}(ds,de). \quad (3.11)$$

In the sequel, we shall show that $\{(Y_t^n, Z_t^n, U_t^n)_{0\leq t\leq T}\}_{n\geq 1}$ is a Cauchy sequence in the Banach space $\mathcal{B}^2(\mathbb{R}^k)$. To this end we define for any integers n,m and $\delta \in \{Y,Z,U\}$, $\widehat{\delta}^{n,m} = \delta^n - \delta^m$. Thus the triplet $(\widehat{Y}_t^{n,m}, \widehat{Z}_t^{n,m}, \widehat{U}_t^{n,m})_{0\leq t\leq T}$ is solution to the MBSDEP with parameters $(0, T, \widehat{g}^{n,m})$:

$$\widehat{Y}_t^{n,m} = \int_t^T \widehat{g}^{n,m}(s, \widehat{Y}_s^{n,m}, V_s, W_s) - \int_t^T \widehat{Z}_s^{n,m} dB_s - \int_t^T \int_E \widehat{U}_s^{n,m}(e)\widetilde{\mu}(ds,de), \quad (3.12)$$

where

$$\widehat{g}^{n,m}(s,y,V_s,W_s) = g_n(s, y+Y_s^m, V_s, W_s) - g_m(s, Y_s^m, V_s, W_s), \; y \in \mathbb{R}^k, \; 0 \leq s \leq T.$$

It follows from (3.9) that for any $y \in \mathbb{R}^k$,

$$\begin{aligned}\langle y, \widehat{g}^{n,m}(t,y,V_t,W_t)\rangle &= \langle y, g_n(t, y+Y_t^m, V_t, W_t) - g_n(t, Y_t^m, V_t, W_t)\rangle \\ &\quad + \langle y, g_n(t, Y_t^m, V_t, W_t) - g_m(t, Y_t^m, V_t, W_t)\rangle \\ &\leq \kappa(|y|^2) + |y||g_n(t, Y_t^m, V_t, W_t) - g_m(t, Y_t^m, V_t, W_t)|.\end{aligned}$$

We deduce that the generator $\widehat{g}^{n,m}$ of the MBSDEP (3.12) satisfies the assumption **(A)** with

$$\psi(u) = \kappa(u), \quad \alpha = 0 \quad \text{and} \quad \varphi_t = |g_n(t, Y_t^m, V_t, W_t) - g_m(t, Y_t^m, V_t, W_t)|.$$

Hence applying Proposition 2.3 (with $u = 0$ and $\theta = 1$), there exists a constant $c > 0$ depending only on T such that for $0 \le t \le T$,

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\le r\le T} |\widehat{Y}_r^{n,m}|^2\right] + \mathbb{E}\left[\int_t^T |\widehat{Z}_s^{n,m}|^2 ds\right] + \mathbb{E}\left[\int_t^T \int_E |\widehat{U}_s^{n,m}(e)|^2 \lambda(de) ds\right] \\
\le \int_t^T \kappa\left(\mathbb{E}\left[\sup_{s\le r\le T} |\widehat{Y}_r^{n,m}|^2\right]\right) ds \\
+ c\mathbb{E}\left[\int_0^T (|g_n(s, Y_s^m, V_s, W_s) - g_m(s, Y_s^m, V_s, W_s)|)^2 ds\right].
\end{aligned} \tag{3.13}$$

On the other part, from eq.(3.5) we have for $n, m \ge 1$ and $0 \le s \le T$,

$$|g_n(s, Y_s^m, V_s, W_s) - g_m(s, Y_s^m, V_s, W_s)| \le \int_{\{u:|u|\le 1\}} h^{n,m,s}(u) \psi(u) du$$

where

$$h^{n,m,s}(u) = |g(s, Y_s^m - u/n, V_s, W_s) - g(s, Y_s^m - u/m, V_s, W_s)|.$$

By **(H2)** and (3.10), we derive that for any $u \in \mathbb{R}^k$,

$$d\mathbb{P} \times dt\text{-a.e.}, \quad h^{n,m,s}(u) \to 0 \quad \text{as} \quad n, m \to \infty.$$

Moreover from (3.6) and (3.10), we have for any $0 \le s \le T$ and $u \in \mathbb{R}^k$ such that $|u| \le 1$, $h^{n,m,s}(u) \le 2(K + \beta K + \varphi(r+1))$. This implies in particular

$$|g_n(s, Y_s^m, V_s, W_s) - g_m(s, Y_s^m, V_s, W_s)| \le 2(K + \beta K + \varphi(r+1)).$$

Applying Lebesgue's dominated convergence theorem twice, we obtain

$$\lim_{n,m\to\infty} \mathbb{E}\left[\int_0^T |g_n(s, Y_s^m, V_s, W_s) - g_m(s, Y_s^m, V_s, W_s)|^2 ds\right] = 0. \tag{3.14}$$

Let us set for $0 \le t \le T$, $f(t) = \limsup_{n,m\to\infty} f^{n,m}(t)$ where $f^{n,m}(t)$ stands for the left hand side of (3.13). Then the function f is well defined by (3.10). Taking the lim sup in (3.13) and combining Fatou's lemma, (3.14) and the properties (essentially monotonicity and continuity) of κ we deduce that

$$\forall 0 \le t \le T, \quad f(t) \le c \int_t^T \kappa(f(s)) ds.$$

Bihari's inequality implies that

$$\lim_{n,m\to\infty} \left\{\mathbb{E}\left[\sup_{t\le r\le T} |\widehat{Y}_r^{n,m}|^2\right] + \mathbb{E}\left[\int_t^T |\widehat{Z}_s^{n,m}|^2 ds\right] + \mathbb{E}\left[\int_t^T \int_E |\widehat{U}_s^{n,m}(e)|^2 \lambda(de) ds\right]\right\} = 0.$$

As a consequence, we derive that $\{(Y_t^n, Z_t^n, U_t^n)_{0\leq t\leq T}\}_{n\geq 1}$ is a Cauchy sequence in the Banach space $\mathcal{B}^2(\mathbb{R}^k)$. Let $(Y_t, Z_t, U_t)_{0\leq t\leq T}$ be the limit process of the sequence $\{(Y_t^n, Z_t^n, U_t^n)_{0\leq t\leq T}\}_{n\geq 1}$ in $\mathcal{B}^2(\mathbb{R}^k)$. Taking the limit in uniform convergence in probability in (3.11) and combining (3.7), (3.10) and assumptions **(H2)**, we deduce by Lebesgue's dominated convergence theorem that (Y_t, Z_t, U_t) solves the MBSDEP (3.3).

Step 2: In this step, we intend to remove the bounded condition required on the process $(V_t, W_t)_{0\leq t\leq T}$ in Step 1. To this end we introduce the condition:
(B2): for a given pair of processes $(V, W) \in \mathcal{M}^2(\mathbb{R}^{k\times d}) \times \mathcal{L}^2(\widetilde{\mu}, \mathbb{R}^k)$, there exists $K > 0$ such that

$$d\mathbb{P}\text{-a.s.}, \quad |\xi| \leq K, \quad \text{and} \quad d\mathbb{P}\times dt\text{-a.e.}, \quad |g(t,0,0,0)| \leq K, \tag{3.15}$$

Let us prove that under condition **(B2)**, there exists a unique solution to MBSDEP (3.3).
For any integer $n \geq 1$ and $z \in \mathbb{R}^{k\times d}$, we consider the function $q_n(z) = zn/(|z| \vee n)$. Then we have

$$\forall (z,u) \in \mathbb{R}^{k\times d} \times \mathbb{R}^k, \quad |q_n(z)| + |q_n(u)| \leq (|z| + |u|) \wedge n.$$

It follows from Step 1 that for any $n \geq 1$, there exists a solution $(Y_t^n, Z_t^n, U_t^n)_{0\leq t\leq T}$ to the following MBSDEP

$$Y_t^n = \xi + \int_t^T g(s, Y_s^n, q_n(V_s), q_n(W_s))ds - \int_t^T Z_s^n dB_s - \int_t^T \int_E U_s^n(e)\widetilde{u}(ds,de). \tag{3.16}$$

Similarly, we shall show that $\{(Y_t^n, Z_t^n, U_t^n)_{0\leq t\leq T}\}_{n\geq 1}$ solution to (3.16) is a Cauchy sequence in $\mathcal{B}^2(\mathbb{R}^k)$. Using the notations introduced in Step 1, we derive that the triplet $(\widehat{Y}_t^{n,m}, \widehat{Z}_t^{n,m}, \widehat{U}_t^{n,m})_{0\leq t\leq T}$ solves the MBSDEP

$$\widehat{Y}_t^{n,m} = \int_t^T \widehat{g}^{n,m}(s, \widehat{Y}_s^{n,m}, V_s, W_s)ds - \int_t^T \widehat{Z}_s^{n,m} dB_s - \int_t^T \int_E \widehat{U}_s^{n,m}(e)\widetilde{u}(ds,de), \tag{3.17}$$

where for any $0 \leq s \leq T$,

$$\widehat{g}^{n,m}(s,y,V_s,W_s) = g(s, y+Y_s^m, q_n(V_s), q_n(W_s)) - g(s, y+Y_s^m, q_m(V_s), q_m(W_s)), \quad y \in \mathbb{R}^k.$$

By standard computations, it follows from **(H1)** and **(H4)** that for any $y \in \mathbb{R}^k, d\mathbb{P}\times dt$ -a.e.,

$$\langle y, \widehat{g}^{n,m}(t,y,V_t,W_t)\rangle \leq \kappa(|y|^2) + \beta|y|[|q_n(V_t) - q_m(V_t)| + |q_n(W_t) - q_m(W_t)|]$$

Hence the generator $\widehat{g}^{n,m}$ of the MBSDEP (3.17) satisfies the assumption **(A)** with

$$\psi(u) = \kappa(u), \quad \alpha = 0 \quad \text{and} \quad \varphi_t = \beta[|q_n(V_t) - q_m(V_t)| + |q_n(W_t) - q_m(W_t)|].$$

It follows from Proposition 2.3 with $u = 0$ and $\theta = 1$ that there exists a constant $c > 0$ depending only on T such that for $0 \leq t \leq T$,

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\leq r\leq T} |\widehat{Y}_r^{n,m}|^2\right] + \mathbb{E}\left[\int_t^T |\widehat{Z}_s^{n,m}|^2 ds\right] + \mathbb{E}\left[\int_t^T \int_E |\widehat{U}_s^{n,m}(e)|^2 \lambda(de) ds\right] \\
\leq \int_t^T \kappa\left(\mathbb{E}\left[\sup_{s\leq r\leq T} |\widehat{Y}_r^{n,m}|^2\right]\right) ds \\
+ c\beta^2 \mathbb{E}\left[\int_0^T (|q_n(V_s) - q_m(V_s)| + |q_n(W_s) - q_m(W_s)|)^2 ds\right].
\end{aligned} \tag{3.18}$$

Since $\kappa \in \mathbf{S}$, applying Gronwall's lemma we obtain for any $0 \le t \le T$ and $n,m \ge 1$,

$$\mathbb{E}\left[\sup_{t\le r\le T}|\widehat{Y}_r^{n,m}|^2\right]+\mathbb{E}\left[\int_t^T|\widehat{Z}_s^{n,m}|^2ds\right]+\mathbb{E}\left[\int_t^T\int_E|\widehat{U}_s^{n,m}(e)|^2\lambda(de)ds\right]$$
$$\le\left(cAT+2c\beta^2\mathbb{E}\left[\int_0^T(|V_s|+|W_s|)^2ds\right]\right)\times e^{cAT}.$$

Taking the lim sup in (3.18) with respect to n,m combined with Fatou's lemma, the monotonicity and continuity of κ and Bihari's inequality, we deduce that $\{(Y_t^n,Z_t^n,U_t^n)_{0\le t\le T}\}_{n\ge 1}$ is a Cauchy sequence in $\mathcal{B}^2(\mathbb{R}^k)$.
Let $(Y_t,Z_t,U_t)_{0\le t\le T}$ be the limit process of the sequence $\{(Y_t^n,Z_t^n,U_t^n)_{0\le t\le T}\}_{n\ge 1}$ in $\mathcal{B}^2(\mathbb{R}^k)$. Taking the limit in uniform convergence in probability in (3.16) in view of **(H2)**, **(H4)**, we deduce by Lebesgue's dominated convergence theorem that $(Y_t,Z_t,U_t)_{0\le t\le T}$ solves the MBSDEP (3.3).

Step 3: We shall prove that given $\xi \in L^2(\mathcal{F}_T,\mathbb{R}^k)$, if condition **(B2)** introduced in (3.15) holds, then there exists a unique solution to the MBSDEP (2.1).
Using Step 2, we consider now the well defined sequence $(Y_t^n,Z_t^n,U_t^n)_{n\ge 0}$ given by

$$\begin{cases}Y_t^0=0,\ Z_t^0=0,\ U_t^0=0,\\ Y_t^n=\xi+\displaystyle\int_t^T g(s,Y_s^n,Z_s^{n-1},U_s^{n-1})ds-\int_t^T Z_s^n dB_s-\int_t^T\int_E U_s^n(e)\widetilde{\mu}(ds,de),\ \ n\ge 1.\end{cases}\tag{3.19}$$

By assumptions **(H1)** and **(H4)**, we have for any $y\in\mathbb{R}^k, d\mathbb{P}\times dt$-a.e,

$$\begin{aligned}\langle y,g(t,y,Z_t^{n-1},U_t^{n-1})\rangle &= \langle y,g(t,y,Z_t^{n-1},U_t^{n-1})-g(t,0,Z_t^{n-1},U_t^{n-1})\rangle\\ &\quad+\langle y,g(t,0,Z_t^{n-1},U_t^{n-1})\rangle\\ &\le \kappa(|y|^2)+|y|(|g(t,0,0,0)|+\beta(|Z_t^{n-1}|+|U_t^{n-1}|))\end{aligned}$$

This implies that the generator g of the MBSDEP (3.19) satisfies the assumption **(A)** with

$$\psi(u)=\kappa(u),\quad \alpha=0\quad\text{and}\quad \varphi_t=|g(t,0,0,0)|+\beta(|Z_t^{n-1}|+|U_t^{n-1}|).$$

Hence applying Proposition 2.3 (with $\theta=32\beta^2$ and $u=0$), we deduce that there exists a constant $c>0$ depending only on β such that for $0\le t\le T$,

$$\mathbb{E}\left[\sup_{t\le r\le T}|Y_r^n|^2\right]+\mathbb{E}\left[\int_t^T|Z_s^n|^2ds\right]+\mathbb{E}\left[\int_t^T\int_E|U_s^n(e)|^2\lambda(de)ds\right]$$
$$\le e^{c(T-t)}M(t)+ce^{c(T-t)}\int_t^T\kappa\left(\mathbb{E}\left[\sup_{s\le r\le T}|Y_r|^2\right]\right)ds$$
$$+\frac{e^{c(T-t)}}{8}\mathbb{E}\left[\int_t^T|Z_s^{n-1}|^2ds+\int_t^T\int_E|U_s^{n-1}(e)|^2\lambda(de)ds\right],$$

where, thanks to condition **(B2)**, we have

$$M(t)=c\mathbb{E}[|\xi|^2]+\frac{1}{16\beta^2}\mathbb{E}\left[\int_t^T|g(s,0,0,0)|^2ds\right]\le K^2\left(c+\frac{T}{16\beta^2}\right):=M.\tag{3.20}$$

Let us set $T_1 = \max\{T - \ln 2/c, T - \ln 2/(2cA), 0\}$. Then for $T_1 \le t \le T$, we have the inequalities

$$e^{c(T-t)} \le 2, \;\; e^{2cA(T-t)} \le 2$$

and for any $n \ge 1$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\le r\le T}|Y_r^n|^2\right] + \mathbb{E}\left[\int_t^T |Z_s^n|^2 ds\right] + \mathbb{E}\left[\int_t^T \int_E |U_s^n(e)|^2 \lambda(de)ds\right] \\
&\le 2M + c\int_t^T \kappa\left(\mathbb{E}\left[\sup_{s\le r\le T}|Y_r|^2\right]\right)ds \\
&+ \frac{1}{4}\mathbb{E}\left[\int_t^T |Z_s^{n-1}|^2 ds + \int_t^T \int_E |U_s^{n-1}(e)|^2 \lambda(de)ds\right].
\end{aligned}$$

Remark 2.1 and Gronwall's lemma yields for $T_1 \le t \le T$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\le r\le T}|Y_r^n|^2\right] + \mathbb{E}\left[\int_t^T |Z_s^n|^2 ds\right] + \mathbb{E}\left[\int_t^T \int_E |U_s^n(e)|^2 \lambda(de)ds\right] \\
&\le \left(2M + 2cAT + \frac{1}{4}\mathbb{E}\left[\int_t^T |Z_s^{n-1}|^2 ds + \int_t^T \int_E |U_s^{n-1}(e)|^2 \lambda(de)ds\right]\right) \times e^{2cA(T-t)} \\
&\le 4M + 4cAT + \frac{1}{2}\mathbb{E}\left[\int_t^T |Z_s^{n-1}|^2 ds + \int_t^T \int_E |U_s^{n-1}(e)|^2 \lambda(de)ds\right],
\end{aligned}$$

which implies by induction that for any $n \ge 1$ and $T_1 \le t \le T$,

$$\mathbb{E}\left[\sup_{t\le r\le T}|Y_r^n|^2\right] + \mathbb{E}\left[\int_t^T |Z_s^n|^2 ds\right] + \mathbb{E}\left[\int_t^T \int_E |U_s^n(e)|^2 \lambda(de)ds\right] \le 8M + 8cAT. \tag{3.21}$$

In the same spirit, we have for $T_1 \le t \le T$,

$$\widehat{Y}_t^{n,m} = \int_t^T \widehat{g}^{n,m}(s, \widehat{Y}_s^{n,m})ds - \int_t^T \widehat{Z}_s^{n,m} dB_s - \int_t^T \int_E \widehat{U}_s^{n,m}(e)\widetilde{\mu}(ds, de) \tag{3.22}$$

where

$$\widehat{g}^{n,m}(s,y) = g(s, y + Y_s^m, Z_s^{n-1}, U_s^{n-1}) - g(s, Y_s^m, Z_s^{m-1}, U_s^{m-1}), \; y \in \mathbb{R}^k, \; 0 \le s \le T.$$

We deduce from assumptions **(H1)** and **(H4)** that for any $y \in \mathbb{R}^k$,

$$\begin{aligned}
\langle y, \widehat{g}^{n,m}(t,y)\rangle &\le \kappa(|y|^2) + \beta|y|(|Z_t^{n-1} - Z_t^{m-1}| + |U_t^{n-1} - U_t^{m-1}|) \\
&= \kappa(|y|^2) + \beta|y|(|\widehat{Z}_t^{n-1,m-1}| + |\widehat{U}_t^{n-1,m-1}|).
\end{aligned}$$

Then the generator $\widehat{g}^{n,m}$ of the MBSDEP (3.22) satisfies the assumption **(A)** with

$$\psi(u) = \kappa(u), \quad \alpha = 0 \quad \text{and} \quad \varphi_t = \beta(|\widehat{Z}_t^{n-1,m-1}| + |\widehat{U}_t^{n-1,m-1}|).$$

It follows from Proposition 2.3 with $\theta = 16\beta^2$ and $u = 0$ that there exists a constant $\gamma > 0$ depending only on β such that for $T_1 \le t \le T$,

$$\mathbb{E}\left[\sup_{t\le r\le T}|\widehat{Y}_r^{n,m}|^2\right]+\mathbb{E}\left[\int_t^T|\widehat{Z}_s^{n,m}|^2ds\right]+\mathbb{E}\left[\int_t^T\int_E|\widehat{U}_s^{n,m}(e)|^2\lambda(de)ds\right]$$
$$\le \gamma e^{2\gamma(T-t)}\int_t^T\kappa\left(\mathbb{E}\left[\sup_{s\le r\le T}|\widehat{Y}_r^{n,m}|^2\right]\right)ds$$
$$+\frac{e^{\gamma(T-t)}}{8}\mathbb{E}\left[\int_t^T|\widehat{Z}_s^{n-1,m-1}|^2ds+\int_t^T\int_E|\widehat{U}_s^{n-1,m-1}(e)|^2\lambda(de)ds\right].$$

We can assume that $\gamma = c$. Let us recall $T_1 = \max\{T-\ln 2/c, T-\ln 2/(2cA), 0\}$. Then for any $T_1 \le t \le T$, we have

$$\mathbb{E}\left[\sup_{t\le r\le T}|\widehat{Y}_r^{n,m}|^2\right]+\mathbb{E}\left[\int_t^T|\widehat{Z}_s^{n,m}|^2ds\right]+\mathbb{E}\left[\int_t^T\int_E|\widehat{U}_s^{n,m}(e)|^2\lambda(de)ds\right]$$
$$\le 2c\int_t^T\kappa\left(\mathbb{E}\left[\sup_{s\le r\le T}|\widehat{Y}_r^{n,m}|^2\right]\right)ds$$
$$+\frac{1}{4}\mathbb{E}\left[\int_t^T|\widehat{Z}_s^{n-1,m-1}|^2ds+\int_t^T\int_E|\widehat{U}_s^{n-1,m-1}(e)|^2\lambda(de)ds\right].$$

Hence in view of (3.21), taking the lim sup in the above inequality and using Fatou's lemma, the monotonicity and continuity of κ and Bihari's inequality, we prove that $\{(Y_t^n, Z_t^n, U_t^n)\}_{n\ge 1}$ is a Cauchy sequence in the Banach space $\mathcal{B}^2(\mathbb{R}^k)$.
Let $(Y_t, Z_t, U_t)_{T_1\le t\le T}$ be the limit of this sequence. Taking the limit in uniform convergence in probability in (3.19), in view of **(H2)-(H4)**, we deduce by Lebesgue's dominated convergence theorem that $(Y_t, Z_t, U_t)_{T_1\le t\le T}$ is a solution to the MBSDEP (2.1) on the interval $[T_1, T]$. Note that $T - T_1$ is a positive number and depends only on β and A. We can repeat the above procedure in finite steps to obtain a solution to the MBSDEP with parameters (ξ, T, g) on $[T_2, T_1], [T_3, T_2], ...,$ and then on $[0, T]$.

Step 4 : We shall prove that for $\xi \in L^2(\mathcal{F}_T, \mathbb{R}^k)$ there exists a unique solution to the MBSDEP (2.1) with parameters (ξ, T, g). To this end we consider the sequences $(g_n)_{n\ge 1}$ and $(\xi_n)_{n\ge 1}$ defined as follows:
For an integer $n \ge 1$, let (where the function $q_n(x)$ is defined in the previous step)

$$\begin{cases}\xi_n = q_n(\xi),\\ g_n(t, Y_t, Z_t, U_t) = g(t, Y_t, Z_t, U_t) - g(t,0,0,0) + q_n(g(t,0,0,0)).\end{cases} \tag{3.23}$$

It easily seen that

$$|\xi_n| \le n, \quad |\xi_n| \le \xi \quad \text{and} \quad |g_n(t,0,0,0)| \le n.$$

By Lebesgue's dominated convergence theorem and assumption **(H5)**, we have

$$\mathbb{E}\left[|\xi_n-\xi|^2\right] \xrightarrow{n\to\infty} 0 \text{ and } \mathbb{E}\left[\int_0^T |q_n(g(t,0,0,0)) - g(t,0,0,0)|^2dt\right] \xrightarrow{n\to\infty} 0. \tag{3.24}$$

By Step 3, for any $n \geq 1$, let $(Y_t^n, Z_t^n, U_t^n)_{T_1 \leq t \leq T}$ denotes the unique solution to the following MBSDEP :

$$Y_t^n = \xi_n + \int_t^T g_n(s, Y_s^n, Z_s^n, U_s^n)ds - \int_t^T Z_s^n dB_s - \int_t^T \int_E U_s^n(e)\widetilde{\mu}(ds, de). \tag{3.25}$$

Putting $\widehat{\xi}^{n,m} = \xi_n - \xi_m$ and using the same notations as in the previous steps, we have for any $0 \leq t \leq T$,

$$\begin{aligned}\widehat{Y_t}^{n,m} &= \widehat{\xi}^{n,m} + \int_t^T \widehat{g}^{n,m}(s, \widehat{Y_s}^{n,m}, \widehat{Z_s}^{n,m}, \widehat{U_s}^{n,m})ds - \int_t^T \widehat{Z_s}^{n,m} dB_s \\ &\quad - \int_t^T \int_E \widehat{U_s}^{n,m}(e)\widetilde{\mu}(ds, de)\end{aligned} \tag{3.26}$$

where for $(y, z, u) \in \mathbb{R}^k \times \mathbb{R}^{k\times d} \times \mathbb{R}^k$ the generator $\widehat{g}^{n,m}$ defined by

$$\widehat{g}^{n,m}(s, y, z, u) = g_n(s, y + Y_s^m, z + Z_s^m, u + U_s^m) - g_m(s, Y_s^m, Z_s^m, U_s^m)$$

can be rewritten as

$$\begin{aligned}\widehat{g}^{n,m}(s, y, z, u) &= g_n(s, y + Y_s^m, z + Z_s^m, u + U_s^m) - g_m(s, y + Y_s^m, z + Z_s^m, u + U_s^m) \\ &\quad + g_m(s, y + Y_s^m, z + Z_s^m, u + U_s^m) - g_m(s, Y_s^m, Z_s^m, U_s^m).\end{aligned} \tag{3.27}$$

Thanks to this representation and eq.(3.23), it follows from **(H1)** and **(H4)**

$$\langle y, \widehat{g}^{n,m}(s, y, z, u)\rangle |y||q_n(g(t, 0, 0, 0)) - q_m(g(t, 0, 0, 0))| + \kappa(|y|^2) + \beta|y|(|z| + |u|)$$

Then the generator $\widehat{g}^{n,m}$ of MBSDEP (3.26) satisfies the assumption **(A)** with

$$\psi(u) = \kappa(u), \ \ \alpha = \beta \quad \text{and} \quad \varphi_t = |q_n(g(t, 0, 0, 0)) - q_m(g(t, 0, 0, 0))|.$$

Applying Proposition 2.3 (with $u = 0$ and $\theta = 1$) that there exists a constant $c > 0$ depending only on T and β such that for $0 \leq t \leq T$,

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\leq r\leq T} |\widehat{Y_r}^{n,m}|^2\right] + \mathbb{E}\left[\int_t^T |\widehat{Z_s}^{n,m}|^2 ds\right] + \mathbb{E}\left[\int_t^T \int_E |\widehat{U_s}^{n,m}(e)|^2 \lambda(de)ds\right] \\ &\qquad \leq \mathbb{E}\left[|\widehat{\xi}^{n,m}|^2\right] + c\int_t^T \kappa\left(\mathbb{E}\left[\sup_{s\leq r\leq T} |\widehat{Y_r}^{n,m}|^2\right]\right)ds \\ &\qquad + c\mathbb{E}\left[\int_0^T (|q_n(g(t, 0, 0, 0)) - q_m(g(t, 0, 0, 0))|)^2 ds\right].\end{aligned} \tag{3.28}$$

Using once again Remark 2.1 and Gronwall's lemma, we deduce that for $0 \leq t \leq T$ and $n, m \geq 1$,

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\leq r\leq T} |\widehat{Y_r}^{n,m}|^2\right] + \mathbb{E}\left[\int_t^T |\widehat{Z_s}^{n,m}|^2 ds\right] + \mathbb{E}\left[\int_t^T \int_E |\widehat{U_s}^{n,m}(e)|^2 \lambda(de)ds\right] \\ &\qquad \leq \left(4c\mathbb{E}[|\xi|^2] + cAT + 4c\mathbb{E}\left[\int_0^T |g(s, 0, 0, 0)|^2 ds\right]\right) \times e^{cAT}.\end{aligned}$$

Hence in view of (3.24), by taking the lim sup in (3.28) with respect to n,m and using Fatou's lemma, the monotonicity and continuity of κ and Bihari's inequality we prove that $\left\{(Y_t^n, Z_t^n, U_t^n)_{0\leq t\leq T}\right\}_{n\geq 1}$ is a Cauchy sequence in the Banach space $\mathcal{B}^2(\mathbb{R}^k)$.
Let $(Y_t, Z_t, U_t)_{0\leq t\leq T}$ be the limit process of the sequence $\{(Y_t^n, Z_t^n, U_t^n)_{0\leq t\leq T}\}_{n\geq 1}$. Taking the limit in uniform convergence in probability in (3.25), and using the assumptions **(H2)-(H4)**, we deduce by Lebesgue's dominated convergence theorem that $(Y_t, Z_t, U_t)_{0\leq t\leq T}$ solves the MBSDEP (2.1) with parameters (ξ, T, g). This completes the proof. □

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