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Addendum to: "*Quasicrystals, Almost Periodic Patterns, Mean Periodic Functions and Irregular Sampling*"

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To Alexander Olevskii with admiration and gratitude

Abstract. This note completes [15]. Sharp results on Poisson summation formula and irregular sampling are announced. In both cases tools from [15] are needed.

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1 Introduction

The readers of the *African Diaspora Journal of Mathematics* might be eager to know the solution to some of the problems which were discussed in [15]. Since *Quasicrystals, almost periodic patterns, mean periodic functions and irregular sampling* was published spectacular advances were achieved by S. Grepstad, N. Lev, and A. Olevskii. These achievements (Theorem 2.2, Theorem 2.3, Theorem 3.11, and Theorem 3.12) will be described and commented in this note. Theorem 3.6, Theorem 3.7, and the last section on open problems are the only original contributions of the author.

A set $\Lambda \subset \mathbb{R}^n$ is called uniformly discrete (u.d.) if

$$d(\Lambda) := \inf_{\lambda \neq \lambda';\, \lambda,\lambda' \in \Lambda} |\lambda - \lambda'| > 0 \tag{1.1}$$

The two following issues were addressed in [15] :

Problem 1.1. *Which uniformly discrete set Λ can support a measure μ whose Fourier transform $\hat{\mu}$ is concentrated on a u.d. set S ?*

In the strong form of Problem 1 $\hat{\mu}$ shall be supported on a u.d. set S. The difference between the two formulations (concentrated vs. supported) will be explained below.

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Problem 1.2. *Let Λ be a Delone set. We are interested in the collection $C(\Lambda)$ of compact sets K such that*

$$C(\sum_{\lambda\in\Lambda}|c(\lambda)|^2)^{1/2} \leq (\int_K |\sum_{\lambda\in\Lambda} c(\lambda)\exp(2\pi i\,\lambda\cdot x)|^2\,dx)^{1/2} \tag{1.2}$$

holds for a positive constant C and every sequence $c(\lambda) \in l^2(\Lambda)$. Does there exist a Delone set Λ such that (1.2) *becomes true when the Lebesgue measure $|K|$ of K is large enough ? Does there exist a Delone set Λ such that $|K| > D^+(\Lambda)$ implies $K \in C(\Lambda)$?*

Delone sets and the upper uniform density $D^+(\Lambda)$ will be defined below.

In the solution to Problem 1 given in Section 4 of [15] μ is supported on a model set Λ and its Fourier transform $\hat{\mu}$ is an atomic measure whose support is dense in $\mathbb{R}^n$. Moreover for any positive ϵ there exists a u.d. set S_ϵ such that the ratio between (a) the mass of $\hat{\mu}$ on $B(x,R)\setminus S_\epsilon$ and (b) the mass of $\hat{\mu}$ on $B(x,R)$ does not exceed ϵ. This holds for every $R \geq 1$ and uniformly in x. We then say that $\hat{\mu}$ is *concentrated* on S_ϵ as ϵ tends to 0. However $\hat{\mu}$ is not *supported* on S_ϵ. This limitation in the construction given in [15] comes from the fact that a function and its Fourier transform cannot be simultaneously compactly supported. N. Lev and A. Olevskii digged much deeper in [9], [10] and proved that the standard Poisson formula is the unique solution to the strong version of Problem 1. This outstanding achievement will be described in Section 2. On the other hand S. Grepstad and N. Lev improved on the partial answer to Problem 2 which was given in [15]. Their spectacular results will be announced in Section 3.

2 Poisson summation formula

This section is a continuation of Section 4 of [15].

Definition 2.1. *We denote by $\mathcal{M}$ the collection of (complex) measures μ which are supported on a u.d. set Λ and which are temperate distributions.*

We then have

$$\mu = \sum_{\lambda\in\Lambda} c(\lambda)\delta_\lambda \tag{2.1}$$

where $c(\lambda) \neq 0$, $d(\Lambda) > 0$. Moreover $|c(\lambda)| \leq C(1+|\lambda|)^N$ for some exponent N. Then the Fourier transform $\hat{\mu}$ of μ exists in the distributional sense. We have

$$\hat{\mu}(x) = \sum_{\lambda\in\Lambda} c(\lambda)\exp(-2\pi i\,\lambda\cdot x) \tag{2.2}$$

In the following two theorems N. Lev and A. Olevskii characterized the measures $\mu \in \mathcal{M}$ whose Fourier transform also belongs to $\mathcal{M}$. Then $\hat{\mu}$ is supported on a u.d. set S and we have :

$$\hat{\mu} = \sum_{s\in S} b(s)\delta_s,\ b(s) \neq 0, d(S) > 0 \tag{2.3}$$

The set S is the spectrum of the measure μ. N. Lev and A. Olevskii proved the following in [9], [10] :

Theorem 2.2. *In the one-dimensional case, let μ be a measure in $\mathcal{M}$ whose Fourier transform also belongs to $\mathcal{M}$. Then the support Λ of μ is contained in a finite union of translates of a certain lattice $\Gamma = h\mathbb{Z}$, $h > 0$. The same is true for S (with the dual lattice $\Gamma^* = h^{-1}\mathbb{Z}$).*

Theorem 2.2 is valid in several dimensions under a positivity assumption.

Theorem 2.3. *Let $\mu \in \mathcal{M}$ be a positive measure satisfying* (2.3). *Then the support Λ of μ is contained in a finite union of translates of a certain lattice $\Gamma = A\mathbb{Z}^n$, A being an invertible matrix.*

Let us sketch the proof of Theorem 2.2 which is given in [9]. We begin with two definitions.

Definition 2.4. *A subset Λ of $\mathbb{R}^n$ is a Delone set if there exist two radii $R_2 > R_1 > 0$ such that*

(a) every ball with radius R_1, whatever be its location, shall contain at most one point in Λ

(b) every ball with radius R_2, whatever be its location, shall contain at least one point in Λ.

Definition 2.5. *A Delone set Λ is a "Meyer set" if there exists a finite set F such that $\Lambda - \Lambda \subset \Lambda + F$.*

Here $\Lambda - \Lambda$ denotes the set of all differences $\lambda - \lambda'$, $\lambda, \lambda' \in \Lambda$. J. Lagarias proved the following result [7] :

Lemma 2.6. *Let Λ be a Delone set such that $\Lambda - \Lambda$ is still a Delone set. Then Λ is a "Meyer set".*

The definition of a model set $\Lambda \subset \mathbb{R}^n$ is given now.

Let $m \in \mathbb{N}, N = n + m, \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$ and consider a lattice $\Gamma \subset \mathbb{R}^N$. If $(x,y) = X \in \mathbb{R}^n \times \mathbb{R}^m$, we write $x = p_1(X)$ and $y = p_2(X)$.

Let $\Gamma^* \subset \mathbb{R}^N$ be the dual lattice and p_1^*, p_2^* be defined as p_1, p_2. Let us assume that $p_1 : \Gamma \to p_1(\Gamma)$ is a one-to-one mapping and that $p_1(\Gamma)$ is a dense subgroup of $\mathbb{R}^n$. We impose the same properties on p_2. Then $p_1^* : \Gamma^* \to p_1^*(\Gamma^*)$ is a one-to-one mapping and $p_1^*(\Gamma^*)$ is a dense subgroup of $\mathbb{R}^n$ and the same holds for p_2^*. A set $W \subset \mathbb{R}^m$ is Riemann integrable if its boundary has a zero Lebesgue measure.

Definition 2.7. *Let W be a Riemann integrable compact subset of $\mathbb{R}^m$ with a positive measure. Then the model set Λ defined by Γ and W is*

(2.4) $$\Lambda = \{\lambda = p_1(\gamma); \gamma \in \Gamma, \, p_2(\gamma) \in W\}$$

The compact set W is named the *window* of the model set Λ.
A model set is *simple* if $m = 1$ and $W = [a,b)$ is a semi-closed interval.

"Meyer sets" are characterized by the following property [13] :

Lemma 2.8. *A Delone set Λ is a "Meyer set" if and only if there exists a lattice or a model set M and a finite set F such that $\Lambda \subset M + F$.*

One more definition is needed. The ball centered at x with radius R is denoted by $B(x,R)$ and $\#E$ denotes the cardinality of a finite set E.

Definition 2.9. *Let Λ be a u.d. set, $R > 0$, and*

(2.5) $$D_R^+(\Lambda) = \sup_{x \in \mathbb{R}^n} |B(x,R)|^{-1} \#(\Lambda \cap B(x,R))$$

The uniform upper density of Λ is defined by

(2.6) $$D^+(\Lambda) = \limsup_{R \to \infty} D_R^+(\Lambda)$$

N. Lev and A. Olevskii improved on Lemma 2.8.6 :

Lemma 2.10. *Let* Λ *be a Delone set. If* $D^{+}(\Lambda - \Lambda)$ *is finite, then* Λ *is a "Meyer set".*

The assumption in Lemma 2.10 is equivalent to the following: there exists a constant $R > 0$ such that $\sup_{x\in\mathbb{R}^n} \#(\Lambda \cap B(x,R)) = C < \infty$. If $\Lambda - \Lambda$ was assumed to be uniformly discrete, then $\Lambda - \Lambda$ would be a Delone set and Lemma 2.10 would follow from Lagarias' theorem.

Let us assume $n = 1$ and sketch the proof of Theorem 2.2 given in [9]. N. Lev and A. Olevskii need to apply Lemma 2.10 to the support Λ of μ. In order to check that $D^{+}(\Lambda - \Lambda)$ is finite they prove a stronger property (2.7) which is defined now. Let $\Lambda_h = \Lambda \cap (\Lambda - h)$, $h \in \Lambda - \Lambda$. The stronger property is the existence of a positive constant c such that for every $h \in \Lambda - \Lambda$ we have

$$D_{\#}(\Lambda_h) \geq c > 0 \tag{2.7}$$

Here the lower density of a u.d. set S is defined as

$$D_{\#}(S) = \liminf_{R\to\infty} |B(0,R)|^{-1} \#(S \cap B(0,R)) \tag{2.8}$$

Fine results on spectral gaps obtained by M. Mitkovski and A. Poltoratski in [16] are needed to prove (2.7). But (2.7) implies that $D^{+}(\Lambda - \Lambda)$ is finite. Finally Λ is a "Meyer set" as Lemma 2.10 shows. Using (2.7) again together with Lemma 2.8, N. Lev and A. Olevskii prove that this "Meyer set" Λ is contained in $L + F$ where L is a lattice and F is finite. This ends the proof of Theorem 2.2.

3 Irregular sampling

Is it possible to adapt the sampling rate of a signal to its sparcity ? Following the paradigm of "compressed sensing" a signal (or an image) is sparse if the Lebesgue measure $|K|$ of its spectrum K is small. The spectrum $K = \sigma(f)$ of $f \in L^2(\mathbb{R}^n)$ is defined in this essay as the closure of the support of the Fourier transform $\hat{f}$ of f. This Fourier transform is normalized by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} \exp(-2\pi i x\cdot\xi) f(x)\, dx \tag{3.1}$$

If K is a compact subset of $\mathbb{R}^n$ the Paley-Wiener space PW_K is defined by

$$PW_K = \{f \in L^2(\mathbb{R}^n);\ \sigma(f) \subset K\} \tag{3.2}$$

The definition of the Paley-Wiener space makes sense when K is a Borel set but Theorem 3.5 cannot hold if K is not compact, as it will be proved below.

Given a (small) positive number β and a compact set K with $|K| \leq \beta$ we want to sample the functions $f \in PW_K$ in the most efficient way. We are given a uniformly discrete set of points $\Lambda \subset \mathbb{R}^n$ and the sample of f is defined by $f(\lambda)$, $\lambda \in \Lambda$. The goal is to recover $f \in PW_K$ from $f(\lambda)$, $\lambda \in \Lambda$. If Λ is uniformly discrete we have $f(\lambda) \in l^2(\Lambda)$.

The problem of recovering f from its sample has two versions, denoted by (a) and (b). In (a) Λ and K are given and we wish to retrieve f from its sample on Λ. In (b) Λ and a "small" constant β are given, K is unknown and the only information which is at our disposal is (i) $f \in PW_K$ with $|K| \leq \beta$ and (ii) the sample of f on Λ. Can f be retrieved from its sample ?

We now consider the first issue (K is given) and follow H.J. Landau [8].

Definition 3.1. *Let K be a compact subset of $\mathbb{R}^n$. A collection of points $\Lambda \subset \mathbb{R}^n$ is a set of stable sampling for PW_K if there exists a constant C such that*

$$f \in PW_K \Rightarrow \|f\|_2^2 \le C \sum_{\lambda\in\Lambda} |f(\lambda)|^2 \tag{3.3}$$

The property of stable sampling is equivalent to the following: every $F \in L^2(K)$ is the sum of a generalized Fourier expansion

$$F(x) = \sum_{\lambda\in\Lambda} c(\lambda)\exp(2\pi i\,\lambda\cdot x) \tag{3.4}$$

where

$$\sum_{\lambda\in\Lambda} |c(\lambda)|^2 \le C \int_K |F(x)|^2\,dx \tag{3.5}$$

and where the series (3.4) converges to F in $L^2(K)$. If Λ is a set of stable sampling for the space PW_K, any "band-limited" $f \in PW_K$ can be reconstructed from its sample $f(\lambda)$, $\lambda \in \Lambda$, and this reconstruction can be achieved by a linear algorithm. This remark follows from some general properties of frames.

We also consider the property of stable interpolation [8].

Definition 3.2. *Let K be a compact subset of $\mathbb{R}^n$. Then $\Lambda \subset \mathbb{R}^n$ is a set of stable interpolation for PW_K if there exists a constant C such that for every square-summable sequence $c(\lambda)$, $\lambda \in \Lambda$, we have*

$$\sum_{\lambda\in\Lambda} |c(\lambda)|^2 \le C \int_K |\sum_{\lambda\in\Lambda} c(\lambda)\exp(2\pi i\,x\cdot\lambda)|^2\,dx \tag{3.6}$$

A set of stable interpolation is uniformly discrete. An equivalent definition of a set of stable interpolation for PW_K is the following property. Every square-summable sequence $a(\lambda)$, $\lambda \in \Lambda$, can be interpolated by a function f in PW_K : $f(\lambda) = a(\lambda)$, $\lambda \in \Lambda$.

If Λ is both a set of stable sampling and of stable interpolation for PW_K, then the family of functions $\exp(2\pi i\lambda\cdot x), \lambda \in \Lambda$, is a Riesz basis of $L^2(K)$.

H.J. Landau discovered necessary conditions for sampling and interpolation [8]. These conditions are relating the upper or lower density of Λ to the measure of K.

If $\Lambda \subset \mathbb{R}^n$ is a collection of points, the upper density of Λ is defined by (2.6) and the lower density is defined similarly :

$$\underline{\mathrm{dens}}\,\Lambda = \liminf_{R\to\infty} |B(x,R)|^{-1} \inf_{x\in\mathbb{R}^n} \#(\Lambda \cap B(x,R)) \tag{3.7}$$

A collection Λ of points has a uniform density (denoted by $\mathrm{dens}\,\Lambda$) if the upper density and the lower density of Λ coincide. Here is Landau's theorem.

Theorem 3.3. *If Λ be a set of stable sampling for PW_K, then*

$$\underline{\text{dens}}\,\Lambda \geq |K| \tag{3.8}$$

Similarly if Λ be a set of stable interpolation for PW_K, then

$$\overline{\text{dens}}\,\Lambda \leq |K| \tag{3.9}$$

These necessary conditions are not sufficient. But they are almost sufficient (i) in the one dimensional case when K is an interval or (ii) in any dimension when Λ is a simple model set. Here "almost" means that in (3.8) and (3.9) large inequalities are replaced by strict ones. Surprisingly these two cases will both be treated using Beurling's theorem. Let us consider (i) first. A. Beurling proved the following [3]

Theorem 3.4. *Let Λ be a uniformly discrete set of real numbers. For any interval J the condition*

$$\underline{\text{dens}}\,\Lambda > |J| \tag{3.10}$$

implies that Λ is a set of stable sampling for PW_J and similarly

$$\overline{\text{dens}}\,\Lambda < |J| \tag{3.11}$$

implies that Λ is a set of stable interpolation for PW_J.

This settles (i) with the exception of the limiting case $\text{dens}\,\Lambda = |J|$. Let us turn to case (ii). A model set is *simple* if $m = 1$ and $W = [a, b)$ is a semi-closed interval. B. Matei and the author proved the following theorem [11] :

Theorem 3.5. *Let Λ be a simple quasicrystal and $K \subset \mathbb{R}^n$ be a compact set. Then Λ is a set of stable sampling for the Hilbert space PW_K whenever $|K| < \text{dens}\,\Lambda$. We now assume that K is Riemann integrable. Then Λ is a set of stable interpolation whenever $|K| > \text{dens}\,\Lambda$.*

If Λ is an ordinary lattice these properties cannot hold and the obstacle is aliasing. To suppress aliasing it suffices to use simple quasicrystals instead of lattices. Theorem 3.5 does not hold in general if K is not compact.

Theorem 3.6. *Let Λ be a simple quasicrystal. For every positive ϵ there exists a closed set F with $|F| \leq \epsilon$ such that Λ is not a set of stable sampling for PW_F.*

Here is the proof. Theorem 3.5 implies that Λ is a set of stable sampling for $B(0, \epsilon/2)$ if ϵ is small enough. Any model set is harmonious [14], [15]. Therefore for every $\eta > 0$ there exists a relatively dense set $M(\eta)$ such that

$$\tau \in M(\eta) \Rightarrow |\exp(2\pi i \tau \cdot \lambda) - 1| \leq \eta \tag{3.12}$$

Then we set $\eta = 2^{-j}$ and pick $\lambda_j \in M(2^{-j})$ with $|\lambda_j| > 2^j$. Next $B_j = B(\lambda_j, \epsilon 2^{-j})$ and finally F is defined as the union $B(0, \epsilon/2) \bigcup_2^\infty B_j$. We have $|F| = \epsilon$. Let g_j be any function in L^2 enjoying the following two properties: (i) the Fourier transform of g_j is supported on the ball $B(0, \epsilon 2^{-j})$ and (ii) $\|g_j\|_2 = 1$. The counter example is $f_j(x) = (\exp(2\pi i \lambda_j \cdot x) - 1) g_j(x)$. Then $f_j \in PW_F$ when $2^{-j} \leq \epsilon/2$ and $\|f_j\|_2 = \sqrt{2}$. We have $\sum_{\lambda \in \Lambda} |g_j(\lambda)|^2 \leq C$ since Λ is a set of stable sampling for $B(0, \epsilon/2)$. This and (3.12) imply $\sum_{\lambda \in \Lambda} |f_j(\lambda)|^2 \leq C 2^{-2j}$ which contradicts the property of stable sampling when j is sufficiently large.

Theorem 3.5 paves the way to the solution of Problem (b).

Theorem 3.7. *Let Λ be a simple quasicrystal. Then every function $f \in L^2(\mathbb{R}^n)$ whose spectrum $\sigma(f)$ is compact and satisfies $|\sigma(f)| \leq \beta < \frac{1}{2}\operatorname{dens}\Lambda$ is uniquely defined by its sample $f(\lambda)$, $\lambda \in \Lambda$, and this result is optimal.*

The proof is straightforward. Let $f_1 \in PW_{K_1}$ and $f_2 \in PW_{K_2}$ satisfy $f_1 = f_2$ on Λ. Then the spectrum of $f = f_1 - f_2$ lies in $K = K_1 \cup K_2$ and we have $f(\lambda) = 0$, $\lambda \in \Lambda$. It implies $f = 0$ by Theorem 3.5. This proof does not give any algorithm for retrieving a sparse signal from its sample.

The optimality of the condition $\beta < \frac{1}{2}\operatorname{dens}\Lambda$ is proved by the argument used in [11]. We claim that for every $\beta > \frac{1}{2}\operatorname{dens}\Lambda$ there exist two compact sets K and L such that $|K| = |L| = \beta$ and two functions g and h such that $g \in PW_K$, $h \in PW_L$, $g(\lambda) = h(\lambda)$, $\forall \lambda \in \Lambda$, and $g \neq h$. Here is the construction. Take any pair K, L consisting of two Riemann integrable compact sets with $|K| = |L| > d/2$ where $d = \operatorname{dens}\Lambda$ and $K \cap L = \emptyset$. Consider the union $K' = K \cup L$. Then $|K'| > d$ which implies that Λ is a set of stable interpolation for $PW_{K'}$. Redundancy in Theorem 3.5 implies that there exists a function $f \in PW_{K'}$ which is not identically 0 and vanishes on Λ. Then f can be written as a sum $f = g + h$ where $g \in PW_K$ and $h \in PW_L$. Finally g and $-h$ are the functions we are looking for.

Theorem 3.5 does not cover the limiting case $|K| = \operatorname{dens}\Lambda$. This was achieved by S. Grepstad and N. Lev [4] and depends on a remarkable theorem by S. A. Avdonin.

3.1 Avdonin's theorem

S. Grepstad and N. Lev used two ingredients to address the limiting case $|K| = \operatorname{dens}\Lambda$. The first one is a duality between stable sampling and stable interpolation. The following lemma is an improved version of an observation which was already used in [11], [15].

Let $\Gamma \subset \mathbb{R}^n \times \mathbb{R}$ be a lattice as in Definition 2.7, $W = I = [a, b)$ be an interval, and let $\Lambda(\Gamma, I) \subset \mathbb{R}^n$ be the simple quasicrystal defined by (2.4). The "dual" quasicrystal $\Lambda^*(\Gamma^*, K) \subset \mathbb{R}$ is defined by

$$\Lambda^*(\Gamma^*, K) = \{\lambda^* = p_2^*(\gamma^*);\ \gamma^* \in \Gamma^*,\ p_1^*(\gamma^*) \in K\} \tag{3.13}$$

This duality is not the one which was studied in [13] and which was consistent with the definition of the dual lattice.

Lemma 3.8. *With these notations $\Lambda(\Gamma, I)$ is a set of stable sampling for PW_K if and only if $\Lambda^*(\Gamma^*, K)$ is a set of stable interpolation for PW_I. Similarly $\Lambda(\Gamma, I)$ is a set of stable interpolation for PW_K if and only if $\Lambda^*(\Gamma^*, K)$ is a set of stable sampling for PW_I.*

This lemma was used in [11] and [15] to reduce the proof of Theorem 3.5 to the one-dimensional case. Indeed Lemma 3.8 and Beurling's theorem immediately imply Theorem 3.5. To treat the limiting case ($|K| = \operatorname{dens}\Lambda$) S. Grepstad and N. Lev followed a similar strategy. Instead of relying on Beurling's theorem, they used Avdonin's theorem which is an outstanding improvement on M. I. Kadec's 1/4 theorem.

Theorem 3.9. *Let λ_j, $j \in \mathbb{Z}$, be a sequence of real numbers, let $\eta_j = \lambda_j - j$, $j \in \mathbb{Z}$ and let us assume that the following conditions* (a), (b) *and* (c) *hold*

(a) $|\lambda_j - \lambda_k| \geq c > 0$, $j \neq k$

(b) $\sup_{j \in \mathbb{Z}} |\lambda_j - j| \leq C$

(c) *Let us assume that there exists a positive integer N such that*

$$\sup_{k\in\mathbb{Z}} \frac{1}{N}\Big|\sum_{j=k}^{k+N-1} \eta_j\Big| = \theta < \frac{1}{4} \tag{3.14}$$

Then the system $\exp(2\pi i x\lambda_j)$, $j \in \mathbb{Z}$, *is a Riesz basis for* $L^2[0,1]$.

In (a) and (b) c and C are two positive constants. Let us stress again that Beurling's theorem suffices for proving Theorem 3.5 while the spectacular results by S. Grepstad and N. Lev are grounded on Avdonin's theorem. It is interesting that the precise value $1/4$ in Avdonin's theorem does not play any role in the proof and could be replaced by any smaller constant. Theorem 3.8 is a special case of the result proved in [1].

3.2 Bounded remainder sets

Bounded remainder sets which are defined in this subsection will play a decisive role in Theorem 2.10.

Let Λ be a simple model set and let Γ be the lattice which enters in the definition of Λ given in Definition 2.7. After a linear change of variables it can be assumed that Γ is given by the following variant of Definition 2.7. We have $\gamma = (\gamma_1, \gamma_2) \in \Gamma$ if and only if $\gamma_1 = m + \beta(\alpha\cdot m - l)$, $\gamma_2 = l - \alpha\cdot m$ where $\alpha, \beta \in \mathbb{R}^n$, $m \in \mathbb{Z}^n$, $l \in \mathbb{Z}$. Here $x\cdot y$ denotes the inner product between x and y. Moreover the definition of model sets implies that the numbers $1, \alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$ and that $\beta_1, \beta_2, \ldots, \beta_n, 1+\alpha\cdot\beta$ are also linearly independent over $\mathbb{Q}$.

To every compact set $K \subset \mathbb{R}^n$ we associate the $\mathbb{Z}^n$ periodic version of its indicator function defined as $\chi_K(x) = \sum_{k\in\mathbb{Z}^n} \mathbf{1}_K(x+k)$.

Definition 3.10. *Let* $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ *and assume that* $1, \alpha_1, \alpha_2, \ldots, \alpha_n$ *are linearly independent over* $\mathbb{Q}$. *Then a compact set K is a bounded remainder set with respect to* α *if there exists a constant* $C = C(\alpha, K)$ *such that*

$$\Big|\sum_{l=0}^{m-1} \chi_K(x + l\alpha) - m|K|\Big| \le C \tag{3.15}$$

holds for every $m \in \mathbb{N}$ *and for almost every* $x \in \mathbb{T}^n$.

3.3 Main facts

S. Grepstad and N. Lev [4] made two spectacular discoveries.

Theorem 3.11. *Let* Λ *be a simple quasicrystal such that* $|I| \notin p_2(\Gamma)$. *Then there does not exist a Riemann integrable compact set K such that* Λ *is simultaneously a set of stable sampling and stable interpolation for* PW_K.

This negative result is completed by the following positive result.

Theorem 3.12. *We now assume* $|I| \in p_2(\Gamma)$ *and that* α *and* β *satisfy the requirements listed above. Let K be a Riemann integrable compact set such that*

(a) K is a bounded remainder set with respect to α

(b) $|K| = |I|$.

Then the simple quasicrystal Λ *is simultaneously a set of stable sampling and of stable interpolation for* PW_K

S. Grepstad and N. Lev characterized these bounded remainder sets in [5]. In one dimension G. Kozma and N. Lev [6] proved the following special case of Theorem 2.10. We begin with some notations. Let $\theta > 1$ be a real number. The integral part $[x]$ of a real number x is the largest integer $k \in \mathbb{Z}$ such that $k \leq x$. Define $\Lambda_\theta \subset \mathbb{Z}$ by $\Lambda_\theta = \{[k\theta], k \in \mathbb{Z}\}$. With these notations we have

Theorem 3.13. *Let us assume that* $\theta > 1$ *is irrational. Let* $S \subset \mathbb{T}$ *be a finite union of intervals* J_m, $1 \leq m \leq M$. *We assume that the sum of the lengths of these intervals equals the density* θ^{-1} *of* Λ_θ *and that the length of each* J_m *belongs to* $\theta^{-1}\mathbb{Z} + \mathbb{Z}$.

Then any square-summable function f *defined on* S *can be uniquely written as a generalized Fourier series*

$$f(x) = \sum_{\lambda \in \Lambda_\theta} c_\lambda \, exp(2\pi i \lambda x) \tag{3.16}$$

where the frequencies λ *belong to* Λ_θ *and the coefficients* c_λ *belong to* $l^2(\Lambda_\theta)$. *This series converges to* f *in* $L^2(S)$.

4 Open problems

We return to Theorem 2.10. S. Grepstad and N. Lev clarified the role played by the condition $|I| \in p_2(\Gamma)$. But we would like to better understand the role played by the condition imposed on K. Is it necessary ?

What would happen if the L^2 norm was replaced by an L^p norm, $1 < p < \infty$? Is the collection of functions $\exp(2\pi i x \cdot \lambda)$, $\lambda \in \Lambda$, a Schauder basis of $L^p(K)$? The proof of the duality lemma (Lemma 3.8) relies on Plancherel theorem and does not seem to extend to L^p.

The next issue deals with the case $p = \infty$. Let us denote by C_Λ the space of all almost periodic functions whose spectrum is contained in Λ. If the conditions of Theorem 2.10 are satisfied, is it true that $\|f\|_\infty$ and $\sup_{x \in K} |f(x)|$ are equivalent norms on C_Λ ?

Our last problem is perhaps the most difficult question. What happens if our model set Λ is defined by a Riemann integrable window which is not an interval ?

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