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On Jacobi Fields Along Eigenmappings of the Tension Field for Mappings into a Symmetric Riemannian Manifold

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Abstract. We prove that the mean value (for some measure $\mu = \chi dx$ with $\chi \geq 0, dx$ = Riemannian measure) of the squared norm of the gradient of the unitary direction of a Jacobi field along an eigenmapping v (associated to an eigenvalue $\lambda \geq 0$) of the tension field, for mappings from a compact Riemannian manifold (M, g) into a symmetric Riemannian manifold (N, h) of positive sectional curvature, is smaller than $c\lambda$, where $c > 0$ depends only on the diameter and upper and lower curvature bounds of (N, h). For negative λ, we prove that there is no nonvanishing Jacobi field along the eigenmappings, under the same assumptions on (M, g) and (N, h).

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1 Introduction

Let (M, g) and (N, h) be Riemannian manifolds, and $u : M \to N$ a smooth mapping. Let $v :]-1, 1[\times]-1, 1[\times M \to N$, $(r, s, x) \mapsto v_{r,s}(x)$ be a smooth mapping such that $v_{0,0} = u$, and $v_{r,s|\partial M} = u_{|\partial M}$ in case $\partial M \neq \emptyset$. The energy of u is

$$E(u) = \frac{1}{2}\int_M \|du\|^2 (x) dx.$$

We have

$$\begin{aligned}\frac{\partial^2}{\partial r \partial s}|_{r=s=0} E(v_{r,s}) &= \int_M \left\langle -[\nabla_{e_i}\nabla_{e_i}V - \nabla_{\nabla_{e_i}e_i}V] - R^N(V, d_{e_i}u)d_{e_i}u, W\right\rangle (x)dx \\ &\quad - \int_M \left\langle \nabla_r \frac{\partial v_{r,s}}{\partial s}|_{r=s=0}, \tau(u)\right\rangle (x)dx,\end{aligned}$$

where

$$V := \frac{\partial v_{r,s}}{\partial r}|_{r=s=0} \text{ and } W := \frac{\partial v_{r,s}}{\partial s}|_{r=s=0}$$

are vector fields along u,

$$\tau(u) := \text{trace } (\nabla du) = \nabla_{e_i} d_{e_i} u - d_{\nabla_{e_i}e_i} u$$

is the tension field of u, and $(e_i)_i$ is a local orthonormal frame.

If one assumes that $\tau(u) = 0$, i.e. u is *harmonic*, then one has

$$\frac{\partial^2}{\partial r \partial s}|_{r=s=0} E(v_{r,s}) = \int_M \left\langle -[\nabla_{e_i}\nabla_{e_i}V - \nabla_{\nabla_{e_i}e_i}V] - R^N(V, d_{e_i}u)d_{e_i}u, W\right\rangle (x)dx.$$

Harmonic mappings between Riemannian manifolds have been introduced by Eells - Sampson in 1964. See [6] for an introductory course.

For a harmonic mapping u, some $V \in \Gamma(u^{-1}(TN))$ is called a Jacobi field along u when

$$\nabla_{e_i}\nabla_{e_i}V - \nabla_{\nabla_{e_i}e_i}V + R^N(V, d_{e_i}u)d_{e_i}u = 0 \text{ on } M.$$

One sees also that when $r \mapsto v_{r,0}$ is a geodesic, then even when u is not harmonic, one has

$$\frac{d^2}{dr^2}|_{r=0} E(v_{r,0}) = \int_M \left\langle -[\nabla_{e_i}\nabla_{e_i}V - \nabla_{\nabla_{e_i}e_i}V] - R^N(V, d_{e_i}u)d_{e_i}u, V\right\rangle (x)dx.$$

The existence of non vanishing Jacobi fields along a harmonic mapping u makes it difficult to say if u is locally energy minimizing or not, and it gives informations about the uniqueness of u in its homotopy class. When (N,h) has nonpositive sectional curvature, it has been proved by Hartman in [3] that such a Jacobi field V satisfies

$$\nabla V = 0 \text{ and } \left\langle R^N(V, d_{e_i}u)d_{e_i}u, V\right\rangle = 0 \text{ on } M.$$

In our work [11] we tried to extend in some way this result of Hartman to cases where the sectional curvature of (N,h) is no more nonpositive, but (N,h) being symmetric. We proved (roughly said) in that work that given such a Jacobi field V, if it is integrable, i.e. there exists $v :]-1,1[\times M \to N$ a smooth mapping such that $v(0,.) = u$, $v(t,.)$ is harmonic, for any $t \in]-1,1[$ and $V(x) = \frac{\partial v(t,x)}{\partial t}|_{t=0}, \forall x \in M$, then $\nabla[\|V\|^{-1} V] = 0$.

In our work [7] we introduced in 2002, together with Prof. Jost, the functional

$$E_\lambda(u) = \frac{1}{2}[\int_M \|du\|^2(x)dx - \lambda \int_M d^2(u(x), w(x))dx]$$

for some fixed $\lambda \in \mathbb{R}$ and $w \in C^\infty(M,N)$, where $d(u(x), w(x))$ is the Riemannian distance between $u(x)$ and $w(x)$ in (N,h). When $\lambda < 0$ and (M,g) and (N,h) are Euclidean spaces, E_λ is the Mumford - Shah functional, which is used in image approximation, see e.g. [1].

The critical points of E_λ are the *eigenmappings of the tension field* τ associated to the eigenvalue λ (for the model mapping w). In [7] we proved that the spectrum of τ in this sense is continuous and the set of eigenvalues and eigenmappings may bifurcate, even when (N,h) has nonpositive sectional curvature. As far as we know, no other authors considered this problem until now. This eigenvalue problem generalizes the one for the Laplace - Beltrami operator Δ for functions defined on (M,g), see e.g. [5], [2], [6] and [19]. In our work [10] we proved some first eigenvalue estimates for τ. In these studies, the case where the model mapping w is harmonic is the most close to the real valued functions case, where the model mapping is constant.

We have to point out that there exist studies for the spectral theory for the nonlinear p - Laplace, which also generalize the case of Δ, see e.g. [16]. There are also many other studies on general nonlinear eigenvalue problems related to Δ, but in the framework of Banach spaces.

One field for the application of the eigenvalue problems related to Δ is the vibration theory. See e.g. [4] for many other applications in ingeneering sciences.

For the eigenmappings of τ an important question is to know if they are unique, or minimize the functional E_λ, exactly as we treated the case $\lambda = 0$ in [11]. In [7] we have seen that when (N,h) has nonpositive sectional curvature and $\lambda \leq 0$, then E_λ is convex (see below for the definition) and the uniqueness problem has the same answers as in the work of Hartman for harmonic mappings. When one removes the nonpositive sectional curvature assumption on (N,h), then E_λ is no more expected to be convex even when $\lambda \leq 0$.

With the above notations: for $v :]-1,1[\times M \to N$ a smooth mapping such that $v(0,.) = u$, and $t \mapsto d^2(v(t,x), w(x))$ is derivable, for any $x \in M$, we have

$$\frac{d}{dt}|_{t=0}\, E_\lambda(v(t,.)) = -\int_M \left\langle \tau(u)(x) - \lambda \exp^{-1}_{u(x)} w(x), V(x) \right\rangle dx,$$

and for

$$\tau(u)(x) - \lambda \exp^{-1}_{u(x)} w(x) = 0,$$

we have

$$\begin{aligned}\frac{d^2}{dt^2}|_{t=0}\, E_\lambda(v(t,.)) \quad &= \quad \int_M [\|\nabla V\|^2(x) - \left\langle R^N(V, d_{e_i}u)d_{e_i}u, V \right\rangle (x) \\ &\quad + \lambda \left\langle \nabla_{V(x)} \exp^{-1}_{.} w(x), V(x) \right\rangle]dx \\ &\quad - \int_M \left\langle \nabla_t \frac{\partial v(t,x)}{\partial t}|_{t=0}, \tau(u)(x) - \lambda \exp^{-1}_{u(x)} w(x) \right\rangle dx,\end{aligned}$$

and

$$\begin{aligned}\frac{d^2}{dt^2}|_{t=0}\, E_\lambda(v(t,.)) \quad &= \quad \int_M [\|\nabla V\|^2(x) - \left\langle R^N(V, d_{e_i}u)d_{e_i}u, V \right\rangle (x) \\ &\quad + \lambda \left\langle \nabla_{V(x)} \exp^{-1}_{.} w(x), V(x) \right\rangle]dx.\end{aligned}$$

We have also

$$\frac{\partial^2}{\partial r \partial s}|_{r=s=0}\, E(v_{r,s}) =$$

$$\begin{aligned}&- \int_M \left\langle \nabla_{e_i}\nabla_{e_i} V - \nabla_{\nabla_{e_i}e_i} V + R^N(V, d_{e_i}u)d_{e_i}u - \lambda \nabla_V \exp^{-1}_{.} w(x), W \right\rangle (x)dx \\ &- \int_M \left\langle \nabla_r \frac{\partial v_{r,s}}{\partial s}|_{r=s=0}, \tau(u) - \lambda \exp^{-1}_{u(x)} w(x) \right\rangle (x)dx \\ = \quad &- \int_M \left\langle \nabla_{e_i}\nabla_{e_i} V - \nabla_{\nabla_{e_i}e_i} V + R^N(V, d_{e_i}u)d_{e_i}u - \lambda \nabla_V \exp^{-1}_{.} w(x), W \right\rangle (x)dx.\end{aligned}$$

It is clear from these formula that: for $Riem^{(N,h)} \leq 0$ and $\lambda \leq 0$, we have

$$\frac{d^2}{dt^2}|_{t=0}\, E_\lambda(v(t,.)) \geq 0, \forall t.$$

So $t \mapsto E_\lambda(v(t,.))$ is convex.

V is called a *Jacobi field* along u, for $\tau(u) - \lambda \exp^{-1}_u w = 0$, when

$$\nabla_{e_i}\nabla_{e_i} V - \nabla_{\nabla_{e_i}e_i} V + R^N(V, d_{e_i}u)d_{e_i}u - \lambda \nabla_V \exp^{-1}_{.} w = 0.$$

Let $K > 0$ (resp. $-K_1 < 0$) be an upper bound (resp. a lower bound) for the sectional curvature of (N,h). Then, we have

$$K_2 \|V\|^2 \leq \left\langle R^N(V, d_{e_i}u)d_{e_i}u, V\right\rangle \leq K_3 \|V\|^2,$$

where

$$K_2 := -K_1 \max_{x\in M} \|du\|^2(x), K_3 := K \max_{x\in M} \|du\|^2(x).$$

From e.g. p. 156 of [6] we have: for $d(x,y) < \frac{\pi}{2\sqrt{K}}$, and $X \in T_yN$:

$$D^2[\frac{1}{2}d^2(a,.)](y)(X,X) = -\left\langle \nabla_X \exp_.^{-1} a, X\right\rangle,$$

and

$$\sqrt{K}d(a,y)\cot(\sqrt{K}d(a,y))\|X\|^2 \leq -\left\langle \nabla_X \exp_.^{-1} a, X\right\rangle \leq \sqrt{K}d(a,y)\coth(\sqrt{K}d(a,y))\|X\|^2. \quad (1.1)$$

It follows

$$-M_2\|V\|^2 \geq \left\langle \nabla_{V(x)} \exp_.^{-1} w(x), V(x)\right\rangle \geq -M_1\|V\|^2,$$

where

$$M_1 := \max_{x\in M} \sqrt{K_1}d(u(x),w(x))\coth[\sqrt{K_1}d(u(x),w(x))]$$

and

$$M_2 := \min_{x\in M} \sqrt{K}d(u(x),w(x))\cot[\sqrt{K}d(u(x),w(x))].$$

For $\lambda \leq 0$ we then get

$$\frac{d^2}{dt^2}|_{t=0} E_\lambda(v(t,.)) \geq \int_M [\|\nabla V\|^2 - (\lambda M_2 + K_3)\|V\|^2](x)dx,$$

and for $\lambda \geq 0$ we have

$$\frac{d^2}{dt^2}|_{t=0} E_\lambda(v(t,.)) \geq \int_M [\|\nabla V\|^2 - (\lambda M_1 + K_3)\|V\|^2](x)dx.$$

From that, one can see that in some simple cases the second derivative is nonnegative. For instance, $\lambda \geq 0$ and V such that

$$(C) \qquad \int_M \|\nabla V\|^2(x)dx \geq (\lambda M_1 + K_3)\int_M \|V\|^2(x)dx,$$

we have

$$\frac{d^2}{dt^2}|_{t=0} E_\lambda(v(t,.)) \geq 0.$$

One can remark that (C) is a Poincaré type inequality.

If one assumes that the upper bound K of the sectional curvature of (N,h) is < 0, and that u has rank ≥ 2, then one has

$$\left\langle R^N(V, d_{e_i}u)d_{e_i}u, V\right\rangle \leq K_3'\|V\|^2 \text{ with } K_3' := \frac{1}{2}K\min_{x\in M}\|du\|^2(x).$$

We conjecture that, there exists $c > 0$ such that: for any λ and u such that $rank(u) \geq 2$ and $\tau(u) - \lambda \exp_u^{-1} w = 0$, we have $\|du\|^2(x) \geq c, \forall x \in M$.

Assuming that, that is true and $K < 0$, $\lambda \geq 0$, we get

$$\frac{d^2}{dt^2}|_{t=0}\, E_\lambda(v(t,.)) \geq \int_M [\|\nabla V\|^2 - (\lambda M_1 + K_3')\|V\|^2](x)dx \geq \int_M [\|\nabla V\|^2 - (\lambda M_1 + \frac{1}{2}Kc)\|V\|^2](x)dx.$$

It follows that,

$$\frac{d^2}{dt^2}|_{t=0}\, E_\lambda(v(t,.)) \geq 0, \text{ for } 0 \leq \lambda \leq -\frac{Kc}{2I'(w)},$$

where $I'(w) := \min_{x\in M} inj(w(x)) > 0$.

Let's point out that, when the model mapping is energy minimizing, we found in [10] a lower bound for those $\lambda > 0$ which have an eigenmapping different from w. But, not all harmonic mappings are energy minimizing ones.

In this work we are only interested in integrable Jacobi fields along the eigenmapping u, i.e. those $V \in \Gamma(u^{-1}(TN))$ such that, there exists $v : [-1,1]\times M \to N$ a smooth mapping such that $v(0,.) = u$,for any $t \in [-1,1]$, $v(t,.)$ satisfies

$$\tau(v(t,.)) - \lambda \exp^{-1}_{v(t,.)} w = 0,$$

and

$$V(x) = \frac{\partial v(t,x)}{\partial t}|_{t=0}, \forall x \in M.$$

Those Jacobi fields are the most important ones.

2 Definitions and results

2.1 Definitions

Let (M,g) and (N,h) be two smooth Riemannian manifolds,.

Let us suppose that (N,h) is isometrically embedded into some Euclidean space $\mathbb{R}^k$. Then

$$W^{1,2}(M,N) := \{v \in W^{1,2}(M,\mathbb{R}^k) \ / \ v(x) \in N \text{ for a.e. } x \in M\}$$

where $W^{1,2}(M,\mathbb{R}^k)$ is the usual Sobolev space of all maps in $L^2(M,\mathbb{R}^k)$ whose derivative in the sense of distributions is a square integrable function.

Let $u \in W^{1,2}(M,N)$.

2.1.1 R^N is the curvature form of (N,h). $Riem^{(N,h)}$ is the sectional curvature tensor of (N,h), and $Ric^{(N,h)}$ is the Ricci curvature tensor of (N,h).

2.1.2 (N,h) is called a *symmetric Riemannian manifold* when : for any $a \in N$, there exits σ_a an isometry of (N,h) such that $\sigma_a(a) = a$ and $d\sigma_a(a) = -id_{T_aN}$.

2.1.3 Let $a,b \in N$ be such that $d(a,b) < \min\{\frac{\pi}{\sqrt{K}}, inj(a)\}$, where d is the *Riemannian distance function*, $inj(a)$ is the injectivity radius of (N,h) at a, and $K > 0$ is an upper bound for the sectional curvature of (N,h). Then $P_{a,b}$ is the *parallel transport* from a to b along the unique geodesic going from a to b.

2.1.4 The *energy* of u is $E(u) := \frac{1}{2}\int_M \|du(x)\|^2\, dx$, where du is the derivative of u in the sense of distributions, and for $\lambda \in \mathbb{R}$ and $w \in W^{1,2}(M,N)$, we set

$$E_\lambda(u) := E(u) - \lambda\frac{1}{2}\int_M d^2(u(x),w(x))dx.$$

w is called the *model* of E_λ.

2.1.5 u is called an *eigenmapping of the tension field* τ associated to λ, when it is a critical point of E_λ i.e. : for any *variation* of v i.e. any map $v : [-1,1]\times M \to N$ such that $v(0,.) = u, t \mapsto v(t,x)$ is C^1 $\forall x \in M$, and $x \mapsto v(t,x) \in W^{1,2}(M,N)$, for any $t \in [-1,1]$, is such that $t \mapsto v(t,x)$ is constant for $x \in \partial M$, we have

$$\frac{d}{dt}E_\lambda(v(t,.))|_{t=0} = 0.$$

It is well known that if w is C^0 and

$$d(w(x),u(x)) < \min\{\frac{\pi}{\sqrt{K}}, inj(w(x))\},$$

for any $x \in M$, then u is an eigenmapping of τ w.r.t. λ iff u is a weak solution of

$$\tau(u) - \lambda \exp_u^{-1} w = 0,$$

where $\tau(u)$ =trace (∇du) and in local coordinates

$$\tau(u)^\alpha(x) = g^{ij}(x)[\frac{\partial^2 u^\alpha}{\partial x^i \partial x^j}(x) - {}^M\Gamma_{ij}^k(x)\frac{\partial u^\alpha}{\partial x^k}(x) + \frac{\partial u^\beta}{\partial x^i}(x)\frac{\partial u^\delta}{\partial x^j}(x)\, {}^N\Gamma_{\beta\delta}^\alpha(u(x))],$$

where ${}^N\Gamma_{\beta\delta}^\alpha$ is the Christofell symbol of (N,h) in the considered local coordinates system.

When u is an eigenmapping of associated to $\lambda = 0$, one says that u is *(weakly) harmonic*.

2.1.6 We will say that a functional F defined on $W^{1,2}(M,N)$ is *convex* (resp. *strictly convex*) when : for any $v : [0,1]\times M \to N$ a map such that, $\forall x \in M, t \mapsto v(t,x)$ is a minimizing geodesic, and $v(t,x) \in W^{1,2}$,for any $t \in [0,1]$, we have

$$F(v(t,.)) \leq \text{ (resp. } < \text{) } (1-t)F(v(0,.)) + tF(v(1,.)), \forall t \in [0,1].$$

2.1.7 We shall say that the *Poincaré inequality* is satisfied on (M,g) when: There exists $C'_P > 0$ depanding only on M such that: for any $\xi \in W_0^{1,2}(M)$ such that $\xi(x_0) = 0$ for some $x_0 \in M \cup \partial M$, we have

$$\int_M \xi^2(x)dx \leq C'_P \int_M \|d\xi\|^2(x)dx.$$

For instance, if $(M,g) = (B(x_0,r),g)$ is such that

$$r < \min\{inj(x_0), \frac{\pi}{2\sqrt{K_M}}\}$$

where $K_M > 0$ is an upper bound of $Riem^{(M,g)}$, then using the fact that $\exp_{x_0}$ is a diffeomorphism of $B(0,r) \subseteq T_{x_0}M$ onto $B(x_0,r)$, one can see easily that the Poincaré inequality is satisfied on (M,g) and on $\Gamma_0(u^{-1}(TN))$. More precisely:

Let $V \in \Gamma_0(u^{-1}(TN))$ vanish at x_0. Let $\gamma : [0,1] \to B(x_0,r)$ be the geodesic from x_0 to some point of ∂M. Let $(e_i)_{1\leq i\leq m}$ be an orthonormal frame which is parallel along γ. Then

$$\forall t \in [0,1], \langle V(\gamma(t)), e_i(\gamma(t))\rangle = \int_0^t \frac{d}{ds}\langle V(\gamma(s)), e_i(\gamma(s))\rangle ds = \int_0^t \left\langle \nabla_{\dot\gamma(s)}V, e_i(\gamma(s))\right\rangle ds,$$

so

$$\|V(\gamma(t))\|^2 \leq r\int_0^1 \|\nabla V\|^2(\gamma(s))\|\dot\gamma(s)\|ds.$$

Integrating this last inequality gives the result.

2.1.8 For $u \in C^2(M,N)$, $u^{-1}(TN)$ is the pullback vector bundle, and $\Gamma_0(u^{-1}(TN))$ is the set of those of its $W^{1,2}$ - sections which vanish at some point $x_0 \in M \cup \partial M$. Let's assume that the Poincaré inequality is satisfied on (M,g). Then, for any $W \in \Gamma_0(u^{-1}(TN))$, by setting $W = \|W\|\, V$ and applying the Poincaré inequality to $\|W\|$, we have

$$\int_M \|W\|^2(x)dx \leq C'_P \int_M \|\nabla W\|^2(x)dx.$$

Assuming that we have: for $K_3 \leq \frac{1}{2C'_P}$ and $C'_P < \frac{1}{\lambda M_1 + K_3}$, i.e. $\lambda M_1 + K_3 < \frac{1}{C'_P}$, one has that (C) is satisfied. It follows the following convexity result for E_λ when $\partial M \neq \emptyset$.

Lemma 2.1. *Let's assume that the Poincaré inequality is true on (M,g), and that u is an eigenmapping of τ associated to $\lambda \geq 0$ such that*

$$d(u(x),w(x)) < \min\{inj(w(x)), \frac{\pi}{2\sqrt{K}} - \alpha\}, \forall x \in M,$$

for some fixed $\alpha \in]0, \frac{\pi}{2\sqrt{K}}[$.
Assuming that $K_3 \leq \frac{1}{2C'_P}$ and $\lambda < \frac{1}{M_3}[\frac{1}{C'_P} - K_3]$, where $M_3 = \min\{\max_{x\in M} inj(w(x)), \frac{\pi}{2} - \alpha\sqrt{K}\}$, there is no non trivial Jacobi field in $\Gamma_0(u^{-1}(TN))$.

2.1.9 One says that u is a minimizer of E_λ when, for any $v \in W^{1,2}(M,N)$ homotopic to u (i.e. $\exists H : [0,1] \times M \to N$ continuous in t such that $H(t,.) \in W^{1,2}(M,N)$,for any $t \in [0,1]$, $H(0,.) = u$ and $H(1,.) = v$) such that $v_{|\partial M} = u_{|\partial M}$ whenever $\partial M \neq \emptyset$, we have $E_\lambda(u) \leq E_\lambda(v)$.

2.1.10 Let u be an eigenmapping of τ associated to some $\lambda \geq 0$ and w. Then, for any $W \in \Gamma_0(u^{-1}(TN))$ we have

$$\int_M \left\langle \tau(u) - \lambda \exp_u^{-1} w, W \right\rangle dx = 0.$$

By taking $W = \exp_u^{-1} w$, easy computations using the fact that

$$|\langle a,b \rangle| \leq \frac{\Lambda_2}{2}\|a\|^2 + \frac{1}{2\Lambda_2}\|b\|^2$$

give us

$$\frac{\Lambda_2}{2}E(u) \leq \lambda \int_M d^2(u(x),w(x))dx + \frac{1}{2\Lambda_2}\int_M \left\|D_{dw}(\exp_u^{-1}.)\right\|^2(x)dx. \tag{2.1}$$

At page 25 of [14], by using regularity theory arguments, we proved the existence of two positive constants c_1 and c_2 (depending only on geometric data) such that:

$$\forall x \in M, \|du\|^2(x) \leq c_1\lambda + c_2 E(u).$$

It then follows: $\exists c_3 > 0$ and $c_4 > 0$ such that

$$\forall x \in M, \|du\|^2(x) \leq c_3\lambda + c_4. \tag{2.2}$$

Let's assume that the boundary ∂M of M is non void, and let $\vec{n}$ be a unit normal vectorfield to ∂M. We say that $u \in W^{1,2}(M,N)$ satisfies the Neuman boundary condition, when $du.\vec{n} = 0$.

2.1.11 We want to include here a lower bound result for the eigenvalues of τ.

Lemma 2.2. *Let u be an eigenmapping of τ associated to $\lambda \in \mathbb{R}$ such that*

$$d(u(x), w(x)) < inj(w(x)), \forall x \in M,$$

and

$$\int_M \left\langle D_{d_{e_i} w(x)}(\exp^{-1}_{u(x)} .), d_{e_i} u(x) \right\rangle dx \geq 0,$$

where $(e_i)_{1 \leq i \leq m}$ is a local frame of orthonormal basis of (M, g). Let $I(w) := \max_{x \in M} inj(w(x)) < +\infty$. Assume that $-K_1 \leq Riem^{(N,h)} \leq 0$, $M \cup \partial M$ is compact and either M is closed, or u satisfies the Neuman boundary condition. Then:

$$1^\circ)\ \textit{If } Ric^{(M,g)} \geq c > 0 \textit{ and } 0 \leq \lambda I(w) \sqrt{K_1} \coth[I(w) \sqrt{K_1}] < c,$$

or

$$2^\circ)\ \textit{If }\ Ric^{(M,g)} \geq 0 \textit{ and } \lambda < 0,$$

we have that u is constant.

In particular, when $(N, h) = \mathbb{R}$, w is constant, $Ric^{(M,g)} \geq c > 0$ and $\lambda < c$, then $\Delta u = -\lambda u \Rightarrow u$ is constant.

Proof. From the classical Bochner formula we have

$$\Delta \|du\|^2 = 2 \|\nabla(du)\|^2 - \left\langle R^N(d_{e_i} u, d_{e_j} u) d_{e_j} u, d_{e_i} u \right\rangle + \left\langle du.Ric^M(e_j), d_{e_j} u \right\rangle + \left\langle \nabla_{e_j} \nabla_{e_i} d_{e_i} u, d_{e_j} u \right\rangle.$$

By taking normal coordinates centered at the point we are computing, we get

$$\nabla_{e_j} \nabla_{e_i} d_{e_i} u = \nabla_{e_j} \tau(u) = \lambda \nabla_{e_j}(\exp^{-1}_{u(.)} w(.)) = \lambda[\nabla_{d_{e_j} u}(\exp^{-1}_{.} w) + D_{d_{e_j} w}(\exp^{-1}_{u} .)].$$

It follows:

$$\begin{aligned} \text{for } \lambda \quad &\geq \quad 0, \Delta \|du\|^2 \geq \lambda \left\langle \nabla_{d_{e_j} u}(\exp^{-1}_{.} w), d_{e_j} u \right\rangle + c \|du\|^2 + \left\langle D_{d_{e_i} w(x)}(\exp^{-1}_{u(x)} .), d_{e_i} u(x) \right\rangle \\ &\geq \quad [c - \lambda I(w) \sqrt{K_1} \coth[I(w) \sqrt{K_1}]] \|du\|^2 + \left\langle D_{d_{e_i} w(x)}(\exp^{-1}_{u(x)} .), d_{e_i} u(x) \right\rangle \end{aligned}$$

and

$$\text{for } \lambda < 0, \Delta \|du\|^2 \geq -\lambda \|du\|^2 + \left\langle D_{d_{e_i} w(x)}(\exp^{-1}_{u(x)} .), d_{e_i} u(x) \right\rangle.$$

Since

$$\int_M \Delta(\|du\|^2)(x) dx = 0,$$

we get the result

Lemma 2.3. *Let's assume that $\partial M \neq \emptyset$, and that the Poincaré inequality is satisfied on (M, g). Let $c_3 > 0$ and $c_4 > 0$ be the geometric constants in (2.2). We assume that $KC'_P c_4 \in [0, 1[$.*

Let u be an eigenmapping of τ associated to some $\lambda \geq 0$ with model mapping w. Then, for

$$\lambda < \frac{1 - KC'_P c_4}{C'_P(Kc_3 + \Lambda_1)},$$

any Jacobi field along u vanishes identically.

Proof. We have

$$W = fV,\ \nabla_{e_i} W = (d_{e_i} f)V + f\nabla_{e_i} V\ ,\ \|\nabla W\|^2 = \|df\|^2 + f^2\,\|\nabla V\|^2\,. \tag{2.3}$$

We have also $d_{e_i} f = \langle \nabla_{e_i} W, V\rangle$, and then

$$\|df\|^2 = \langle \nabla_{e_i} W, V\rangle^2 \leq \|\nabla W\|^2\,.$$

W is a Jacobi field is equivalent to

$$\nabla_{e_i}\nabla_{e_i} W - \nabla_{\nabla_{e_i} e_i} W + R^N(W, d_{e_i}u)d_{e_i}u - \lambda\nabla_W \exp_{\cdot}^{-1} w = 0.$$

By taking W as test function, we get

$$\begin{aligned}\int_M \|\nabla W\|^2\,dx &= \int_M [\left\langle R^N(W, d_{e_i}u)d_{e_i}u, W\right\rangle - \lambda\left\langle \nabla_W \exp_{\cdot}^{-1} w, W\right\rangle]dx\\ &\leq [K(c_3\lambda + c_4) + \lambda\Lambda_1]\int_M f^2(x)dx.\end{aligned}$$

So, from the Poincaré inequality and (2.3) we get

$$\frac{1}{C'_P}\int_M f^2 dx \leq \int_M \|df\|^2\,dx \leq [K(c_3\lambda + c_4) + \lambda\Lambda_1]\int_M f^2(x)dx.$$

This gives the result.

Lemma 2.4. *We assume $K < 0$, rank$(u) \geq 0$, where u is an eigenmapping of τ associated to some $\lambda \geq 0$. Then, if*

$$\lambda \leq -\frac{K}{2\Lambda_1}\min_{x\in M}\|du\|^2(x),$$

any Jacobi field along u is parallel.

Proof. With the same notations as in the proof just before, we have

$$\begin{aligned}\int_M \|\nabla W\|^2\,dx &= \int_M [\left\langle R^N(W, d_{e_i}u)d_{e_i}u, W\right\rangle - \lambda\left\langle \nabla_W \exp_{\cdot}^{-1} w, W\right\rangle]dx\\ &\leq \int_M [K\left\|V \wedge d_{e_i}u\right\|^2(x) + \lambda\Lambda_1]\,\|W\|^2\,dx\\ &\leq \int_M [\frac{1}{2}K\,\|du\|^2(x) + \lambda\Lambda_1]\,\|W\|^2\,dx \leq 0\end{aligned}$$

Remark 2.5. (N,h) is not assumed symmetric in Lemma i for $i \in \{1,2,3,4\}$. We included these results here since they cannot constitute a separate paper.

2.2 Results

Theorem 2.6. *Let (M,g) be a compact Riemannian manifold, and (N,h) a symmetric compact Riemannian manifold such that $0 < K_1 \leq Riem^{(N,h)} \leq K$. Let $w \in C^2(M,N)$, $\lambda \in \mathbb{R}$, and E_λ be defined with model w. Let $u \in W^{1,2}(M,N)$ and $[0,1]\times M \ni (t,x) \mapsto v_t(x) := v(t,x) \in N$ a variation of u such that, for any $x \in M, t \mapsto v_t(x)$ is a geodesic , and, for some $\alpha \in]0, \frac{\pi}{2\sqrt{K}}[$ we have*

$$\max\{d(v_0(x), w(x)), d(v_1(x), w(x))\} < \min\{inj(w(x)), \frac{\pi}{2\sqrt{K}} - \alpha\}, \forall x \in M.$$

Let

$$\begin{aligned} V(t,x) \quad &: \quad = \left\| \frac{\partial v_t}{\partial t} |_{t=0} (x) \right\|^{-1} \frac{\partial v_t}{\partial t} |_{t=0} (x), \forall x \in M \textit{ such that } v_0(x) \neq v_1(x) \\ V(t,x) \quad &: \quad = 0_{T_{v_0(x)}N}, \textit{ when } v_0(x) = v_1(x) \\ \Lambda_1 \quad &: \quad = 2D\sqrt{K_1} \coth[2D\sqrt{K_1}], \end{aligned}$$

where

$$D := \frac{\pi}{2\sqrt{K}} - \alpha,$$

and

$$\Lambda_2 := (\pi - 2\sqrt{K}\alpha)\cot(\pi - 2\sqrt{K}\alpha).$$

Then:

$1°$) *if* $\lambda < 0$*, then any Jacobi field along an eigenmapping associated to* λ *vanishes identically;*

$2°$) *if* v_t *is an eigenmapping of* τ *associated to* $\lambda \geq 0$*, for any* $t \in [0,1]$*, we have*

$$\int_M d^2(v_0(x), v_t(x)) \|\nabla V(t,.)\|^2 (x) dx \leq \frac{\lambda}{\Lambda_2}(2+\Lambda_1) \int_M d^2(v_0(x), v_t(x)) dx, \forall t \in [0,1].$$

Remark 2.7. The assumption saying that, for any $x \in M, t \mapsto v_t(x)$ is a geodesic is not necessary for the conclusion in $1°$), as it can be seen from Corollary 2 in [11].

As in [11], the result of theorem 1 seems to be true when w in C^2 only outside some closed $A \subseteq M$ which has Hausdorff dimension $\leq \dim(M) - 2$.

Remark 2.8. As we have seen in Theorem 3 of [10], the assumption

$$\max\{d(v_0(x), w(x)), d(v_1(x), w(x))\} < \min\{inj(w(x)), \frac{\pi}{2\sqrt{K}}\}, \forall x \in M,$$

implies that, if w is class of C^k at some point $x_0 \in M$, then so does any critical mapping u of E_λ, for any $\lambda \in \mathbb{R}$. So we may suppose that all our mappings are C^2 like the model w.

Remark 2.9. Since our mappings are continuous and M is compact, the constant Λ_2 may be taken independent of w, v_0 and v_1. Furthermore, the case $\lambda = 0$ gives one result of [11].

Remark 2.10. It seems that we have to make the same assumptions on (N,h) in [11] as we do here.

3 Proofs of the results

3.1 Proof of the theorem

It is easy to see that we may suppose w.o.l.g that the length of the geodesic $t \mapsto v_t(x)$ is less than $\frac{\pi}{4\sqrt{K}}$, for any $x \in M$.

Step 1:

From the formula (3.1) of [11] we have: $\forall t \in [0,1]$,

$$\begin{aligned} \|dv_t\|^2(x) \quad \leq \quad & (1-t)\|dv_0\|^2(x) + t\|dv_1\|^2(x) - t(1-t)\|\nabla d(v_0(.), v_1(.))\|^2(x) \\ & -Kt(1-t)d^2(v_0(x), v_1(x)) \sum_{1\leq i\leq m} C_i(x) \left\|d_{e_i}v_0\right\|^2(x), \end{aligned}$$

where

$$C_i(x) = \frac{-1}{\sin^2 \sqrt{K} d(v_0(x), A_i(x))},$$

$A_i(x) := \lim_{s\to 0} A_i(x, x+te_i)$, $A_i(x,y)$ is one of the two points where, the two geodesics through $(v_0(x), v_1(x))$ and $(v_0(y), v_1(y))$ meet, by the representation of the quadrilateral $(v_0(x), v_0(y), v_1(y), v_1(x))$ on the two dimensional sphere of radius $\frac{1}{\sqrt{K}}$. $d(v_0(x), A_i(x))$ is actually the distance on this two - sphere. It follows that

$$\begin{aligned} E(v_t) &\leq (1-t)E(v_0) + tE(v_1) - t(1-t)\int_M \|\nabla d(v_0(.), v_1(.))\|^2(x)dx \qquad (3.1)\\ &\quad -Kt(1-t)\int_M d^2(v_0(x), v_1(x))\sum_{1\leq i\leq m} C_i(x)\left\|d_{e_i}v_0\right\|^2(x)dx \end{aligned}$$

One has also: $\forall t\in[0,1]$,

$$\begin{aligned} \|dv_t\|^2(x) &\leq (1-t)\|dv_0\|^2(x) + t\|dv_1\|^2(x) - t(1-t)\|\nabla d(v_0(.), v_1(.))\|^2(x) \qquad (3.2)\\ &\quad -Kt(1-t)d^2(v_0(x), v_1(x))\sum_{1\leq i\leq m} C_i'(x)\left\|d_{e_i}v_1\right\|^2(x), \end{aligned}$$

where

$$C_i'(x) = \frac{-1}{\sin^2\sqrt{K}d(v_1(x), -A_i(x))},$$

and $-A_i(x)$ is the opposite of $A_i(x)$ on the two sphere of $\mathbb{R}^3$. If $d(v_0(x), A_i(x)) \geq d(v_1(x), -A_i(x))$ we will consider (3.1), and if not we will consider (3.2). Since

$$d(v_0(x), A_i(x)) + d(v_0(x), v_1(x)) + d(v_1(x), -A_i(x)) = \frac{\pi}{\sqrt{K}},$$

for any $x\in M$ and any $1\leq i\leq m$, there exists some constant $\varepsilon > 0$ such that:

$$d_\infty(v_0, v_1) < \varepsilon \Rightarrow K_1 Q_i(x)[1+\ln d(v_0(x), v_1(x))] \geq -K\max\{C_i(x), C_i'(x)\}, \qquad (3.3)$$

for any $x\in M$ and any $1\leq i\leq m$, where $Q_i(x)$ appears in the formula (3.6) below.

From now on we may assume that $d_\infty(v_0, v_1) < \varepsilon$ by replacing v_1 by v_t, and w.l.o.g $\max\{C_i(x), C_i'(x)\} = C_i(x)$.

An easy computation gives us: $\forall x\in M$

$$\|\nabla d(u(.), v(.))\|^2(x) = \sum_{1\leq i\leq m} D_{e_i}[d(u(.), v(.))D_{e_i}d(u(.), v(.))](x)$$

$$\begin{aligned} &-\sum_{1\leq i\leq m} D_{e_i}[[\ln d(u(.), v(.))][\left\langle \exp^{-1}_{v(.)}u(.), d_{e_i}v(.)\right\rangle + \left\langle \exp^{-1}_{u(.)}v(.), d_{e_i}u(.)\right\rangle]](x)\\ &+\sum_{1\leq i\leq m}[1+\ln d(u(x), v(x))][D_{e_i}\left\langle \exp^{-1}_{v(.)}u(.), d_{e_i}v(.)\right\rangle + D_{e_i}\left\langle \exp^{-1}_{u(.)}v(.), d_{e_i}u(.)\right\rangle](x). \end{aligned}$$

The divergence theorem then gives

$$\begin{aligned} \int_M \|\nabla d(v_0(.), v_1(.))\|^2(x)dx &= \int_M [1+\ln d(v_0(x), v_1(x))]D_{e_i}[\left\langle \exp^{-1}_{v_1(.)}v_0(.), d_{e_i}v_1\right\rangle + \qquad (3.4)\\ &\quad \left\langle \exp^{-1}_{v_0(.)}v_1(.), d_{e_i}v_0\right\rangle](x)dx. \end{aligned}$$

Let $Z^i_{v_0}(x)$ and $Z^i_{v_1}(x)$ be defined by

$$\nabla^N_{Z^i_{v_0}(x)}\exp^{-1}_{.}v_1(x) = d_{e_i}v_0(x) \text{ and } \nabla^N_{Z^i_{v_1}(x)}\exp^{-1}_{.}v_0(x) = d_{e_i}v_1(x), \forall x\in M, 1\leq i\leq m.$$

Then, using the two lemmas in the appendix one has that

$$D_{e_i}[\left\langle \exp^{-1}_{v_1(.)} v_0(.), d_{e_i}v_1 \right\rangle + \left\langle \exp^{-1}_{v_0(.)} v_1(.), d_{e_i}v_0 \right\rangle](x) = \tag{3.5}$$

$$\begin{aligned}
= \; & \left\langle \nabla^N_{B_i(x)} \exp^{-1}_{.} v_1(x), B_i(x) \right\rangle + \left\langle \exp^{-1}_{v_1(x)} v_0(x), \tau(v_1)(x) \right\rangle + \left\langle \exp^{-1}_{v_0(x)} v_1(x), \tau(v_0)(x) \right\rangle \\
& + \left\langle R^N(Z^i_{v_0}(x), \exp^{-1}_{v_0(x)} v_1(x)) P_{v_1(x)v_0(x)} d_{e_i} v_1(x), \exp^{-1}_{v_0(x)} v_1(x) \right\rangle \\
& + \left\langle R^N(Z^i_{v_1}(x), \exp^{-1}_{v_1(x)} v_0(x)) P_{v_0(x)v_1(x)} d_{e_i} v_0(x), \exp^{-1}_{v_1(x)} v_0(x) \right\rangle \\
& + \left\langle \nabla^N_{\exp^{-1}_{v_0(x)} v_1(x)} \nabla^N_{Z^i_{v_0}(x)} [P_{v_1(.)v_0(.)} d_{e_i} v_1], \exp^{-1}_{v_0(x)} v_1(x) \right\rangle \\
& + \left\langle \nabla^N_{\exp^{-1}_{v_1(x)} v_0(x)} \nabla^N_{Z^i_{v_1}(x)} [P_{v_0(.)v_1(.)} d_{e_i} v_0], \exp^{-1}_{v_1(x)} v_0(x) \right\rangle
\end{aligned}$$

$$\begin{aligned}
= \; & \left\langle \nabla^N_{B_i(x)} \exp^{-1}_{.} v_1(x), B_i(x) \right\rangle + \lambda[\left\langle \exp^{-1}_{v_1(x)} v_0(x), \exp^{-1}_{v_1(x)} w(x) \right\rangle \\
& + \left\langle \exp^{-1}_{v_0(x)} v_1(x), \exp^{-1}_{v_0(x)} w(x) \right\rangle] + \Omega
\end{aligned}$$

since v_0 and v_1 are critical points of E_λ, where $B_i(x) = d_{e_i}v_0(x) - P_{v_1(x)v_0(x)} d_{e_i} v_1(x)$, and Ω denotes the last four terms.

We are going now to look at how things depend on $d_\infty(v_1, v_0)$: some kind of power series development.

We have

$$\left\langle R^N(Z^i_u(x), \exp^{-1}_{u(x)} v(x)) P_{v(x)u(x)} d_{e_i} v(x), \exp^{-1}_{u(x)} v(x) \right\rangle =$$

$$\begin{aligned}
= \; & \left\langle R^N(d_{e_i}u(x), \exp^{-1}_{u(x)} v(x)) d_{e_i} u(x), \exp^{-1}_{u(x)} v(x) \right\rangle \\
& + \left\langle R^N(Z^i_u(x) - d_{e_i}u(x), \exp^{-1}_{u(x)} v(x)) d_{e_i} u(x), \exp^{-1}_{u(x)} v(x) \right\rangle \\
& + \left\langle R^N(d_{e_i}u(x), \exp^{-1}_{u(x)} v(x)) [P_{v(x)u(x)} d_{e_i} v(x) - d_{e_i}u(x)], \exp^{-1}_{u(x)} v(x) \right\rangle \\
& + \left\langle R^N(Z^i_u(x) - d_{e_i}u(x), \exp^{-1}_{u(x)} v(x)) [P_{v(x)u(x)} d_{e_i} v(x) - d_{e_i}u(x)], \exp^{-1}_{u(x)} v(x) \right\rangle,
\end{aligned}$$

It follows: there exists $C(v_1(x), v_0(x)) \in \mathbb{R}$ such that $\|C(v_1(x), v_0(x))\|_\infty \to 0$ as $d_\infty(v_1, v_0) \to 0$, and

$$\begin{aligned}
& \left\langle R^N(Z^i_{v_0}(x), \exp^{-1}_{v_0(x)} v_1(x)) P_{v_1(x)v_0(x)} d_{e_i} v_1(x), \exp^{-1}_{v_0(x)} v_1(x) \right\rangle \\
= \; & \left\langle R^N(d_{e_i}v_0(x), \exp^{-1}_{v_0(x)} v_1(x)) d_{e_i} v_0(x), \exp^{-1}_{v_0(x)} v_1(x) \right\rangle \\
& + C(v_1(x), v_0(x)) d(v_1(x), v_0(x))^2.
\end{aligned}$$

In what will follow the functions $C(v_1(.), v_0(.))$ are not necessarily the same.

We have

$$\begin{aligned}
\nabla^N_{Z^i_{v_0}(x)} [P_{v_1(.)v_0(.)} d_{e_i} v_1] \; = \; & \nabla^N_{Z^i_{v_0}(x) - d_{e_i}v_0(x)} [P_{v_1(.)v_0(.)} d_{e_i} v_1 - d_{e_i} v_0] + \nabla^N_{Z^i_{v_0}(x) - d_{e_i}v_0(x)} d_{e_i} v_0 \\
& + \nabla^N_{d_{e_i}v_0(x)} [P_{v_1(.)v_0(.)} d_{e_i} v_1 - d_{e_i} v_0] + \nabla^N_{d_{e_i}v_0(x)} d_{e_i} v_0, \forall x, i.
\end{aligned}$$

All the terms converge to zero as $v_1(x) \to v_0(x)$, apart from $\nabla^N_{d_{e_i}v_0(x)} d_{e_i} v_0$.

Since v_0 is a critical point of E_λ we have in local coordinates $(y^\alpha)_\alpha$ on N that

$$\nabla^N_{d_{e_i}v_0(x)} d_{e_i}v_0 = \frac{\partial v_0^\beta}{\partial e_i}(x)\frac{\partial v_0^\gamma}{\partial e_i}(x)\Gamma^\alpha_{\beta\gamma}(v_0(x))\frac{\partial}{\partial y^\alpha}(v_0(x)) = -(\Delta v_0^\alpha)(x)\frac{\partial}{\partial y^\alpha}(v_0(x)) + \lambda \exp^{-1}_{v_0(x)} w(x).$$

So

$$\nabla^N_{\exp^{-1}_{v_0(x)} v_1(x)} \nabla^N_{d_{e_i}v_0(x)} d_{e_i}v_0 = -(\Delta v_0^\alpha)(x)\nabla^N_{\exp^{-1}_{v_0(x)} v_1(x)} \frac{\partial}{\partial y^\alpha} + \lambda\nabla^N_{\exp^{-1}_{v_0(x)} v_1(x)} \exp^{-1}_{v_0(x)} w(x) = 0$$

by choosing $(y^\alpha)_\alpha$ to be normal coordinates centered at the point we are computing.

Since

$$\begin{aligned} &\left\langle R^N(d_{e_i}v_0(x), \exp^{-1}_{v_0(x)} v_1(x))d_{e_i}v_0(x), \exp^{-1}_{v_0(x)} v_1(x)\right\rangle \\ \leq \quad & -K_1 Q_i(x)\left\|d_{e_i}v_0(x)\right\|^2 d^2(v_1(x), v_0(x)), \end{aligned}$$

for some generic $Q_i(x) > 0$ depending only on geometric data, the above computations give us

$$[1+\ln(d(v_1(x), v_0(x)))]D_{e_i}[\left\langle \exp^{-1}_{v_1(.)} v_0(.), d_{e_i}v_1\right\rangle + \left\langle \exp^{-1}_{v_0(.)} v_1(.), d_{e_i}v_0\right\rangle](x)$$

$$\begin{aligned} \geq \quad & [1+\ln(d(v_1(x), v_0(x)))]\left\langle \nabla^N_{B_i(x)} \exp^{-1}_{.} v_1(x), B_i(x)\right\rangle \\ & +K_1 Q_i(x)[1+\ln(d(v_1(x), v_0(x)))]d^2(v_1(x), v_0(x))[\left\|d_{e_i}v_0(x)\right\|^2 + \\ & \left\|d_{e_i}v_1(x)\right\|^2] + \lambda[1+\ln(d(v_1(x), v_0(x)))][\left\langle \exp^{-1}_{v_1(x)} v_0(x), \exp^{-1}_{v_1(x)} w(x)\right\rangle \\ & + \left\langle \exp^{-1}_{v_0(x)} v_1(x), \exp^{-1}_{v_0(x)} w(x)\right\rangle] + C(v_1(x), v_0(x))d(v_1(x), v_0(x))^2 \end{aligned} \tag{3.6}$$

$$\begin{aligned} \geq \quad & [1+\ln(d(v_1(x), v_0(x)))]\left\langle \nabla^N_{B_i(x)} \exp^{-1}_{.} v_1(x), B_i(x)\right\rangle \\ & -KQ_i(x)d^2(v_1(x), v_0(x))\left\|d_{e_i}v_0(x)\right\|^2 \\ & +\lambda[1+\ln(d(v_1(x), v_0(x)))][\left\langle \exp^{-1}_{v_1(x)} v_0(x), \exp^{-1}_{v_1(x)} w(x)\right\rangle \\ & + \left\langle \exp^{-1}_{v_0(x)} v_1(x), \exp^{-1}_{v_0(x)} w(x)\right\rangle] + C(v_1(x), v_0(x))d(v_1(x), v_0(x))^2, \text{ by (3.3)}, \end{aligned}$$

where we are assuming w.o.l.g $1+\ln(d(v_1(x), v_0(x))) < 0, \forall x \in M$. By putting this into (3.4) we get

$$\int_M \|\nabla d(v_0(.), v_1(.))\|^2(x)dx \geq \tag{3.7}$$

$$\begin{aligned} \geq \quad & \int_M [[1+\ln(d(v_1(x), v_0(x)))]\left\langle \nabla^N_{B_i(x)} \exp^{-1}_{.} v_1(x), B_i(x)\right\rangle \\ & -KQ_i(x)d^2(v_1(x), v_0(x))\left\|d_{e_i}v_0(x)\right\|^2 \\ & +\lambda[1+\ln(d(v_1(x), v_0(x)))][\left\langle \exp^{-1}_{v_1(x)} v_0(x), \exp^{-1}_{v_1(x)} w(x)\right\rangle \\ & + \left\langle \exp^{-1}_{v_0(x)} v_1(x), \exp^{-1}_{v_0(x)} w(x)\right\rangle] + C(v_1(x), v_0(x))d(v_1(x), v_0(x))^2]dx. \end{aligned}$$

Step 2:

From (1.1) we get the following convexity result for the L^2 - distance functional: $\forall t \in [0,1]$,

$$\begin{aligned}\int_M d^2(v_t(x),w(x))dx &\geq (1-t)\int_M d^2(v_0(x),w(x))dx+t\int_M d^2(v_1(x),w(x))dx\\ &\quad -t(1-t)\Lambda_1\int_M d^2(v_1(x),v_0(x))dx,\end{aligned}$$

and

$$\begin{aligned}\int_M d^2(v_t(x),w(x))dx &\leq (1-t)\int_M d^2(v_0(x),w(x))dx+t\int_M d^2(v_1(x),w(x))dx\\ &\quad -t(1-t)\Lambda_2\int_M d^2(v_1(x),v_0(x))dx.\end{aligned}$$

It follows from (3.1): $\forall t\in[0,1]$,

1°) For $\lambda\geq 0$,

$$\begin{aligned}E_\lambda(v_t) &\leq (1-t)E_\lambda(v_0)+tE_\lambda(v_1)-t(1-t)[\int_M \|\nabla d(v_0(.),v_1(.))\|^2(x)dx\\ &\quad +K\int_M d^2(v_0(x),v_1(x))\sum_{1\leq i\leq m}C_i(x)\left\|d_{e_i}v_0\right\|^2(x)dx-\lambda\Lambda_1\int_M d^2(v_1(x),v_0(x))dx];\end{aligned}$$

2°) For $\lambda<0$,

$$\begin{aligned}E_\lambda(v_t) &\leq (1-t)E_\lambda(v_0)+tE_\lambda(v_1)-t(1-t)[\int_M \|\nabla d(v_0(.),v_1(.))\|^2(x)dx\\ &\quad +K\int_M d^2(v_0(x),v_1(x))\sum_{1\leq i\leq m}C_i(x)\left\|d_{e_i}v_0\right\|^2(x)dx-\lambda\Lambda_2\int_M d^2(v_1(x),v_0(x))dx].\end{aligned}$$

Since v_0 and v_1 are critical points of E_λ, the function $t\mapsto E_\lambda(v_t)$ cannot be majorized by a strictly convex function, so we have:

1°) For $\lambda\geq 0$,

$$\int_M \|\nabla d(v_0(.),v_1(.))\|^2(x)dx\leq \tag{3.8}$$

$$-K\int_M d^2(v_0(x),v_1(x))\sum_{1\leq i\leq m}C_i(x)\left\|d_{e_i}v_0\right\|^2(x)dx+\lambda\Lambda_1\int_M d^2(v_1(x),v_0(x))dx,$$

and

2°) For $\lambda<0$,

$$\int_M \|\nabla d(v_0(.),v_1(.))\|^2(x)dx\leq$$

$$-K\int_M d^2(v_0(x),v_1(x))\sum_{1\leq i\leq m}C_i(x)\left\|d_{e_i}v_0\right\|^2(x)dx+\lambda\Lambda_2\int_M d^2(v_1(x),v_0(x))dx.$$

This together with (3.7) and (3.3) gives us:

1°) For $\lambda\geq 0$,

$$\left|\int_M[1+\ln(d(v_1(x),v_0(x)))]\left\langle\nabla^N_{B_i(x)}\exp_.^{-1}v_1(x),B_i(x)\right\rangle dx\right|\leq$$

$$C_1\int_M |1+\ln(d(v_1(x),v_0(x)))|d^2(v_1(x),v_0(x))[\|dv_0\|^2(x)+\|dv_1\|^2(x)]dx$$
$$+\lambda\int_M |1+\ln(d(v_1(x),v_0(x)))|[\left\langle \exp^{-1}_{v_0(x)} v_1(x), \exp^{-1}_{v_0(x)} w(x)\right\rangle$$
$$+\left\langle \exp^{-1}_{v_1(x)} v_0(x), \exp^{-1}_{v_1(x)} w(x)\right\rangle]dx + \int_M C(v_1(x),v_0(x))d(v_1(x),v_0(x))^2dx$$
$$+\lambda\Lambda_1\int_M d^2(v_1(x),v_0(x))dx,$$

and

2°) For $\lambda < 0$, we have the same formula, where this time the last term becomes

$$-\lambda\Lambda_2\int_M d^2(v_1(x),v_0(x))dx.$$

We conclude as in the appendix that

$$\lim_{d_{L_2}(v_0,v_1)\to 0}\int_M \left\|dv_0 - P_{v_1v_0}dv_1\right\|^2(x)dx = 0,$$

where $d_{L_2}(v_0,v_1) := [\int_M d^2(v_0(x),v(x)dx]^{1/2}$. In particular, we have

$$\forall x\in M,\ \lim_{d_{L_2}(v_0,v_1)\to 0}\left\|dv_0 - P_{v_1v_0}dv_1\right\|^2(x) = 0. \tag{3.9}$$

In fact: Let $0 < t_n \to 0$ and $v_n := v_{t_n}, \forall n\in\mathbb{N}$. Then the sequence $(v_n)_n$ admits a subsequence which converges in L^2 to v_0, since the sequence $(E(v_n))_n$ is bounded by (2.1).

Step 3:

We will prove the following claim later.

Claim 3.1. For $f(t) := \left\langle \exp^{-1}_{v_t(x)} v_0(x), \exp^{-1}_{v_t(x)} w(x)\right\rangle + \left\langle \exp^{-1}_{v_0(x)} v_t(x), \exp^{-1}_{v_0(x)} w(x)\right\rangle$, we have $f(0) = f'(0) = 0$ and $f''(0) = 2d^2(v_0(x),v_1(x))$.

Combinating this with (3.8), one gets:

1°) For $\lambda \geq 0$,

$$\int_M |1+\ln[d(v_1(x),v_0(x))]|[\sqrt{K}d(v_1(x),v_0(x))\cot[\sqrt{K}d(v_1(x),v_0(x))] \tag{3.10}$$
$$\|B_i(x)\|^2 - \lambda(2+\Lambda_1)]d^2(v_1(x),v_0(x))dx$$
$$\leq \int_M C(v_1(x),v_0(x))d^2(v_1(x),v_0(x))dx,$$

and

2°) For $\lambda < 0$,

$$\int_M |1+\ln[d(v_1(x),v_0(x))]|[\sqrt{K}d(v_1(x),v_0(x))\cot[\sqrt{K}d(v_1(x),v_0(x))]$$
$$\|B_i(x)\|^2 - \lambda(2+\Lambda_2)]d^2(v_1(x),v_0(x))dx$$
$$\leq \int_M C(v_1(x),v_0(x))d^2(v_1(x),v_0(x))dx.$$

By studying the function $t \mapsto \left\| P_{v_0(x)v_t(x)} d_{e_i} v_0(x) - d_{e_i} v_t(x) \right\|^2$, we get

$$\frac{\partial^2}{\partial t^2} \left\| P_{v_0(x)v_t(x)} d_{e_i} v_0(x) - d_{e_i} v_t(x) \right\|^2 = \tag{3.11}$$

$$\begin{aligned} &= -2 \langle \nabla_t \nabla_t d_{e_i} v_t(x), P_{v_0(x)v_t(x)} d_{e_i} v_0(x) - d_{e_i} v_t(x) \rangle + 2 \left\| \nabla_{e_i} \frac{\partial v_t}{\partial t}(x) \right\|^2 \\ &= -2 \left\langle R^N \left(\frac{\partial v_t}{\partial t}(x), d_{e_i} v_t(x)\right) \frac{\partial v_t}{\partial t}(x), P_{v_0(x)v_t(x)} d_{e_i} v_0(x) - d_{e_i} v_t(x) \right\rangle \\ &\quad + 2 \left\| \nabla_{e_i} \frac{\partial v_t}{\partial t}(x) \right\|^2 \end{aligned}$$

since $\nabla_t \frac{\partial v_t}{\partial t}(x)$ vanishes. We have

$$\frac{\partial v_t}{\partial t}(x) = P_{v_0(x)v_t(x)} \exp^{-1}_{v_0(x)} v_1(x) = d(v_0(x), v_1(x)) V(t,x),$$

and then

$$\nabla_{e_i} \frac{\partial v_t}{\partial t}(x) = D_{e_i}[d(v_0(.), v_1(.))](x) V(t,x) + d(v_0(x), v_1(x)) \nabla_{e_i} V(t,x).$$

Since $\|V(t,x)\| \equiv 1$, we have that $\langle V(t,x), \nabla_{e_i} V(t,x) \rangle = 0$ and then

$$\left\| \nabla_{e_i} \frac{\partial v_t}{\partial t}(x) \right\|^2 \geq d^2(v_0(x), v_1(x)) \left\| \nabla_{e_i} V(t,x) \right\|^2, \forall x, t.$$

Since $V(t,x) = P_{v_0(x)v_t(x)} V(0,x)$, we have

$$\nabla_{e_i} V(t,x) = (\nabla_{d_{e_i} v_0(x)} P_{.v_t(x)}) V(0,x) + (\nabla_{d_{e_i} v_t(x)} P_{v(x).}) V(0,x) + P_{v_0(x)v_t(x)} \nabla_{e_i} V(0,x).$$

It follows

$$\begin{aligned} \left\| \nabla_{e_i} V(t,x) \right\|^2 &= \left\| \nabla_{e_i} V(0,x) \right\|^2 + \left\| (\nabla_{d_{e_i} v_t(x)} P_{0(x).}) V(0,x) \right\|^2 + \\ &\quad \left\| (\nabla_{d_{e_i} v_0(x)} P_{.v_t(x)}) V(0,x) \right\|^2 + \\ &\quad 2 \left\langle (\nabla_{d_{e_i} v_0(x)} P_{.v_t(x)}) V(0,x), (\nabla_{d_{e_i} v_t(x)} P_{v_0(x).}) V(0,x) \right\rangle \\ &\quad + 2 \left\langle (\nabla_{d_{e_i} v_0(x)} P_{.v_t(x)}) V(0,x), P_{v_0(x)v_t(x)} \nabla_{e_i} V(0,x) \right\rangle \\ &\quad + 2 \left\langle (\nabla_{d_{e_i} v_t(x)} P_{v_0(x).}) V(0,x), P_{v_0(x)v_t(x)} \nabla_{e_i} V(0,x) \right\rangle. \end{aligned}$$

It follows from the lemmas in the appendix that

$$\left\| \nabla_{e_i} V(t,x) \right\|^2 \geq \frac{1}{2} \left\| \nabla_{e_i} V(0,x) \right\|^2 - C_2 d^2(v_0(x), v_t(x)) [\left\| d_{e_i} v_0(x) \right\|^2 + \left\| d_{e_i} v_t(x) \right\|^2].$$

Since $f(0) = f'(0)$ we have $\int_0^1 \int_0^t f''(s) ds dt = f(1)$, and then (3.11) and (3.9) give us

$$\|B_i(x)\|^2 \geq d^2(v_1(x), v_0(x)) \left\| \nabla_{e_i} V(0,.)(x) \right\|^2 + C(v_1(x), v_0(x)) d^2(v_1(x), v_0(x)), \forall x \in M.$$

So (3.10) gives, for $\lambda \geq 0$

$$\begin{aligned} &\int_M |1 + \ln[d(v_1(x), v_0(x))]| \, [\Lambda_2 \|\nabla V(0,.)(x)\|^2 - \lambda(2 + \Lambda_1)] d^2(v_1(x), v_0(x)) dx \\ \leq &\int_M C(v_1(x), v_0(x)) d^2(v_1(x), v_0(x)) dx, \end{aligned}$$

and for $\lambda < 0$ we have

$$\begin{aligned}
&\int_M |1+\ln[d(v_1(x),v_0(x))]|\,[\Lambda_2\,\|\nabla V(0,.)(x)\|^2-\lambda(2+\Lambda_2)]d^2(v_1(x),v_0(x))dx \\
\leq\ &\int_M C(v_1(x),v_0(x))d^2(v_1(x),v_0(x))dx.
\end{aligned}$$

By putting v_t at the place of v_1 and dividing both sides of the inequality by t^2, one gets: For $B=\Lambda_1$ or Λ_2 depending on the sign of λ,

$$\begin{aligned}
&\int_M |1+\ln t+\ln[d(v_1(x),v_0(x))]|\,[\Lambda_2\,\|\nabla V(0,.)(x)\|^2-\lambda(2+B)]d^2(v_1(x),v_0(x))dx \\
\leq\ &\int_M C_t(v_1(x),v_0(x))d^2(v_1(x),v_0(x))dx,
\end{aligned}$$

where $\lim\limits_{t\to 0}|\ln t|^{-1}\,\|C_t(v_1(x),v_0(x))\|_\infty = 0$.

The compactness of (N,h) gives: $\exists t_0$ such that: $\forall t\leq t_0$ we have

$$|\ln t|\leq|1+\ln t+\ln[d(v_1(x),v_0(x))]|\leq 3\,|\ln t|,\ \forall x\in M.$$

It follows: $\forall t\leq t_0$

$$\begin{aligned}
&\int_M [\Lambda_2\,\|\nabla V(0,.)(x)\|^2-\lambda(2+B)]d^2(v_1(x),v_0(x))dx \\
\leq\ &|\ln t|^{-1}\int_M C_t(v_1(x),v_0(x))d^2(v_1(x),v_0(x))dx.
\end{aligned}$$

Taking the limit as $t\to 0$ we get

$$\int_M [\Lambda_2\,\|\nabla V(0,.)(x)\|^2-\lambda(2+B)]d^2(v_1(x),v_0(x))dx\leq 0.$$

It follows that the case $\lambda<0$ is impossible, and that for $\lambda\geq 0$ one has:

$$\int_M \|\nabla V(0,.)(x)\|^2\,d^2(v_1(x),v_0(x))dx\leq\frac{\lambda(2+\Lambda_1)}{\Lambda_2}\int_M d^2(v_1(x),v_0(x))dx.$$

By again replacing v_1 by v_t one gets the announced result in the theorem.

It remains now to prove the claim.

<u>Proof of the Claim</u>:

We have: $\forall t\in[0,1]$,

$$\begin{aligned}
f'(t) &= \left\langle\frac{d}{dt}\exp^{-1}_{v_0(x)}v_t(x),\exp^{-1}_{v_0(x)}w(x)\right\rangle+\left\langle\nabla_t\exp^{-1}_{v_t(x)}v_0(x),\exp^{-1}_{v_t(x)}w(x)\right\rangle \\
&\quad+\left\langle\exp^{-1}_{v_t(x)}v_0(x),\nabla_t\exp^{-1}_{v_t(x)}w(x)\right\rangle \\
&= \left\langle D(\exp^{-1}_{v_0(x)}.)(v_t(x)).\dot v_t(x),\exp^{-1}_{v_0(x)}w(x)\right\rangle+\left\langle\nabla^N(\exp^{-1}_{.}v_0(x)).\dot v_t(x),\exp^{-1}_{v_t(x)}w(x)\right\rangle \\
&\quad+\left\langle\exp^{-1}_{v_t(x)}v_0(x),\nabla^N(\exp^{-1}_{.}w(x)).\dot v_t(x)\right\rangle.
\end{aligned}$$

We have $f'(0)=0$ because of page 119 of [10]. For any $t\in[0,1]$, we have

$$\begin{aligned} f''(t) &= \left\langle D^2(\exp^{-1}_{v_0(x)}.)(v_t(x))(\dot v_t(x),\dot v_t(x)) + D(\exp^{-1}_{v_0(x)}.)(v_t(x)).\nabla_t \dot v_t(x), \exp^{-1}_{v_0(x)} w(x)\right\rangle \\ &\quad +2\left\langle \nabla^N(\exp^{-1}_{.} v_0(x)).\dot v_t(x), \nabla^N(\exp^{-1}_{.} w(x)).\dot v_t(x)\right\rangle + \\ &\quad \left\langle \nabla^{N2}(\exp^{-1}_{.} v_0(x)).(\dot v_t(x),\dot v_t(x)) + \nabla^N(\exp^{-1}_{.} v_0(x)).\nabla_t \dot v_t(x), \exp^{-1}_{v_t(x)} w(x)\right\rangle \\ &\quad + \left\langle \exp^{-1}_{v_t(x)} v_0(x), \nabla^{N2}(\exp^{-1}_{.} w(x)).(\dot v_t(x),\dot v_t(x)) + \nabla^N(\exp^{-1}_{.} w(x)).\nabla_t \dot v_t(x)\right\rangle. \end{aligned}$$

We have $\nabla_t \dot v_t(x) = 0$ since we have a geodesic. We have

$$D^2(\exp^{-1}_{v_0(x)}.)(v_t(x))(\dot v_t(x),\dot v_t(x))\,|_{t=0} = 0$$

since $\exp^{-1}_{v_0(x)}$ defines the normal coordinates centered at $v_0(x)$. We have seen in [10] that: If $X \in \chi(N)$ then

$$\nabla^N_{X(z)}(\exp^{-1}_{.} w(x)) = -X(z) + X(z)^\alpha (\exp^{-1}_z w(x))^\beta \Gamma^\rho_{\alpha\beta} e_\rho(z), \forall z \in N,$$

where $(e_a)_\alpha$ is some frame of orthonormal basis of (N,h). It follows

$$\begin{aligned} &\left\langle \nabla^N_{\dot v_t(x)}(\exp^{-1}_{.} v_0(x)), \nabla^N_{\dot v_t(x)}(\exp^{-1}_{.} w(x))\right\rangle |_{t=0} \\ = &\left\langle -\dot v_t(x) + \dot v_t(x)^\alpha (\exp^{-1}_{v_t(x)} v_0(x))^\beta \Gamma^\rho_{\alpha\beta}(v_t(x)) e_\rho(v_t(x)),\right. \\ &\left. -\dot v_t(x) + \dot v_t(x)^\alpha (\exp^{-1}_{v_t(x)} w(x))^\beta \Gamma^\rho_{\alpha\beta}(v_t(x)) e_\rho(v_t(x))\right\rangle |_{t=0} = \|\dot v_t(x)\|^2 |_{t=0}, \end{aligned}$$

by taking normal coordinates centered at the considered point. So

$$f''(0) = 2\,\|\dot v_t(x)\|^2 |_{t=0} + \left\langle \nabla^{N2}(\exp^{-1}_{.} v_0(x)).(\dot v_t(x),\dot v_t(x)), \exp^{-1}_{v_t(x)} w(x)\right\rangle |_{t=0}.$$

We have

$$\nabla^N_{\dot v_t(x)}(\exp^{-1}_{.} v_0(x)) = -\dot v_t(x) + \dot v_t(x)^\alpha (\exp^{-1}_{v_t(x)} v_0(x))^\beta \Gamma^\rho_{\alpha\beta}(v_t(x)) e_\rho(v_t(x)),$$

so

$$\nabla^{N2}(\exp^{-1}_{.} v_0(x)).(\dot v_t(x),\dot v_t(x)) = \nabla_t[\nabla^N_{\dot v_t(x)}(\exp^{-1}_{.} v_0(x))]$$

$$\begin{aligned} = \quad & -\nabla_t \dot v_t(x) + \frac{d}{dt}[\dot v_t(x)^\alpha (\exp^{-1}_{v_t(x)} v_0(x))^\beta]\Gamma^\rho_{\alpha\beta}(v_t(x)) e_\rho(v_t(x)) + \\ & \dot v_t(x)^\alpha (\exp^{-1}_{v_t(x)} v_0(x))^\beta \nabla_t[\Gamma^\rho_{\alpha\beta}(v_t(x)) e_\rho(v_t(x))]. \end{aligned}$$

It follows that

$$\nabla^{N2}(\exp_{.} v_0(x)).(\dot v_t(x),\dot v_t(x))\,|_{t=0} = 0,$$

and then

$$f''(0) = 2\,\|\dot v_t(x)\|^2_{|t=0} = 2d^2(v_0(x), v_1(x)).$$

In this way the Claim is proved

We conclude that the theorem is proved

4 Appendix

In order to make this work a little self contained, we include here the most important of the tools from [9] we used. The constants and notations here are not the same as in this work.

Theorem 4.1. *Let (M,g) and (N,h) be compact Riemannian manifolds. Suppose that (N,h) is symmetric and has its sectional curvature bounded from above by $K > 0$. Let $\alpha \in]0, \frac{\pi}{2\sqrt{K}}[$. Then, there exist $C_1 > 0$ and $C_2 > 0$ such that: $\forall u, v \in W^{1,2}((M,g),(N,h))$ harmonic such that: $d_\infty(u,v) \leq \frac{\pi}{2\sqrt{K}} - \alpha$ and $u_{|\partial M} = v_{|\partial M}$ if $\partial M \neq \emptyset$, we have:*

$$\left|\int_M [1 + \ln(d(u(x),v(x))] \left\langle \nabla^N_{B_i(x)} \exp_.^{-1} v(x), B_i(x) \right\rangle dx\right| \leq$$

$$C_1 \int_M |1 + \ln(d(u(x),v(x))| d^2(u(x),v(x))[||\nabla u||^2(x) + ||\nabla v||^2(x)]dx$$
$$+KC_2 \int_M d^2(u(x),v(x))[||\nabla u||^2(x) + ||\nabla v||^2(x)]dx$$

where $B_i(x) := D_{e_i}u(x) - P_{v(x)u(x)}D_{e_i}v(x)$.

Corollary 4.2. *Let $\alpha \in]0, \frac{\pi}{4\sqrt{K}}[$ and $(u_n)_n$ be a sequence in $W^{1,2}((M,g),(N,h))$ of harmonic mappings which converges in L^2 to a mapping u, and satisfies:*

a) the sequence $(E(u_n))_n$ is bounded

b) $u_{n|\partial M} = u_{|\partial M}, \forall n$ when $\partial M \neq \emptyset$

c) $d_\infty(u,v) \leq \frac{\pi}{4\sqrt{K}} - \alpha$.

Then we have

$\lim\limits_{n\to+\infty} \int_M \left\|\nabla u - P_{u_n u}\nabla u_n\right\|^2(x)dx = 0$, *and then* $\lim\limits_{n\to\infty} E(u_n) = E(u)$.

Lemma 4.3. *Let $a, x \in N$ be such that $d(a,x) < \frac{\pi}{\sqrt{K}}$. Then: $\forall h, k \in T_xN$ and $Y_0 \in T_aN$, there exist $Y, Z \in \Gamma(TN)$ such that $Y(a) = Y_0, \nabla^N_Z(\exp_.^{-1} a)(x) = k$ and*

$$\langle(\nabla_k P_{.a})(x).h, Y_0\rangle = -\left\langle R^N(Z, \exp_x^{-1} a)P_{ax}Y_0, h\right\rangle - \left\langle \nabla^N_{\exp_x^{-1} a}\nabla^N_Z Y(x), h\right\rangle.$$

Lemma 4.4. *Let (N,h) be a symmetric Riemannian manifold such that $Riem^{(N,h)} \leq K, K > 0$, and $a, b \in N$ be such that $d(a,b) < \frac{\pi}{\sqrt{K}}$. Let $h, k \in T_aN$ and $h' = P_{ab}(h), k' = P_{ab}(k)$. Then we have*

$$\left\langle \nabla^N_h \exp_.^{-1} b, k\right\rangle = \left\langle \nabla^N_{h'} \exp_.^{-1} a, k'\right\rangle.$$

Proof of Lemma 4.3:

Let $[0,1] \ni t \mapsto x(t)$ be the geodesic such that $x(0) = x$ and $\dot{x}(0) = k$. Let H be the parallel vectorfield along $x(t)$ such that $H(x) = h$, and Y the parallel vectorfield along γ_{xa} such that $Y(a) = Y_0$. We have

$$\begin{aligned}(\nabla_k P_{.a})(x).h &= \nabla^N_X(P_{.a}.H) \text{ (since } \nabla^N_X H = 0 \text{), and}\\ \left\langle \nabla^N_k(P_{.a}.H), Y_0\right\rangle &= D_k\langle P_{.a}.H, Y_0\rangle.\end{aligned}$$

X is a vectorfield such that $X(x) = k$. Since $Y(x(t)) = P_{ax(t)}Y_0$ we have

$$\langle P_{x(t)a}H, Y_0\rangle = \langle P_{x(t)a}H, P_{x(t)a}Y(x(t))\rangle = \langle H(x(t)), Y(x(t))\rangle, \forall t, \text{ and then}$$

$$D_k \langle P_{x(t)a} H, Y_0 \rangle (x) = D_k \langle H, Y \rangle (x) = \left\langle H, \nabla^N_k Y \right\rangle (x). \tag{4.1}$$

We have also, for any $t \in [0,1]$, $\nabla_{\exp^{-1}_{x(t)} a} Y = 0$, and then

$$0 = [\nabla_t \nabla_{\exp^{-1}_{x(t)} a} Y](t=0) = R^N(k, \exp^{-1}_x a)Y + \nabla^N_{\exp^{-1}_x a} \nabla^N_X Y + \nabla^N_{[X, \exp^{-1}_{.} a](x)} Y, \tag{4.2}$$

where we define the vectorfield X first along $x(t)$ and then along $\gamma_{x(t)a}$ by parallel transport of k. We have

$$\begin{aligned} [X, \exp^{-1}_{.} a](x) &= \nabla^N_k \exp^{-1}_{.} a, \text{ and (4.2) becomes} \\ \nabla^N_{\nabla^N_k \exp^{-1}_{.} a} Y &= -R^N(k, \exp^{-1}_x a)Y - \nabla^N_{\exp^{-1}_x a} \nabla^N_X Y. \end{aligned}$$

Since $\exp^{-1}_{.} a = \frac{1}{2} grad[d^2(.,a)]$ we have:

$$\forall Z \in T_x N, Z \neq 0, \left\langle \nabla^N_Z \exp^{-1}_{.} a, Z \right\rangle = D^2[-\frac{1}{2} d^2(.,a)](Z,Z) < 0.$$

So $\mathrm{Ker} \nabla^N_{.}(\exp^{-1}_{.} a) = \{0\}$, and then there exists a unique $Z_0 \in T_x N$, which we prolonge to a vectorfield $Z \in \Gamma(TN)$ such that $\nabla^N_Z (\exp^{-1}_{.})(x) = k$. By replacing X by Z in (4.2) we get

$$\begin{aligned} \nabla^N_k Y &= -R^N(Z, \exp^{-1}_x a)Y - \nabla^N_{\exp^{-1}_x a} \nabla^N_Z Y \text{ , and then (4.1) gives us} \\ \langle (\nabla_k P_{.a})(x).h, Y_0 \rangle &= -\left\langle R^N(Z, \exp^{-1}_x a)Y_0, h \right\rangle - \left\langle \nabla^N_{\exp^{-1}_x a} \nabla^N_Z Y(x), h \right\rangle \end{aligned}$$

<u>Proof of Lemma 4.4</u>:

Let $c = \exp_a(\frac{1}{2} \exp^{-1}_a b)$ and σ the symmetry with center c. Then for x close enough to b, we have

$$d\sigma(b). \exp^{-1}_b x = \exp^{-1}_a \sigma(x), \tag{4.3}$$

since σ is an isometry. So $[0,1] \ni t \overset{\gamma}{\mapsto} \sigma[\exp_b(t \exp^{-1}_b x)]$ is the minimal geodesic such that $\dot{\gamma}(0) = \exp^{-1}_a \sigma(x)$. In particular $d\sigma(b). \exp^{-1}_b a = \exp^{-1}_a b$. By derivating this equality w.r.t b we get

$$\begin{aligned} (\nabla^N_{h'} d\sigma)(b). \exp^{-1}_b a + d\sigma(b) \nabla^N_{h'} \exp^{-1}_{.} a &= D_{h'} \exp^{-1}_a ., \text{ and then} \\ d\sigma(b) \nabla^N_{h'} \exp^{-1}_{.} a &= D_{h'} \exp^{-1}_a ., \text{ since} \\ \nabla d\sigma &= 0 \text{ (} \sigma \text{ being an isometry).} \end{aligned}$$

$$\text{So } \nabla^N_{h'} \exp^{-1}_{.} a = d\sigma(a) D_{h'} \exp^{-1}_a .. \tag{4.4}$$

Let $(a_1, ..., a_m)$ be an orthonormal basis of $T_a N$, and $(a'_1, ..., a'_m)$ its parallel transport to the point b along the geodesic γ_{ab}. Let's set $\exp^{-1}_b x = \sum_{1 \le i \le m} f_i(x) a'_i$. Then

$$\begin{aligned} d\sigma(b) \exp^{-1}_b x &= \sum_{1 \le i \le m} f_i(x) d\sigma(b) a'_i = - \sum_{1 \le i \le m} f_i(x) a_i, \text{ since} \\ d\sigma(a'_i) &= -a_i, \text{ because of } \nabla d\sigma = 0. \\ \text{(4.3) implies } \exp^{-1}_a \sigma(x) &= - \sum_{1 \le i \le m} f_i(x) a_i. \end{aligned}$$

By derivating this last inequality w.r.t x one gets

$$\begin{aligned} D_h[\exp^{-1}_a \sigma(.)](x) &= - \sum_{1 \le i \le m} (D_h f_i)(x) a_i, \text{ and then} \\ D_{d\sigma(a)h}(\exp^{-1}_a .) &= d\sigma(b) \sum_{1 \le i \le m} (D_h f_i)(x) a'_i, \\ -D_{h'}(\exp^{-1}_a .) &= d\sigma(b).D_h[\sum_{1 \le i \le m} f_i(x) a'_i] = d\sigma(b) D_h \exp^{-1}_b .. \end{aligned}$$

Otherwise said $d\sigma(a)D_{h'}(\exp_a^{-1}.) = -D_h \exp_b^{-1}..$

$$\begin{aligned}
(4.4) \text{ implies } \nabla^N_{h'} \exp_.^{-1} a &= -D_h \exp_b^{-1}. = -d\sigma(a)\nabla^N_h \exp_.^{-1} b. \\
\text{So } \left\langle \nabla^N_{h'} \exp_.^{-1} a, k' \right\rangle &= -\left\langle d\sigma(a)\nabla^N_h \exp_.^{-1} b, k' \right\rangle \\
&= \left\langle d\sigma(a)\nabla^N_h \exp_.^{-1} b, d\sigma(a)k \right\rangle \\
&= \left\langle \nabla^N_h \exp_.^{-1} b, k \right\rangle
\end{aligned}$$

Using these two lemma one proves formula (3.5). And then one uses (3.1), (3.4) and (3.5), and the same argument of convexity for E_0 that we used here for E_λ, to prove the theorem.

4.1 Proof of the Corollary 4.2

Since $u_n \xrightarrow{L^2} u$ and M has finite measure, the theorem of Egorov gives us the existence of a subsequence $(u_{n_k})_k$ which converges μ - almost uniformly to u, where $\mu := dx$.

Let $S(u, v, B)$ denote the righthand side of the inequality in the theorem with $B \subseteq M$ at the place of M.

Since the sequence $(E(u_n))_{n\in\mathbb{N}}$ is bounded and $\mu(M)$ is finite, one can prove as in Claim T1 of [13] that, there exists a subsequence of $(u_{n_k})_k$ which we denote again by $(u_{n_k})_k$ such that

$$\int_{\{x\in M/\left\|\nabla u_{n_k}\right\|\geq m\}} \left\|\nabla u_{n_k}\right\|^2 d\mu \underset{m\to\infty}{\rightarrow} 0 \text{ uniformly in } k.$$

Since $x \mapsto |1 + \ln d(u_n(x), u_m(x))| d^2(u_m(x), u_n(x))$ is uniformly bounded in n and m, we have: $\forall\varepsilon > 0, \exists\delta > 0$ such that, for any measurable $B \subseteq M$ such that $\mu(B) < \delta$,

$$\left|S(u_{n_k}, u_{n_l}, B)\right| < \varepsilon \text{ and } \int_B \left\|\nabla u_{n_k}\right\|^2 d\mu < \varepsilon, \forall k, l.$$

Let's fix ε and δ.

Because of the μ - almost uniform convergence of $(u_{n_k})_k$, there exists $A_\delta \subseteq M$ measurable such that $\mu(A_\delta) < \delta$ and $(u_{n_k})_k$ converges uniformly to u on $M\backslash A_\delta$. So, $\exists k_0$ such that

$$\begin{aligned}
k, l &\geq k_0 \Rightarrow d(u_{n_k}(x), u_{n_l}(x)) < \varepsilon, \forall x \in M\backslash A_\delta, \text{ and also} \\
k, l &\geq k_0 \Rightarrow \left|1 + \ln d(u_{n_k}(x), u_{n_l}(x))\right| d(u_{n_k}(x), u_{n_l}(x)) < \varepsilon, \forall x \in M\backslash A_\delta.
\end{aligned}$$

It folows

$$\begin{aligned}
k, l \geq k_0 \Rightarrow S(u_{n_k}, u_{n_l}, M\backslash A_\delta) &\leq C_1\varepsilon^2 \int_M [\left\|\nabla u_{n_k}\right\|^2 + \left\|\nabla u_{n_l}\right\|^2] d\mu \\
&+\varepsilon^2 K C_2(E(u_{n_k}) + E(u_{n_l})).
\end{aligned}$$

For ε small enough we get

$$\begin{aligned}
k, l &\geq k_0 \Rightarrow S(u_{n_k}, u_{n_l}, M\backslash A_\delta) \leq \varepsilon \\
&\Rightarrow S(u_{n_k}, u_{n_l}, M) \leq S(u_{n_k}, u_{n_l}, A_\delta) + S(u_{n_k}, u_{n_l}, M\backslash A_\delta) \leq 2\varepsilon.
\end{aligned}$$

It follows

$$\lim_{l,k\to+\infty} \int_M (1 + \ln d(u_{n_k}(x), u_{n_l}(x))) \left\langle \nabla^N_{A_{i,k,l}(x)} \exp_.^{-1} u_{n_k}(x), A_{i,k,l}(x) \right\rangle dx = 0,$$

where $A_{i,k,l}(x) := d_{e_i} u_{n_l}(x) - P_{u_{n_k}(x), u_{n_l}(x)} d_{e_i} u_{nk(x)}$.

Claim 4.5. $\lim_{l,k\to+\infty}\int_M \left\|A_{i,k,l}(x)\right\|^2 dx = 0.$

We will prove that later.

By embedding (N,h) into some $\mathbb{R}^q$ we have

$$\begin{aligned}\left\|d_{e_i}u_{n_k}(x) - d_{e_i}u_{n_l}(x)\right\|^2 &\leq 2\left\|d_{e_i}u_{n_l}(x) - P_{u_{n_k}u_{n_l}} d_{e_i}u_{n_k}(x)\right\|^2 + \\ &\quad 2\left\|d_{e_i}u_{n_k}(x) - P_{u_{n_k}u_{n_l}} d_{e_i}u_{n_k}(x)\right\|^2,\end{aligned}$$

and one gets as at the beginning of this proof that

$$\lim_{k,l\to+\infty}\int_M \left\|d_{e_i}u_{n_k}(x) - d_{e_i}u_{n_l}(x)\right\|^2 dx = 0.$$

Therefore $(u_{n_k})_k$ is a Cauchy sequence in $W^{1,2}((M,g),\mathbb{R}^q)$ which is complete. So $(u_{n_k})_k$ converges in $W^{1,2}$ to some $v \in W^{1,2}((M,g),(N,h))$. Since $(u_{n_k})_k$ converges to u in L^2 we have that $u = v$.

In this way, any subsequence of $(u_n)_n$ has a subsequence which converges to u in $W^{1,2}((M,g),\mathbb{R}^q)$. We conclude that $(u_n)_n$ converges in $W^{1,2}$ to u. It follows easily that

$$\lim_{n\to+\infty} E(u_n) = E(u) \text{ and } \lim_{n\to+\infty}\int_M \left\|\nabla u - P_{u_n u}\nabla u_n\right\|^2(x)dx = 0.$$

Proof of the Claim 4.5:

From (1.1) we have:

$$\left\langle \nabla^N_{A_{i,k,l}} \exp^{-1}_{.} u_{n_k}(x), A_{i,k,l}(x)\right\rangle = -\frac{1}{2}D^2[d^2(., u_{n_k}(x))](A_{i,k,l}(x), A_{i,k,l}(x)) \leq 0.$$

$$\text{Let's set } P_{k,l} := \{x \in M \ / \ d(u_{n_k}(x), u_{n_l}(x)) \geq \frac{1}{2}e^{-1}\}.$$

Since $d(u_{n_k}(x), u_{n_l}(x)) < \frac{\pi}{2\sqrt{K}}$ we have: $\forall x \in P_{k,l}$

$$-\ln 2 \leq 1 + \ln d(u_{n_k}(x), u_{n_l}(x)) \leq 1 + \ln(\frac{\pi}{2\sqrt{K}}). \text{ Let's set}$$

$$B_{i,k,l}(x) := [1 + \ln d(u_{n_k}(x), u_{n_l}(x))]\left\langle \nabla^N_{A_{i,k,l}(x)} \exp^{-1}_{.} u_{n_k}(x), A_{i,k,l}(x)\right\rangle.$$

Since N is compact, $\exists C_7 > 0$ depending only on the geometry of (N,h) such that: $\forall x \in P_{k,l}$

$$\left|B_{i,k,l}(x)\right| \leq C_7 \max\{\ln 2, \left|1 + \ln(\frac{\pi}{2\sqrt{K}})\right|\}[\left\|d_{e_i}u_{n_k}\right\|^2(x) + \left\|d_{e_i}u_{n_l}\right\|^2(x)].$$

For $\varepsilon < \frac{1}{2}e^{-1}$ we have : $\forall k,l \geq k_0, d(u_{n_k}(x), u_{n_l}(x)) < \frac{1}{2}e^{-1}, \forall x \in M\backslash A_\delta$, so $P_{k,l} \subseteq A_\delta$.

Therefore: $\exists C_8 > 0$ such that, $\forall k,l \geq k_0$ we have

$$\left|\int_{P_{k,l}} \sum_i B_{i,k,l}(x)dx\right| \leq C_8 \int_{A_\delta} [\left\|\nabla u_{n_k}\right\|^2(x) + \left\|\nabla u_{n_l}\right\|^2(x)] \leq C_8\varepsilon.$$

$\forall k,l \geq k_0$ and $\forall x \in M\backslash P_{k,l}$ we have $1 + \ln d(u_{n_k}(x), u_{n_l}(x)) < -\ln 2$, so (1.1) gives us

$$B_{i,k,l}(x) \geq$$

$$-(1+\ln d(u_{n_k}(x),u_{n_l}(x)))\,\sqrt{K}d(u_{n_k}(x),u_{n_l}(x))\cot[\,\sqrt{K}d(u_{n_k}(x),u_{n_l}(x))]\left\|A_{i,k,l}(x)\right\|^2$$

$$\geq C_9\left\|A_{i,k,l}(x)\right\|^2,\text{ for some constant } C_9>0.$$

$$\begin{aligned}\text{We have }\int_{M\setminus P_{k,l}} B_{i,k,l}(x)dx &\leq \left|\int_{P_{k,l}} B_{i,k,l}(x)dx\right|+\left|\int_M B_{i,k,l}(x)dx\right|,\text{ so}\\ \int_{M\setminus P_{k,l}}\left\|A_{i,k,l}(x)\right\|^2 dx &\leq \frac{1}{C_9}[\left|\int_{P_{k,l}} B_{i,k,l}(x)dx\right|+\left|\int_M B_{i,k,l}(x)dx\right|].\end{aligned}$$

Since $\left\|A_{i,k,l}(x)\right\|^2\leq 2[\left\|d_{e_i}u_{n_k}\right\|^2(x)+\left\|d_{e_i}u_{n_l}\right\|^2(x)]$ we have

$$\int_{P_{k,l}}\left\|A_{i,k,l}(x)\right\|^2 dx\leq 2\int_{A_\delta}[\left\|\nabla u_{n_k}\right\|^2(x)+\left\|\nabla u_{n_l}\right\|^2(x)]dx<2\varepsilon,\forall k,l\geq k_0.$$

In this way, $\forall k,l\geq k_0$ we have

$$\begin{aligned}\int_M\sum_i\left\|A_{i,k,l}(x)\right\|^2 dx &= \int_{M\setminus P_{k,l}}\sum_i\left\|A_{i,k,l}(x)\right\|^2 dx+\int_{P_{k,l}}\sum_i\left\|A_{i,k,l}(x)\right\|^2 dx\\ &\leq 2\varepsilon+\frac{C_8}{C_9}\varepsilon+\frac{1}{C_9}\left|\int_M\sum_i B_{i,k,l}(x)dx\right|.\end{aligned}$$

And finally

$$\lim_{l,k\to+\infty}\int_M\sum_i\left\|A_{i,k,l}(x)\right\|^2 dx=0.$$

References

[1] A. Braides & G. Dal Maso, Nonlocal approximation of the Mumford - Shah functional, Calc. of Variations n°5 (1997), 293 - 322

[2] D. Gilbarg & N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Grundlehren der math. Wiss. 224, Springer-Verlag, 1983

[3] P. Hartman, On homotopic harmonic maps, Can. J. Math. 19 (1967), 673 - 687

[4] Heinrich Voss, Nonlinear eigenvalue problems, Preprint (Available on the net), Hamburg university of technology

[5] I. Chavel, Eigenvalues in Riemannian geometry, Academic press, pure and applied mathematics, 1984

[6] J. Jost, Riemannian geometry and geometric analysis, Universitext, Springer (2002)

[7] J. Jost & M. Kourouma, A nonlinear eigenvalue problem for mappings between Riemannian manifolds, Afrika Matematika, série 3 vol. 13 (2002) 87-109

[8] M. Kourouma, Convexité de l'énergie pour les applications à valeurs dans un espace de longueur (X,d) à courbure d'Alexandrov $\leq K$, K>0, Afrika Matematika, série 4, volume 17, p. 91-108 (2005)

[9] M. Kourouma, Continuité de l'énergie pour les applications harmoniques à valeurs dans une variété Riemannienne symétrique, Revue Ivoiroirienne des Sciences et Technol. 06, p. 113 - 127 (2005)

[10] M. Kourouma, First eigenvalue estimates for the tension field, Afrika Matematika, série 3, vol. 18 (2007), p. 109 - 128

[11] M. Kourouma, On Jacobi fields along harmonic maps with values into a symmetric Riemannnian manifold, Annales Mathématiques Africaines, série 1, Vol. 1 (2010), p. 21 - 36

[12] M. Kourouma, Regularity of weakly harmonic sections, Differential Geometry and Applications 18 (2003), 1 - 19

[13] M. Kourouma, Reduction of the dimension of the singular set of energy minimizing maps with values in a compact metric space with upper curvature bound, Afrika Mat. série 3 vol. 16 (2005), 55-86

[14] M. Kourouma, Harmonische Schnitte Riemannscher Faserbündel, Thèse de doctorat, 1994 Ruhr - Universität Bochum, Allemagne

[15] L. Lemaire & J. C. Wood, Jacobi fields along harmonic 2 - spheres in $\mathbb{CP}^2$ are integrable, Journal London math. soc. 66, 2 (2002) 468 - 486

[16] Peter Lindqwist, A nonlinear eigenvalue problem, Preprint (Available on the net), Departement of mathematics, Norwegen university of science and technology

[17] Yu G. Reshetnyak, Inextensible mappings in a space of curvature no greater than K, Sib. Math. J. 9 (1968), 683 - 689

[18] R. Schoen & K. Uhlenbeck, Regularity theory for harmonic maps, J. Diff. Geometry 17 (1982), 307 - 335

[19] R. Schoen & S. T. Yau, Lectures on Differential Geometry, Conference Proceedings and Lecture Notes in Geometry and Topology, vol. 1, International Press, 1994

[20] Shui-Nee Chow, Jack K. Hale, Methods of bifurcation theory, Grundlehren der mathematischen Wiss. 251, Springer 1996.