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L-Modules, L-Comodules and Hom-Lie Quasi-Bialgebras

Ibrahima BAKAYOKO*
Département de Mathématiques,
UJNK/Centre Universitaire de N'Zérékoré
BP 50 N'Zérékoré, Guinée

Abstract. In this paper, we discuss A-modules and L-modules (resp. L-comodules) for Hom-Lie algebras (resp. Hom-Lie coalgebras). We show that for a given Hom-associative algebra A (resp. Hom-coassociative coalgebra), the A-module (resp. comodule) extends to $L(A)$-module (resp. comodule), where $L(A)$ is the associated Lie algebra (resp. Lie coalgebra), with the same structure map. We also prove that L-modules become L_α-modules, where L_α is the Hom-Lie algebra obtained from the Lie algebra L by stwisting the Lie bracket. Then we introduce Hom-Lie quasi-bialgebras and prove that a Lie quasi-bialgebra turns to a Hom-Lie quasi-bialgebra by stwisting the Lie quasi-bialgebra structure by an endomorphism. Moreover, we show that an exact Lie quasi-bialgebra extends to an exact Hom-Lie quasi-bialgebra.

Résumé. Nous montrons que l'on peut passer des modules sur les algèbres Hom-associatives (resp. coalgèbres Hom-coassociatives) aux modules sur les algèbres de Hom-Lie (resp. coalgèbres de Hom-Lie). Nous montrons aussi que les L-modules deviennent des L_α-modules, où L_α est obtenue de l'algèbre de Lie L en modifiant le crochet de Lie. Puis, nous introduisons les quasi-bigèbres de Hom-Lie et nous montrons qu'une quasi-bigèbre de Lie devient une quasi-bigèbre de Hom-Lie via la modification de la structure de quasi-bigèbre de Lie par un endomorphisme. Ensuite, nous montrons qu'une quasi-bigèbre de Lie exacte s'étend en une quasi-bigèbre de Hom-Lie exacte.

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1 Introduction

Hom-Lie algebras originate from [7] but Hom-associative algebras was introduced first in [11]. They are a generalization of algebras. It is shown in [11] that the commutator bracket of Hom-associative algebras gives rise to Hom-Lie algebra i.e. $[x,y] = \mu(x,y) - \mu(y,x)$. This class of Hom-Lie algebras will play a central role in subsection 2.2. Many examples of Hom-Lie algebras can be found in [21]. Given an algebra A and an algebra endomorphism α, one obtains a Hom-associative algebra structure on A with multiplication $\mu_\alpha = \alpha \circ \mu$. The same procedure can be applied to coalgebras, bialgebras and Lie coalgebras to obtain respectively Hom-coalgebras [14], Hom-bialgebras [24]

*e-mail address: ibrahimabakayoko@yahoo.fr

and Hom-Lie coalgebras (Proposition 3.5). Hom-type analogues of quantum groups, Lie bialgebras and infinitesimal bialgebras are studied in [27]-[30].

Modules over algebras arise often in algebraic topology, quantum groups [9], Lie and Hopf algebras theories [16], [18] and group representations [1]. For example, the singular modulo p cohomology $H^*(X,\mathbf{Z}/p)$ of a topological space X is an $\mathcal{A}_p$-module algebra, where $\mathcal{A}_p$ is a Steenrod algebra associated to the prime p [6]. Likewise the complex cobordism $MU^*(X)$ of a topological space X is a S-module algebra, where S is Landweber-Novikov algebra [10], [17] of stable cobordism operations.

Modules over Hom-associative algebras are discussed in [25]. They are modules over Hom-type of associative algebras, and are obtained by twisting the module structures. It is proved in [25], Lemma 2.5 that we can deduce modules over Hom-associative algebras from a given one via an algebra endomorphism.

L-modules, introduced and called Hom-L-modules in [21], appear as a generalization of Lie modules. They are obtained by twisting the Lie modules structures by the endomorphisms. The role of Lie modules in the construction of Lie bialgebras is exposed in [15].

Dualizing the preceding notions we otain the following ones. Hom-coasso-ciative coalgebras are dual to Hom-associative algebras and generalize coassociative coalgebras. Comodules over Hom-coassociative coalgebras are
studied in [24], [19].

As in the case of associative algebras, it is shown in [31], that one can associate a Lie coalgebra to a given coassociative coalgebra. We show that this construction can be extended to Hom-Lie coalgebra i.e. given a Hom-coassociative coalgebra (A,Δ,α), we proved that the triple (A,γ,α), where $\gamma = \Delta - \Delta^{op}$, is a Hom-Lie coalgebra. This class of Hom-Lie coalgebra will be the foundation of subsection 3.2.

Comodules over Lie coalgebras, also called Lie comodules are studied in [31]. L-comodules are obtained by twisting the Lie comodule structure [31] by the endomorphisms. The application of Lie comodules in the construction of Lie bialgebras is treated in [31].

The purpose of this paper is to construct a theory on modules and comodules over Hom-Lie algebras and Hom-Lie coalgebras respectively. More precisely, since to a Hom-associative algebra corresponds a Hom-Lie algebra [11], given a module over a Hom-associative algebra we associate a module over the corresponding Hom-Lie algebra [11]. Then to a given comodule over a Hom-coassociative coalgebra, we associate a comodule over the corresponding Hom-Lie coalgebra. Then we give a construction of Hom-Lie quasi-bialgebras from a given Lie quasi-bialgebra.

The paper is organized as follows. In section 2, we present some constructions of L-modules. For example we show that L-modules are closed under direct sum. To a given module over a Hom-associative algebra we associate a L-module, where the Lie bracket is the commutator of the Hom-associative multiplication. As corollaries, we provide other constructions of L-modules; first by twisting the multiplication bracket by an algebra endomorphism, then by twisting the L-module structure map.

In section 3, we point out that starting from comodule over Hom-coasso-ciative coalgebra we get a L-comodule, where the Hom-Lie coalgebra is deduced from the Hom-coassociative multiplication as described below. Then we give other constructions of L-comodules by twisting the comultiplication and the comodule structure map for the Hom-coassociative coalgebras.

Section 4 is devoted to Hom-Lie quasi-bialgebras, which is the Hom-type of Lie quasi-bialgebras [5], [3]. We mainly prove that we can obtain a Hom-Lie quasi-bialgebra, under certain conditions, by twisting a Lie quasi-bialgebra structure.

We fix the following notations and conventions.

- **K** will be a field of caracteristic different from 2.
- We will write $\Delta(c) = \sum c_1 \otimes c_2$ (Sweedlers' notation).
- $\oint$ means cyclic summation.

2 L-modules

In this section, we recall basic definitions and present some constructions of L-modules.

2.1 Hom-Lie algebras

We recall the definitions of Hom-associative algebras, Hom-Lie algebras and their connection.

Definition 2.1. ([11]) A Hom-associative algebra is a triple (A,μ,α) consisting of a linear space A, a **K**-bilinear map $\mu : A\times A \longrightarrow A$ and a linear space map $\alpha : A \longrightarrow A$ satisfying

$$\mu(\alpha(x),\mu(y,z)) = \mu(\mu(x,y),\alpha(z)) \quad \text{(Hom-associativity).} \tag{2.1}$$

Or

$$\alpha(x)yz = (xy)\alpha(z),$$

where $\mu(x,y) = xy$.
If in addition α satisfies

$$\alpha(\mu(x,y)) = \mu(\alpha(x),\alpha(y)) \quad \text{(multiplicativity),} \tag{2.2}$$

then (A,μ,α) is said to be multiplicative.

When $\alpha = Id_A$, (A,μ,Id_A), simply denoted (A,μ), is an associative algebra.

The Lemma below allows to get a Hom-associative algebra from an associative algebra and an algebra endomorphism.

Lemma 2.2. *([21]) Let (A,μ) be an associative algebra and $\alpha : A \longrightarrow A$ be an algebra endomorphism. Then the triple (A,μ_α,α), where $\mu_\alpha = \alpha\circ\mu$, is a multiplicative Hom-associative algebra.*

Definition 2.3. ([7]) A Hom-Lie algebra is a triple $(V,[\cdot,\cdot],\alpha)$ consisting of a linear space V, a bilinear map $[\cdot,\cdot] : V\times V \longrightarrow V$ and a linear space map $\alpha : V \longrightarrow V$ satisfying

$$[x,y] = -[y,x] \quad \text{(skew-symmetry)} \tag{2.3}$$

$$\oint[\alpha(x),[y,z]] = 0 \quad \text{(Hom-Jacobi identity)} \tag{2.4}$$

When $\alpha = Id_V$, we obtain the definition of Lie algebras.

The following result is the Lie-version of Lemma 2.2.

Proposition 2.4. *([22]) Let $(L,[\cdot,\cdot])$ be a Lie algebra and α a Lie algebra endomorphism. Then $(L,[\cdot,\cdot]_\alpha,\alpha)$ is a Hom-Lie algebra where $[\cdot,\cdot]_\alpha = \alpha\circ[\cdot,\cdot]$.*

The following Lemma, on which lies the subsection 2.2, connects Hom-associative algebras to Hom-Lie algebras i.e. to any Hom-associative algebra A one may associate a Hom-Lie algebra $L(A)$.

Lemma 2.5. *([11]) Let (A,μ_A,α_A) be a Hom-associative algebra. Then $(L(A),[\cdot,\cdot],\alpha_L)$ is a Hom-Lie algebra, where $L(A) = A$ as vector space, $[x,y] = \mu_A(x,y)-\mu_A(y,x)$ for all $x,y\in A$ and $\alpha_L = \alpha_A$.*

2.2 $L(A)$-modules

In this subsection, we say module for left-module.

Let us give some definitions.

Definition 2.6. ([25]) A Hom-module is a pair (M,α) in which M is a vector space and $\alpha : M \longrightarrow M$ is a linear map.

Definition 2.7. ([25]) Let (A,μ_A,α_A) be a Hom-associative algebra and (M,α_M) be a Hom-module. An A-module structure on M consists of a $\mathbf{K}$-bilinear map $\mu_M : A\otimes M \longrightarrow M$ such that

$$\alpha_M\circ\mu_M = \mu_M\circ(\alpha_A\otimes\alpha_M) \tag{2.5}$$

$$\mu_M\circ(\alpha_A\otimes\mu_M) = \mu_M\circ(\mu_A\otimes\alpha_M). \tag{2.6}$$

Remark 2.8. The conditions (2.5) and (2.6) can be rewritten respectively

$$\alpha_M(x\star m) = \alpha_A(x)\star\alpha_M(m) \tag{2.7}$$

and

$$\alpha_A(x)\star(y\star m) = (x\cdot y)\star\alpha_M(m) \tag{2.8}$$

where we put $\mu_M(x\otimes m) = x\star m$, $\mu_A(x,y) = x\cdot y$ for $x,y\in A$ and $m\in M$.

Example 2.9. a) A multiplicative Hom-associative algebra (A,μ,α) is a module over itself.

b) Let (V,μ_V,α_V) and (W,μ_W,α_W) be two modules over a Hom-associative algebra (A,μ,α). Then the direct product $M = V\times W$ is a module over the associative algebra A with structure maps $\mu_M : A\otimes M\to M$ and $\alpha_M : M\to M$ defined by $\mu_M(a,(v,w)) = (\mu_V(a,v),\mu_W(a,w))$ and $\alpha_M(v,w) = (\alpha_V(v),\alpha_W(w))$.
In particular, when A is a multiplicative Hom-associative algebra, then $A^2 = A\times A$ is also an A-module.

Twisting a module structure map by an algebra endomorphism, we get another one as stated in the following Lemma.

Lemma 2.10. *([25]) Let (A,μ_A,α_A) be a multiplicative Hom-associative algebra and M an A-module with structure map $\mu_M : A\otimes M \longrightarrow M$. Define the map*

$$\tilde{\mu}_M = \mu_M\circ(\alpha_A^2\otimes Id_M) : A\otimes M \longrightarrow M. \tag{2.9}$$

Then $\tilde{\mu}_M$ is a structure map of another A-module structure map on M.

Proof. We have to prove the relations (2.5) and (2.6) for $\tilde{\mu}_M$. For any $x,y\in A$ and $m\in M$, we have

$$\begin{aligned}\alpha_M(\tilde{\mu}_M(x\otimes m)) &= \alpha_M(\mu_M(\alpha_A^2\otimes Id_M)(x\otimes m)) = \alpha_M(\mu_M(\alpha_A^2(x)\otimes m))\\ &= \mu_M(\alpha_A^3(x)\otimes\alpha_M(m)) = \mu_M(\alpha_A^2(\alpha_A(x)),\alpha_M(m)) \quad \text{(by (2.5))}\\ &= \mu_M(\alpha_A^2\otimes Id_M)(\alpha_A(x)\otimes\alpha_M(m)) = \tilde{\mu}_M(\alpha_A(x)\otimes\alpha_M(m)).\end{aligned}$$

And,

$$\begin{aligned}
(\tilde{\mu}_M\circ(\alpha_A\otimes\tilde{\mu}_M))(x\otimes y\otimes m) &= \tilde{\mu}_M(\alpha_A(x)\otimes\tilde{\mu}_M(y\otimes m))\\
&= \tilde{\mu}_M(\alpha_A(x)\otimes\mu_M(\alpha_A^2(y)\otimes m))\\
&= \mu_M(\alpha_A^3(x)\otimes\mu_M(\alpha_A^2(y)\otimes m))\\
&= \mu_M(\mu_A(\alpha_A^2(x)\otimes\alpha_A^2(y))\otimes\alpha_M(m)) \quad \text{(by (2.6))}\\
&= \mu_M(\alpha_A^2(\mu_A(x\otimes y))\otimes\alpha_M(m)) \quad \text{(by (2.2))}\\
&= \mu_M(\alpha_A^2\otimes Id_M)(\mu_A(x\otimes y)\otimes\alpha_M(m))\\
&= \tilde{\mu}_M(\mu_A\otimes\alpha_M)(x\otimes y\otimes m).
\end{aligned}$$

Now we define L-module.

Definition 2.11. ([21]) Let $(L,[\cdot,\cdot],\alpha_L)$ be a Hom-Lie algebra and (M,α_M) be a Hom-module. A L-module on M consists of a $\mathbf{K}$-bilinear map $\mu_M: L\times M\to M$ such that for any $m\in M, x,y\in L$,

$$\alpha_M(\mu_M(x,m)) = \mu_M(\alpha_L(x),\alpha_M(m)). \tag{2.10}$$

$$\mu_M([x,y],\alpha_M(m)) = \mu_M(\alpha_L(x),\mu_M(y,m))-\mu_M(\alpha_L(y),\mu_M(x,m)) \tag{2.11}$$

Remark 2.12. When $\alpha_M = Id_M$ and $\alpha = Id_L$, we recover the definition of Lie modules [8].

The following statement is the Lie-type of Lemma 2.10.

Proposition 2.13. *Let $(L,[\cdot,\cdot],\alpha_L)$ be a Hom-Lie algebra and M be a L-module with structure map $\mu_M: L\otimes M\longrightarrow M$. Define the map*

$$\tilde{\mu}_M = \mu_M\circ(\alpha_L^2\otimes Id_M): L\otimes M\longrightarrow M. \tag{2.12}$$

Then $\tilde{\mu}_M$ is a structure map of another L-module structure map on M.

Proof. By the proof of Lemma 2.10, we only need to prove (2.11) for $\tilde{\mu}_M$.
For any $x,y\in L, m\in M$,

$$\begin{aligned}
\tilde{\mu}_M([x,y],\alpha_M(m)) &= \mu_M(\alpha_L^2\otimes Id_M)([x,y],\alpha_M(m))\\
&= \mu_M([\alpha_L^2(x),\alpha_L^2(y)],\alpha_M(m))\\
&= \mu_M(\alpha_L^3(x),\mu_M(\alpha_L^2(y),m))-\mu_M(\alpha_L^3(y),\mu_M(\alpha_L^2(x),m))\\
&= \mu_M(\alpha_L^3(x),\mu_M(\alpha_L^2\otimes Id_M)(y\otimes m))\\
&\quad -\mu_M(\alpha_L^3(y),\mu_M(\alpha_L^2\otimes Id_M)(x\otimes m))\\
&= \mu_M(\alpha_L^2(\alpha_L(x),\tilde{\mu}_M(y\otimes m))-\mu_M(\alpha_L^2(\alpha_L(y)),\tilde{\mu}_M(x\otimes m))\\
&= \mu_M(\alpha_L^2\otimes Id_M)(\alpha_L(x)\otimes\tilde{\mu}_M(y\otimes m))\\
&\quad -\mu_M(\alpha_L^2\otimes Id_M)(\alpha_L(y)\otimes\tilde{\mu}_M(x\otimes m))\\
&= \tilde{\mu}_M(\alpha_L(x)\otimes\tilde{\mu}_M(y\otimes m))-\tilde{\mu}_M(\alpha_L(y)\otimes\tilde{\mu}_M(x\otimes m)).
\end{aligned}$$

Hence the conclusion holds.

Here are some examples of L-modules.

Example 2.14. ([21])

a) One can consider L itself as a L-module in which the L-action is the bracket $[\cdot,\cdot]$.

b) If L is a Lie algebra and M is a module in the usual sense, then (M, Id_M) is a L-module.

For simplicity we write "$\star$", "$\bullet$" and "$\diamond$" for the module structure maps.

Example 2.15. Let (A, μ_A, α_A) be a Hom-associative algebra, $(V, \star, \alpha_V)$ and $(W, \bullet, \alpha_W)$ be two $L(A)$-modules. The direct sum $M = V \oplus W$ with $\alpha_M = \alpha_V \oplus \alpha_W$ is $L(A)$-module for the operation

$$x \diamond (v \oplus w) = x \star v + x \bullet w, \quad \forall x \in L(A), \forall v \in V, \forall w \in W.$$

Proof. For $x, y \in L(A), v \in V, w \in W$,

$$\begin{aligned}
[x,y] \diamond \alpha_M(v+w) &= [x,y] \diamond (\alpha_V(v) + \alpha_W(w)) \\
&= [x,y] \star \alpha_V(v) + [x,y] \bullet \alpha_W(w) \\
&= \alpha_L(x) \star (y \star v) - \alpha_L(y) \star (x \star v) \\
&\quad + \alpha_L(x) \bullet (y \bullet w) - \alpha_L(y) \bullet (x \bullet w) \\
&= \alpha_L(x) \star (y \star v) + \alpha_L(x) \bullet (y \bullet w) \\
&\quad - \alpha_L(y) \star (x \star v) - \alpha_L(y) \bullet (x \bullet w) \\
&= \alpha_L(x) \diamond (y \diamond (v+w)) - \alpha_L(y) \diamond (x \diamond (v+w)),
\end{aligned}$$

and

$$\begin{aligned}
\alpha_M(x \diamond (v+w)) &= \alpha_M(x \star v + x \bullet w) = \alpha_M(x \star v) + \alpha_M(x \bullet w) \\
&= \alpha_V(x \star v) + \alpha_W(x \bullet w) \\
&= \alpha_L(x) \star \alpha_V(v) + \alpha_L(x) \bullet \alpha_W(w) \\
&= \alpha_L(x) \diamond (\alpha_V(v) + \alpha_W(w)) \\
&= \alpha_L(x) \diamond \alpha_M(v+w). \qquad (2.13)
\end{aligned}$$

The following result shows that A-modules extend to $L(A)$-modules for the same module structure map.

Theorem 2.16. *Let (A, μ_A, α_A) be a Hom-associative algebra and (M, μ_M, α_M) be an A-module. Then, M is a $L(A)$-module for the structure map μ_M.*

Proof. In fact, it suffices to show the relation (2.11). For any $x, y \in A, m \in M$, we have

$$\begin{aligned}
&\mu_M(\alpha_L \otimes \mu_M)(x \otimes y \otimes m) - \mu_M(\alpha_L \otimes \mu_M)(\tau \otimes Id_M)(x \otimes y \otimes m) = \\
= \ & \alpha_L(x) \star (y \star m) - \alpha_L(y) \star (x \star m) \\
= \ & \mu_A(x,y) \star \alpha_M(m) - \mu_A(y,x) \star \alpha_M(m) \quad (\text{ by } (2.8)) \\
= \ & (\mu_A(x,y) - \mu_A(y,x)) \star \alpha_M(m) \\
= \ & [x,y] \star \alpha_M(m) \\
= \ & \mu_M([\cdot,\cdot] \otimes \alpha_M)(x \otimes y \otimes m).
\end{aligned}$$

Remark 2.17. Let (A, μ_A, α_A) be a Hom-associative algebra and (M, μ_M, α_M) be a $L(A)$-module i.e. the condition (2.10) is satisfied and

$$\mu_M(\mu_A(x,y), \alpha_M(m)) - \mu_M(\alpha_A(x), \mu_M(y,m)) = \mu_M(\mu_A(y,x), \alpha_M(m)) - \mu_M(\alpha_A(y), \mu_M(x,m)).$$

It follows that a $L(A)$-module M is an A-module if and only if

$$\mu_M(\mu_A(y,x), \alpha_M(m)) - \mu_M(\alpha_A(y), \mu_M(x,m)) = 0,$$

for all $x, y \in A$ and $m \in M$ that is M is an A-module.

Example 2.18. Let (A,μ,α_A) be a multiplicative Hom-associative algebra. Then A is a $L(A)$-module and $L(A)$ is an A-module because A is an A-module (Example 2.9).

The corollaries below give a large class of examples of $L(A)$-modules.

Corollary 2.19. *Let $A_\alpha = (A,\mu_\alpha,\alpha)$ be a multiplicative Hom-associative algebra as in Lemma (2.2) and (M,μ_M,α_M) be an A_α-module. Then, M is a $L(A)$-module for the structure map μ_M.*

Proof. Prove (2.11). Indeed, for $x,y\in L(A), m\in M$, we have

$$\begin{aligned}
&\mu_M(\alpha_L\otimes\mu_M)(x\otimes y\otimes m)-\mu_M(\alpha_L\otimes\mu_M)(\tau\otimes Id_M)(x\otimes y\otimes m)=\\
=\;&\alpha_L(x)\star(y\star m)-\alpha_L(y)\star(x\star m)\\
=\;&(\mu_{\alpha_L}(x,y))\star\alpha_M(m)-(\mu_{\alpha_L}(y,x))\star\alpha_M(m)\quad(\text{ by (2.8)})\\
=\;&(\mu_{\alpha_L}(x,y)-\mu_{\alpha_L}(y,x))\star\alpha_M(m)\\
=\;&[x,y]\star\alpha_M(m)\\
=\;&\mu_M([\cdot,\cdot]\otimes\alpha_M)(x\otimes y\otimes m).
\end{aligned}$$

Corollary 2.20. *Let (A,μ_A,α_A) be a multiplicative Hom-associative algebra and (M,α_M) be an A-module for the structure map μ_M. Put*

$$\tilde\mu_M=\mu_M\circ(\alpha_A^2\otimes Id_M).$$

Then M is a $L(A)$-module for the structure map $\tilde\mu_M$.

Proof. We know from Lemma (2.10) that $\tilde\mu_M$ is a structure map of A-module. Thus it suffices to prove (2.11). For $x,y\in L(A), m\in M$, one has

$$\begin{aligned}
&\tilde\mu_M(\alpha_L\otimes\tilde\mu_M)(x\otimes y\otimes m)-\tilde\mu_M(\alpha_L\otimes\tilde\mu_M)(\tau\otimes Id_M)(x\otimes y\otimes m)=\\
=\;&\alpha_L^2(\alpha_L(x))\star(\alpha_L^2(y)\star m)-\alpha_L^2(\alpha_L(y))\star(\alpha_L^2(x)\star m)\\
=\;&\alpha_L(\alpha_L^2(x))\star(\alpha_L^2(y)\star m)-\alpha_L(\alpha_L^2(y))\star(\alpha_L^2(x)\star m)\\
=\;&\mu_A(\alpha_L^2(x),\alpha_L^2(y))\star\alpha_M(m)-\mu_A(\alpha_L^2(y),\alpha_L^2(x))\star\alpha_M(m)\quad(\text{ by (2.8)})\\
=\;&\alpha_L^2(\mu_A(x,y)-\mu_A(y,x))\star\alpha_M(m)\quad(\alpha_L\text{ being a morphism})\\
=\;&\alpha_A^2([x,y])\star\alpha_M(m)\\
=\;&\tilde\mu_M([\cdot,\cdot]\otimes\alpha_M)(x\otimes y\otimes m).
\end{aligned}$$

3 L-comodules

In this section, we recall basic definitions and we give some dual results of the preceding section. Sometimes we omit the summation symbol for simplicity.

3.1 Hom-Lie coalgebras

We recall the definitions of Hom-coassociative coalgebras, Hom-Lie coalgebras and their connection.

Definition 3.1. ([14]) A Hom-coassociative coalgebra is a triple (C,Δ,α) in which C is a vector space, $\Delta: C\longrightarrow C\otimes C$ and $\alpha: C\longrightarrow C$ are linear maps such that :

1) $\Delta\circ\alpha=\alpha^{\otimes 2}\circ\Delta$ (comultiplicativity)

2) $(\alpha\otimes\Delta)\circ\Delta=(\Delta\otimes\alpha)\otimes\Delta$ (Hom-coassociativity)

In the Sweedler's notation, the above conditions mean that

1') $\sum\alpha(x)_1\otimes\alpha(x)_2=\sum\alpha(x_1)\otimes\alpha(x_2)$

2') $\sum\alpha(x_1)\otimes x_{21}\otimes x_{22}=\sum x_{11}\otimes x_{12}\otimes\alpha(x_2)$.

The following result is dual to Lemma 2.2.

Lemma 3.2. *([13]) Let (C,Δ) be a coassociative coalgebra and $\alpha: C\longrightarrow C$ be a coalgebra endomorphism. Define the map*

$$\Delta_\alpha=\Delta\circ\alpha: C\longrightarrow C\otimes C \tag{3.1}$$

Then (C,Δ_α,α) is a Hom-coassociative coalgebra.

The following definition is the Hom-type of the one defined in [31] in the case of Lie coalgebra.

Definition 3.3. ([14]) A Hom-Lie coalgebra is a triple (L,γ,α) in which L is a vector space, $\gamma: L\longrightarrow L\otimes L$ and $\alpha: L\longrightarrow L$ are linear maps such that

1) $\gamma=-\tau\circ\gamma$ (skew-cocommutativity)

2) $\gamma\circ\alpha=\alpha^{\otimes 2}\circ\gamma$ (comultiplicativity)

3) $\oint(\alpha\otimes\gamma)\circ\gamma=0$ (Hom-co-Jacobi identity)

where $\tau: L\otimes L\rightarrow L\otimes L$ is the twist isomorphism i.e. $\tau(x\otimes y)=y\otimes x$.

The following proposition, on which lies the subsection 3.2, connects Hom-coassociative coalgebras C to Hom-Lie coalgebras $L(C)$.

Lemma 3.4. *Let (C,Δ_C,α_C) be a Hom-coassociative coalgebra and $\gamma_C: C\longrightarrow C\otimes C$ be a linear map defined by*

$$\gamma_C(x)=x_1\otimes x_2-x_2\otimes x_1 \quad \textit{with} \quad \Delta_C(x)=x_1\otimes x_2.$$

Then $(L(C),\gamma_C,\alpha_C)$ is a Hom-Lie coalgebra, where $L(C)=C$ as vector space.

Proof. *First verify the skew-cocommutativity of γ_C*

$$\begin{aligned}\gamma_C(x) &= x_1\otimes x_2-x_2\otimes x_1=-(x_2\otimes x_1-x_1\otimes x_2)\\ &= -(\tau(x_1\otimes x_2)-\tau(x_2\otimes x_1))\\ &= -\tau(x_1\otimes x_2-x_2\otimes x_1)=-\tau\circ\gamma_C(x).\end{aligned}$$

Now verify the Hom-co-Jacobi identity

$$\begin{aligned}(\alpha\otimes\gamma_C)\gamma_C(x) &= (\alpha\otimes\gamma_C)(x_1\otimes x_2-x_2\otimes x_1)\\ &= \alpha(x_1)\otimes\gamma_C(x_2)-\alpha(x_2)\otimes\gamma_C(x_1)\\ &= \alpha(x_1)\otimes(x_{21}\otimes x_{22}-x_{22}\otimes x_{21})-\alpha(x_2)\otimes(x_{11}\otimes x_{12}-x_{12}\otimes x_{11})\\ &= \alpha(x_1)\otimes x_{21}\otimes x_{22}-\alpha(x_1)\otimes x_{22}\otimes x_{21}\\ &\quad -\alpha(x_2)\otimes x_{11}\otimes x_{12}+\alpha(x_2)\otimes x_{12}\otimes x_{11}.\end{aligned}$$

So,

$$\oint(\alpha\otimes\gamma_C)\gamma_C(x) = \underbrace{\alpha(x_1)\otimes x_{21}\otimes x_{22}}_{1} - \underbrace{\alpha(x_1)\otimes x_{22}\otimes x_{21}}_{2} - \underbrace{\alpha(x_2)\otimes x_{11}\otimes x_{12}}_{3}$$
$$+\underbrace{\alpha(x_2)\otimes x_{12}\otimes x_{11}}_{4} + \underbrace{x_{21}\otimes x_{22}\otimes\alpha(x_1)}_{5} - \underbrace{x_{22}\otimes x_{21}\otimes\alpha(x_1)}_{6}$$
$$-\underbrace{x_{11}\otimes x_{12}\otimes\alpha(x_2)}_{7} + \underbrace{x_{12}\otimes x_{11}\otimes\alpha(x_2)}_{8} + \underbrace{x_{22}\otimes\alpha(x_1)\otimes x_{21}}_{9}$$
$$-\underbrace{x_{21}\otimes\alpha(x_1)\otimes x_{22}}_{10} - \underbrace{x_{12}\otimes\alpha(x_2)\otimes x_{11}}_{11} + \underbrace{x_{11}\otimes\alpha(x_2)\otimes x_{12}}_{12}.$$

According to the Hom-coassociativity and the skew-cocommutativity of Δ_C, the 1st and 7th, 2nd and 8th, 3th and 5th, 4th and 6th, 9th and 11th, 10th and 12th terms cancel pairwise.

The following Proposition is the Lie-type of Lemma 3.2 and dualizes corollary 2.6 *in [21].*

Proposition 3.5. *([4]) Let (L,γ) be a Lie coalgebra and α a coalgebra endomorphism. Then $L_\alpha = (L,\gamma_\alpha = \gamma\circ\alpha,\alpha)$ is a Hom-Lie coalgebra.*

Proof. Since (L,γ) is a Lie coalgebra, to show that L_α is a Hom-Lie coalgebra, we need to prove two things : (i) γ_α is skew-symmetric, (ii) γ_α satisfies the Hom-co-Jacobi identity.

For (i), we know that α commutes with γ and γ is skew-symmetric, so,

$$\gamma_\alpha = \gamma\circ\alpha = (-\tau\circ\gamma)\circ\alpha = -\tau\circ(\gamma\circ\alpha) = -\tau\circ\gamma_\alpha.$$

For (ii), we have for all $x\in L_\alpha$

$$\begin{aligned}(\alpha\otimes\gamma_\alpha)\gamma_\alpha(x) &= (\alpha\otimes\gamma_\alpha)\gamma(\alpha(x)) = (\alpha\otimes\gamma)(\alpha^{\otimes 2}\gamma(x)) = (\alpha^2\otimes\gamma_\alpha\circ\alpha)\gamma(x)\\ &= (\alpha^2\otimes\gamma\circ\alpha^2)\gamma(x) = (\alpha^2\otimes(\alpha^2)^{\otimes 2}\gamma)\gamma(x) = (\alpha^2)^{\otimes 3}(Id\otimes\gamma)\gamma(x)\end{aligned}$$

Thus $\oint(\alpha\otimes\gamma_\alpha)\gamma_\alpha(x) = \oint(\alpha^2)^{\otimes 3}(Id\otimes\gamma)\gamma(x) = (\alpha^2)^{\otimes 3}\oint(Id\otimes\gamma)\gamma(x) = 0.$ Which means that L_α is a Hom-Lie coalgebra.

3.2 $L(C)$-comodules

In this section, we dualize the notions introduced in section 2.2.

Definition 3.6. ([19]) Let (C,Δ_C,α_C) be a Hom-coassociative coalgebra and (M,α_M) be a Hom-module. A C-comodule structure on M consists of a linear map $\Delta_M : M \longrightarrow C\otimes M, m\mapsto \sum m_{(-1)}\otimes m_{(0)}$ such that

$$\Delta_M\circ\alpha_M = (\alpha_C\otimes\alpha_M)\circ\Delta_M \tag{3.2}$$
$$(\alpha_C\otimes\Delta_M)\circ\Delta_M = (\Delta_C\otimes\alpha_M)\circ\Delta_M. \tag{3.3}$$

Example 3.7. A Hom-coassociative coalgebra is a comodule over itself.

Remark 3.8. The conditions (3.2) and (3.3) can be rewritten respectively

$$(\alpha_M(m))_{(-1)}\otimes(\alpha_M(m))_{(0)} = \alpha_C(m_{(-1)})\otimes\alpha_M(m_{(0)}), \tag{3.4}$$

and

$$\alpha_C(m_{(-1)})\otimes m_{(0)(-1)}\otimes m_{(0)(0)} = m_{(-1)1}\otimes m_{(-1)2}\otimes\alpha_M(m_{(0)}). \tag{3.5}$$

The following result dualizes the Lemma 2.10.

Lemma 3.9. *([19]) Let (C,Δ_C,α_C) be a Hom-coassociative coalgebra and (M,α_M) be a C-comodule with structure map $\Delta_M : M \to C\otimes M$. Define the map*

$$\tilde{\Delta}_M = (\alpha_C^2 \otimes Id_M)\circ \Delta_M : M \longrightarrow C\otimes M \tag{3.6}$$

Then $\tilde{\Delta}_M$ is a structure map of another C-comodule structure on M.

Proof. For all $m \in M$, we have

$$\begin{aligned}(\tilde{\Delta}_M\circ\alpha_M)(m) &= ((\alpha_C\otimes Id_M)\circ\Delta_M\circ\alpha_M)(m)\\ &= ((\alpha_C\otimes Id_M)\circ(\alpha_C\otimes\alpha_M)\circ\Delta_M)(m) \quad (by\, (3.2))\\ &= ((\alpha_C\otimes\alpha_M)\circ(\alpha_C^2\otimes Id_M)\circ\Delta_M)(m)\\ &= ((\alpha_C\otimes\alpha_M)\circ\tilde{\Delta}_M)(m).\end{aligned}$$

$$\begin{aligned}((\alpha_C\otimes\tilde{\Delta}_M)\circ\tilde{\Delta}_M)(m) &= ((\alpha_C\otimes\tilde{\Delta}_M)\circ(\alpha_C\otimes Id_M)\Delta_M)(m)\\ &= (\alpha_C\otimes\tilde{\Delta}_M)(\alpha_C^2(m_{(-1)})\otimes m_{(0)})\\ &= \alpha_C^3(m_{(-1)})\otimes\alpha_C^2(m_{(0)(-1)})\otimes m_{(0)(0)}\\ &= (\alpha_C^2(m_{(-1)}))_1\otimes(\alpha_C^2(m_{(-1)}))_2\otimes\alpha_M(m_{(0)}) \quad (by\,(3.5))\\ &= \Delta_C(\alpha_C^2(m_{(-1)}))\otimes\alpha_M(m_{(0)})\\ &= ((\Delta_C\otimes\alpha_M)\circ(\alpha_C^2\otimes Id_M))(m_{(-1)}\otimes m_{(0)})\\ &= ((\Delta_C\otimes\alpha_M)\circ\tilde{\Delta}_M)(m).\end{aligned}$$

Therefore, $\tilde{\Delta}_M$ is a structure map of another C-comodule structure on M.

The following definition is the Hom-type of the one given in [31].

Definition 3.10. Let (L,γ,α_L) be a Hom-Lie coalgebra and (M,α_M) be a Hom-module. If there exists a linear map $\gamma_M : M \to C\otimes M$ such that

$$\gamma_M\circ\alpha_M = (\alpha_L\otimes\alpha_M)\circ\gamma_M \tag{3.7}$$

and

$$(\gamma\otimes\alpha_M)\gamma_M = (\alpha_L\otimes\gamma_M)\gamma_M - (\tau\otimes Id_M)(\alpha_L\otimes\gamma_M)\gamma_M \tag{3.8}$$

then M is called a L-comodule.

Remark 3.11. When $\alpha_L = Id_L$ and $\alpha_M = Id_M$, we recover the definition of Lie comodules [31].

Example 3.12. Let (C,Δ_C,α_C) be a Hom-coassociative coalgebra. Then C is a $L(C)$-comodule.

We have the following remark.

Remark 3.13. Let (C,Δ_C,α_C) be a Hom-coassociative coalgebra and (M,Δ_M,α_M) be a $L(C)$-comodule. A $L(C)$-comodule M is a C-comodule if and only if M is a C-comodule.

Proposition 3.14. *Let (L,γ) be a Lie coalgebra, M be a Lie comodule for the structure map γ_M. Then the Hom-module (M,Id_M) is a L_α-comodule for γ_M.*

Proof. M being a Lie comodule for the structure map γ_M, we have for any $m \in M$,

$$(\gamma\otimes Id_M)\gamma_M(m)-(Id_L\otimes\gamma_M)\gamma_M(m)+(\tau\otimes Id_M)(Id_L\otimes\gamma_M)\gamma_M(m)=0.$$

Or

$$[(\gamma\otimes Id_M)-(Id_L\otimes\gamma_M)+(\tau\otimes Id_M)(Id_L\otimes\gamma_M)]\gamma_M(m)=0.$$

As $\gamma_M \not\equiv 0$, we have,

$$[(\gamma\otimes Id_M)-(Id_L\otimes\gamma_M)+(\tau\otimes Id_M)(Id_L\otimes\gamma_M)]\equiv 0.$$

But,

$$(\alpha\otimes\gamma_M)\gamma_M(m)=(Id_L\otimes\gamma_M)(\alpha\otimes Id_M)\gamma_M(m),$$

and

$$(\gamma_\alpha\otimes Id_M)\gamma_M(m)=(\gamma\circ\alpha\otimes Id_M)\gamma_M(m)=(\gamma\otimes Id_M)(\alpha\otimes Id_M)\gamma_M(m).$$

According to Proposition 3.5, L_α is a Hom-Lie coalgebra, so

$$\begin{aligned}
&(\gamma_\alpha\otimes Id_M)\gamma_M(m)-(\alpha\otimes\gamma_M)\gamma_M(m)+(\tau\otimes Id_M)(\alpha\otimes\gamma_M)\gamma_M(m)=\\
&=(\gamma\otimes Id_M)(\alpha\otimes Id_M)\gamma_M(m)-(Id_L\otimes\gamma_M)(\alpha\otimes Id_M)\gamma_M(m)\\
&\quad+(\tau\otimes Id_M)(Id_L\otimes\gamma_M)(\alpha\otimes Id_M)\gamma_M(m)\\
&=[(\gamma\otimes Id_M)-(Id_L\otimes\gamma_M)+(\tau\otimes Id_M)(Id_L\otimes\gamma_M)](\alpha\otimes Id_M)\gamma_M(m)=0.
\end{aligned}$$

Theorem 3.15. *Let (C,Δ_C,α_C) be Hom-coassociative coalgebra and (M,α_M) be a C-comodule for the structure map Δ_M. Then M is a $L(C)$-comodule for $\gamma_M=\Delta_M$.*

Proof. The relation (3.7) holds because M is a C-comodule. Verify that (3.8) holds also. For any $m \in M$, we have :

$$\begin{aligned}
&(\alpha_C\otimes\gamma_M)\gamma_M(m)-(\tau\otimes Id_M)(\alpha_C\otimes\gamma_M)\gamma_M(m)=\\
&=(\alpha_C\otimes\gamma_M)(m_{(-1)}\otimes m_{(0)})-(\tau\otimes Id_M)(\alpha_C(m_{(-1)})\otimes\gamma_M(m_{(0)}))\\
&=\alpha_C(m_{(-1)})\otimes m_{(0)(-1)}\otimes m_{(0)(0)}-(\tau\otimes Id_M)(\alpha_C(m_{(-1)})\otimes m_{(0)(-1)}\otimes m_{(0)(0)})\\
&=\alpha_C(m_{(-1)})\otimes m_{(0)(-1)}\otimes m_{(0)(0)}-m_{(0)(-1)}\otimes\alpha_C(m_{(-1)})\otimes m_{(0)(0)}.
\end{aligned}$$

According to the C-comodule structure of M (3.5),

$$\begin{aligned}
&(\alpha_C\otimes\gamma_M)\gamma_M(m)-(\tau\otimes Id_M)(\alpha_C\otimes\gamma_M)\gamma_M(m)=\\
&=m_{(-1)1}\otimes m_{(-1)2}\otimes\alpha_M(m_{(0)})-m_{(-1)2}\otimes m_{(-1)1}\otimes\alpha_M(m_{(0)}).
\end{aligned}$$

Now,

$$\begin{aligned}
(\gamma\otimes\alpha_M)\gamma_M(m) &= (\gamma\otimes Id_M)(m_{(-1)}\otimes m_{(0)})=\gamma(m_{(-1)})\otimes m_{(0)}\\
&= [m_{(-1)1}\otimes m_{(-1)2}-m_{(-1)2}\otimes m_{(-1)1}]\otimes\alpha_M(m_{(0)})\\
&= m_{(-1)1}\otimes m_{(-1)2}\otimes\alpha_M(m_{(0)})-m_{(-1)2}\otimes m_{(-1)1}\otimes\alpha_M(m_{(0)}).
\end{aligned}$$

Thus the equality follows immediatly.

Corollary 3.16. *Let (C,Δ_C,α_C) be a Hom-coassociative coalgebra, and (M,α_M) be a C-comodule for the structure map Δ_M. Put*

$$\tilde{\Delta}_M=(\alpha_C^2\otimes Id_M)\Delta_M.$$

Then M is a $L(C)$-comodule for the structure map $\tilde{\Delta}_M$.

Corollary 3.17. *Let $C_\alpha=(C,\Delta_\alpha,\alpha)$ be a Hom-coassociative coalgebra as in Lemma 3.2 and (M,α_M) be a C_α-comodule for the structure map Δ_M. Then, M is a $L(C)$-comodule for the structure map Δ_M.*

4 Hom-Lie quasi-bialgebras

In this section we recall basic notions on cohomology of Hom-Lie algebra and we introduce Hom-Lie quasi-bialgebras.
We extend some definitions relative to the cohomology of Hom-Lie algebras [12], with values in $\mathcal{G}$, to the analogues one for the cohomology of Hom-Lie algebras with values in $\Lambda^2\mathcal{G}$.

Let $(\mathcal{G},\mu,\alpha)$ be a Hom-Lie algebra. For any entiger $k \geq 1$, the set of k-Hom-cochains on $\mathcal{G}$ with values in $\Lambda^2\mathcal{G}$ is the set of k-linear alterning maps

$$C^k(\mathcal{G},\Lambda^2\mathcal{G}) = \{\varphi : \mathcal{G}^k \to \Lambda^2\mathcal{G}\},$$

where $\mathcal{G}^k = \mathcal{G}\times\mathcal{G}\times\cdots\times\mathcal{G}$ (k times).

Definition 4.1. Let $(\mathcal{G},\mu,\alpha)$ be a Hom-Lie algebra. A 1-Hom-cochain, with values in $\Lambda^2\mathcal{G}$, is a map f, where $f \in C^1(\mathcal{G},\Lambda^2\mathcal{G})$ satisfying

$$\alpha^{\otimes 2} \circ f = f \circ \alpha.$$

We extend 1-coboundary operator [12], with values in $\mathcal{G}$, for Hom-Lie algebras to a 1-coboundary operator, with values in $\Lambda^2\mathcal{G}$ as follows.

Definition 4.2. (i) We call 1-coboundary operator of a Hom-Lie algebra $\mathcal{G}$ with values in $\Lambda^2\mathcal{G}$ the map

$$\delta^1_{HL} : C^1(\mathcal{G},\Lambda^2\mathcal{G}) \to C^2(\mathcal{G},\Lambda^2\mathcal{G}), f \mapsto \delta^1_{HL}f$$

defined by

$$\delta^1_{HL}f(x,y) = f(\mu(x,y)) + y\cdot f(x) - x\cdot f(y)$$

where

$$x\cdot(y_1\otimes y_2) = \mu(\alpha(x),y_1)\otimes\alpha(y_2) + \alpha(y_1)\otimes\mu(\alpha(x),y_2)$$

(ii) A 1-Hom-cochain f is called 1-cocycle if $\delta^1_{HL}f = 0$.

Now, for any element x of a Hom-Lie algebra $(\mathcal{G},\mu,\alpha)$ and any integer $k \geq 2$, one defines the adjoint action of $\mathcal{G}$ on $\mathcal{G}\otimes\cdots\otimes\mathcal{G}$ (k times) by

$$ad_x^{\mu,\alpha}(y_1\otimes y_2\otimes\cdots\otimes y_k) = \sum_{i=1}^{k}\alpha(y_1)\otimes\cdots\otimes\alpha(y_{i-1})\otimes\mu(x,y_i)\otimes\alpha(y_{i+1})\otimes\cdots\otimes\alpha(y_k),$$

for any $y_1,\ldots,y_k \in \mathcal{G}$.

Definition 4.3. A Hom-Lie quasi-bialgebra is a quintuple $(\mathcal{G},\mu,\gamma,\phi,\alpha)$ where $(\mathcal{G},\mu,\alpha)$ is a Hom-Lie algebra, $\gamma : \mathcal{G} \longrightarrow \Lambda^2\mathcal{G}$ is a 1-cocycle and $\phi \in \Lambda^3\mathcal{G}$ such that :

$$Alt(\gamma\otimes\alpha)\gamma(x) = ad_x^{\mu,\alpha}\phi, \quad (4.1)$$

$$Alt(\gamma\otimes\alpha\otimes\alpha)\phi = 0 \quad (4.2)$$

where $Alt(x\otimes y\otimes z) = x\otimes y\otimes z + y\otimes z\otimes x + z\otimes x\otimes y$ and γ is a 1-cocycle means that $\gamma(\mu(x,y)) = x\cdot\gamma(y) - y\cdot\gamma(x)$ for any $x,y \in \mathcal{G}$.
If in addition α commutes with μ and γ, we say that the Hom-Lie quasi-bialgebra $(\mathcal{G},\mu,\gamma,\phi,\alpha)$ is multiplicative.

Remark 4.4. When $\alpha = Id_{\mathcal{G}}$, we recover the definition of Lie quasi-bialgebras [5].

Theorem 4.5. *If $(\mathcal{G},\mu,\gamma,\phi)$ is a Lie quasi-bialgebra and α is a morphism with resect to μ and γ, and which commutes with ad_x^{μ} for any $x\in\mathcal{G}$, then*

$$(\mathcal{G},\mu_\alpha=\alpha\circ\mu,\gamma_\alpha=\gamma\circ\alpha,\phi_\alpha=\alpha^{\otimes 3}\phi)$$

is a Hom-Lie quasi-bialgebra.

Proof. *According to Lemma 2.2, $(\mathcal{G},\mu_\alpha,\alpha)$ is a Hom-Lie algebra. Let us show that γ_α is a 1-cocycle i.e. $\gamma_\alpha(\mu_\alpha(x,y))=x\cdot\gamma_\alpha(y)-y\cdot\gamma_\alpha(x)$ for any $x,y\in\mathcal{G}$. We have,*

$$\begin{aligned}
&\gamma_\alpha(\mu_\alpha(x,y))-x\cdot\gamma_\alpha(y)+y\cdot\gamma_\alpha(x)\\
&=\gamma(\alpha^2(\mu(x,y)))-x\cdot\gamma(\alpha(y))+y\cdot\gamma(\alpha(x))\\
&=(\alpha^2)^{\otimes 2}\gamma(\mu(x,y))-x\cdot(\alpha^{\otimes 2}\gamma(y))+y\cdot(\alpha^{\otimes 2}\gamma(x))\\
&=(\alpha^2)^{\otimes 2}\gamma(\mu(x,y))-x\cdot(\alpha(y_1)\otimes\alpha(y_2))+y\cdot(\alpha(x_1)\otimes\alpha(x_2))\\
&=(\alpha^2)^{\otimes 2}\gamma(\mu(x,y))-\mu_\alpha(\alpha(x),\alpha(y_1))\otimes\alpha^2(y_2))-\alpha^2(y_1)\otimes\mu_\alpha(\alpha(x),\alpha(y_2))\\
&\quad+\mu_\alpha(\alpha(y),\alpha(x_1))\otimes\alpha^2(x_2))+\alpha^2(x_1)\otimes\mu_\alpha(\alpha(y),\alpha(x_2))\\
&=(\alpha^2)^{\otimes 2}(\gamma(\mu(x,y))-\mu(x,y_1)\otimes y_2-y_1\otimes\mu(x,y_2)+\mu(y,x_1)\otimes x_2+x_1\otimes\mu(y,x_2))\\
&=(\alpha^2)^{\otimes 2}(\gamma(\mu(x,y))-x\cdot\gamma(y)+y\cdot\gamma(x))\\
&=0.
\end{aligned}$$

This prove that γ_α is a 1-cocycle.

It remains to prove relations (4.1) and (4.2) for the corresponding maps. We have

$$\begin{aligned}
Alt(\gamma_\alpha\otimes\alpha)\gamma_\alpha(x) &= Alt(\gamma\circ\alpha\otimes\alpha)(\gamma(\alpha(x))=Alt(\gamma\circ\alpha\otimes\alpha)(\alpha^{\otimes 2}\gamma(x))\\
&= Alt((\alpha^2)^{\otimes 2}\gamma\otimes\alpha^2)\gamma(x)=(\alpha^2)^{\otimes 3}(\gamma\otimes Id)\gamma(x)\\
&= (\alpha^2)^{\otimes 3}ad_x^{\mu}\phi
\end{aligned}$$

To conclude (4.1), put $\phi=x_1\wedge x_2\wedge x_3$. So

$$\begin{aligned}
(\alpha^2)^{\otimes 3}ad_x^{\mu}\phi &= (\alpha^2)^{\otimes 3}(\mu(x,x_1)\wedge x_1\wedge x_3-\mu(x,x_2)\wedge x_1\wedge x_3+\mu(x,x_3)\wedge x_1\wedge x_2)\\
&= (\alpha^2)^{\otimes 3}(ad_x^{\mu}x_1\wedge x_1\wedge x_3-ad_x^{\mu}x_2\wedge x_1\wedge x_3+ad_x^{\mu}x_3\wedge x_1\wedge x_2)\\
&= (\alpha)^{\otimes 3}[\alpha(ad_x^{\mu}x_1)\wedge\alpha(x_1)\wedge\alpha(x_3)-\alpha(ad_x^{\mu}x_2)\wedge\alpha(x_1)\wedge\alpha(x_3)\\
&\quad+\alpha(ad_x^{\mu}x_3)\wedge\alpha(x_1)\wedge\alpha(x_2)]\\
&= (\alpha)^{\otimes 3}[(\alpha\circ ad_x^{\mu})x_1\wedge\alpha(x_1)\wedge\alpha(x_3)-(\alpha\circ ad_x^{\mu})x_2\wedge\alpha(x_1)\wedge\alpha(x_3)\\
&\quad+(\alpha\circ ad_x^{\mu})x_3\wedge\alpha(x_1)\wedge\alpha(x_2)]\\
&= (\alpha)^{\otimes 3}[(ad_x^{\mu}\circ\alpha)x_1\wedge\alpha(x_1)\wedge\alpha(x_3)-(ad_x^{\mu}\circ\alpha)x_2\wedge\alpha(x_1)\wedge\alpha(x_3)\\
&\quad+(ad_x^{\mu}\circ\alpha)x_3\wedge\alpha(x_1)\wedge\alpha(x_2)]\\
&= (\alpha)^{\otimes 3}[ad_x^{\mu}(\alpha x_1)\wedge\alpha(x_1)\wedge\alpha(x_3)-(ad_x^{\mu}(\alpha x_2)\wedge\alpha(x_1)\wedge\alpha(x_3)\\
&\quad+ad_x^{\mu}(\alpha x_3)\wedge\alpha(x_1)\wedge\alpha(x_2)]\\
&= (\alpha)^{\otimes 3}(\mu(x,(\alpha x_1))\wedge\alpha(x_1)\wedge\alpha(x_3)-\mu(x,\alpha(x_2))\wedge\alpha(x_1)\wedge\alpha(x_3)\\
&\quad+\mu(x,\alpha(x_3))\wedge\alpha(x_1)\wedge\alpha(x_2))\\
&= (\mu(\alpha(x),\alpha^2(x_1))\wedge\alpha^2(x_1)\wedge\alpha^2(x_3)-\mu(\alpha(x),\alpha\alpha^2(x_2))\wedge\alpha^2(x_1)\wedge\alpha^2(x_3)\\
&\quad+\mu(\alpha(x),\alpha^2(x_3))\wedge\alpha^2(x_1)\wedge\alpha^2(x_2))\\
&= (\mu_\alpha(x,\alpha(x_1))\wedge\alpha^2(x_1)\wedge\alpha^2(x_3)-\mu_\alpha(x,\alpha(x_2))\wedge\alpha^2(x_1)\wedge\alpha^2(x_3)\\
&\quad+\mu_\alpha(x,\alpha(x_3))\wedge\alpha^2(x_1)\wedge\alpha^2(x_2))\\
&= ad_x^{\mu_\alpha,\alpha}(\alpha(x_1)\wedge\alpha(x_2)\wedge\alpha(x_3))=ad_x^{\mu_\alpha,\alpha}(\alpha^{\otimes 3}\phi)\\
&= ad_x^{\mu_\alpha,\alpha}(\phi_\alpha).
\end{aligned}$$

For the relation (4.2) we have,

$$\begin{aligned} Alt(\gamma_\alpha \otimes \alpha \otimes \alpha)\phi_\alpha &= Alt(\gamma \circ \alpha \otimes \alpha \otimes \alpha)(\alpha^{\otimes 3}\phi) = Alt(\gamma \circ \alpha^2 \otimes \alpha^2 \otimes \alpha^2)(\phi) \\ &= Alt((\alpha^2)^{\otimes 2}\gamma \otimes \alpha^2 \otimes \alpha^2)(\phi) \\ &= Alt(\alpha^2)^{\otimes 4}[(\gamma \otimes Id_{\mathcal{G}} \otimes Id_{\mathcal{G}})(\phi)] \\ &= (\alpha^2)^{\otimes 4}[Alt(\gamma \otimes Id_{\mathcal{G}} \otimes Id_{\mathcal{G}})(\phi)] \\ &= 0. \end{aligned}$$

Therefore $(\mathcal{G}, \mu_\alpha, \gamma_\alpha, \phi_\alpha, \alpha)$ is a Hom-Lie quasi-bialgebra.

Example 4.6. Let G be a diagonal matrix Lie group and let $\mathcal{G}$ be the Lie algebra of G. Let $\gamma : \mathcal{G} \longrightarrow \Lambda^2\mathcal{G}$ be a linear map and $\phi \in \Lambda^3\mathcal{G}$ such that $(\mathcal{G}, \mu, \gamma, \phi)$ be a Lie quasi-bialgebra. According to [21] Example 2.14, the map $Ad_x : \mathcal{G} \longrightarrow \mathcal{G}, g \mapsto xgx^{-1},\ x \in G$ is a morphism of Lie algebra. Moreover, suppose that Ad_x commutes with γ. Then the quintuple $(\mathcal{G}, \mu_{Ad_x}, \gamma_{Ad_x}, \phi_{Ad_x}, Ad_x)$ is a Hom-Lie quasi-bialgebra.

Definition 4.7. A Hom-Lie quasi-bialgebra $(\mathcal{G}, \mu, \gamma, \phi, \alpha)$ is said to be exact if there exists $a \in \Lambda^2\mathcal{G}$ such that

$$\alpha^{\otimes 2}(a) = a \quad \text{and} \quad \gamma = ad^{\mu,\alpha}(a)$$

Proposition 4.8. *Let $(\mathcal{G}, \mu, \gamma, a, \phi)$ be an exact Lie quasi-bialgebra and α be a Lie algebra morphism such that $\alpha^{\otimes 2}(a) = a$. Then $(\mathcal{G}, \mu_\alpha, \gamma_\alpha, a, \phi_\alpha, \alpha)$ is an exact Hom-Lie quasi-bialgebra.*

A similar analysis may be made for modules over color Hom-Lie algebras. Also, the procedure to twist classical algebraic structures to obtain the Hom-version may be applied to color Poisson algebras and to other contructions in the case of Lie quasi-bialgebras like the twisting relation for Hom-Lie quasi-bialgebras. One may again think of the Laplacian of Hom-Lie quasi-bialgebras. We hope to return to these questions elswhere.

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