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Abstract. We obtain the expression of Ricci tensor for a *GCR*-lightlike submanifold of indefinite complex space form and discuss the properties of Ricci tensor on totally geodesic *GCR*-lightlike submanifold of an indefinite complex space form.

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1 Introduction

As a generalization of complex and totally real submanifolds of Kähler manifolds, the geometry of *CR*-submanifolds of Kähler manifolds was initiated by Bejancu [1]. Since there are significant uses of lightlike geometry in mathematical physics and relativity, Duggal and Bejancu [2], introduced the notion of *CR*-lightlike submanifolds of indefinite Kähler manifolds, which did not include complex and totally real subcases. Therefore Duggal and Sahin [3], introduced *SCR*-lightlike submanifolds of indefinite Kähler manifolds but there was no inclusion relation between *CR* and *SCR*-cases. Then

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as an umbrella over real hypersurfaces, invariant, screen real and *CR*-lightlike submanifolds, Duggal and Sahin [4], introduced *GCR*-lightlike submanifolds of indefinite Kähler manifolds and further studied by [8, 9, 10, 12]. Recently Sangeet et al. [11] introduced *GCR*-lightlike submanifolds of indefinite nearly Kähler manifolds and obtain their existence in indefinite nearly Kähler manifolds of constant holomorphic sectional curvature c and of constant type α. In this paper, we obtain the expression of Ricci tensor for a *GCR*-lightlike submanifold of indefinite complex space form and discuss the properties of Ricci tensor on totally geodesic *GCR*-lightlike submanifold of an indefinite complex space form.

2 Lightlike Submanifolds

Let $(\bar{M},\bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m,n \geq 1$, $1 \leq q \leq m+n-1$ and (M,g) be an m-dimensional submanifold of $\bar{M}$ and g the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M, then M is called a lightlike submanifold of $\bar{M}$, (for detail see [2]). For a degenerate metric g on M $TM^{\perp}$ is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $RadT_xM = T_xM \cap T_xM^{\perp}$, which is known as radical (null) subspace. If the mapping $RadTM : x \in M \longrightarrow RadT_xM$, defines a smooth distribution on M of rank $r > 0$, then the submanifold M of $\bar{M}$ is called an r-lightlike submanifold and $RadTM$ is called the radical distribution on M.

Screen distribution $S(TM)$ is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, that is, $TM = RadTM \perp S(TM)$, and $S(TM^{\perp})$ is a complementary vector subbundle to $RadTM$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}\mid_M$ and to $RadTM$ in $S(TM^{\perp})^{\perp}$ respectively. Then we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}), \tag{2.1}$$

$$T\bar{M}\mid_M = TM \oplus tr(TM) = (RadTM \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{2.2}$$

Let u be a local coordinate neighborhood of M and consider the local quasi-orthonormal fields of frames of $\bar{M}$ along M, on u as $\{\xi_1,...,\xi_r,W_{r+1},...,W_n,N_1,...,N_r,X_{r+1},...,X_m\}$, where $\{\xi_1,...,\xi_r\}$, $\{N_1,...,N_r\}$ are local lightlike bases of $\Gamma(RadTM\mid_u)$, $\Gamma(ltr(TM)\mid_u)$, respectively and $\{W_{r+1},...,W_n\}$, $\{X_{r+1},...,X_m\}$ are local orthonormal bases of $\Gamma(S(TM^{\perp})\mid_u)$, $\Gamma(S(TM)\mid_u)$, respectively. For this quasi-orthonormal fields of frames, we have

Theorem 2.1. *([2]). Let $(M,g,S(TM),S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M},\bar{g})$. Then, there exists a complementary vector bundle $ltr(TM)$ of $RadTM$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM)\mid_{\mathrm{u}})$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}\mid_{\mathrm{u}}$, where* u *is a coordinate neighborhood of M such that*

$$\bar{g}(N_i,\xi_j) = \delta_{ij}, \quad \bar{g}(N_i,N_j) = 0, \textit{for any} \quad i,j \in \{1,2,..,r\},$$

where $\{\xi_1,...,\xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$, then according to the decomposition (2.2), the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \tag{2.3}$$

for any $X,Y \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_U X\}$ and $\{h(X,Y), \nabla^{\perp}_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called second fundamental form, A_U is a linear operator on M and known as shape operator.

According to (2.1), considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively, then (2.3) becomes

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \tag{2.4}$$

$$\bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{2.5}$$

where we put $h^l(X,Y) = L(h(X,Y)), h^s(X,Y) = S(h(X,Y)), D^l_X U = L(\nabla^{\perp}_X U)$, $D^s_X U = S(\nabla^{\perp}_X U)$. As h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued respectively, therefore they are called as the lightlike second fundamental form and the screen second fundamental form on M. In particular

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N), \tag{2.6}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W), \tag{2.7}$$

where $X \in \Gamma(TM), N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Using (2.4)-(2.7) we obtain

$$\bar{g}(h^s(X,Y),W) + \bar{g}(Y,D^l(X,W)) = g(A_W X,Y), \tag{2.8}$$

$$\bar{g}(D^s(X,N),W) = \bar{g}(N,A_W X), \tag{2.9}$$

$$\bar{g}(A_N X,N') + \bar{g}(N,A_{N'}X) = 0, \tag{2.10}$$

for any $\xi \in \Gamma(RadTM)$, $W \in \Gamma(S(TM^{\perp}))$ and $N,N' \in \Gamma(ltr(TM))$.

Let $\bar{P}$ be the projection morphism of TM on $S(TM)$, we can induce some new geometric objects on the screen distribution $S(TM)$ on M as

$$\nabla_X \bar{P}Y = \nabla^*_X \bar{P}Y + h^*(X,\bar{P}Y), \quad \nabla_X \xi = -A^*_{\xi} X + \nabla^{*t}_X \xi, \tag{2.11}$$

for any $X,Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, where $\{\nabla^*_X \bar{P}Y, A^*_{\xi} X\}$ and $\{h^*(X,\bar{P}Y), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(RadTM)$, respectively. ∇^* and ∇^{*t} are linear connections on complementary distributions $S(TM)$ and $RadTM$, respectively. h^* and A^* are $\Gamma(RadTM)$-valued and $\Gamma(S(TM))$-valued bilinear forms and known as the second fundamental forms of distributions $S(TM)$ and $RadTM$, respectively. Using (2.4), (2.5) and (2.11), we obtain

$$\bar{g}(h^*(X,\bar{P}Y),N) = \bar{g}(A_N X,\bar{P}Y), \tag{2.12}$$

for any $X,Y \in \Gamma(TM), \xi \in \Gamma(Rad(TM))$ and $N \in \Gamma(ltr(TM))$.

Denote by $\bar{R}$ and R the curvature tensors of $\bar{\nabla}$ and ∇, respectively, then by straightforward calculations ([2]), we have

$$\begin{aligned}\bar{R}(X,Y)Z \quad &= R(X,Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y \\ &-A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y,Z) - (\nabla_Y h^l)(X,Z) \\ &+D^l(X,h^s(Y,Z)) - D^l(Y,h^s(X,Z)) + (\nabla_X h^s)(Y,Z) \\ &-(\nabla_Y h^s)(X,Z) + D^s(X,h^l(Y,Z)) - D^s(Y,h^l(X,Z)),\end{aligned} \tag{2.13}$$

where

$$(\nabla_X h^s)(Y,Z) = \nabla^s_X h^s(Y,Z) - h^s(\nabla_X Y,Z) - h^s(Y,\nabla_X Z),$$

$$(\nabla_X h^l)(Y,Z) = \nabla^l_X h^l(Y,Z) - h^l(\nabla_X Y,Z) - h^l(Y,\nabla_X Z).$$

Gray [6], defined nearly Kähler manifolds as

Definition 2.2. Let $(\bar{M}, \bar{J}, \bar{g})$ be an indefinite almost Hermitian manifold and $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ with respect to $\bar{g}$. Then $\bar{M}$ is called an indefinite nearly Kähler manifold if

$$(\bar{\nabla}_X \bar{J})Y + (\bar{\nabla}_Y \bar{J})X = 0, \quad \text{or} \quad (\bar{\nabla}_X \bar{J})X = 0 \tag{2.14}$$

for any $X, Y \in \Gamma(T\bar{M})$.

Nearly Kähler manifold of constant holomorphic curvature c is denoted by $\bar{M}(c)$ and its curvature tensor field $\bar{R}$ is given by, [16]

$$\begin{aligned} \bar{R}(X,Y,Z,W) &= \frac{c}{4}\{\bar{g}(X,W)\bar{g}(Y,Z) - \bar{g}(X,Z)\bar{g}(Y,W) + \bar{g}(X,\bar{J}W)\bar{g}(Y,\bar{J}Z) \\ &\quad -\bar{g}(X,\bar{J}Z)\bar{g}(Y,\bar{J}W) - 2\bar{g}(X,\bar{J}Y)\bar{g}(Z,\bar{J}W)\} \\ &\quad +\frac{1}{4}\{\bar{g}((\bar{\nabla}_X\bar{J})(W),(\bar{\nabla}_Y\bar{J})(Z)) - \bar{g}((\bar{\nabla}_X\bar{J})(Z),(\bar{\nabla}_Y\bar{J})(W)) \\ &\quad -2\bar{g}((\bar{\nabla}_X\bar{J})(Y),(\bar{\nabla}_Z\bar{J})(W))\} \end{aligned} \tag{2.15}$$

for X, Y, Z vector fields on $\bar{M}$. Using (2.15) and (2.13), we obtain

$$\begin{aligned} g(R(X,Y)Z,W) &= \frac{c}{4}\{\bar{g}(X,W)\bar{g}(Y,Z) - \bar{g}(X,Z)\bar{g}(Y,W) + \bar{g}(X,\bar{J}W)\bar{g}(Y,\bar{J}Z) \\ &\quad -\bar{g}(X,\bar{J}Z)\bar{g}(Y,\bar{J}W) - 2\bar{g}(X,\bar{J}Y)\bar{g}(Z,\bar{J}W)\} \\ &\quad +\frac{1}{4}\{\bar{g}((\bar{\nabla}_X\bar{J})(W),(\bar{\nabla}_Y\bar{J})(Z)) - \bar{g}((\bar{\nabla}_X\bar{J})(Z),(\bar{\nabla}_Y\bar{J})(W)) \\ &\quad -2\bar{g}((\bar{\nabla}_X\bar{J})(Y),(\bar{\nabla}_Z\bar{J})(W))\} - g(A_{h^l(X,Z)}Y,W) \\ &\quad +g(A_{h^l(Y,Z)}X,W) - g(A_{h^s(X,Z)}Y,W) \\ &\quad +g(A_{h^s(Y,Z)}X,W) - \bar{g}((\nabla_X h^l)(Y,Z),W) \\ &\quad +\bar{g}((\nabla_Y h^l)(X,Z),W) - \bar{g}(D^l(X,h^s(Y,Z)),W) \\ &\quad +\bar{g}(D^l(Y,h^s(X,Z)),W). \end{aligned} \tag{2.16}$$

A nearly Kähler manifold is said to be of constant type α [6], if there exists a real valued C^∞ function α on $\bar{M}$ such that

$$\|(\bar{\nabla}_X\bar{J})(Y)\|^2 = \alpha\{\|X\|^2\|Y\|^2 - g(X,Y)^2 - g(X,\bar{J}Y)^2\}.$$

Lemma 2.3. *([16]). If $\bar{M}$ is a nearly Kähler manifold, then $(\bar{\nabla}_X\bar{J})\bar{J}Y = -\bar{J}(\bar{\nabla}_X\bar{J})Y$ and $(\bar{\nabla}_{\bar{J}X}\bar{J})Y = -\bar{J}(\bar{\nabla}_X\bar{J})Y$.*

Definition 2.4. ([11]). Let $(M, g, S(TM))$ be a real lightlike submanifold of an indefinite nearly Kähler manifold $(\bar{M}, \bar{g}, J)$ then M is called a generalized Cauchy-Riemann (*GCR*)-lightlike submanifold if the following conditions are satisfied

(A) There exist two subbundles D_1 and D_2 of $Rad(TM)$ such that

$$Rad(TM) = D_1 \oplus D_2, \quad \bar{J}(D_1) = D_1, \quad \bar{J}(D_2) \subset S(TM).$$

(B) There exist two subbundles D_0 and D' of $S(TM)$ such that

$$S(TM) = \{\bar{J}D_2 \oplus D'\} \perp D_0, \quad \bar{J}(D_0) = D_0, \quad \bar{J}(D') = L_1 \perp L_2,$$

where D_0 is a non-degenerate distribution on M, L_1 and L_2 are vector subbundle of $ltr(TM)$ and $S(TM)^\perp$, respectively.

Then the tangent bundle TM of M is decomposed as $TM = D \perp D'$, where $D = Rad(TM) \oplus D_0 \oplus \bar{J}D_2$. M is called a proper GCR-lightlike submanifold if $D_1 \neq \{0\}, D_2 \neq \{0\}, D_0 \neq \{0\}$ and $L_2 \neq \{0\}$.

Example 2.5. ([5]) Let M be a submanifold of R^{14}_4 given by the equations

$$x_1 = x_{14}, \quad x_2 = -x_{13}, \quad x_3 = x_{12}, \quad x_7 = \sqrt{1 - x_8^2}.$$

Then TM is spanned by $Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}$, where

$$Z_1 = \partial x_1 + \partial x_{14}, \quad Z_2 = \partial x_2 - \partial x_{13}, \quad Z_3 = \partial x_3 + \partial x_{12},$$

$$Z_4 = \partial x_4, \quad Z_5 = \partial x_5, \quad Z_6 = \partial x_6, \quad Z_7 = -x_8 \partial x_7 + x_7 \partial x_8,$$

$$Z_8 = \partial x_9, \quad Z_9 = \partial x_{10}, \quad Z_{10} = \partial x_{11}.$$

Clearly M is 3-lightlike with $Rad(TM) = Span\{Z_1, Z_2, Z_3\}$ and $\bar{J}Z_1 = Z_2$. Therefore $D_1 = Span\{Z_1, Z_2\}$. On the other hand, $\bar{J}Z_3 = Z_4 - Z_{10} \in \Gamma(S(TM))$ implies that $D_2 = Span\{Z_3\}$. Since $\bar{J}Z_5 = Z_6$ and $\bar{J}Z_8 = Z_9$ therefore $D_0 = Span\{Z_5, Z_6, Z_8, Z_9\}$. By direct calculations, $S(TM^{\perp}) = Span\{W = x_7 \partial x_7 + x_8 \partial x_8\}$. Thus $\bar{J}Z_7 = -W$. Hence $L_2 = S(TM^{\perp})$. On the other hand, the lightlike transversal bundle $ltr(TM)$ is spanned by

$$\{N_1 = \frac{1}{2}(-\partial x_1 + \partial x_{14}), N_2 = \frac{1}{2}(-\partial x_2 - \partial x_{13}), N_3 = \frac{1}{2}(-\partial x_3 + \partial x_{12})\},$$

therefore $Span\{N_1, N_2\}$ is invariant with respect to $\bar{J}$ and $\bar{J}N_3 = -\frac{1}{2}Z_4 - \frac{1}{2}Z_{10}$. Hence $L_1 = Span\{N_3\}$ and $D' = Span\{\bar{J}N_3, \bar{J}W\}$. Thus M is a proper GCR-lightlike submanifold of R^{14}_4.

Let Q, P_1 and P_2 be the projections on D, $\bar{J}(L_1) = M_1 \subset D'$ and $\bar{J}(L_2) = M_2 \subset D'$, respectively. Then for any $X \in \Gamma(TM)$, we have

$$X = QX + P_1X + P_2X = QX + PX,$$

applying J to above equation, we obtain

$$\bar{J}X = TX + wP_1X + wP_2X = TX + wX,$$

where TX and wX are the tangential and transversal components of JX, respectively. Similarly

$$\bar{J}V = BV + CV,$$

for any $V \in \Gamma(tr(TM))$, where BV and CV are the sections of TM and $tr(TM)$, respectively.

3 Ricci Tensor of a *GCR*-lightlike Submanifold of an indefinite Complex Space Form

Let $\{E_1, E_2, ..., E_m\}$ be a local orthonormal frame field on M such that $\{E_1, E_2, ..., E_p, E_{p+1} = \bar{J}E_1, E_{p+2} = \bar{J}E_2, ..., E_{2p} = \bar{J}E_p\}$, $\{\xi_1, \xi_2, ..., \xi_s, \xi_{s+1} = \bar{J}\xi_1, \xi_{s+2} = \bar{J}\xi_2, ..., \xi_{2s} = \bar{J}\xi_s\}$, $\{\xi_{2s+1}, \xi_{2s+2}, ..., \xi_r\}$ and $\{\bar{J}\xi_{2s+1}, \bar{J}\xi_{2s+2}, ..., \bar{J}\xi_r\}$ be local frame fields on D_0, D_1, D_2 and $\bar{J}D_2$, respectively and $\{F_1, F_2, ..., F_q\}$ be a local frame field on D', then by direct computation, we have

$$\sum_{i=1}^{m} g(U, E_i) g(E_i, V) = g(U, V), \tag{3.1}$$

$$\sum_{i=r+1}^{m} g(U,E_i)g(E_i,V) = g(\bar{P}U,\bar{P}V), \tag{3.2}$$

$$\sum_{i=1}^{m-q} g(U,E_i)g(E_i,V) = g(QU,QV) \tag{3.3}$$

and

$$\sum_{i=1}^{q} g(U,E_i)g(E_i,V) = g(PU,PV), \tag{3.4}$$

for any $U,V\in\Gamma(TM)$ and $\bar{P}$ is the projection morphism of TM on $S(TM)$. Then the Ricci tensor is given by

$$Ric(U,V) = \sum_{a=1}^{r} \bar{g}(R(U,\xi_a)V,N_a) + \sum_{b=r+1}^{m} \bar{g}(R(U,U_b)V,U_b). \tag{3.5}$$

Using (2.16), we obtain

$$\begin{aligned}
\textstyle\sum_{a=1}^{r} \bar{g}(R(U,\xi_a)V,N_a) \quad &= -\frac{c}{4}\sum_{a=1}^{r}\bar{g}(\bar{J}V,\xi_a)\bar{g}(\bar{J}U,N_a) - \frac{cr}{4}g(U,V)\\
&-\frac{c}{4}\sum_{a=1}^{r}\bar{g}(\bar{J}U,V)\bar{g}(\bar{J}\xi_a,N_a) - \frac{c}{2}\sum_{a=1}^{r}\bar{g}(\bar{J}U,\xi_a)\bar{g}(\bar{J}V,N_a)\\
&+\sum_{a=1}^{r}\bar{g}(A_{h^l(\xi_a,V)}U,N_a) - \sum_{a=1}^{r}\bar{g}(A_{h^l(U,V)}\xi_a,N_a)\\
&+\sum_{a=1}^{r}\bar{g}(A_{h^s(\xi_a,V)}U,N_a) - \sum_{a=1}^{r}\bar{g}(A_{h^s(U,V)}\xi_a,N_a)\\
&+\frac{1}{4}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_U\bar{J})(N_a),(\bar{\nabla}_{\xi_a}\bar{J})(V))\\
&-\frac{1}{4}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_U\bar{J})(V),(\bar{\nabla}_{\xi_a}\bar{J})(N_a))\\
&-\frac{1}{2}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_U\bar{J})(\xi_a),(\bar{\nabla}_V\bar{J})(N_a))\}
\end{aligned} \tag{3.6}$$

Now, using equation (2.30) of [2] at page 158, for any $U\in\Gamma(T(M))$, define a differential 1-form as

$$\eta_a(U) = \bar{g}(U,N_a), \forall a\in\{1,2,...,r\},$$

then any vector field U on M can be expressed as

$$U = \bar{P}U + \sum_{a=1}^{r}\eta_a(U)\xi_a, \tag{3.7}$$

Therefore, we have

$$\bar{g}(U,\bar{J}V) = \bar{g}(\bar{P}U,\bar{J}V) + \sum_{a=1}^{r}\bar{g}(U,N_a)g(\xi_a,\bar{J}V). \tag{3.8}$$

Also, using (2.9), (2.10) and (3.8) in (3.6), we obtain

$$\begin{aligned}\sum_{a=1}^{r}\bar{g}(R(U,\xi_a)V,N_a) &= -\frac{c(r+3)}{4}g(U,V)+\frac{3c}{4}g(TU,TV)\\ &-\frac{c}{4}\sum_{a=1}^{r}\bar{g}(\bar{J}U,V)\bar{g}(\bar{J}\xi_a,N_a)-\sum_{a=1}^{r}\bar{g}(A_{N_a}U,h^l(\xi_a,V))\\ &+\sum_{a=1}^{r}\bar{g}(A_{N_a}\xi_a,h^l(U,V))+\sum_{a=1}^{r}\bar{g}(D^s(U,N_a),h^s(\xi_a,V))\\ &-\sum_{a=1}^{r}\bar{g}(D^s(\xi_a,N_a),h^s(U,V))+\frac{1}{4}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_U\bar{J})(N_a),(\bar{\nabla}_{\xi_a}\bar{J})(V))\\ &-\frac{1}{4}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_U\bar{J})(V),(\bar{\nabla}_{\xi_a}\bar{J})(N_a))\\ &-\frac{1}{2}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_U\bar{J})(\xi_a),(\bar{\nabla}_V\bar{J})(N_a))\}. \end{aligned} \tag{3.9}$$

Using (2.8), (2.12), (2.16) and (3.2), we obtain

$$\begin{aligned}\sum_{b=r+1}^{m}g(R(U,U_b)V,U_b) &= -\frac{c}{2}g(\bar{P}U,\bar{P}V)-\frac{(m-r)c}{4}g(U,V)+\sum_{b=r+1}^{m}\{\bar{g}(h^l(U_b,V),h^*(U,U_b))\\ &-\bar{g}(h^l(U,V),h^*(U_b,U_b))+\bar{g}(h^s(U_b,V),h^s(U,U_b))\\ &-\bar{g}(h^s(U,V),h^s(U_b,U_b))\}\\ &+\frac{3}{4}\sum_{b=r+1}^{m}\bar{g}((\bar{\nabla}_U\bar{J})(U_b),(\bar{\nabla}_{U_b}\bar{J})(V)). \end{aligned} \tag{3.10}$$

Thus subsituting (3.9) and (3.10) in (3.5), we obtain the expression of Ricci tensor of a *GCR*-lightlike submanifold as

$$\begin{aligned}Ric(U,V) &= -\frac{(m+3)c}{4}g(U,V)+\frac{3c}{4}g(TU,TV)-\frac{c}{2}g(\bar{P}U,\bar{P}V)\\ &-\frac{c}{4}\sum_{a=1}^{r}g(\bar{J}U,V)\bar{g}(\bar{J}\xi_a,N_a)+\sum_{a=1}^{r}\bar{g}(A_{N_a}\xi_a,h^l(U,V))\\ &-\sum_{a=1}^{r}\bar{g}(A_{N_a}U,h^l(\xi_a,V))+\sum_{a=1}^{r}\bar{g}(D^s(U,N_a),h^s(\xi_a,V))\\ &-\sum_{a=1}^{r}\bar{g}(D^s(\xi_a,N_a),h^s(U,V))-\sum_{b=r+1}^{m}\bar{g}(h^l(U,V),h^*(U_b,U_b))\\ &+\sum_{b=r+1}^{m}\bar{g}(h^l(U_b,V),h^*(U,U_b))-\sum_{b=r+1}^{m}\bar{g}(h^s(U,V),h^s(U_b,U_b))\\ &+\sum_{b=r+1}^{m}\bar{g}(h^s(U_b,V),h^s(U,U_b))+\frac{1}{4}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_U\bar{J})(N_a),(\bar{\nabla}_{\xi_a}\bar{J})(V))\\ &-\frac{1}{4}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_U\bar{J})(V),(\bar{\nabla}_{\xi_a}\bar{J})(N_a))-\frac{1}{2}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_U\bar{J})(\xi_a),(\bar{\nabla}_V\bar{J})(N_a))\}\\ &+\frac{3}{4}\sum_{b=r+1}^{m}\bar{g}((\bar{\nabla}_U\bar{J})(U_b),(\bar{\nabla}_{U_b}\bar{J})(V)). \end{aligned}$$

Using (2.14), we obtain

$$\begin{aligned} Ric(U,V)-Ric(V,U) &= -\frac{c}{4}\sum_{a=1}^{r} g(\bar{J}U,V)\bar{g}(\bar{J}\xi_a,N_a)+\frac{c}{4}\sum_{a=1}^{r} g(\bar{J}V,U)\bar{g}(\bar{J}\xi_a,N_a) \\ &-\sum_{a=1}^{r}\bar{g}(A_{N_a}U,h^l(\xi_a,V))+\sum_{a=1}^{r}\bar{g}(A_{N_a}V,h^l(\xi_a,U)) \\ &+\sum_{a=1}^{r}\bar{g}(D^s(U,N_a),h^s(\xi_a,V))-\sum_{a=1}^{r}\bar{g}(D^s(V,N_a),h^s(\xi_a,U)) \\ &+\sum_{b=r+1}^{m}\bar{g}(h^l(U_b,V),h^*(U,U_b))-\sum_{b=r+1}^{m}\bar{g}(h^l(U_b,U),h^*(V,U_b)) \\ &+\frac{1}{4}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_U\bar{J})(N_a),(\bar{\nabla}_V\bar{J})(\xi_a)) \\ &-\frac{1}{4}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_U\bar{J})(\xi_a),(\bar{\nabla}_V\bar{J})(N_a)) \\ &-\frac{1}{2}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_U\bar{J})(V),(\bar{\nabla}_{\xi_a}\bar{J})(N_a))\}. \end{aligned} \tag{3.11}$$

It should be noted that the induced Ricci tensor of a lightlike submanifold M is not always symmetric because the induced connection ∇ is not a metric connection. Therefore from (3.11), it is clear that the induced Ricci tensor of a *GCR*-lightlike submanifold M of indefinite nearly K*ä*hler manifold is not symmetric. In [4], Duggal and Sahin proved a theorem for non-existence of totally umbilical proper *GCR*-lightlike submanifolds in a complex space form as:

Theorem 3.1. *There exist no totally umbilical proper GCR- lightlike submanifold of an indefinite complex space form $\bar{M}(c)$, such that $c \neq 0$.*

In [12], Sangeet et al. proved the following theorem.

Theorem 3.2. *Let M be a proper totally umbilical GCR-lightlike submanifold of an indefinite Kähler manifold $\bar{M}$, then M is a totally geodesic GCR-lightlike submanifold.*

Thus using the Theorems (3.1) and (3.2) with (3.11), we have the following theorem.

Theorem 3.3. *Let M be a proper totally umbilical GCR-lightlike submanifold of an indefinite Kähler manifold $\bar{M}$, then the induce Ricci tensor on M is symmetric.*

Next, using orthonormal frame fields on D', D_0, $\bar{J}D_2$ and $Rad(TM)$, we can also define Ricci tensors as

$$\begin{aligned} Ric(U,V) &= \sum_{i=1}^{q} g(R(U,F_i)V,F_i)+\sum_{k=1}^{2p} g(R(U,E_k)V,E_k) \\ &+\sum_{l=2s+1}^{r}\bar{g}(R(U,\bar{J}\xi_l)V,\bar{J}N_l)+\sum_{a=1}^{r}\bar{g}(R(U,\xi_a)V,N_a), \end{aligned}$$

therefore, we have

$$
\begin{aligned}
Ric_D(U,V) &= \sum_{k=1}^{2p} g(R(U,E_k)V,E_k) + \sum_{l=2s+1}^{r} \bar{g}(R(U,\bar{J}\xi_l)V,\bar{J}N_l) \\
&\quad + \sum_{a=1}^{r} \bar{g}(R(U,\xi_a)V,N_a), \qquad (3.12)
\end{aligned}
$$

and

$$
Ric_{D'}(U,V) = \sum_{i=1}^{q} g(R(U,F_i)V,F_i).
$$

Using (2.9), (2.12), (2.16) and (3.3), we obtain

$$
\begin{aligned}
\textstyle\sum_{k=1}^{2p} g(R(U,E_k)V,E_k) &= -\frac{c}{2} g(QU,QV) - \frac{pc}{2} g(U,V) + \sum_{k=1}^{2p} \bar{g}(h^l(E_k,V),h^*(U,E_k)) \\
&\quad - \sum_{k=1}^{2p} \bar{g}(h^l(U,V),h^*(E_k,E_k)) + \sum_{k=1}^{2p} \bar{g}(h^s(E_k,V),h^s(U,E_k)) \\
&\quad - \sum_{k=1}^{2p} \bar{g}(h^s(U,V),h^s(E_k,E_k)) \\
&\quad + \frac{3}{4} \sum_{k=1}^{2p} \bar{g}((\bar{\nabla}_U \bar{J})(E_k),(\bar{\nabla}_{E_k} \bar{J})(V)). \qquad (3.13)
\end{aligned}
$$

and using the Lemma 2.3, we get

$$
\begin{aligned}
\textstyle\sum_{l=2s+1}^{r} \bar{g}(R(U,\bar{J}\xi_l)V,\bar{J}N_l) &= -\frac{c(r-2s)}{4} g(U,V) + \frac{c}{4} \sum_{l=2s+1}^{r} \bar{g}(\bar{J}\xi_l,V)\bar{g}(U,\bar{J}N_l) \\
&\quad - \sum_{l=2s+1}^{r} \bar{g}(A_{h^l(U,V)}\bar{J}\xi_l,\bar{J}N_l) + \sum_{l=2s+1}^{r} \bar{g}(A_{h^l(\bar{J}\xi_l,V)}U,\bar{J}N_l) \\
&\quad - \sum_{l=2s+1}^{r} \bar{g}(A_{h^s(U,V)}\bar{J}\xi_l,\bar{J}N_l) + \sum_{l=2s+1}^{r} \bar{g}(A_{h^s(\bar{J}\xi_l,V)}U,\bar{J}N_l) \\
&\quad - \frac{1}{4} \sum_{l=2s+1}^{r} \bar{g}((\bar{\nabla}_U \bar{J})(N_l),(\bar{\nabla}_V \bar{J})(\xi_l)) \\
&\quad - \frac{1}{4} \sum_{l=2s+1}^{r} \bar{g}((\bar{\nabla}_U \bar{J})(V),(\bar{\nabla}_{N_l} \bar{J})(\xi_l)) \\
&\quad - \frac{1}{2} \sum_{l=2s+1}^{r} \bar{g}((\bar{\nabla}_U \bar{J})(\xi_l),(\bar{\nabla}_V \bar{J})(N_l)). \qquad (3.14)
\end{aligned}
$$

Using (3.9), (3.13) and (3.14) in (3.12), we obtain

$$\begin{aligned}
Ric_D(U,V) &= \frac{3c}{4}g(TU,TV) - \frac{c(2p+2r-2s+3)}{4}g(U,V) - \frac{c}{2}g(QU,QV) \\
&\quad -\frac{c}{4}\sum_{a=1}^{r}\bar{g}(\bar{J}U,V)\bar{g}(\bar{J}\xi_a,N_a) + \frac{c}{4}\sum_{l=2s+1}^{r}\bar{g}(\bar{J}\xi_l,V)\bar{g}(U,\bar{J}N_l) \\
&\quad +\sum_{a=1}^{r}\bar{g}(A_{N_a}\xi_a,h^l(U,V)) - \sum_{a=1}^{r}\bar{g}(A_{N_a}U,h^l(\xi_a,V)) \\
&\quad -\sum_{a=1}^{r}\bar{g}(D^s(\xi_a,N_a),h^s(U,V)) + \sum_{a=1}^{r}\bar{g}(D^s(U,N_a),h^s(\xi_a,V)) \\
&\quad +\sum_{k=1}^{2p}\bar{g}(h^l(E_k,V),h^*(U,E_k)) - \sum_{k=1}^{2p}\bar{g}(h^l(U,V),h^*(E_k,E_k)) \\
&\quad +\sum_{k=1}^{2p}\bar{g}(h^s(E_k,V),h^s(U,E_k)) - \sum_{k=1}^{2p}\bar{g}(h^s(U,V),h^s(E_k,E_k)) \\
&\quad -\sum_{l=2s+1}^{r}\bar{g}(A_{h^l(U,V)}\bar{J}\xi_l,\bar{J}N_l) + \sum_{l=2s+1}^{r}\bar{g}(A_{h^l(\bar{J}\xi_l,V)}U,\bar{J}N_l) \\
&\quad -\sum_{l=2s+1}^{r}\bar{g}(A_{h^s(U,V)}\bar{J}\xi_l,\bar{J}N_l) + \sum_{l=2s+1}^{r}\bar{g}(A_{h^s(\bar{J}\xi_l,V)}U,\bar{J}N_l) \\
&\quad +\frac{1}{4}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_U\bar{J})(N_a),(\bar{\nabla}_{\xi_a}\bar{J})(V)) - \frac{1}{4}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_U\bar{J})(V),(\bar{\nabla}_{\xi_a}\bar{J})(N_a)) \\
&\quad -\frac{1}{2}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_U\bar{J})(\xi_a),(\bar{\nabla}_V\bar{J})(N_a)) + \frac{3}{4}\sum_{k=1}^{2p}\bar{g}((\bar{\nabla}_U\bar{J})(E_k),(\bar{\nabla}_{E_k}\bar{J})(V)) \\
&\quad -\frac{1}{4}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_U\bar{J})(N_l),(\bar{\nabla}_V\bar{J})(\xi_l)) - \frac{1}{4}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_U\bar{J})(V),(\bar{\nabla}_{N_l}\bar{J})(\xi_l)) \\
&\quad -\frac{1}{2}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_U\bar{J})(\xi_l),(\bar{\nabla}_V\bar{J})(N_l)).
\end{aligned}$$

Also using (2.8), (2.12), (2.16) and (3.4), we obtain

$$\begin{aligned}
Ric_{D'}(U,V) &= -\frac{c}{2}g(PU,PV) - \frac{qc}{4}g(U,V) - \sum_{i=1}^{q}\bar{g}(h^l(U,V),h^*(F_i,F_i)) \\
&\quad +\sum_{i=1}^{q}\bar{g}(h^l(F_i,V),h^*(U,F_i)) - \sum_{i=1}^{q}\bar{g}(h^s(F_i,F_i),h^s(U,V)) \\
&\quad +\sum_{i=1}^{q}\bar{g}(h^s(F_i,V),h^s(U,F_i)) + \frac{3}{4}\sum_{i=1}^{q}\bar{g}((\bar{\nabla}_U\bar{J})(F_i),(\bar{\nabla}_{F_i}\bar{J})(V)).
\end{aligned}$$

Let $X,Y \in \Gamma(D)$ and $Z,W \in \Gamma(D')$, then particularly, we have

$$\begin{aligned} Ric_{D'}(X,Y) &= -\frac{qc}{4}g(X,Y) - \sum_{i=1}^{q}\bar{g}(h^l(X,Y),h^*(F_i,F_i)) \\ &+ \sum_{i=1}^{q}\bar{g}(h^l(F_i,Y),h^*(X,F_i)) - \sum_{i=1}^{q}\bar{g}(h^s(F_i,F_i),h^s(X,Y)) \\ &+ \sum_{i=1}^{q}\bar{g}(h^s(F_i,Y),h^s(X,F_i)) + \frac{3}{4}\sum_{i=1}^{q}\bar{g}((\bar{\nabla}_X\bar{J})(F_i),(\bar{\nabla}_{F_i}\bar{J})(Y)), \end{aligned} \tag{3.15}$$

$$\begin{aligned} Ric_D(X,Y) &= \frac{c(s-r-p-1)}{2}g(X,Y) - \frac{c}{4}\sum_{a=1}^{r}\bar{g}(\bar{J}X,Y)\bar{g}(\bar{J}\xi_a,N_a) \\ &+ \sum_{a=1}^{r}\bar{g}(A_{N_a}\xi_a,h^l(X,Y)) - \sum_{a=1}^{r}\bar{g}(A_{N_a}X,h^l(\xi_a,Y)) \\ &- \sum_{a=1}^{r}\bar{g}(D^s(\xi_a,N_a),h^s(X,Y)) + \sum_{a=1}^{r}\bar{g}(D^s(X,N_a),h^s(\xi_a,Y)) \\ &+ \sum_{k=1}^{2p}\bar{g}(h^l(E_k,Y),h^*(X,E_k)) - \sum_{k=1}^{2p}\bar{g}(h^l(X,Y),h^*(E_k,E_k)) \\ &+ \sum_{k=1}^{2p}\bar{g}(h^s(E_k,Y),h^s(X,E_k)) - \sum_{k=1}^{2p}\bar{g}(h^s(X,Y),h^s(E_k,E_k)) \\ &- \sum_{l=2s+1}^{r}\bar{g}(A_{h^l(X,Y)}\bar{J}\xi_l,\bar{J}N_l) + \sum_{l=2s+1}^{r}\bar{g}(A_{h^l(\bar{J}\xi_l,Y)}X,\bar{J}N_l) \\ &- \sum_{l=2s+1}^{r}\bar{g}(A_{h^s(X,Y)}\bar{J}\xi_l,\bar{J}N_l) + \sum_{l=2s+1}^{r}\bar{g}(A_{h^s(\bar{J}\xi_l,Y)}X,\bar{J}N_l) \\ &+ \frac{1}{4}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(N_a),(\bar{\nabla}_{\xi_a}\bar{J})(Y)) - \frac{1}{4}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(Y),(\bar{\nabla}_{\xi_a}\bar{J})(N_a)) \\ &- \frac{1}{2}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(\xi_a),(\bar{\nabla}_Y\bar{J})(N_a)) + \frac{3}{4}\sum_{k=1}^{2p}\bar{g}((\bar{\nabla}_X\bar{J})(E_k),(\bar{\nabla}_{E_k}\bar{J})(Y)) \\ &- \frac{1}{4}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(N_l),(\bar{\nabla}_Y\bar{J})(\xi_l)) - \frac{1}{4}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(Y),(\bar{\nabla}_{N_l}\bar{J})(\xi_l)) \\ &- \frac{1}{2}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(\xi_l),(\bar{\nabla}_Y\bar{J})(N_l)). \end{aligned} \tag{3.16}$$

$$\begin{aligned} Ric_{D'}(X,Z) &= -\sum_{i=1}^{q}\bar{g}(h^l(X,Z),h^*(F_i,F_i)) + \sum_{i=1}^{q}\bar{g}(h^l(F_i,Z),h^*(X,F_i)) \\ &- \sum_{i=1}^{q}\bar{g}(h^s(F_i,F_i),h^s(X,Z)) + \sum_{i=1}^{q}\bar{g}(h^s(F_i,Z),h^s(X,F_i)) \\ &+ \frac{3}{4}\sum_{i=1}^{q}\bar{g}((\bar{\nabla}_X\bar{J})(F_i),(\bar{\nabla}_{F_i}\bar{J})(Z)), \end{aligned} \tag{3.17}$$

$$
\begin{aligned}
Ric_D(X,Z) \quad &= \frac{c}{4}\sum_{l=2s+1}^{r}\bar{g}(\bar{J}\xi_l,Z)\bar{g}(X,\bar{J}N_l)+\sum_{a=1}^{r}\bar{g}(A_{N_a}\xi_a,h^l(X,Z))\\
&-\sum_{a=1}^{r}\bar{g}(A_{N_a}X,h^l(\xi_a,Z))-\sum_{a=1}^{r}\bar{g}(D^s(\xi_a,N_a),h^s(X,Z))\\
&+\sum_{a=1}^{r}\bar{g}(D^s(X,N_a),h^s(\xi_a,Z))+\sum_{k=1}^{2p}\bar{g}(h^l(E_k,Z),h^*(X,E_k))\\
&-\sum_{k=1}^{2p}\bar{g}(h^l(X,Z),h^*(E_k,E_k))+\sum_{k=1}^{2p}\bar{g}(h^s(E_k,Z),h^s(X,E_k))\\
&-\sum_{k=1}^{2p}\bar{g}(h^s(X,Z),h^s(E_k,E_k))-\sum_{l=2s+1}^{r}\bar{g}(A_{h^l(X,Z)}\bar{J}\xi_l,\bar{J}N_l)\\
&+\sum_{l=2s+1}^{r}\bar{g}(A_{h^l(\bar{J}\xi_l,Z)}X,\bar{J}N_l)-\sum_{l=2s+1}^{r}\bar{g}(A_{h^s(X,Z)}\bar{J}\xi_l,\bar{J}N_l)\\
&+\sum_{l=2s+1}^{r}\bar{g}(A_{h^s(\bar{J}\xi_l,Z)}X,\bar{J}N_l)+\frac{1}{4}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(N_a),(\bar{\nabla}_{\xi_a}\bar{J})(Z))\\
&-\frac{1}{4}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(Z),(\bar{\nabla}_{\xi_a}\bar{J})(N_a))-\frac{1}{2}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(\xi_a),(\bar{\nabla}_Z\bar{J})(N_a))\\
&+\frac{3}{4}\sum_{k=1}^{2p}\bar{g}((\bar{\nabla}_X\bar{J})(E_k),(\bar{\nabla}_{E_k}\bar{J})(Z))-\frac{1}{4}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(N_l),(\bar{\nabla}_Z\bar{J})(\xi_l))\\
&-\frac{1}{4}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(Z),(\bar{\nabla}_{N_l}\bar{J})(\xi_l))-\frac{1}{2}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(\xi_l),(\bar{\nabla}_Z\bar{J})(N_l)) \quad (3.18)
\end{aligned}
$$

and

$$
\begin{aligned}
Ric_{D'}(Z,W) \quad &= -\frac{(q+2)c}{4}g(Z,W)-\sum_{i=1}^{q}\bar{g}(h^l(Z,W),h^*(F_i,F_i))\\
&+\sum_{i=1}^{q}\bar{g}(h^l(F_i,W),h^*(Z,F_i))-\sum_{i=1}^{q}\bar{g}(h^s(F_i,F_i),h^s(Z,W))\\
&+\sum_{i=1}^{q}\bar{g}(h^s(F_i,W),h^s(Z,F_i))+\frac{3}{4}\sum_{i=1}^{q}\bar{g}((\bar{\nabla}_Z\bar{J})(F_i),(\bar{\nabla}_{F_i}\bar{J})(W)) \qquad (3.19)
\end{aligned}
$$

$$
\begin{aligned}
Ric_D(Z,W) = & -\frac{c(2p+2r-2s+3)}{4}g(Z,W) + \sum_{a=1}^{r}\bar{g}(A_{N_a}\xi_a, h^l(Z,W)) \\
& - \sum_{a=1}^{r}\bar{g}(A_{N_a}Z, h^l(\xi_a,W)) - \sum_{a=1}^{r}\bar{g}(D^s(\xi_a,N_a), h^s(Z,W)) \\
& + \sum_{a=1}^{r}\bar{g}(D^s(Z,N_a), h^s(\xi_a,W)) + \sum_{k=1}^{2p}\bar{g}(h^l(E_k,W), h^*(Z,E_k)) \\
& - \sum_{k=1}^{2p}\bar{g}(h^l(Z,W), h^*(E_k,E_k)) + \sum_{k=1}^{2p}\bar{g}(h^s(E_k,W), h^s(Z,E_k)) \\
& - \sum_{k=1}^{2p}\bar{g}(h^s(Z,W), h^s(E_k,E_k)) - \sum_{l=2s+1}^{r}\bar{g}(A_{h^l(Z,W)}\bar{J}\xi_l, \bar{J}N_l) \\
& + \sum_{l=2s+1}^{r}\bar{g}(A_{h^l(\bar{J}\xi_l,W)}Z, \bar{J}N_l) - \sum_{l=2s+1}^{r}\bar{g}(A_{h^s(Z,W)}\bar{J}\xi_l, \bar{J}N_l) \\
& + \sum_{l=2s+1}^{r}\bar{g}(A_{h^s(\bar{J}\xi_l,W)}Z, \bar{J}N_l) + \frac{1}{4}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_Z\bar{J})(N_a), (\bar{\nabla}_{\xi_a}\bar{J})(W)) \\
& - \frac{1}{4}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_Z\bar{J})(W), (\bar{\nabla}_{\xi_a}\bar{J})(N_a)) - \frac{1}{2}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_Z\bar{J})(\xi_a), (\bar{\nabla}_W\bar{J})(N_a)) \\
& + \frac{3}{4}\sum_{k=1}^{2p}\bar{g}((\bar{\nabla}_Z\bar{J})(E_k), (\bar{\nabla}_{E_k}\bar{J})(W)) - \frac{1}{4}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_Z\bar{J})(N_l), (\bar{\nabla}_W\bar{J})(\xi_l)) \\
& - \frac{1}{4}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_Z\bar{J})(W), (\bar{\nabla}_{N_l}\bar{J})(\xi_l)) - \frac{1}{2}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_Z\bar{J})(\xi_l), (\bar{\nabla}_W\bar{J})(N_l)). \quad (3.20)
\end{aligned}
$$

Definition 3.4. A *GCR*-lightlike submanifold of an indefinite nearly Kähler manifold is called

(i) totally geodesic *GCR*-lightlike submanifold if its second fundamental form h vanishes, that is, $h(X,Y) = 0$, for any $X, Y \in \Gamma(TM)$.

(ii) D-geodesic *GCR*-lightlike submanifold if $h(X,Y) = 0$, for any $X, Y \in \Gamma(D)$.

(iii) D'-geodesic *GCR*-lightlike submanifold $h(X,Y) = 0$, for any $X, Y \in \Gamma(D')$.

(iv) mixed-geodesic *GCR*-lightlike submanifold if $h(X,Y) = 0$, for any $X \in \Gamma(D)$ and $Y \in \Gamma(D')$.

Thus from (3.15) to (3.20), we have the following results.

Theorem 3.5. *Let M be a totally geodesic GCR-lightlike submanifold of an indefinite complex* *space form* $\bar{M}(c)$*, then for any* $X,Y \in \Gamma(D)$ *and* $Z,W \in \Gamma(D')$

$$\begin{aligned} Ric_D(X,Y) &= \frac{c(s-r-p-1)}{2}g(X,Y) - \frac{c}{4}\sum_{a=1}^{r}\bar{g}(\bar{J}X,Y)\bar{g}(\bar{J}\xi_a,N_a) \\ &+\frac{1}{4}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(N_a),(\bar{\nabla}_{\xi_a}\bar{J})(Y)) - \frac{1}{4}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(Y),(\bar{\nabla}_{\xi_a}\bar{J})(N_a)) \\ &-\frac{1}{2}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(\xi_a),(\bar{\nabla}_Y\bar{J})(N_a)) + \frac{3}{4}\sum_{k=1}^{2p}\bar{g}((\bar{\nabla}_X\bar{J})(E_k),(\bar{\nabla}_{E_k}\bar{J})(Y)) \\ &-\frac{1}{4}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(N_l),(\bar{\nabla}_Y\bar{J})(\xi_l)) - \frac{1}{4}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(Y),(\bar{\nabla}_{N_l}\bar{J})(\xi_l)) \\ &-\frac{1}{2}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(\xi_l),(\bar{\nabla}_Y\bar{J})(N_l)), \end{aligned}$$

$$\begin{aligned} Ric_D(X,Z) &= \frac{c}{4}\sum_{l=2s+1}^{r}\bar{g}(\bar{J}\xi_l,Z)\bar{g}(X,\bar{J}N_l) + \frac{1}{4}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(N_a),(\bar{\nabla}_{\xi_a}\bar{J})(Z)) \\ &-\frac{1}{4}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(Z),(\bar{\nabla}_{\xi_a}\bar{J})(N_a)) - \frac{1}{2}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(\xi_a),(\bar{\nabla}_Z\bar{J})(N_a)) \\ &+\frac{3}{4}\sum_{k=1}^{2p}\bar{g}((\bar{\nabla}_X\bar{J})(E_k),(\bar{\nabla}_{E_k}\bar{J})(Z)) - \frac{1}{4}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(N_l),(\bar{\nabla}_Z\bar{J})(\xi_l)) \\ &-\frac{1}{4}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(Z),(\bar{\nabla}_{N_l}\bar{J})(\xi_l)) - \frac{1}{2}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(\xi_l),(\bar{\nabla}_Z\bar{J})(N_l)), \end{aligned}$$

$$\begin{aligned} Ric_D(Z,W) &= -\frac{c(2p+2r-2s+3)}{4}g(Z,W) + \frac{1}{4}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_Z\bar{J})(N_a),(\bar{\nabla}_{\xi_a}\bar{J})(W)) \\ &-\frac{1}{4}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_Z\bar{J})(W),(\bar{\nabla}_{\xi_a}\bar{J})(N_a)) - \frac{1}{2}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_Z\bar{J})(\xi_a),(\bar{\nabla}_W\bar{J})(N_a)) \\ &+\frac{3}{4}\sum_{k=1}^{2p}\bar{g}((\bar{\nabla}_Z\bar{J})(E_k),(\bar{\nabla}_{E_k}\bar{J})(W)) - \frac{1}{4}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_Z\bar{J})(N_l),(\bar{\nabla}_W\bar{J})(\xi_l)) \\ &-\frac{1}{4}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_Z\bar{J})(W),(\bar{\nabla}_{N_l}\bar{J})(\xi_l)) - \frac{1}{2}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_Z\bar{J})(\xi_l),(\bar{\nabla}_W\bar{J})(N_l)) \end{aligned}$$

and

$$Ric_{D'}(X,Y) = -\frac{qc}{4}g(X,Y) + \frac{3}{4}\sum_{i=1}^{q}\bar{g}((\bar{\nabla}_X\bar{J})(F_i),(\bar{\nabla}_{F_i}\bar{J})(Y)),$$

$$Ric_{D'}(X,Z) = \frac{3}{4}\sum_{i=1}^{q}\bar{g}((\bar{\nabla}_X\bar{J})(F_i),(\bar{\nabla}_{F_i}\bar{J})(Z)),$$

$$Ric_{D'}(Z,W) = -\frac{(q+2)c}{4}g(Z,W) + \frac{3}{4}\sum_{i=1}^{q}\bar{g}((\bar{\nabla}_Z\bar{J})(F_i),(\bar{\nabla}_{F_i}\bar{J})(W)).$$

Theorem 3.6. *Let M be a D-geodesic GCR-lightlike submanifold of an indefinite complex space form $\bar{M}(c)$, then for any $X,Y \in \Gamma(D)$*

$$\begin{aligned}
Ric_D(X,Y) &= \frac{c(s-r-p-1)}{2}g(X,Y) - \frac{c}{4}\sum_{a=1}^{r}\bar{g}(\bar{J}X,Y)\bar{g}(\bar{J}\xi_a,N_a)\\
&+\frac{1}{4}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(N_a),(\bar{\nabla}_{\xi_a}\bar{J})(Y)) - \frac{1}{4}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(Y),(\bar{\nabla}_{\xi_a}\bar{J})(N_a))\\
&-\frac{1}{2}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(\xi_a),(\bar{\nabla}_Y\bar{J})(N_a)) + \frac{3}{4}\sum_{k=1}^{2p}\bar{g}((\bar{\nabla}_X\bar{J})(E_k),(\bar{\nabla}_{E_k}\bar{J})(Y))\\
&-\frac{1}{4}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(N_l),(\bar{\nabla}_Y\bar{J})(\xi_l)) - \frac{1}{4}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(Y),(\bar{\nabla}_{N_l}\bar{J})(\xi_l))\\
&-\frac{1}{2}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(\xi_l),(\bar{\nabla}_Y\bar{J})(N_l)),
\end{aligned}$$

and

$$\begin{aligned}
Ric_{D'}(X,Y) &= -\frac{qc}{4}g(X,Y) + \sum_{i=1}^{q}\bar{g}(h^l(F_i,Y),h^*(X,F_i))\\
&+\sum_{i=1}^{q}\bar{g}(h^s(F_i,Y),h^s(X,F_i)) + \frac{3}{4}\sum_{i=1}^{q}\bar{g}((\bar{\nabla}_X\bar{J})(F_i),(\bar{\nabla}_{F_i}\bar{J})(Y)).
\end{aligned}$$

Theorem 3.7. *Let M be a D'-geodesic GCR-lightlike submanifold of an indefinite complex space form $\bar{M}(c)$, then for any $Z,W \in \Gamma(D')$*

$$\begin{aligned}
Ric_D(Z,W) &= -\frac{c(2p+2r-2s+3)}{4}g(Z,W) - \sum_{a=1}^{r}\bar{g}(A_{N_a}Z,h^l(\xi_a,W))\\
&+\sum_{a=1}^{r}\bar{g}(D^s(Z,N_a),h^s(\xi_a,W)) + \sum_{k=1}^{2p}\bar{g}(h^l(E_k,W),h^*(Z,E_k))\\
&+\sum_{k=1}^{2p}\bar{g}(h^s(E_k,W),h^s(Z,E_k)) + \sum_{l=2s+1}^{r}\bar{g}(A_{h^l(\bar{J}\xi_l,W)}Z,\bar{J}N_l)\\
&+\sum_{l=2s+1}^{r}\bar{g}(A_{h^s(\bar{J}\xi_l,W)}Z,\bar{J}N_l) - \frac{1}{2}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_Z\bar{J})(\xi_a),(\bar{\nabla}_W\bar{J})(N_a))\\
&+\frac{1}{4}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_Z\bar{J})N_a,(\bar{\nabla}_{\xi_a}\bar{J})W) - \frac{1}{4}\sum_{a=1}^{r}\bar{g}((\bar{\nabla}_Z\bar{J})W,(\bar{\nabla}_{\xi_a}\bar{J})N_a)\\
&+\frac{3}{4}\sum_{k=1}^{2p}\bar{g}((\bar{\nabla}_Z\bar{J})(E_k),(\bar{\nabla}_{E_k}\bar{J})(W)) - \frac{1}{4}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_Z\bar{J})(N_l),(\bar{\nabla}_W\bar{J})(\xi_l))\\
&-\frac{1}{4}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_Z\bar{J})(W),(\bar{\nabla}_{N_l}\bar{J})(\xi_l)) - \frac{1}{2}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_Z\bar{J})(\xi_l),(\bar{\nabla}_W\bar{J})(N_l))
\end{aligned}$$

and

$$\begin{aligned} Ric_{D'}(Z,W) &= -\frac{(q+2)c}{4}g(Z,W)+\sum_{i=1}^{q}\bar{g}(h^l(F_i,W),h^*(Z,F_i)) \\ &+\sum_{i=1}^{q}\bar{g}(h^s(F_i,W),h^s(Z,F_i))+\frac{3}{4}\sum_{i=1}^{q}\bar{g}((\bar{\nabla}_X\bar{J})(F_i),(\bar{\nabla}_{F_i}\bar{J})(Y)). \end{aligned}$$

Theorem 3.8. *Let M be a mixed-geodesic GCR-lightlike submanifold of an indefinite complex space form $\bar{M}(c)$, then for any $X\in\Gamma(D)$ and $Z\in\Gamma(D')$*

$$\begin{aligned} Ric_D(X,Z) &= \frac{c}{4}\sum_{l=2s+1}^{r}\bar{g}(\bar{J}\xi_l,Z)\bar{g}(X,\bar{J}N_l)+\frac{3}{4}\sum_{k=1}^{2p}\bar{g}((\bar{\nabla}_X\bar{J})(E_k),(\bar{\nabla}_{E_k}\bar{J})(Z)) \\ &-\frac{1}{4}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(N_l),(\bar{\nabla}_Z\bar{J})(\xi_l))-\frac{1}{4}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(Z),(\bar{\nabla}_{N_l}\bar{J})(\xi_l)) \\ &-\frac{1}{2}\sum_{l=2s+1}^{r}\bar{g}((\bar{\nabla}_X\bar{J})(\xi_l),(\bar{\nabla}_Z\bar{J})(N_l)), \end{aligned}$$

and

$$\begin{aligned} Ric_{D'}(X,Z) &= \sum_{i=1}^{q}\bar{g}(h^l(F_i,Z),h^*(X,F_i))+\sum_{i=1}^{q}\bar{g}(h^s(F_i,Z),h^s(X,F_i)) \\ &+\frac{3}{4}\sum_{i=1}^{q}\bar{g}((\bar{\nabla}_X\bar{J})(F_i),(\bar{\nabla}_{F_i}\bar{J})(Y)). \end{aligned}$$

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