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Abstract

We prove an alternative method in order to obtain generation and analyticity results for the semigroups generated by some degenerate second order differential operators linked to the hypergeometric equation.

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1 Introduction

A particular case of the hypergeometric equation is

$$x(1-x)u''(x)+(c-x)u'(x)=0, \tag{1.1}$$

where c is a real or complex parameter. The related operator

$$Au(x):=x(1-x)u''(x)+(\alpha+\beta x)u'(x), \tag{1.2}$$

with $\alpha=\beta=0$, arose in genetics, and was studied by Feller in 1952 in a famous paper [8]. By equipping A with Wentzell boundary conditions ($Au(x)=0$ for $x=0,1$), Feller

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showed that A (with $\alpha = \beta = 0$) generates a strongly continuous contraction semigroup on $X := C[0,1]$, equipped with the sup-norm $\|\cdot\|_\infty$. That is, for the solution of the resolvent equation

$$\lambda v - Av = h \tag{1.3}$$

(and the general solution of this ordinary equation was known in the nineteenth century), Feller obtained the estimate

$$\|v\|_\infty \leq \frac{\|h\|_\infty}{Re\,\lambda}$$

for $Re\,\lambda > 0$. The natural question arising from Feller's result was whether the semigroup generated by A is an analytic semigroup. This question was affirmatively answered by G. Metafune in 1998 (see [12]).

The result followed from Metafune's estimate

$$\|v\|_\infty \leq M\frac{\|h\|_\infty}{|\lambda|}$$

for v satisfying (1.3) and $Re\,\lambda > K$, where M and K are suitable positive constants. Our proofs make use of known properties of hypergeometric functions, so assuming the reader knows some of this background, our proofs can be regarded as simpler, more direct and extendable to other cases.

These arguments allow us to save the above analyticity results even in other cases (see Section 3). For analyticity results related to degenerate operators with Wentzell boundary conditions in $C[0,1]$ see, e.g., [1, 2],[5- 7] and [10] .

2 The semigroup generated by the operator $Au = x(1-x)u''$ with Wentzell boundary conditions

We would like to study the realization of the operator $Au := x(1-x)u''$ in $C[0.1]$ with domain

$$D(A) := \{u \in C[0,1] \cap C^2(0,1) : \lim_{x \to j} Au(x) = 0,\ j = 0,1\}. \tag{2.1}$$

We first consider the homogeneous equation

$$x(1-x)u''(x) - \lambda u(x) = 0 \tag{2.2}$$

where $Re\,\lambda > 0$, $|\lambda|$ large. We seek for a solution to (2.2) in $C[0,1] \cap C^2(0,1)$ under the form of a power series

$$u_1(x) = \phi(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{2.3}$$

We find that

$$a_0 = 0, \quad a_2 = \frac{\lambda}{2}a_1, \quad a_{n+1} = \frac{\lambda + n(n-1)}{n(n+1)}a_n, \quad n = 1,2,...$$

Take $a_1 = 1$, so that

$$a_{n+1} = \frac{\lambda + n(n-1)}{(n+1)n} \cdot \frac{\lambda + (n-1)(n-2)}{n(n-1)} \cdots \frac{\lambda + 4\cdot 3}{5\cdot 4} \cdot \frac{\lambda + 3\cdot 2}{4\cdot 3} \cdot \frac{\lambda + 2\cdot 1}{3\cdot 2} \cdot \frac{\lambda + 1\cdot 0}{2\cdot 1}$$

$$= \frac{(\lambda+n(n-1))(\lambda+(n-1)(n-2))\cdots(\lambda+4\cdot 3)(\lambda+3\cdot 2)(\lambda+2\cdot 1)(\lambda+1\cdot 0)}{(n+1)!\,n!}.$$

Notice that

$$a_1+a_2 = \frac{\lambda+2}{2} = \prod_{k=1}^{1}\frac{\lambda+(k+1)k}{(k+1)\cdot k},$$

$$a_1+a_2+a_3 = \frac{\lambda+2}{2}+\frac{\lambda+2}{2\cdot 3}\cdot\frac{\lambda}{2} = \frac{\lambda+2}{2}\left(\frac{\lambda+2\cdot 3}{2\cdot 3}\right)$$

$$= \prod_{k=1}^{2}\frac{\lambda+(k+1)k}{(k+1)k},$$

$$a_1+a_2+\cdots+a_n = \prod_{k=1}^{n-1}\frac{\lambda+(k+1)k}{(k+1)k}, \quad n\ge 2,$$

so that

$$\sum_{n=1}^{\infty} a_n = \prod_{k=1}^{\infty}\frac{\lambda+(k+1)k}{(k+1)k}.$$

On the other hand,

$$|a_1|+|a_2| = \frac{|\lambda|+2}{2}$$

$$|a_1+|a_2|+|a_3| \le \frac{(|\lambda|+1\cdot 2)(|\lambda|+2\cdot 3)}{(1\cdot 2)(2\cdot 3)}$$

$$|a_1+|a_2|+|a_3|+|a_4| \le \frac{(|\lambda|+1\cdot 2)(|\lambda|+2\cdot 3)(|\lambda|+3\cdot 4)}{3!\,4!},$$

$$\sum_{k=1}^{n}|a_k| \le \prod_{k=1}^{n-1}\frac{|\lambda|+(k+1)k}{(k+1)k},$$

and therefore

$$\sum_{n=1}^{\infty}|a_n| \le \prod_{n=1}^{\infty}\frac{|\lambda|+(n+1)n}{(n+1)n}.$$

Hence

$$\frac{\sum_{n=1}^{\infty}|a_n|}{|\sum_{n=1}^{\infty}a_n|} \le \frac{\prod_{n=1}^{\infty}\frac{|\lambda|+(n+1)n}{(n+1)n}}{\prod_{n=1}^{\infty}\frac{|\lambda+(n+1)n|}{(n+1)n}} \le C, \tag{2.4}$$

where C is independent of λ. In fact, we observe that $|\phi(1)| \le \sum_{n=1}^{\infty}|a_n|$ converges. To show this, we use the Raabe test. Let $\lambda = x+iy$, so that

$$n\left(|\frac{a_n}{a_{n+1}}|-1\right) = n\left(\frac{n(n+1)-\sqrt{(x+n(n-1))^2+y^2}}{\sqrt{(x+n(n-1))^2+y^2}}\right)$$

$$= n\,\frac{n(n+1)-n(n-1)\sqrt{1+\frac{2x}{n(n-1)}+\frac{x^2+y^2}{n^2(n-1)^2}}}{n(n-1)\sqrt{1+\frac{2x}{n(n-1)}+\frac{x^2+y^2}{n^2(n-1)^2}}}$$

$$\sim n\,\frac{n(n+1)-n(n-1)-x-\frac{x^2+y^2}{2n(n-1)}}{n(n-1)}$$

$$= n\,\frac{2n-x-\frac{x^2+y^2}{2n(n-1)}}{n(n-1)} \to 2 > 1$$

as $n\to\infty$.

Next, we observe that $\psi(x)=\phi(1-x)$ is another solution to equation (2.2). Compute the Wronskian W of ϕ and ψ:

$$W(x)=\begin{vmatrix} \phi(x) & \phi(1-x) \\ \phi'(x) & -\phi'(1-x) \end{vmatrix} = -\phi(x)\phi'(1-x)-\phi'(x)\phi(1-x).$$

Since

$$W'(x)=\begin{vmatrix} \phi(x) & \phi(1-x) \\ \phi''(x) & \phi''(1-x) \end{vmatrix} = 0,$$

for $x\in(0,1)$, we conclude that

$$W(x)=W(\frac{1}{2})=-2\phi(\frac{1}{2})\phi'(\frac{1}{2}).$$

Now, if $x\in(0,1)$, then $\phi(x)$ coincides with

$$x(1-x)F\left(\frac{3+\sqrt{1-4\lambda}}{2},\frac{3-\sqrt{1-4\lambda}}{2},2;x\right),$$

where F is the hypergeometric series

$$F(a,b,c;z)=1+\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(1)_n(c)_n}z^n,$$

with

$$(a)_n := a(a+1)\cdots(a+n-1),$$

(see e.g. [11, p.296]). On the other hand, from [11, p.305], we have

$$\frac{d}{dz}F(a,b,c;z)=\frac{ab}{c}F(a+1,b+1,c+1;z),$$

so that

$$2\phi(\frac{1}{2})\phi'(\frac{1}{2}) = \frac{\lambda+2}{16}F\left(\frac{3+\sqrt{1-4\lambda}}{2},\frac{3-\sqrt{1-4\lambda}}{2},2;\frac{1}{2}\right)\cdot$$

$$F\left(\frac{5+\sqrt{1-4\lambda}}{2},\frac{5-\sqrt{1-4\lambda}}{2},3;\frac{1}{2}\right). \qquad (2.5)$$

Let us compute $F\left(\frac{3+\sqrt{1-4\lambda}}{2},\frac{3-\sqrt{1-4\lambda}}{2},2;\frac{1}{2}\right)$ and $F\left(\frac{5+\sqrt{1-4\lambda}}{2},\frac{5-\sqrt{1-4\lambda}}{2},3;\frac{1}{2}\right)$.

We have

$$F\left(\frac{3+\sqrt{1-4\lambda}}{2},\frac{3-\sqrt{1-4\lambda}}{2},2;\frac{1}{2}\right)=1+\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{(\lambda+2\cdot1)(\lambda+3\cdot2)\cdots(\lambda+(n+1)n)}{(n+1)!n!}$$

and

$$F\left(\frac{5+\sqrt{1-4\lambda}}{2},\frac{5-\sqrt{1-4\lambda}}{2},3;\frac{1}{2}\right)=1+\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{(\lambda+2\cdot 3)(\lambda+3\cdot 4)\cdots(\lambda+(n+1)(n+2))}{(n+2)!n!}.$$

Recall that

$$\sum_{n=1}^{\infty}a_n=\prod_{n=1}^{\infty}\frac{\lambda+(n+1)n}{(n+1)n}$$

and observe that $\sum_{n=1}^{\infty}\frac{1}{2^n}=1$. Then (2.3) implies that

$$|W(\frac{1}{2})|\sim|\lambda+2||\sum_{n=1}^{\infty}a_n|^2. \tag{2.6}$$

Notice that, if ϕ is positive, then [3, Lemma 3] applied to ϕ and ψ enables us to affirm that $I-A$ is surjective. Next we consider $f\in C[0,1]$ and indicate a particular solution to the inhomogeneous equation

$$x(1-x)u''-\lambda u=f. \tag{2.7}$$

Precisely, let

$$\overline{u}_f(x)=-\frac{1}{W(\frac{1}{2})}\int_0^1 f(t)K(x,t)\,dt,$$

where

$$K(x,t):=\begin{cases}\frac{\phi(t)\phi(1-x)}{t(1-t)}, & 0<t<x,\\ \frac{\phi(1-t)\phi(x)}{t(1-t)}, & x<t<1.\end{cases}$$

Then

$$\overline{u}_f(x)=-\frac{1}{W(\frac{1}{2})}\int_x^1\frac{f(t)\phi(1-t)\phi(x)}{t(1-t)}\,dt-\frac{1}{W(\frac{1}{2})}\int_0^x\frac{f(t)\phi(t)\phi(1-x)}{t(1-t)}\,dt.$$

Notice that, for $t\in[0,1]$

$$|\phi(t)|\le\sum_{n=1}^{\infty}|a_n|t,\quad |\phi(1-t)|\le\sum_{n=1}^{\infty}|a_n|(1-t).$$

Hence, for any $x\in(0,1)$,

$$\begin{aligned}|\int_x^1\frac{f(t)\phi(1-t)\phi(x)}{t(1-t)}\,dt| &\le\int_x^1\frac{|f(t)|}{t}\,dt|\phi(x)|\sum_{n=1}^{\infty}|a_n|\\ &\le\|f\|_\infty\int_x^1\frac{dt}{t}|\phi(x)|\sum_{n=1}^{\infty}|a_n|\\ &\le\|f\|_\infty(x|\log x|)(\sum_{n=1}^{\infty}|a_n|)^2.\end{aligned} \tag{2.8}$$

This also shows that

$$\int_x^1\frac{f(t)\phi(1-t)\phi(x)}{t(1-t)}\,dt\to 0$$

as $x \to 0^+$. In a similar way

$$|\int_0^x \frac{f(t)\phi(t)\phi(1-x)}{t(1-t)}\,dt| \le \int_0^x \frac{|f(t)|}{1-t}\,dt|\phi(1-x)| \sum_{n=1}^{\infty}|a_n|$$

$$\le \|f\|_\infty \int_0^x \frac{dt}{1-t}|\phi(1-x)| \sum_{n=1}^{\infty}|a_n|$$

$$\le \|f\|_\infty(1-x)\,|\log(1-x)|(\sum_{n=1}^{\infty}|a_n|)^2, \tag{2.9}$$

implies that $\overline{u}_f$ satisfies (2.7) and

$$\overline{u}_f(0) = 0 = \overline{u}_f(1).$$

Observe that these results can be compared with those by Clément and Timmermans [3], since, according to Feller classification of the boundary points, 0 and 1 are exit points. For Feller classification of the boundary points see e.g. [4, VI Section 4, p.396]. In addition, if we set $M := \sup_{x\in[0,1]} x|\log x|$, as a consequence of (2.8), (2.9), (2.6) and (2.4), we obtain

$$\|\overline{u}_f\|_\infty \le \frac{M}{W(\frac{1}{2})}\|f\|_\infty(\sum_{n=1}^{\infty}|a_n|)^2 \le \frac{M}{|\lambda+2|}\|f\|_\infty\left(\frac{\sum_{n=1}^{\infty}|a_n|}{|\sum_{n=1}^{\infty}a_n|}\right)^2$$

$$\le \frac{MC^2}{|\lambda+2|}\|f\|_\infty \le \frac{C'}{|\lambda|}\|f\|_\infty, \tag{2.10}$$

for $Re\,\lambda > 0, |\lambda|$ large. Therefore, the mapping $f \to \overline{u}_f$ defines a linear bounded operator L_f acting on $C[0,1]$ such that $\|L_f\| \le \frac{C'}{|\lambda|}$ with C' independent of λ. The results above imply that the general solution u to (2.7) is expressed by

$$u(x) = c_1\phi(x) + c_2\phi(1-x) + \overline{u}_f(x).$$

It follows that

$$x(1-x)[c_1\phi''(x) + c_2\phi''(1-x) + \overline{u}''_f(x)] = c_1\lambda\phi(x) + c_2\lambda\phi(1-x) + \lambda\overline{u}_f(x) + f(x).$$

Now, $\overline{u}_f(x)$ tends to 0 as $x \to 0, 1$ and

$$\phi(0) = 0, \quad \phi(1) = \sum_{n=1}^{\infty} a_n.$$

Therefore,

$$\lim_{x\to 0^+} x(1-x)[c_1\phi''(x) + c_2\phi''(1-x) + \overline{u}''_f(x)] = \lambda c_2\phi(1) + f(0),$$

$$\lim_{x\to 1^-} x(1-x)[c_1\phi''(x) + c_2\phi''(1-x) + \overline{u}''_f(x)] = \lambda c_1\phi(1) + f(1).$$

This implies that necessarily

$$c_1 = \frac{-f(1)}{\lambda \sum_{n=1}^{\infty} a_n}, \quad c_2 = \frac{-f(0)}{\lambda \sum_{n=1}^{\infty} a_n}. \tag{2.11}$$

Let $\overline{\overline{u}}_f$ denote the solution of (2.7) corresponding to the coefficients c_1, c_2 as in (2.11). Then, by (2.4) and (2.10) we have

$$\begin{aligned}
\|\overline{\overline{u}}_f\|_\infty &\leq \frac{|f(1)|}{|\lambda| |\sum_{n=1}^{\infty} a_n|}\|\phi\|_\infty + \frac{|f(0)|}{|\lambda| |\sum_{n=1}^{\infty} a_n|}\|\phi(1-\cdot)\|_\infty + \|\overline{u}_f\|_\infty \\
&\leq \frac{|f(1)| \sum_{n=1}^{\infty} |a_n|}{|\lambda| |\sum_{n=1}^{\infty} a_n|} + \frac{|f(0)| \sum_{n=1}^{\infty} |a_n|}{|\lambda| |\sum_{n=1}^{\infty} a_n|} + \frac{C'}{|\lambda|}\|f\|_\infty \\
&\leq \frac{C''}{|\lambda|}\|f\|_\infty,
\end{aligned}$$

where $C'' := \max\{C', 2C\}$.

Hence we have proved the affirmation as follows.

Theorem 2.1. *The operator A with domain*

$$D(A) := \{u \in C[0,1] \cap C^2(0,1) : \lim_{x\to j} Au(x) = 0,\ j = 0,1\}$$

generates a Feller semigroup on $C[0,1]$ *which is analytic of angle* $\frac{\pi}{2}$.

We remark that for this theorem we have provided an alternative proof to a result due to Metafune [12].

3 The semigroup generated by the operator $A_1 u = x(1-x)u'' + (\overline{x} - x)u'$ with Wentzell boundary conditions

Next aim is to show that our method can be useful also for more general degenerate second order differential operators with Wentzell boundary conditions.

Let us fix $\overline{x} \in (0,1)$ and define the operator $A_1 u := x(1-x)u'' + (\overline{x} - x)u'$ with domain

$$D(A_1) := \{u \in C[0,1] \cap C^2(0,1) : \lim_{x\to j} A_1 u(x) = 0,\ j = 0,1\}.$$

Then the following result holds.

Proposition 3.1. *(i) Every solution of the equation* $x(1-x)u + (\overline{x} - x)u' - \lambda u = 0$ *is bounded near* 0 *and* 1. *(ii) The operator* $(A_1, D(A_1))$ *generates a Feller semigroup on* $C[0,1]$ *which is analytic of angle* $\frac{\pi}{2}$.

Proof. For simplicity, let us consider $\overline{x} = \frac{1}{2}$ and evaluate

$$W_1(x) := exp\left(-\int_{\frac{1}{2}}^{x} \frac{\frac{1}{2} - s}{s(1-s)}\, ds\right)$$

$$Q(x) := \frac{1}{x(1-x)W_1(x)} \int_{\frac{1}{2}}^{x} W_1(s)\,ds$$

$$R(x) := W_1(x) \int_{\frac{1}{2}}^{x} \frac{1}{s(1-s)W_1(s)}\,ds.$$

Then $W_1(x) = \frac{1}{2\sqrt{x(1-x)}}$. Direct calculations show also that

$$Q \in L^1(0,\frac{1}{2}), \quad R \in L^1(0,\frac{1}{2})$$

thus 0 is regular. Moreover

$$Q \in L^1(\frac{1}{2},1), \quad R \in L^1(\frac{1}{2},1)$$

hence 1 is regular too. Then, according to the results by Clément and Timmermans [3] (see also [4] Theorem 4.14 p. 396), the assertion (i) follows. The assertion (ii) is a consequence of [4] Theorem 4.18 p. 398 and [4] Theorem 4.21 p.401.

Example. Take $\overline{x} = \frac{1}{2}$. By arguing as in the previous section, we deduce that, if $\phi_1(x) = \sum_{n=0}^{\infty} b_n x^n$ is a solution of

$$x(1-x)u'' + (\frac{1}{2} - x)u' - \lambda u = 0, \tag{3.1}$$

for $Re\,\lambda > 0$, $|\lambda|$ large, then

$$b_{n+1} = \frac{n^2 + \lambda}{(n+1)(n+\frac{1}{2})}\, b_n \quad n \geq 1, \tag{3.2}$$

provided that $b_0 = \frac{1}{2\lambda}$ and $b_1 = 1$.

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