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Biharmonic Hypersurfaces in E^5 with Zero Scalar Curvature

Deepika*
University School of Basic and Applied Sciences,
Guru Gobind Singh Indraprastha University,
Sector-16C, Dwarka, New Delhi-110078, India.

Ram Shankar Gupta†
Associate Professor, Department of Mathematics,
Central University of Jammu,
Sainik colony, Jammu-180011, India.
Assistant Professor, University School of Basic and Applied Sciences,
Guru Gobind Singh Indraprastha University,
Sector-16C, Dwarka, New Delhi-110078, India.

Abstract. We prove non-existence of proper biharmonic hypersurfaces of zero scalar curvature in Euclidean space E^5.

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1 Introduction

The notion of poly-harmonic maps was introduced in [6] as a natural generalization of the well-known harmonic maps. Thus, while harmonic maps between Riemannian manifolds are critical points of the energy functional, the biharmonic maps are critical points of the bienergy functional.

The study of biharmonic submanifolds in Euclidean spaces was initiated by B. Y. Chen in the middle of 1980s. In particular, he proved that there exist no proper biharmonic surfaces in Euclidean 3-spaces [1]. There are many related to non-existence results in Euclidean spaces developed by I. Dimitric [4, 5]. Also in [8], non-existence of proper biharmonic hypersurfaces in Euclidean 4-spaces was proved. Recently, non-existence of proper biharmonic hypersurfaces with three distinct principal curvatures [9, 10, 11] and also for $\delta(2)$-ideal and $\delta(3)$-ideal hypersurfaces in Euclidean space [3] were obtained. The global version of Chen's conjecture for biharmonic submanifolds in Euclidean space was studied in [7]. For more works in this field, please see [2].

*e-mail address: sdeep2007@gmail.com
†e-mail address: ramshankar.gupta@gmail.com

In view of the above mentioned development, it is natural to investigate existence/non-existence of proper biharmonic hypersurfaces in Euclidean space $E^n (n \geq 5)$ with number of distinct principal curvatures greater or equal than 4.

Consequently, in this paper, we study biharmonic hypersurfaces with up to four distinct principal curvatures in Euclidean space E^5 using the technique developed in [8].

2 Preliminaries

Let (M, g) be a hypersurface isometrically immersed in a 5-dimensional Euclidean space $(E^5, \overline{g})$ and $g = \overline{g}_{|M}$.

Let $\overline{\nabla}$ and ∇ denote linear connections on E^5 and M, respectively. Then, the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \ \ \forall \ \ X, Y \in \Gamma(TM), \tag{2.1}$$

$$\overline{\nabla}_X \xi = -A_\xi X, \tag{2.2}$$

where ξ is the unit normal vector to M, h is the second fundamental form and A is the shape operator. It is well known that the second fundamental form h and shape operator A are related by

$$\overline{g}(h(X, Y), \xi) = g(A_\xi X, Y). \tag{2.3}$$

The mean curvature vector is given by

$$H = \frac{1}{4} traceA. \tag{2.4}$$

The Gauss and Codazzi equations are given by

$$R(X, Y)Z = g(AY, Z)AX - g(AX, Z)AY, \tag{2.5}$$

$$(\nabla_X A)Y = (\nabla_Y A)X, \tag{2.6}$$

respectively, where R is the curvature tensor and

$$(\nabla_X A)Y = \nabla_X AY - A(\nabla_X Y), \tag{2.7}$$

for all $X, Y, Z \in \Gamma(TM)$.

A biharmonic submanifold in a Euclidean space is called proper biharmonic if it is not minimal. The necessary and sufficient conditions for M to be biharmonic in E^5 are

$$\triangle H + H traceA^2 = 0, \tag{2.8}$$

$$AgradH + 2HgradH = 0, \tag{2.9}$$

where H denotes the mean curvature. Also, the Laplace operator $\triangle$ of a scalar valued function f is given by [1]

$$\triangle f = -\sum_{i=1}^{4} (e_i e_i f - \nabla_{e_i} e_i f), \tag{2.10}$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal local tangent frame on M.

3 Biharmonic hypersurfaces of zero scalar curvature

(a) *Four distinct principal curvatures*

In this section we study biharmonic hypersurfaces M with shape operator diagonal. We also assume that mean curvature is not constant. From (2.9), it is easy to see that grad$H$ is an eigenvector of the shape operator A with the corresponding principal curvature $-2H$. Without loss of generality, we choose e_1 in the direction of grad$H$ and therefore shape operator A of hypersurfaces will take the following form with respect to a suitable frame $\{e_1, e_2, e_3, e_4\}$

$$A_H = \begin{pmatrix} -2H & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{pmatrix}. \tag{3.1}$$

The grad$H$ can be expressed as

$$gradH = \sum_{i=1}^{4} e_i(H)e_i. \tag{3.2}$$

As we have taken e_1 parallel to grad$H$, consequently

$$e_1(H) \neq 0, e_2(H) = 0, e_3(H) = 0, e_4(H) = 0. \tag{3.3}$$

We express

$$\nabla_{e_i} e_j = \sum_{k=1}^{4} \omega_{ij}^k e_k, \qquad i, j = 1, 2, 3, 4. \tag{3.4}$$

Using (3.4) and the compatibility conditions $(\nabla_{e_k} g)(e_i, e_i) = 0$ and $(\nabla_{e_k} g)(e_i, e_j) = 0$, we obtain

$$\omega_{ki}^i = 0, \qquad \omega_{ki}^j + \omega_{kj}^i = 0, \tag{3.5}$$

for $i \neq j$, and $i, j, k = 1, 2, 3, 4$.

Taking $X = e_i, Y = e_j$ in (2.7) and using (3.1), (3.4), we get

$$(\nabla_{e_i} A)e_j = e_i(\lambda_j)e_j + \textstyle\sum_{k=1}^{n} \omega_{ij}^k e_k(\lambda_j - \lambda_k).$$

Putting the value of $(\nabla_{e_i} A)e_j$ in (2.6), we find

$$e_i(\lambda_j)e_j + \textstyle\sum_{k=1}^{n} \omega_{ij}^k e_k(\lambda_j - \lambda_k) = e_j(\lambda_i)e_i + \sum_{k=1}^{n} \omega_{ji}^k e_k(\lambda_i - \lambda_k),$$

whereby for $i \neq j = k$ and $i \neq j \neq k$, we obtain

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j = (\lambda_j - \lambda_i)\omega_{jj}^i, \tag{3.6}$$

$$(\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j, \tag{3.7}$$

respectively, for distinct $i, j, k = 1, 2, 3, 4$.

Since $\lambda_1 = -2H$, from (3.3), we get

$$e_1(\lambda_1) \neq 0, e_2(\lambda_1) = 0, e_3(\lambda_1) = 0, e_4(\lambda_1) = 0. \tag{3.8}$$

Using (3.3), (3.4) and the fact that $[e_i\ e_j](H) = 0 = \nabla_{e_i}e_j(H) - \nabla_{e_j}e_i(H) = \omega_{ij}^1 e_1(H) - \omega_{ji}^1 e_1(H)$, for $i \neq j$ and $i, j = 2, 3, 4$, we find

$$\omega_{ij}^1 = \omega_{ji}^1. \tag{3.9}$$

Now, we show that $\lambda_j \neq \lambda_1, j = 2, 3, 4$. In fact, if $\lambda_j = \lambda_1$ for $j \neq 1$, from (3.6), we find

$$e_1(\lambda_j) = (\lambda_1 - \lambda_j)\omega_{j1}^j = 0, \tag{3.10}$$

which contradicts the first expression of (3.8).

Since M has four distinct principal curvatures, from (2.4), we obtain that

$$\lambda_2 + \lambda_3 + \lambda_4 = 6H. \tag{3.11}$$

Putting $i \neq 1, j = 1$ in (3.6) and using (3.8) and (3.5), we find

$$\omega_{1i}^1 = 0, \qquad i = 1, 2, 3, 4. \tag{3.12}$$

Putting $k = 1, j \neq i$, and $i, j = 2, 3, 4$ in (3.7), and using (3.9), we get

$$\omega_{ij}^1 = \omega_{ji}^1 = \omega_{1i}^j = \omega_{i1}^j = 0, \qquad j \neq i, and\ \ i, j = 2, 3, 4. \tag{3.13}$$

Now, using (3.4), (3.12) and (3.13), we have:

Lemma 3.1. *Let M be a biharmonic hypersurface of non-constant mean curvature with four distinct principal curvatures in Euclidean space E^5, having the shape operator given by (3.1) with respect to suitable orthonormal frame $\{e_1, e_2, e_3, e_4\}$. Then,*

$$\nabla_{e_1}e_1 = \nabla_{e_1}e_2 = \nabla_{e_1}e_3 = \nabla_{e_1}e_4 = 0, \tag{3.14}$$

$$\nabla_{e_2}e_1 = -\omega_{22}^1 e_2, \nabla_{e_2}e_2 = \omega_{22}^1 e_1 + \omega_{22}^3 e_3 + \omega_{22}^4 e_4, \nabla_{e_2}e_3 = \omega_{23}^2 e_2 + \omega_{23}^4 e_4, \nabla_{e_2}e_4 = \omega_{24}^2 e_2 + \omega_{24}^3 e_3 \tag{3.15}$$

$$\nabla_{e_3}e_1 = -\omega_{33}^1 e_3, \nabla_{e_3}e_2 = \omega_{32}^3 e_3 + \omega_{32}^4 e_4, \nabla_{e_3}e_3 = \omega_{33}^1 e_1 + \omega_{33}^2 e_2 + \omega_{33}^4 e_4, \nabla_{e_3}e_4 = \omega_{34}^2 e_2 + \omega_{34}^3 e_3 \tag{3.16}$$

$$\nabla_{e_4}e_1 = -\omega_{44}^1 e_4, \nabla_{e_4}e_2 = \omega_{42}^4 e_4 + \omega_{42}^3 e_3, \nabla_{e_4}e_3 = \omega_{43}^4 e_4 + \omega_{43}^2 e_2, \nabla_{e_4}e_4 = \omega_{44}^1 e_1 + \omega_{44}^2 e_2 + \omega_{44}^3 e_3, \tag{3.17}$$

where ω_{ij}^i satisfy (3.5) and (3.6) for $i, j = 1, 2, 3, 4$.

Evaluating $g(R(X,Y)Z,W)$, using Lemma 3.1, and (2.5) and (3.1), we find the following (3.18)~(3.35):

•$g(R(e_1, e_2)e_1, e_2)$,

$$e_1(\omega_{22}^1) - (\omega_{22}^1)^2 = -2H\lambda_2. \tag{3.18}$$

•$g(R(e_1, e_3)e_1, e_3)$,

$$e_1(\omega_{33}^1) - (\omega_{33}^1)^2 = -2H\lambda_3. \tag{3.19}$$

•$g(R(e_1, e_4)e_1, e_4)$,

$$e_1(\omega_{44}^1) - (\omega_{44}^1)^2 = -2H\lambda_4. \tag{3.20}$$

•$g(R(e_1, e_2)e_2, e_3)$,

$$e_1(\omega_{22}^3) - \omega_{22}^3\omega_{22}^1 = 0. \tag{3.21}$$

•$g(R(e_1, e_2)e_2, e_4)$,

$$e_1(\omega_{22}^4) - \omega_{22}^4\omega_{22}^1 = 0. \tag{3.22}$$

•$g(R(e_1,e_3)e_3,e_2)$,

$$e_1(\omega^2_{33})-\omega^2_{33}\omega^1_{33}=0. \tag{3.23}$$

•$g(R(e_1,e_3)e_3,e_4)$,

$$e_1(\omega^4_{33})-\omega^4_{33}\omega^1_{33}=0. \tag{3.24}$$

•$g(R(e_1,e_4)e_4,e_2)$,

$$e_1(\omega^2_{44})-\omega^2_{44}\omega^1_{44}=0. \tag{3.25}$$

•$g(R(e_1,e_4)e_4,e_3)$,

$$e_1(\omega^3_{44})-\omega^3_{44}\omega^1_{44}=0. \tag{3.26}$$

•$g(R(e_2,e_3)e_2,e_3)$,

$$e_2(\omega^2_{33})+e_3(\omega^3_{22})-\omega^1_{22}\omega^1_{33}-\omega^4_{22}\omega^4_{33}-(\omega^3_{22})^2-(\omega^2_{33})^2+\omega^4_{32}\omega^4_{23}-\omega^2_{34}\omega^2_{43}-\omega^3_{42}\omega^3_{24}=\lambda_2\lambda_3. \tag{3.27}$$

•$g(R(e_2,e_3)e_2,e_1)$,

$$e_3(\omega^1_{22})+\omega^3_{22}\omega^1_{33}-\omega^3_{22}\omega^1_{22}=0. \tag{3.28}$$

•$g(R(e_2,e_4)e_2,e_4)$,

$$e_2(\omega^2_{44})+e_4(\omega^4_{22})-\omega^1_{22}\omega^1_{44}-\omega^3_{22}\omega^3_{44}-(\omega^4_{22})^2-(\omega^2_{44})^2+\omega^3_{42}\omega^3_{24}-\omega^2_{34}\omega^2_{43}-\omega^4_{32}\omega^4_{23}=\lambda_2\lambda_4. \tag{3.29}$$

•$g(R(e_2,e_4)e_2,e_1)$,

$$e_4(\omega^1_{22})+\omega^4_{22}\omega^1_{44}-\omega^4_{22}\omega^1_{22}=0. \tag{3.30}$$

•$g(R(e_3,e_4)e_3,e_4)$,

$$e_3(\omega^3_{44})+e_4(\omega^4_{33})-\omega^1_{33}\omega^1_{44}-\omega^2_{33}\omega^2_{44}-(\omega^4_{33})^2-(\omega^3_{44})^2+\omega^2_{34}\omega^2_{43}-\omega^3_{24}\omega^3_{42}-\omega^4_{23}\omega^4_{32}=\lambda_3\lambda_4. \tag{3.31}$$

•$g(R(e_3,e_4)e_3,e_1)$,

$$e_4(\omega^1_{33})+\omega^4_{33}\omega^1_{44}-\omega^4_{33}\omega^1_{33}=0. \tag{3.32}$$

•$g(R(e_2,e_3)e_3,e_1)$,

$$e_2(\omega^1_{33})+\omega^2_{33}\omega^1_{22}-\omega^2_{33}\omega^1_{33}=0. \tag{3.33}$$

•$g(R(e_2,e_4)e_4,e_1)$,

$$e_2(\omega^1_{44})+\omega^2_{44}\omega^1_{22}-\omega^2_{44}\omega^1_{44}=0. \tag{3.34}$$

•$g(R(e_3,e_4)e_4,e_1)$,

$$e_3(\omega^1_{44})+\omega^3_{44}\omega^1_{33}-\omega^3_{44}\omega^1_{44}=0. \tag{3.35}$$

Now, evaluating scalar curvature of the hypersurface, using (2.5) and (3.1), we get

$$\lambda_2^2+\lambda_3^2+\lambda_4^2=12H^2, \tag{3.36}$$

assuming that the scalar curvature of the hypersurface is zero.

Using (3.1) and (3.36), we find

$$\text{trace}A^2=4H^2+\lambda_2^2+\lambda_3^2+\lambda_4^2=16H^2$$

Using Lemma 3.1, equations (2.10) and (3.3) and putting the value of traceA^2 in (2.8), we obtain

$$-e_1e_1(H) + (\omega_{22}^1 + \omega_{33}^1 + \omega_{44}^1)e_1(H) + 16H^3 = 0. \tag{3.37}$$

Using (3.3), Lemma 3.1, and the fact that $[e_i\ e_1](H) = 0 = \nabla_{e_i}e_1(H) - \nabla_{e_1}e_i(H)$, for $i = 2,3,4$, we find

$$e_ie_1(H) = 0. \tag{3.38}$$

Also, using (3.38) and the fact that $[e_i\ e_1](e_1(H)) = 0 = \nabla_{e_i}e_1(e_1(H)) - \nabla_{e_1}e_i(e_1(H))$, we obtain

$$e_ie_1e_1(H) = 0, \qquad i = 2,3,4. \tag{3.39}$$

Differentiating (3.37) with e_i, and using (3.3) and (3.39), we get

$$e_i(\omega_{22}^1 + \omega_{33}^1 + \omega_{44}^1) = 0, \qquad i = 2,3,4. \tag{3.40}$$

Now, we have:

Lemma 3.2. *Let M be a biharmonic hypersurface of zero scalar curvature with four distinct principal curvatures in Euclidean space E^5, having the shape operator given by (3.1) with respect to suitable orthonormal frame $\{e_1,e_2,e_3,e_4\}$. Then,* $e_2(\lambda_3) = 0 = e_2(\lambda_4)$, $e_3(\lambda_2) = 0 = e_3(\lambda_4)$, $e_4(\lambda_3) = 0 = e_4(\lambda_2)$ *and* $\omega_{44}^2 = 0 = \omega_{33}^2$, $\omega_{22}^3 = 0 = \omega_{44}^3$, $\omega_{22}^4 = 0 = \omega_{33}^4$.

Proof. The proof can be divided in three parts:

(i) *Proof of* $e_2(\lambda_4) = 0 = e_2(\lambda_3)$ ***and*** $\omega_{44}^2 = 0 = \omega_{33}^2$.

Differentiating (3.36) and (3.11) with e_2 and using (3.3), we get

$$\lambda_2e_2(\lambda_2) + \lambda_3e_2(\lambda_3) + \lambda_4e_2(\lambda_4) = 0, \tag{3.41}$$

and

$$e_2(\lambda_2) + e_2(\lambda_3) + e_2(\lambda_4) = 0, \tag{3.42}$$

respectively.

Eliminating $e_2(\lambda_2)$ from (3.41) and (3.42), we find

$$(\lambda_3 - \lambda_2)e_2(\lambda_3) + (\lambda_4 - \lambda_2)e_2(\lambda_4) = 0. \tag{3.43}$$

Putting the value of $e_2(\lambda_3)$ and $e_2(\lambda_4)$ from (3.6) in (3.43), we obtain

$$(\lambda_2 - \lambda_4)^2\omega_{44}^2 + (\lambda_2 - \lambda_3)^2\omega_{33}^2 = 0. \tag{3.44}$$

Differentiating (3.44) along e_1 and using (3.6), (3.23), (3.25) and (3.44), we have

$$\begin{aligned}[2(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)\omega_{22}^1 + (2\lambda_1 + \lambda_2 - 3\lambda_4)(\lambda_2 - \lambda_3)\omega_{44}^1 \\ -(2\lambda_1 + \lambda_2 - 3\lambda_3)(\lambda_2 - \lambda_4)\omega_{33}^1]\omega_{44}^2 = 0.\end{aligned} \tag{3.45}$$

Similarly, differentiating (3.11) with e_1 and e_2 successively and using (3.38), we find

$$e_2e_1(\lambda_2) + e_2e_1(\lambda_3) + e_2e_1(\lambda_4) = 0. \tag{3.46}$$

Using (3.6) in (3.46), we get

$$e_2((\lambda_2 - \lambda_1)\omega_{22}^1) + e_2((\lambda_3 - \lambda_1)\omega_{33}^1) + e_2((\lambda_4 - \lambda_1)\omega_{44}^1) = 0. \tag{3.47}$$

Now, putting the value of $e_2(\lambda_2)$ and $e_2(\omega^1_{22})$ from (3.42) and (3.40) in (3.47), and thereafter using (3.6), (3.33), (3.34) and (3.44), we obtain

$$[(\lambda_4-\lambda_3)\omega^1_{22}+(\lambda_3-\lambda_2)\omega^1_{44}+(\lambda_2-\lambda_4)\omega^1_{33}]\omega^2_{44}=0. \tag{3.48}$$

In equations (3.45) and (3.48) we get that at least one of the factors ω^2_{44}, or the expressions between square brackets, has to vanish. We claim that ω^2_{44}=0. In fact, if $\omega^2_{44}\neq 0$, then

$$\begin{aligned}2(\lambda_1-\lambda_2)(\lambda_3-\lambda_4)\omega^1_{22}+(2\lambda_1+\lambda_2-3\lambda_4)(\lambda_2-\lambda_3)\omega^1_{44}\\-(2\lambda_1+\lambda_2-3\lambda_3)(\lambda_2-\lambda_4)\omega^1_{33}=0.\end{aligned} \tag{3.49}$$

$$(\lambda_4-\lambda_3)\omega^1_{22}+(\lambda_3-\lambda_2)\omega^1_{44}+(\lambda_2-\lambda_4)\omega^1_{33}=0. \tag{3.50}$$

Eliminating ω^1_{33}, from (3.49) and (3.50), we get

$$(\lambda_2-\lambda_3)(\lambda_4-\lambda_3)(\omega^1_{22}-\omega^1_{44})=0, \tag{3.51}$$

which shows that

$$\omega^1_{22}=\omega^1_{44}, \tag{3.52}$$

which is not possible as from (3.18) and (3.20), it gives $\lambda_2=\lambda_4$, a contradiction. Therefore, $\omega^2_{44}=0$, which gives $\omega^2_{33}=0$ in view of (3.44). Consequently, from (3.6), we find $e_2(\lambda_3)=e_2(\lambda_4)=0$.

(ii)*Proof of* $e_3(\lambda_2)=0=e_3(\lambda_4)$ ***and*** $\omega^3_{22}=0=\omega^3_{44}$.

Differentiating (3.36) and (3.11) with e_3 and using (3.3), we get

$$\lambda_2e_3(\lambda_2)+\lambda_3e_3(\lambda_3)+\lambda_4e_3(\lambda_4)=0, \tag{3.53}$$

and

$$e_3(\lambda_2)+e_3(\lambda_3)+e_3(\lambda_4)=0, \tag{3.54}$$

respectively.

Eliminating $e_3(\lambda_3)$ from (3.53) and (3.54), we find

$$(\lambda_2-\lambda_3)e_3(\lambda_2)+(\lambda_4-\lambda_3)e_3(\lambda_4)=0. \tag{3.55}$$

Putting the value of $e_3(\lambda_2)$ and $e_3(\lambda_4)$ from (3.6) in (3.55), we obtain

$$(\lambda_3-\lambda_4)^2\omega^3_{44}+(\lambda_3-\lambda_2)^2\omega^3_{22}=0. \tag{3.56}$$

Differentiating (3.56) along e_1 and using (3.6), (3.21), (3.26) and (3.56), we have

$$\begin{aligned}[2(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)\omega^1_{33}+(2\lambda_1+\lambda_3-3\lambda_4)(\lambda_3-\lambda_2)\omega^1_{44}\\-(2\lambda_1+\lambda_3-3\lambda_2)(\lambda_3-\lambda_4)\omega^1_{22}]\omega^3_{44}=0.\end{aligned} \tag{3.57}$$

Similarly, differentiating (3.11) with e_1 and e_3 successively and using (3.38), we find

$$e_3e_1(\lambda_2)+e_3e_1(\lambda_3)+e_3e_1(\lambda_4)=0. \tag{3.58}$$

Using (3.6) in (3.58), we get

$$e_3((\lambda_2-\lambda_1)\omega^1_{22})+e_3((\lambda_3-\lambda_1)\omega^1_{33})+e_3((\lambda_4-\lambda_1)\omega^1_{44})=0. \tag{3.59}$$

Now, putting the value of $e_3(\lambda_3)$ and $e_3(\omega_{33}^1)$ from (3.54) and (3.40) in (3.59), and thereafter using (3.6), (3.28), (3.35) and (3.56), we obtain

$$[(\lambda_4-\lambda_2)\omega_{33}^1+(\lambda_2-\lambda_3)\omega_{44}^1+(\lambda_3-\lambda_4)\omega_{22}^1]\omega_{44}^3=0. \tag{3.60}$$

Equations (3.57) and (3.60) show that at least one of the factors ω_{44}^3, or the expressions between square brackets, has to vanish. We claim that ω_{44}^3=0. In fact, if $\omega_{44}^3\neq 0$, then

$$\begin{aligned}2(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)\omega_{33}^1+(2\lambda_1+\lambda_3-3\lambda_4)(\lambda_3-\lambda_2)\omega_{44}^1\\ -(2\lambda_1+\lambda_3-3\lambda_2)(\lambda_3-\lambda_4)\omega_{22}^1=0.\end{aligned} \tag{3.61}$$

$$(\lambda_4-\lambda_2)\omega_{33}^1+(\lambda_2-\lambda_3)\omega_{44}^1+(\lambda_3-\lambda_4)\omega_{22}^1=0. \tag{3.62}$$

Eliminating ω_{22}^1 from (3.61) and (3.62), we get

$$(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)(\omega_{33}^1-\omega_{44}^1)=0, \tag{3.63}$$

which shows that

$$\omega_{33}^1=\omega_{44}^1, \tag{3.64}$$

which is not possible as from (3.19) and (3.20), it gives $\lambda_3=\lambda_4$, a contradiction. Therefore, $\omega_{44}^3=0$, which gives $\omega_{22}^3=0$ in view of (3.56). Consequently, from (3.6), we find $e_3(\lambda_2)=e_3(\lambda_4)=0$.

(iii) ***Proof of*** $e_4(\lambda_3)=0=e_4(\lambda_2)$ ***and*** $\omega_{22}^4=0=\omega_{33}^4$.

Differentiating (3.36) and (3.11) with e_4 and using (3.3), we get

$$\lambda_2e_4(\lambda_2)+\lambda_3e_4(\lambda_3)+\lambda_4e_4(\lambda_4)=0, \tag{3.65}$$

and

$$e_4(\lambda_2)+e_4(\lambda_3)+e_4(\lambda_4)=0, \tag{3.66}$$

respectively.

Eliminating $e_4(\lambda_4)$ from (3.65) and (3.66), we find

$$(\lambda_3-\lambda_4)e_4(\lambda_3)+(\lambda_2-\lambda_4)e_4(\lambda_2)=0. \tag{3.67}$$

Putting the value of $e_4(\lambda_3)$ and $e_4(\lambda_2)$ from (3.6) in (3.67), we obtain

$$(\lambda_4-\lambda_2)^2\omega_{22}^4+(\lambda_4-\lambda_3)^2\omega_{33}^4=0. \tag{3.68}$$

Differentiating (3.68) along e_1 and using (3.6), (3.22), (3.24) and (3.68), we have

$$\begin{aligned}[2(\lambda_1-\lambda_4)(\lambda_3-\lambda_2)\omega_{44}^1+(2\lambda_1+\lambda_4-3\lambda_2)(\lambda_4-\lambda_3)\omega_{22}^1\\ -(2\lambda_1+\lambda_4-3\lambda_3)(\lambda_4-\lambda_2)\omega_{33}^1]\omega_{22}^4=0.\end{aligned} \tag{3.69}$$

Similarly, differentiating (3.11) with e_1 and e_4 successively and using (3.38), we find

$$e_4e_1(\lambda_2)+e_4e_1(\lambda_3)+e_4e_1(\lambda_4)=0. \tag{3.70}$$

Using (3.6) in (3.70), we get

$$e_4((\lambda_2-\lambda_1)\omega_{22}^1)+e_4((\lambda_3-\lambda_1)\omega_{33}^1)+e_4((\lambda_4-\lambda_1)\omega_{44}^1)=0. \tag{3.71}$$

Now, putting the value of $e_4(\lambda_4)$ and $e_4(\omega_{44}^1)$ from (3.66) and (3.40) in (3.71), and thereafter using (3.6), (3.30), (3.32) and (3.68), we obtain

$$[(\lambda_2-\lambda_3)\omega_{44}^1+(\lambda_3-\lambda_4)\omega_{22}^1+(\lambda_4-\lambda_2)\omega_{33}^1]\omega_{22}^4=0. \tag{3.72}$$

Equations (3.69) and (3.72) show that at least one of the factors ω_{22}^4, or the expressions between square brackets, has to vanish. We claim that ω_{22}^4=0. In fact, if $\omega_{22}^4\neq 0$, then

$$\begin{aligned}2(\lambda_1-\lambda_4)(\lambda_3-\lambda_2)\omega_{44}^1+(2\lambda_1+\lambda_4-3\lambda_2)(\lambda_4-\lambda_3)\omega_{22}^1\\ -(2\lambda_1+\lambda_4-3\lambda_3)(\lambda_4-\lambda_2)\omega_{33}^1=0.\end{aligned} \tag{3.73}$$

$$(\lambda_2-\lambda_3)\omega_{44}^1+(\lambda_3-\lambda_4)\omega_{22}^1+(\lambda_4-\lambda_2)\omega_{33}^1=0. \tag{3.74}$$

Eliminating ω_{33}^1, from (3.73) and (3.74), we get

$$(\lambda_4-\lambda_3)(\lambda_2-\lambda_3)(\omega_{22}^1-\omega_{44}^1)=0, \tag{3.75}$$

which shows that

$$\omega_{22}^1=\omega_{44}^1, \tag{3.76}$$

which is not possible as from (3.18) and (3.20), it gives $\lambda_2=\lambda_4$, a contradiction. Therefore, $\omega_{22}^4=0$, which gives $\omega_{33}^4=0$ in view of (3.68). Consequently, from (3.6), we find $e_4(\lambda_3)=e_4(\lambda_2)=0$.

Combining (i), (ii) and (iii), the proof of Lemma 3.2, is complete.

Now, we have:

Lemma 3.3. *Let M be a biharmonic hypersurface of zero scalar curvature with four distinct principal curvatures in Euclidean space E^5, having the shape operator given by (3.1) with respect to suitable orthonormal frame $\{e_1,e_2,e_3,e_4\}$. Then, $\omega_{23}^4=\omega_{32}^4=\omega_{42}^3=\omega_{24}^3=\omega_{34}^2=\omega_{43}^2=0$.*

Proof. In view of $\omega_{44}^2=0=\omega_{33}^2$, $\omega_{22}^3=0=\omega_{44}^3$, $\omega_{22}^4=0=\omega_{33}^4$, from (3.15) and (3.16), we have

$$\nabla_{e_2}e_3=\omega_{23}^4e_4,\quad \nabla_{e_3}e_2=\omega_{32}^4e_4. \tag{3.77}$$

Now, evaluating $g(R(e_1,e_3)e_2,e_4)$ and $g(R(e_1,e_2)e_3,e_4)$, and using (2.5), (3.1), (3.14) and (3.77), we find

$$e_1(\omega_{32}^4)-\omega_{32}^4\omega_{33}^1=0. \tag{3.78}$$

$$e_1(\omega_{23}^4)-\omega_{23}^4\omega_{22}^1=0. \tag{3.79}$$

Putting $j=4,k=2,i=3$ in (3.7), we get

$$(\lambda_2-\lambda_4)\omega_{32}^4=(\lambda_3-\lambda_4)\omega_{23}^4. \tag{3.80}$$

Differentiating (3.80) with e_1, and using (3.78) and (3.79), we find

$$(e_1(\lambda_2)-e_1(\lambda_4))\omega_{32}^4+(\lambda_2-\lambda_4)\omega_{32}^4\omega_{33}^1=(e_1(\lambda_3)-e_1(\lambda_4))\omega_{23}^4+(\lambda_3-\lambda_4)\omega_{23}^4\omega_{22}^1. \tag{3.81}$$

Putting the values of $e_1(\lambda_2),e_1(\lambda_3)$ and $e_1(\lambda_4)$ from (3.6) in (3.81), we obtain

$$\omega_{32}^4(\omega_{22}^1-\omega_{44}^1)=\omega_{23}^4(\omega_{33}^1-\omega_{44}^1). \tag{3.82}$$

Substituting the value of ω_{23}^4 from (3.80) in (3.82), we find

$$\omega_{32}^4[(\lambda_2-\lambda_3)\omega_{44}^1+(\lambda_4-\lambda_2)\omega_{33}^1+(\lambda_3-\lambda_4)\omega_{22}^1]=0. \tag{3.83}$$

As from (3.74), we have seen that assuming

$$(\lambda_2-\lambda_3)\omega_{44}^1+(\lambda_4-\lambda_2)\omega_{33}^1+(\lambda_3-\lambda_4)\omega_{22}^1=0,$$

leads to contradiction, therefore from (3.83), we obtain $\omega_{32}^4=0$, which together with (3.80) gives $\omega_{23}^4=0$. Also, from (3.5), we get $\omega_{34}^2=-\omega_{32}^4$ and $\omega_{23}^4=-\omega_{24}^3$. Consequently, we obtain $\omega_{34}^2=0$, and $\omega_{24}^3=0$, which together with (3.7) gives $\omega_{43}^2=0$, and $\omega_{42}^3=0$, whereby the proof of Lemma 3.3 is complete.

In view of Lemma 3.2 and Lemma 3.3, the equations (3.27), (3.29) and (3.31) reduce to

$$-\omega_{22}^1\omega_{33}^1=\lambda_2\lambda_3,\quad or\quad \frac{e_1(\lambda_2)e_1(\lambda_3)}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}=-\lambda_2\lambda_3, \tag{3.84}$$

$$-\omega_{22}^1\omega_{44}^1=\lambda_2\lambda_4,\quad or\quad \frac{e_1(\lambda_2)e_1(\lambda_4)}{(\lambda_2-\lambda_1)(\lambda_4-\lambda_1)}=-\lambda_2\lambda_4, \tag{3.85}$$

$$-\omega_{33}^1\omega_{44}^1=\lambda_3\lambda_4,\quad or\quad \frac{e_1(\lambda_4)e_1(\lambda_3)}{(\lambda_4-\lambda_1)(\lambda_3-\lambda_1)}=-\lambda_4\lambda_3, \tag{3.86}$$

respectively.

Differentiating (3.11) along e_1 and putting the values of $e_1(\lambda_2), e_1(\lambda_3), e_1(\lambda_4)$, and using (3.6), we find

$$(\lambda_2-\lambda_1)\omega_{22}^1+(\lambda_3-\lambda_1)\omega_{33}^1+(\lambda_4-\lambda_1)\omega_{44}^1=6e_1(H). \tag{3.87}$$

Again, differentiating (3.87) with e_1 and putting the values of $e_1(\omega_{22}^1), e_1(\omega_{33}^1), e_1(\omega_{44}^1)$, and using (3.18)~(3.20) and (3.6), we obtain

$$\begin{aligned}2(\lambda_2-\lambda_1)(\omega_{22}^1)^2+2(\lambda_3-\lambda_1)(\omega_{33}^1)^2+2(\lambda_4-\lambda_1)(\omega_{44}^1)^2-e_1(\lambda_1)(\omega_{22}^1+\omega_{33}^1+\omega_{44}^1)\\ -2H(\lambda_2^2+\lambda_3^2+\lambda_4^2-\lambda_2\lambda_1-\lambda_3\lambda_1-\lambda_4\lambda_1)=6e_1e_1(H).\end{aligned} \tag{3.88}$$

Using (3.36), (3.11) and the fact that $\lambda_1=-2H$ in (3.88), we find

$$\begin{aligned}2(\lambda_2-\lambda_1)(\omega_{22}^1)^2+2(\lambda_3-\lambda_1)(\omega_{33}^1)^2+2(\lambda_4-\lambda_1)(\omega_{44}^1)^2\\ +2e_1(H)(\omega_{22}^1+\omega_{33}^1+\omega_{44}^1)-48H^3=6e_1e_1(H).\end{aligned} \tag{3.89}$$

Putting the values of $\omega_{22}^1,\quad \omega_{33}^1,\quad \omega_{44}^1$ and using (3.6) in (3.89), we obtain

$$3e_1e_1(H)-\Big(\frac{e_1^2(\lambda_2)}{\lambda_2-\lambda_1}+\frac{e_1^2(\lambda_3)}{\lambda_3-\lambda_1}+\frac{e_1^2(\lambda_4)}{\lambda_4-\lambda_1}\Big)-e_1(H)\Big(\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}+\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\frac{e_1(\lambda_4)}{\lambda_4-\lambda_1}\Big)=-24H^3. \tag{3.90}$$

Eliminating $e_1e_1(H)$ from (3.37) and (3.90), we find

$$\Big(\frac{e_1^2(\lambda_2)}{\lambda_2-\lambda_1}+\frac{e_1^2(\lambda_3)}{\lambda_3-\lambda_1}+\frac{e_1^2(\lambda_4)}{\lambda_4-\lambda_1}\Big)-2e_1(H)\Big(\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}+\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\frac{e_1(\lambda_4)}{\lambda_4-\lambda_1}\Big)=72H^3. \tag{3.91}$$

Now, squaring (3.11) and using (3.36), we get

$$\lambda_2\lambda_3+\lambda_4\lambda_2+\lambda_3\lambda_4=12H^2. \tag{3.92}$$

Also, using (3.11), we can express $e_1^2(\lambda_2)$ as follows:
$e_1^2(\lambda_2)=e_1(\lambda_2)(6e_1(H)-e_1(\lambda_3)-e_1(\lambda_4))=6e_1(H)e_1(\lambda_2)-e_1(\lambda_2)e_1(\lambda_3)-e_1(\lambda_2)e_1(\lambda_4),$

or,

$$\frac{e_1^2(\lambda_2)}{\lambda_2-\lambda_1}=\frac{6e_1(H)e_1(\lambda_2)}{\lambda_2-\lambda_1}-\frac{e_1(\lambda_2)e_1(\lambda_3)}{\lambda_2-\lambda_1}-\frac{e_1(\lambda_2)e_1(\lambda_4)}{\lambda_2-\lambda_1},$$

which by using (3.84) and (3.85), can be written as

$$\frac{e_1^2(\lambda_2)}{\lambda_2-\lambda_1} = \frac{6e_1(H)e_1(\lambda_2)}{\lambda_2-\lambda_1} + \lambda_2\lambda_3(\lambda_3-\lambda_1) + \lambda_2\lambda_4(\lambda_4-\lambda_1). \tag{3.93}$$

Similarly, we can show that

$$\frac{e_1^2(\lambda_3)}{\lambda_3-\lambda_1} = \frac{6e_1(H)e_1(\lambda_3)}{\lambda_3-\lambda_1} + \lambda_3\lambda_2(\lambda_2-\lambda_1) + \lambda_3\lambda_4(\lambda_4-\lambda_1), \tag{3.94}$$

$$\frac{e_1^2(\lambda_4)}{\lambda_4-\lambda_1} = \frac{6e_1(H)e_1(\lambda_4)}{\lambda_4-\lambda_1} + \lambda_4\lambda_3(\lambda_3-\lambda_1) + \lambda_4\lambda_2(\lambda_2-\lambda_1). \tag{3.95}$$

On addition of (3.93)~(3.95), we find

$$\begin{aligned}\tfrac{e_1^2(\lambda_2)}{\lambda_2-\lambda_1} + \tfrac{e_1^2(\lambda_3)}{\lambda_3-\lambda_1} + \tfrac{e_1^2(\lambda_4)}{\lambda_4-\lambda_1} = 6e_1(H)(\tfrac{e_1(\lambda_2)}{\lambda_2-\lambda_1} + \tfrac{e_1(\lambda_3)}{\lambda_3-\lambda_1} + \tfrac{e_1(\lambda_4)}{\lambda_4-\lambda_1}) + (\lambda_2\lambda_3+\lambda_4\lambda_3)(\lambda_3-\lambda_1)\\ +(\lambda_2\lambda_4+\lambda_4\lambda_3)(\lambda_4-\lambda_1) + (\lambda_2\lambda_3+\lambda_4\lambda_2)(\lambda_2-\lambda_1).\end{aligned} \tag{3.96}$$

Using (3.92) in (3.96), we get

$$\begin{aligned}\tfrac{e_1^2(\lambda_2)}{\lambda_2-\lambda_1} + \tfrac{e_1^2(\lambda_3)}{\lambda_3-\lambda_1} + \tfrac{e_1^2(\lambda_4)}{\lambda_4-\lambda_1} &= 6e_1(H)(\tfrac{e_1(\lambda_2)}{\lambda_2-\lambda_1} + \tfrac{e_1(\lambda_3)}{\lambda_3-\lambda_1} + \tfrac{e_1(\lambda_4)}{\lambda_4-\lambda_1}) + (12H^2-\lambda_2\lambda_4)(\lambda_3-\lambda_1)\\ &\quad +(12H^2-\lambda_2\lambda_3)(\lambda_4-\lambda_1) + (12H^2-\lambda_4\lambda_3)(\lambda_2-\lambda_1)\\ &= 6e_1(H)(\tfrac{e_1(\lambda_2)}{\lambda_2-\lambda_1} + \tfrac{e_1(\lambda_3)}{\lambda_3-\lambda_1} + \tfrac{e_1(\lambda_4)}{\lambda_4-\lambda_1}) + 12H^2(\lambda_2+\lambda_3+\lambda_4-3\lambda_1)\\ &\quad -3\lambda_2\lambda_3\lambda_4 + \lambda_1(\lambda_2\lambda_3+\lambda_3\lambda_4+\lambda_2\lambda_4).\end{aligned} \tag{3.97}$$

Using (3.92), (3.11) and the fact that $\lambda_1 = -2H$ in (3.97), we find

$$\tfrac{e_1^2(\lambda_2)}{\lambda_2-\lambda_1} + \tfrac{e_1^2(\lambda_3)}{\lambda_3-\lambda_1} + \tfrac{e_1^2(\lambda_4)}{\lambda_4-\lambda_1} = 6e_1(H)(\tfrac{e_1(\lambda_2)}{\lambda_2-\lambda_1} + \tfrac{e_1(\lambda_3)}{\lambda_3-\lambda_1} + \tfrac{e_1(\lambda_4)}{\lambda_4-\lambda_1}) + 120H^3 - 3\lambda_2\lambda_3\lambda_4. \tag{3.98}$$

From (3.91) and (3.98), we obtain

$$e_1(H)(\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1} + \frac{e_1(\lambda_3)}{\lambda_3-\lambda_1} + \frac{e_1(\lambda_4)}{\lambda_4-\lambda_1}) = -12H^3 + \frac{3\lambda_2\lambda_3\lambda_4}{4}. \tag{3.99}$$

Using (3.37) and (3.99), we have

$$e_1e_1(H) = 4H^3 + \frac{3\lambda_2\lambda_3\lambda_4}{4}. \tag{3.100}$$

On the other hand, using (3.6), we can write

$$\begin{aligned}e_1(\lambda_2\lambda_3\lambda_4) &= e_1(\lambda_2)\lambda_3\lambda_4 + e_1(\lambda_3)\lambda_2\lambda_4 + e_1(\lambda_4)\lambda_3\lambda_2\\ &= \omega_{22}^1(\lambda_2-\lambda_1)\lambda_3\lambda_4 + \omega_{33}^1(\lambda_3-\lambda_1)\lambda_2\lambda_4 + \omega_{44}^1(\lambda_4-\lambda_1)\lambda_3\lambda_2\\ &= \lambda_2\lambda_3\lambda_4(\omega_{22}^1+\omega_{33}^1+\omega_{44}^1) - \lambda_1(\lambda_3\lambda_4\omega_{22}^1 + \lambda_2\lambda_4\omega_{33}^1 + \lambda_3\lambda_2\omega_{44}^1),\end{aligned} \tag{3.101}$$

which on using (3.6) and (3.84)~(3.86), gives

$$e_1(\lambda_2\lambda_3\lambda_4) = \lambda_2\lambda_3\lambda_4(\tfrac{e_1(\lambda_2)}{\lambda_2-\lambda_1} + \tfrac{e_1(\lambda_3)}{\lambda_3-\lambda_1} + \tfrac{e_1(\lambda_4)}{\lambda_4-\lambda_1}) - 6H(\tfrac{e_1(\lambda_2)}{\lambda_2-\lambda_1})(\tfrac{e_1(\lambda_3)}{\lambda_3-\lambda_1})(\tfrac{e_1(\lambda_4)}{\lambda_4-\lambda_1}). \tag{3.102}$$

Now, using (3.18)~(3.20), we compute

$$\begin{aligned}&e_1(\omega_{22}^1\omega_{33}^1\omega_{44}^1) = e_1(\omega_{22}^1)\omega_{33}^1\omega_{44}^1 + e_1(\omega_{33}^1)\omega_{22}^1\omega_{44}^1) + e_1(\omega_{44}^1)\omega_{33}^1\omega_{22}^1\\ &= [(\omega_{22}^1)^2 - 2H\lambda_2]\omega_{33}^1\omega_{44}^1 + [(\omega_{33}^1)^2 - 2H\lambda_3]\omega_{22}^1\omega_{44}^1 + [(\omega_{44}^1)^2 - 2H\lambda_4]\omega_{33}^1\omega_{22}^1,\end{aligned} \tag{3.103}$$

which on using (3.84)~(3.86) gives

$$e_1(\omega_{22}^1\omega_{33}^1\omega_{44}^1) = \lambda_2^2\lambda_3\lambda_4 + \lambda_3^2\lambda_2\lambda_4 + \lambda_4^2\lambda_3\lambda_2 + 6H\lambda_2\lambda_3\lambda_4 = \lambda_2\lambda_3\lambda_4(\lambda_2 + \lambda_3 + \lambda_4 + 6H) \tag{3.104}$$

Using (3.11) and (3.6) in (3.104), we find

$$12H\lambda_2\lambda_3\lambda_4 = e_1(\frac{e_1(\lambda_2)}{\lambda_2 - \lambda_1}\frac{e_1(\lambda_3)}{\lambda_3 - \lambda_1}\frac{e_1(\lambda_4)}{\lambda_4 - \lambda_1}). \tag{3.105}$$

Also, multiplying (3.84), (3.85) and (3.86), we get

$$(\lambda_2\lambda_3\lambda_4)^2 = -(\frac{e_1(\lambda_2)}{\lambda_2 - \lambda_1}\frac{e_1(\lambda_3)}{\lambda_3 - \lambda_1}\frac{e_1(\lambda_4)}{\lambda_4 - \lambda_1})^2. \tag{3.106}$$

Differentiating (3.106) along e_1, and using (3.105), we obtain

$$e_1(\lambda_2\lambda_3\lambda_4) = -12H(\frac{e_1(\lambda_2)}{\lambda_2 - \lambda_1}\frac{e_1(\lambda_3)}{\lambda_3 - \lambda_1}\frac{e_1(\lambda_4)}{\lambda_4 - \lambda_1}). \tag{3.107}$$

Using (3.102) in (3.107), we have

$$e_1(\lambda_2\lambda_3\lambda_4) = 2\lambda_2\lambda_3\lambda_4(\frac{e_1(\lambda_2)}{\lambda_2 - \lambda_1} + \frac{e_1(\lambda_3)}{\lambda_3 - \lambda_1} + \frac{e_1(\lambda_4)}{\lambda_4 - \lambda_1}). \tag{3.108}$$

Differentiating (3.99) with e_1, we find

$$\begin{aligned} &e_1e_1(H)(\tfrac{e_1(\lambda_2)}{\lambda_2-\lambda_1} + \tfrac{e_1(\lambda_3)}{\lambda_3-\lambda_1} + \tfrac{e_1(\lambda_4)}{\lambda_4-\lambda_1}) + e_1(H)(e_1(\tfrac{e_1(\lambda_2)}{\lambda_2-\lambda_1}) + e_1(\tfrac{e_1(\lambda_3)}{\lambda_3-\lambda_1}) + e_1(\tfrac{e_1(\lambda_4)}{\lambda_4-\lambda_1})) \\ &= -36H^2e_1(H) + \tfrac{3e_1(\lambda_2\lambda_3\lambda_4)}{4}. \end{aligned} \tag{3.109}$$

Putting the values of $e_1e_1(H)$ and $e_1(\lambda_2\lambda_3\lambda_4)$ from (3.100) and (3.108) in (3.109), we get

$$\begin{aligned} &(4H^3 - \tfrac{3\lambda_2\lambda_3\lambda_4}{4})(\tfrac{e_1(\lambda_2)}{\lambda_2-\lambda_1} + \tfrac{e_1(\lambda_3)}{\lambda_3-\lambda_1} + \tfrac{e_1(\lambda_4)}{\lambda_4-\lambda_1}) + e_1(H)(e_1(\tfrac{e_1(\lambda_2)}{\lambda_2-\lambda_1}) + e_1(\tfrac{e_1(\lambda_3)}{\lambda_3-\lambda_1}) + e_1(\tfrac{e_1(\lambda_4)}{\lambda_4-\lambda_1})) \\ &= -36H^2e_1(H). \end{aligned} \tag{3.110}$$

Using (3.18)~(3.20) in (3.110) and using (3.11), we have

$$\begin{aligned} &(4H^3 - \tfrac{3\lambda_2\lambda_3\lambda_4}{4})(\tfrac{e_1(\lambda_2)}{\lambda_2-\lambda_1} + \tfrac{e_1(\lambda_3)}{\lambda_3-\lambda_1} + \tfrac{e_1(\lambda_4)}{\lambda_4-\lambda_1}) + e_1(H)((\tfrac{e_1(\lambda_2)}{\lambda_2-\lambda_1})^2 + (\tfrac{e_1(\lambda_3)}{\lambda_3-\lambda_1})^2 + (\tfrac{e_1(\lambda_4)}{\lambda_4-\lambda_1})^2 - 12H^2) \\ &= -36H^2e_1(H). \end{aligned} \tag{3.111}$$

Also, from (3.84) ~ (3.86), we find

$$(\frac{e_1(\lambda_2)}{\lambda_2 - \lambda_1})^2 = -\lambda_2^2, \quad (\frac{e_1(\lambda_3)}{\lambda_3 - \lambda_1})^2 = -\lambda_3^2, \quad (\frac{e_1(\lambda_4)}{\lambda_4 - \lambda_1})^2 = -\lambda_4^2. \tag{3.112}$$

Using (3.112) and (3.36) in (3.111), we obtain

$$(4H^3 - \tfrac{3\lambda_2\lambda_3\lambda_4}{4})(\tfrac{e_1(\lambda_2)}{\lambda_2-\lambda_1} + \tfrac{e_1(\lambda_3)}{\lambda_3-\lambda_1} + \tfrac{e_1(\lambda_4)}{\lambda_4-\lambda_1}) = -12H^2e_1(H). \tag{3.113}$$

Eliminating $e_1(H)$ from (3.99) and (3.113), we find

$$(4H^3 - \tfrac{3\lambda_2\lambda_3\lambda_4}{4})(\tfrac{e_1(\lambda_2)}{\lambda_2-\lambda_1} + \tfrac{e_1(\lambda_3)}{\lambda_3-\lambda_1} + \tfrac{e_1(\lambda_4)}{\lambda_4-\lambda_1})^2 = -12H^2(-12H^3 + \tfrac{3\lambda_2\lambda_3\lambda_4}{4}). \tag{3.114}$$

Also, from (3.112), (3.92) and (3.84) ~ (3.86), we obtain

$$(\tfrac{e_1(\lambda_2)}{\lambda_2-\lambda_1} + \tfrac{e_1(\lambda_3)}{\lambda_3-\lambda_1} + \tfrac{e_1(\lambda_4)}{\lambda_4-\lambda_1})^2 = -36H^2. \tag{3.115}$$

From (3.115) and (3.114), we get

$$3(4H^3 - \frac{3\lambda_2\lambda_3\lambda_4}{4}) = -12H^3 + \frac{3\lambda_2\lambda_3\lambda_4}{4},$$

or,

$$\lambda_2\lambda_3\lambda_4 = 8H^3. \tag{3.116}$$

Differentiating (3.113) along e_1, we find

$$\begin{aligned}(4H^3 - \tfrac{3\lambda_2\lambda_3\lambda_4}{4})e_1(\tfrac{e_1(\lambda_2)}{\lambda_2-\lambda_1} + \tfrac{e_1(\lambda_3)}{\lambda_3-\lambda_1} + \tfrac{e_1(\lambda_4)}{\lambda_4-\lambda_1}) + (12H^2 e_1(H)\\ -\tfrac{3e_1(\lambda_2\lambda_3\lambda_4)}{4})(\tfrac{e_1(\lambda_2)}{\lambda_2-\lambda_1} + \tfrac{e_1(\lambda_3)}{\lambda_3-\lambda_1} + \tfrac{e_1(\lambda_4)}{\lambda_4-\lambda_1}) = -12H^2 e_1e_1(H) - 24He_1^2(H).\end{aligned} \tag{3.117}$$

Using (3.18)~(3.20), (3.112) and (3.11), we can show that

$$e_1(\tfrac{e_1(\lambda_2)}{\lambda_2-\lambda_1} + \tfrac{e_1(\lambda_3)}{\lambda_3-\lambda_1} + \tfrac{e_1(\lambda_4)}{\lambda_4-\lambda_1}) = -24H^2. \tag{3.118}$$

Using (3.99), (3.100), (3.108), (3.115) and (3.118) in (3.117), we find

$$\begin{aligned}(4H^3 - \tfrac{3\lambda_2\lambda_3\lambda_4}{4})(-24H^2) + [12H^2(-12H^3 + \tfrac{3\lambda_2\lambda_3\lambda_4}{4}) - \tfrac{3\lambda_2\lambda_3\lambda_4}{2}(-36H^2)]\\ = -12H^2(4H^3 + \tfrac{3\lambda_2\lambda_3\lambda_4}{4}) - 24He_1^2(H),\end{aligned} \tag{3.119}$$

which on simplifying, gives

$$90H^2\lambda_2\lambda_3\lambda_4 - 192H^5 = -24He_1^2(H). \tag{3.120}$$

On the other hand from (3.99) and (3.113), we get

$$-12H^2e_1^2(H) = (-12H^3 + \frac{3\lambda_2\lambda_3\lambda_4}{4})(4H^3 - \frac{3\lambda_2\lambda_3\lambda_4}{4}). \tag{3.121}$$

Now, eliminating $e_1^2(H)$ from (3.120) and (3.121), we obtain

$$x^2 + 44xH^3 - 48H^6 = 0, \tag{3.122}$$

where $x = \frac{3\lambda_2\lambda_3\lambda_4}{4}$.

On solving (3.122), we get

$$x = \frac{3\lambda_2\lambda_3\lambda_4}{4} = (-22 \pm 2\sqrt{133})H^3. \tag{3.123}$$

Hence, from (3.116) and (3.123), we get that H must be zero.

(b) *Three distinct principal curvatures*

Suppose that M is a biharmonic hypersurface with three distinct principal curvatures and zero scalar curvature with shape operator diagonal. We also assume that mean curvature is not constant. From (2.9), it is easy to see that grad$H$ is an eigenvector of the shape operator A with the corresponding principal curvature $-2H$. Without loss of generality, we choose e_1 in the direction of grad$H$ and therefore shape operator A of hypersurface will take the following form with respect to a suitable frame $\{e_1, e_2, e_3, e_4\}$

$$A_H = \begin{pmatrix} -2H & & & \\ & \lambda & & \\ & & \lambda & \\ & & & \lambda_4 \end{pmatrix}. \tag{3.124}$$

From (3.11), (3.36) and (3.124), we get

$$2\lambda + \lambda_4 = 6H \tag{3.125}$$

$$2\lambda^2 + \lambda_4^2 = 12H^2 \tag{3.126}$$

From (3.125) and (3.126), we find

$$\lambda = \lambda_4 = 2H, \tag{3.127}$$

which contradicts the assumption of three distinct principal curvatures. Hence, there exist no biharmonic hypersurface with three distinct principal curvatures of zero scalar curvature in E^5.

(c) *Two distinct principal curvatures*

Suppose that M is a nonminimal biharmonic hypersurface with two distinct principal curvatures and zero scalar curvature with shape operator diagonal. From (2.9), it is easy to see that grad$H$ is an eigenvector of the shape operator A with the corresponding principal curvature $-2H$. Without loss of generality, we choose e_1 in the direction of grad$H$ and therefore shape operator A of hypersurface will take the following form with respect to a suitable frame $\{e_1, e_2, e_3, e_4\}$

$$A_H = \begin{pmatrix} -2H & & & \\ & \lambda & & \\ & & \lambda & \\ & & & \lambda \end{pmatrix}. \tag{3.128}$$

From (3.11), (3.36) and (3.128), we get

$$\lambda = 2H \tag{3.129}$$

Also, from (3.6) and (3.128), we obtain

$$\omega_{22}^1 = \omega_{33}^1 = \omega_{44}^1 = \frac{e_1(H)}{2H} \tag{3.130}$$

Also, from (2.5), $R(e_1, e_2, e_1, e_2)$ shows that

$$e_1(\omega_{22}^1) = (\omega_{22}^1)^2 - 4H^2. \tag{3.131}$$

Using (3.130) and (3.131), we find

$$e_1 e_1(H) = \frac{3e_1^2(H)}{2H} - 8H^3. \tag{3.132}$$

On the other hand, from (2.8), (2.10), (3.128), and (3.130), we have

$$e_1 e_1(H) = \frac{3e_1^2(H)}{2H} + 16H^3. \tag{3.133}$$

From (3.132) and (3.133), we get that H must be zero, which is a contradiction.

Therefore, we conclude that:

Theorem 3.4. *There exists no proper biharmonic hypersurface M in the Euclidean space E^5 of zero scalar curvature.*

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