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# FUNCTIONAL IMPLICIT HYPERBOLIC FRACTIONAL ORDER DIFFERENTIAL EQUATIONS WITH DELAY

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**Abstract**

In this paper we investigate the existence and uniqueness of solutions for the initial value problems (IVP for short), for some classes of functional hyperbolic differential equations with finite and infinite delay by using some fixed point theorems.

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**Keywords**: Partial hyperbolic differential equation; fractional order; left-sided mixed Riemann-Liouville integral; mixed regularized derivative; solution; finite delay; infinite delay; fixed point.

## 1 Introduction

The Darboux problem for partial hyperbolic differential equations was studied in the papers of Abbas *et al.* [1, 6], Vityuk [48], Vityuk and Mykhailenko [49, 50] and by other authors. We can find numerous applications in rheology, control, porous media, viscoelasticity, electrochemistry, electromagnetism, etc. [24, 27, 36, 39, 43, 47]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [6], Kilbas *et al.* [37], Miller and Ross [42], Samko *et al.* [45], the papers of Abbas *et al.* [1, 4, 5, 7, 8, 9, 10, 11, 12] and the references therein.

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In the present paper, we investigate the existence and uniqueness of solutions of some classes of fractional order implicit hyperbolic differential equations with delay. First, we consider the following fractional order IVP for the system

$$\overline{D}_{\theta}^{r}u(x,y) = f(x,y,u_{(x,y)},\overline{D}_{\theta}^{r}u(x,y));\ \text{if } (x,y) \in J := [0,a]\times[0,b], \tag{1.1}$$

$$u(x,y) = \phi(x,y);\ \text{if } (x,y) \in \tilde{J} := [-\alpha,a]\times[-\beta,b]\backslash(0,a]\times(0,b], \tag{1.2}$$

$$\begin{cases} u(x,0) = \varphi(x);\ x \in [0,a], \\ u(0,y) = \psi(y);\ y \in [0,b], \end{cases} \tag{1.3}$$

where $a,b,\alpha,\beta > 0$, $\theta = (0,0)$, $\overline{D}_{\theta}^{r}$ is the mixed regularized derivative of order $r = (r_1,r_2) \in (0,1]\times(0,1]$, $f : J\times C\times\mathbb{R}^n \to \mathbb{R}^n$ is a given function, $\phi \in C(\tilde{J})$, $\varphi : [0,a] \to \mathbb{R}^n$, $\psi : [0,b] \to \mathbb{R}^n$ are given absolutely continuous functions with $\varphi(x) = \phi(x,0)$, $\psi(y) = \phi(0,y)$ for each $x \in [0,a]$, $y \in [0,b]$, and $C := C([-\alpha,0]\times[-\beta,0])$ is the space of continuous functions on $[-\alpha,0]\times[-\beta,0]$.

We denote by $u_{(x,y)}$ the element of $C$ defined by

$$u_{(x,y)}(s,t) = u(x+s,y+t);\ (s,t) \in [-\alpha,0]\times[-\beta,0],$$

here $u_{(x,y)}(.,.)$ represents the history of the state from time $(x-\alpha, y-\beta)$ up to the present time $(x,y)$.

Next, we consider the following fractional order IVP for the system

$$\overline{D}_{\theta}^{r}u(x,y) = f(x,y,u_{(x,y)},\overline{D}_{\theta}^{r}u(x,y));\ \text{if } (x,y) \in J := [0,a]\times[0,b], \tag{1.4}$$

$$u(x,y) = \phi(x,y),\ \text{if } (x,y) \in \tilde{J}' := (-\infty,a]\times(-\infty,b]\backslash(0,a]\times(0,b], \tag{1.5}$$

$$\begin{cases} u(x,0) = \varphi(x);\ x \in [0,a], \\ u(0,y) = \psi(y);\ y \in [0,b], \end{cases} \tag{1.6}$$

where $\varphi,\psi$ are as in problem (1.1)-(1.3) and $\phi \in C(\tilde{J}')$, $f : J\times\mathcal{B}\times\mathbb{R}^n \to \mathbb{R}^n$ is a given continuous function, and $\mathcal{B}$ is called a phase space that will be specified in Section 3.

Later we deal with the existence and uniqueness of solutions to fractional order IVP, for the system

$$\overline{D}_{\theta}^{r}u(x,y) = f(x,y,u_{(\rho_1(x,y,u_{(x,y)}),\rho_2(x,y,u_{(x,y)}))},\overline{D}_{\theta}^{r}u(x,y));\ \text{if } (x,y) \in J, \tag{1.7}$$

$$u(x,y) = \phi(x,y);\ \text{if } (x,y) \in \tilde{J}, \tag{1.8}$$

$$u(x,0) = \varphi(x),\ u(0,y) = \psi(y);\ x \in [0,a],\ y \in [0,b], \tag{1.9}$$

where $f : J\times C\times\mathbb{R}^n \to \mathbb{R}^n$, $\rho_1,\rho_2 : J\times C \to \mathbb{R}$ are given functions.

Finally we consider the following initial value problem for partial functional differential equations

$$\overline{D}_{\theta}^{r}u(x,y) = f(x,y,u_{(\rho_1(x,y,u_{(x,y)}),\rho_2(x,y,u_{(x,y)}))},\overline{D}_{\theta}^{r}u(x,y));\ \text{if } (x,y) \in J, \tag{1.10}$$

$$u(x,y) = \phi(x,y); \text{ if } (x,y) \in \tilde{J}', \tag{1.11}$$

$$u(x,0) = \varphi(x), \ u(0,y) = \psi(y); \ \ x \in [0,a], \ y \in [0,b], \tag{1.12}$$

where $f : J\times\mathcal{B}\times\mathbb{R}^n \to \mathbb{R}^n$, $\rho_1,\rho_2 : J\times\mathcal{B} \to \mathbb{R}$ are given functions.

Differential delay equations, or functional differential equations, have been used in modeling scientific phenomena for many years. Often, it has been assumed that the delay is either a fixed constant or is given as an integral in which case it is called a distributed delay, see for instance the books [30, 38]. Over the past several years it has become apparent that equations with state-dependent delay arise also in several areas such as in classical electrodynamics [26], in population models [13, 15, 18, 19], in models of commodity price fluctuations [16, 40], and in models of blood cell productions [17, 20, 21, 41].

There exists an extensive literature for integer order differential equations with state-dependent delay equations, see among another works, Aiello *et al.* [13], Arino *et al.* [14], Domoshnitsky *et al.* [25], Hartung *et al.* [31, 32, 33, 34], Hernandez *et al.* [35], Schumakher [46], Qesmi and Walther [44], and Walther [51].

We present two results for each of our problems, the first one is based on Banach's contraction principle and the second one on the nonlinear alternative of Leray-Schauder type. The main results of the present paper extend those considered without delay [10, 50], and those when the nonlinearity does not depend on the fractional derivative [2, 3, 10, 12].

## 2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C(J)$ we denote the Banach space of all continuous functions from $J$ into $\mathbb{R}^n$ with the norm

$$\|w\|_\infty = \sup_{(x,y)\in J} \|w(x,y)\|,$$

where $\|.\|$ denotes a suitable complete norm on $\mathbb{R}^n$.
As usual, by $AC(J)$ we denote the space of absolutely continuous functions from $J$ into $\mathbb{R}^n$ and $L^1(J)$ is the space of Lebesgue-integrable functions $w : J \to \mathbb{R}^n$ with the norm

$$\|w\|_1 = \int_0^a \int_0^b \|w(x,y)\| dy dx.$$

**Definition 2.1.** [37, 45] Let $\alpha \in (0,\infty)$ and $u \in L^1(J)$. The partial Riemann-Liouville integral of order $\alpha$ of $u(x,y)$ with respect to $x$ is defined by the expression

$$I^\alpha_{0,x} u(x,y) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u(s,y) ds, \text{ for almost all } x \in [0,a] \text{ and all } y \in [0,b],$$

where $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\varsigma) = \int_0^\infty t^{\varsigma-1} e^{-t} dt;\ \varsigma > 0$.

Analogously, we define the integral

$$I_{0,y}^{\alpha}u(x,y)=\frac{1}{\Gamma(\alpha)}\int_0^y (y-s)^{\alpha-1}u(x,s)ds, \text{ for all } x\in[0,a] \text{ and almost all } y\in[0,b].$$

**Definition 2.2.** [37, 45] Let $\alpha\in(0,1]$ and $u\in L^1(J)$. The Riemann-Liouville fractional derivative of order $\alpha$ of $u(x,y)$ with respect to $x$ is defined by

$$(D_{0,x}^{\alpha}u)(x,y)=\frac{\partial}{\partial x}I_{0,x}^{1-\alpha}u(x,y), \text{ for almost all } x\in[0,a] \text{ and all } y\in[0,b].$$

Analogously, we define the derivative

$$(D_{0,y}^{\alpha}u)(x,y)=\frac{\partial}{\partial y}I_{0,y}^{1-\alpha}u(x,y), \text{ for all } x\in[0,a] \text{ and almost all } y\in[0,b].$$

**Definition 2.3.** [37, 45] Let $\alpha\in(0,1]$ and $u\in L^1(J)$. The Caputo fractional derivative of order $\alpha$ of $u(x,y)$ with respect to $x$ is defined by the expression

$${}^cD_{0,x}^{\alpha}u(x,y)=I_{0,x}^{1-\alpha}\frac{\partial}{\partial x}u(x,y), \text{ for almost all } x\in[0,a] \text{ and all } y\in[0,b].$$

Analogously, we define the derivative

$${}^cD_{0,y}^{\alpha}u(x,y)=I_{0,y}^{1-\alpha}\frac{\partial}{\partial y}u(x,y), \text{ for all } x\in[0,a] \text{ and almost all } y\in[0,b].$$

**Definition 2.4.** [48] Let $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$, $\theta=(0,0)$ and $u\in L^1(J)$. The left-sided mixed Riemann-Liouville integral of order $r$ of $u$ is defined by

$$(I_\theta^r u)(x,y)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}u(s,t)dtds.$$

In particular,

$$(I_\theta^\theta u)(x,y)=u(x,y),\ (I_\theta^\sigma u)(x,y)=\int_0^x\int_0^y u(s,t)dtds; \text{ for almost all } (x,y)\in J,$$

where $\sigma=(1,1)$.
For instance, $I_\theta^r u$ exists for all $r_1,r_2\in(0,\infty)$, when $u\in L^1(J)$. Note also that when $u\in C(J)$, then $(I_\theta^r u)\in C(J)$, moreover

$$(I_\theta^r u)(x,0)=(I_\theta^r u)(0,y)=0;\ x\in[0,a],\ y\in[0,b].$$

**Example 2.5.** Let $\lambda,\omega\in(-1,0)\cup(0,\infty)$ and $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$, then

$$I_\theta^r x^\lambda y^\omega=\frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)}x^{\lambda+r_1}y^{\omega+r_2}, \text{ for almost all } (x,y)\in J.$$

By $1-r$ we mean $(1-r_1,1-r_2)\in(0,1]\times(0,1]$. Denote by $D_{xy}^2:=\frac{\partial^2}{\partial x\partial y}$, the mixed second order partial derivative.

**Definition 2.6.** [50] Let $r \in (0,1] \times (0,1]$ and $u \in L^1(J)$. The mixed fractional Riemann-Liouville derivative of order $r$ of $u$ is defined by the expression $D^r_\theta u(x,y) = (D^2_{xy} I^{1-r}_\theta u)(x,y)$ and the Caputo fractional-order derivative of order $r$ of $u$ is defined by the expression

$$^cD^r_\theta u(x,y) = (I^{1-r}_\theta D^2_{xy} u)(x,y) = \frac{1}{\Gamma(1-r_1)\Gamma(1-r_2)} \int_0^x \int_0^y \frac{D^2_{st} u(s,t)}{(x-s)^{r_1}(y-t)^{r_2}} dtds.$$

The case $\sigma = (1,1)$ is included and we have

$$(D^\sigma_\theta u)(x,y) = (^cD^\sigma_\theta u)(x,y) = (D^2_{xy} u)(x,y), \text{ for almost all } (x,y) \in J.$$

**Example 2.7.** Let $\lambda, \omega \in (-1,0) \cup (0,\infty)$ and $r = (r_1, r_2) \in (0,1] \times (0,1]$, then

$$D^r_\theta x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} x^{\lambda-r_1} y^{\omega-r_2}, \text{ for almost all } (x,y) \in J.$$

**Definition 2.8.** [50] For a function $u : J \to \mathbb{R}^n$, we set

$$q(x,y) = u(x,y) - u(x,0) - u(0,y) + u(0,0).$$

By the mixed regularized derivative of order $r = (r_1, r_2) \in (0,1] \times (0,1]$ of a function $u(x,y)$, we name the function

$$\overline{D}^r_\theta u(x,y) = D^r_\theta q(x,y).$$

The function

$$\overline{D}^{r_1}_{0,x} u(x,y) = D^{r_1}_{0,x}[u(x,y) - u(0,y)],$$

is called the partial $r_1$−order regularized derivative of the function $u(x,y) : J \to \mathbb{R}^n$ with respect to the variable $x$. Analogously, we define the derivative

$$\overline{D}^{r_2}_{0,y} u(x,y) = D^{r_2}_{0,y}[u(x,y) - u(x,0)].$$

## 3 The phase space $\mathcal{B}$

The notation of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato (see [29]). For any $(x,y) \in J$ denote $E_{(x,y)} := [0,x] \times \{0\} \cup \{0\} \times [0,y]$, furthermore in case $x = a,\ y = b$ we write simply $E$. Consider the space $(\mathcal{B}, \|(.,.)\|_\mathcal{B})$ is a seminormed linear space of functions mapping $(-\infty,0] \times (-\infty,0]$ into $\mathbb{R}^n$, and satisfying the following fundamental axioms which were adapted from those introduced by Hale and Kato for ordinary differential functional equations:

$(A_1)$ If $z : (-\infty,a] \times (-\infty,b] \to \mathbb{R}^n$ continuous on $J$ and $z_{(x,y)} \in \mathcal{B}$, for all $(x,y) \in E$, then there are constants $H, K, M > 0$ such that for any $(x,y) \in J$ the following conditions hold:

$(i)$ $z_{(x,y)}$ is in $\mathcal{B}$;

$(ii)$ $\|z(x,y)\| \leq H\|z_{(x,y)}\|_{\mathcal{B}}$,

$(iii)$ $\|z_{(x,y)}\|_{\mathcal{B}} \leq K \sup_{(s,t)\in[0,x]\times[0,y]} \|z(s,t)\| + M \sup_{(s,t)\in E_{(x,y)}} \|z_{(s,t)}\|_{\mathcal{B}}$,

$(A_2)$ For the function $z(.,.)$ in $(A_1)$, $z_{(x,y)}$ is a $\mathcal{B}$-valued continuous function on $J$.

$(A_3)$ The space $\mathcal{B}$ is complete.

Now, we present some examples of phase spaces [22, 23].

**Example 3.1.** Let $\mathcal{B}$ be the set of all functions $\phi : (-\infty,0]\times(-\infty,0] \to \mathbb{R}^n$ which are continuous on $[-\alpha,0]\times[-\beta,0]$, $\alpha,\beta \geq 0$, with the seminorm

$$\|\phi\|_{\mathcal{B}} = \sup_{(s,t)\in[-\alpha,0]\times[-\beta,0]} \|\phi(s,t)\|.$$

Then we have $H = K = M = 1$. The quotient space $\widehat{\mathcal{B}} = \mathcal{B}/\|.\|_{\mathcal{B}}$ is isometric to the space $C([-\alpha,0]\times[-\beta,0],\mathbb{R}^n)$ of all continuous functions from $[-\alpha,0]\times[-\beta,0]$ into $\mathbb{R}^n$ with the supremum norm, this means that partial differential functional equations with finite delay are included in our axiomatic model.

**Example 3.2.** Let $\gamma \in \mathbb{R}$, $C_\gamma$ be the set of all continuous functions $\phi : (-\infty,0]\times(-\infty,0] \to \mathbb{R}^n$ for which a limit $\lim_{\|(s,t)\|\to\infty} e^{\gamma(s+t)}\phi(s,t)$ exists, with the norm

$$\|\phi\|_{C_\gamma} = \sup_{(s,t)\in(-\infty,0]\times(-\infty,0]} e^{\gamma(s+t)}\|\phi(s,t)\|.$$

Then we have $H = 1$ and $K = M = \max\{e^{-\gamma(a+b)}, 1\}$.

**Example 3.3.** Let $\alpha,\beta,\gamma \geq 0$ and let

$$\|\phi\|_{CL_\gamma} = \sup_{(s,t)\in[-\alpha,0]\times[-\beta,0]} \|\phi(s,t)\| + \int_{-\infty}^{0}\int_{-\infty}^{0} e^{\gamma(s+t)}\|\phi(s,t)\|dtds.$$

be the seminorm for the space $CL_\gamma$ of all functions $\phi : (-\infty,0]\times(-\infty,0] \to \mathbb{R}^n$ which are continuous on $[-\alpha,0]\times[-\beta,0]$ measurable on $(-\infty,-\alpha]\times(-\infty,0]\cup(-\infty,0]\times(-\infty,-\beta]$, and such that $\|\phi\|_{CL_\gamma} < \infty$. Then

$$H = 1,\ K = \int_{-\alpha}^{0}\int_{-\beta}^{0} e^{\gamma(s+t)}dtds,\ M = 2.$$

# 4 Existence Results with Finite Delay

Let us start by defining what we mean by a solution of the problem (1.1)-(1.3).

**Definition 4.1.** A function $u \in C_{(\alpha,\beta)} := C([-\alpha,a]\times[-\beta,b])$ such that $u(x,y)$, $\overline{D}_{0,x}^{r_1}u(x,y)$, $\overline{D}_{0,y}^{r_2}u(x,y), \overline{D}_{\theta}^{r}u(x,y)$ are continuous for $(x,y) \in J$ and $I_\theta^{1-r}u(x,y) \in AC(J)$ is said to be a solution of (1.1)-(1.3) if $u$ satisfies equations (1.1),(1.3) on $J$ and the condition (1.2) on $\tilde{J}$.

In the sequel, we need the following lemma

**Lemma 4.2.** *[50] Let a function $f(x,y,u,z): J\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ be continuous. Then problem (1.1)-(1.2) is equivalent to the problem of the solution of the equation*

$$g(x,y) = f(x,y,\mu(x,y)+I_\theta^r g(x,y), g(x,y)),$$

*and if $g(x,y)\in C(J)$ is the solution of this equation, then $u(x,y)=\mu(x,y)+I_\theta^r g(x,y)$, where*

$$\mu(x,y) = \varphi(x)+\psi(y)-\varphi(0).$$

Further, we present conditions for the existence and uniqueness of a solution of problem (1.1)-(1.3).

**Theorem 4.3.** *Assume*

$(H_1)$ *The function $f: J\times C\times\mathbb{R}^n \to \mathbb{R}^n$ is continuous,*

$(H_2)$ *For any $u,v\in C,\ w,z\in\mathbb{R}^n$ and $(x,y)\in J$, there exist constants $\ell>0$ and $0<l<1$ such that*

$$\|f(x,y,u,z)-f(x,y,v,w)\| \le \ell\|u-v\|_C + l\|z-w\|.$$

*If*

$$\frac{\ell a^{r_1} b^{r_2}}{(1-l)\Gamma(1+r_1)\Gamma(1+r_2)} < 1, \tag{4.1}$$

*then there exists a unique solution for IVP (1.1)-(1.3) on $[-\alpha,a]\times[-\beta,b]$.*

*Proof.* Transform the problem (1.1)-(1.2) into a fixed point problem. Consider the operator $N: C_{(\alpha,\beta)} \to C_{(\alpha,\beta)}$ defined by,

$$(Nu)(x,y) = \begin{cases}\Phi(x,y);\ (x,y)\in\tilde{J},\\ \mu(x,y)+I_\theta^r g(x,y);\ (x,y)\in J,\end{cases} \tag{4.2}$$

where $g\in C(J)$ such that

$$g(x,y) = f(x,y,u_{(x,y)}, g(x,y)).$$

From Lemma 4.2, the problem of finding the solutions of the $IVP$ (1.1)-(1.3) is reduced to finding the solutions of the operator equation $N(u)=u$.

Let $v,w\in C_{(\alpha,\beta)}$. Then, for $(x,y)\in J$, we have

$$\|N(v)(x,y)-N(w)(x,y)\| \le \int_0^x\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\|g(s,t)-h(s,t)\|dtds, \tag{4.3}$$

where $g,h\in C(J)$ such that

$$g(x,y) = f(x,y,v_{(x,y)}, g(x,y))$$

and

$$h(x,y) = f(x,y,w_{(x,y)}, h(x,y)).$$

By $(H_2)$, we get

$$\|g(x,y)-h(x,y)\| \leq \ell\|v_{(x,y)}-w_{(x,y)}\|_C + l\|g(x,y)-h(x,y)\|.$$

Then

$$\begin{aligned}\|g(x,y)-h(x,y)\| &\leq \frac{\ell}{1-l}\|v_{(x,y)}-w_{(x,y)}\|_C \\ &\leq \frac{\ell}{1-l}\|v-w\|_\infty.\end{aligned}$$

Thus, (6.1) implies that

$$\begin{aligned}\|N(v)-N(w)\|_\infty &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\frac{\ell}{1-l}\|v-w\|_\infty dtds \\ &\leq \frac{\ell a^{r_1}b^{r_2}}{(1-l)\Gamma(1+r_1)\Gamma(1+r_2)}\|v-w\|_\infty.\end{aligned}$$

Hence

$$\|N(v)-N(w)\|_\infty \leq \frac{\ell a^{r_1}b^{r_2}}{(1-l)\Gamma(1+r_1)\Gamma(1+r_2)}\|v-w\|_\infty.$$

By (4.1), $N$ is a contraction, and hence $N$ has a unique fixed point by Banach's contraction principle.

**Theorem 4.4.** *[28] (*Nonlinear alternative of Leray-Schauder type*) Let $X$ be a Banach space and $C$ a nonempty convex subset of $X$. Let $U$ a nonempty open subset of $C$ with $0 \in U$ and $T : \overline{U} \to C$ continuous and compact operator.*
*Then either*

- *(a) $T$ has fixed points. Or*
- *(b) There exist $u \in \partial U$ and $\lambda \in [0,1]$ with $u = \lambda T(u)$.*

**Theorem 4.5.** *Assume $(H_1)$ and the following hypothesis hold*

$(H_3)$ *There exist $p,q,d \in C(J,\mathbb{R}_+)$ such that*

$$\|f(x,y,u,z)\| \leq p(x,y)+q(x,y)\|u\|_C + d(x,y)\|z\|$$

*for $(x,y) \in J$ and each $u \in C,\ z \in \mathbb{R}^n$.*

*If*

$$d^* + \frac{q^* a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1, \tag{4.4}$$

*where $d^* = \sup_{(x,y)\in J} d(x,y)$ and $q^* = \sup_{(x,y)\in J} q(x,y)$,*
*then the IVP (1.1)-(1.3) has at least one solution on $[-\alpha,a]\times[-\beta,b]$.*

*Proof.* Transform the problem (1.1)-(1.3) into a fixed point problem. Consider the operator $N$ defined in (4.2). We shall show that the operator $N$ is continuous and compact.

**Step 1:** *N is continuous.*
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in $C_{(\alpha,\beta)}$. Let $\eta > 0$ be such that $\|u_n\| \le \eta$. Then

$$\|(Nu_n)(x,y)-(Nu)(x,y)\| \le \int_0^x\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\|g_n(s,t)-g(s,t)\|dtds, \tag{4.5}$$

where $g_n, g \in C(J)$ such that

$$g_n(x,y) = f(x,y,u_{n(x,y)},g_n(x,y))$$

and

$$g(x,y) = f(x,y,u_{(x,y)},g(x,y)).$$

Since $u_n \to u$ as $n \to \infty$ and $f$ is a continuous function, we get

$$g_n(x,y) \to g(x,y) \text{ as } n\to\infty, \text{ for each } (x,y)\in J.$$

Hence, (4.5) gives

$$\|N(u_n)-N(u)\|_\infty \le \frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\|g_n-g\|_\infty \to 0 \text{ as } n\to\infty.$$

**Step 2:** *N maps bounded sets into bounded sets in $C_{(\alpha,\beta)}$.*
Indeed, it is enough show that for any $\eta^* > 0$, there exists a positive constant $M^*$ such that, for each $u \in B_{\eta^*} = \{u \in C_{(\alpha,\beta)} : \|u\|_\infty \le \eta^*\}$, we have $\|N(u)\|_\infty \le M^*$. For $(x,y)\in J$, we have

$$\|(Nu)(x,y)\| \le \|\mu(x,y)\| + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\|g(s,t)\|dtds, \tag{4.6}$$

where $g\in C(J)$ such that

$$g(x,y) = f(x,y,u_{(x,y)},g(x,y)).$$

By $(H_3)$ we have for each $(x,y)\in J$,

$$\begin{aligned}\|g(x,y)\| &\le p(x,y)+q(x,y)\|\mu(x,y)+I_\theta^r g(x,y)\|+d(x,y)\|g(x,y)\|\\ &\le p^*+q^*\Big(\|\mu\|_\infty+\frac{a^{r_1}b^{r_2}\|g(x,y)\|}{\Gamma(1+r_1)\Gamma(1+r_2)}\Big)+d^*\|g(x,y)\|,\end{aligned}$$

where $p^* = \sup\limits_{(x,y)\in J} p(x,y)$.
Then, by (4.4) we have

$$\|g(x,y)\| \le \frac{p^*+q^*\|\mu\|_\infty}{1-d^*-\frac{q^*a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}} := M.$$

Thus, (4.6) implies that

$$\|N(u)\|_\infty \le \|\mu\|_\infty + \frac{Ma^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} := M^*.$$

**Step 3:** *N maps bounded sets into equicontinuous sets in $C_{(\alpha,\beta)}$.*
Let $(x_1,y_1),(x_2,y_2)\in J$, $x_1<x_2$, $y_1<y_2$, $B_{\eta^*}$ be a bounded set of $C_{(\alpha,\beta)}$ as in Step 2, and let $u\in B_{\eta^*}$. Then

$$\begin{aligned}
&\|(Nu)(x_2,y_2)-(Nu)(x_1,y_1)\|\\
\le\ &\|\mu(x_2,y_2)-\mu(x_1,y_1)\|\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{x_1}\int_0^{y_1}[(x_2-s)^{r_1-1}(y_2-t)^{r_2-1}-(x_1-s)^{r_1-1}(y_1-t)^{r_2-1}]\|g(s,t)\|dtds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}(x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|g(s,t)\|dtds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{x_1}\int_{y_1}^{y_2}(x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|g(s,t)\|dtds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_0^{y_1}(x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|g(s,t)\|dtds,
\end{aligned}$$

where $g\in C(J)$ such that

$$g(x,y)=f(x,y,u(x,y),g(x,y)).$$

But $\|g(x,y)\|\le M$. Thus

$$\begin{aligned}
\|(Nu)(x_2,y_2)-(Nu)(x_1,y_1)\| \le\ &\|\mu(x_2,y_2)-\mu(x_1,y_1)\|\\
+\ &\frac{M}{\Gamma(1+r_1)\Gamma(1+r_2)}[2y_2^{r_2}(x_2-x_1)^{r_1}+2x_2^{r_1}(y_2-y_1)^{r_2}\\
+\ &x_1^{r_1}y_1^{r_2}-x_2^{r_1}y_2^{r_2}-2(x_2-x_1)^{r_1}(y_2-y_1)^{r_2}].
\end{aligned}$$

As $x_1\to x_2$, $y_1\to y_2$ the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem, we can conclude that $N$ is continuous and completely continuous.

**Step 4**: *A priori bounds.*
We now show there exists an open set $U\subseteq C_{(\alpha,\beta)}$ with $u\neq\lambda N(u)$, for $\lambda\in(0,1)$ and $u\in\partial U$. Let $u\in C_{(\alpha,\beta)}$ and $u=\lambda N(u)$ for some $0<\lambda<1$. Thus, for each $(x,y)\in J$, we have

$$u(x,y)=\lambda\mu(x,y)+\frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)dtds.$$

This implies by $(H_3)$ and as in step 2 that, $\|u\|\le M^*$. Hence, for each $(x,y)\in[-\alpha,a]\times[-\beta,b]$, we have

$$\|u\|_\infty\le\max(\|\phi\|_C,M^*):=R.$$

Set

$$U=\{u\in C_{(\alpha,\beta)}:\|u\|_\infty<R+1\}.$$

By our choice of $U$, there is no $u\in\partial U$ such that $u=\lambda N(u)$, for $\lambda\in(0,1)$. As a consequence of Theorem 4.4, we deduce that $N$ has a fixed point $u$ in $\overline{U}$ which is a solution to problem (1.1)-(1.2).

# 5 Existence Results for Infinite Delay

Let us start by defining what we mean by a solution of the problem (1.4)-(1.6). Let the space

$$\Omega := \{u : (-\infty,a]\times(-\infty,b] \to \mathbb{R}^n : \ u_{(x,y)} \in \mathcal{B} \text{ for } (x,y)\in E \ \text{ and } u|_J \in C(J,\mathbb{R}^n)\}.$$

**Definition 5.1.** A function $u\in\Omega$ is said to be a solution of (1.4)-(1.6) if $u$ satisfies equations (1.4) and (1.6) on $J$ and the condition (1.5) on $\tilde{J}'$.

Our first existence result for the IVP (1.4)-(1.6) is based on the Banach contraction principle.

**Theorem 5.2.** *Assume that the following hypotheses hold:*

$(H_1')$ *There exist constants $\ell' > 0$ and $0 < l' < 1$such that*

$$\|f(x,y,u,z)-f(x,y,v,w)\| \le \ell'\|u-v\|_{\mathcal{B}} + l'\|z-w\|,$$

*for any $u,\ v\in\mathcal{B},\ z,w\in\mathbb{R}^n,$ and $(x,y)\in J$.*

*If*

$$\frac{K\ell' a^{r_1} b^{r_2}}{(1-l')\Gamma(1+r_1)\Gamma(1+r_2)} < 1, \tag{5.1}$$

*then there exists a unique solution for IVP (1.4)-(1.6) on $(-\infty,a]\times(-\infty,b]$.*

*Proof.* Consider the operator $N : \Omega\to\Omega$ defined by,

$$(Nu)(x,y) = \begin{cases} \phi(x,y); & (x,y)\in\tilde{J}', \\ \mu(x,y) + I_\theta^r g(x,y); & (x,y)\in J, \end{cases} \tag{5.2}$$

where

$$g(x,y) = f(x,y,u_{(x,y)},g(x,y))); \ (x,y)\in J.$$

Let $v(.,.) : (-\infty,a]\times(-\infty,b]\to\mathbb{R}^n$ be a function defined by,

$$v(x,y) = \begin{cases} \phi(x,y), & (x,y)\in\tilde{J}', \\ \mu(x,y), & (x,y)\in J. \end{cases}$$

Then $v_{(x,y)} = \phi$ for all $(x,y)\in E$. For each $w\in C(J)$ with $w(x,y) = 0$ for each $(x,y)\in E$ we denote by $\overline{w}$ the function defined by

$$\overline{w}(x,y) = \begin{cases} 0, & (x,y)\in\tilde{J}', \\ w(x,y) & (x,y)\in J. \end{cases}$$

If $u(.,.)$ satisfies the integral equation,

$$u(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)dtds,$$

we can decompose $u(.,.)$ as $u(x,y)=\overline{w}(x,y)+v(x,y);\ (x,y)\in J$, which implies $u_{(x,y)}=\overline{w}_{(x,y)}+v_{(x,y)}$, for every $(x,y)\in J$, and the function $w(.,.)$ satisfies

$$w(x,y)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)dtds,$$

where

$$g(x,y)=f(x,y,\overline{w}_{(x,y)}+v_{(x,y)},g(x,y));\ (x,y)\in J.$$

Set

$$C_0=\{w\in C(J):\ w(x,y)=0\ \text{ for } (x,y)\in E\},$$

and let $\|.\|_{(a,b)}$ be the seminorm in $C_0$ defined by

$$\|w\|_{(a,b)}=\sup_{(x,y)\in E}\|w_{(x,y)}\|_{\mathcal{B}}+\sup_{(x,y)\in J}\|w(x,y)\|=\sup_{(x,y)\in J}\|w(x,y)\|,\ w\in C_0.$$

$C_0$ is a Banach space with norm $\|.\|_{(a,b)}$. Let the operator $P:\ C_0\to C_0$ be defined by

$$(Pw)(x,y)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,\overline{w}_{(s,t)}+v_{(s,t)},g(s,t))dtds,\tag{5.3}$$

where

$$g(x,y)=f(x,y,\overline{w}_{(x,y)}+v_{(x,y)},g(x,y));\ (x,y)\in J.$$

Then the operator $N$ has a fixed point is equivalent to $P$ has a fixed point, and so we turn to proving that $P$ has a fixed point. We shall show that $P:\ C_0\to C_0$ is a contraction map. Indeed, consider $w,w^*\in C_0$. Then we have for each $(x,y)\in J$

$$\|(Pw)(x,y)-(Pw^*)(x,y)\|\le\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}\|g(s,t)-g^*(s,t)\|dtds,$$

where

$$g^*(x,y)=f(x,y,\overline{w^*}_{(x,y)}+v_{(x,y)}).$$

But, for each $(x,y)\in J$, we have

$$\|g(x,y)-g^*(x,y)\|\le\frac{\ell'}{1-l'}\|\overline{w}_{(x,y)}-\overline{w^*}_{(x,y)}\|_{\mathcal{B}}.$$

Thus, we obtain that, for each $(x,y)\in J$

$$\begin{aligned}\|(Pw)(x,y)&-(Pw^*)(x,y)\|\le\frac{\ell'}{(1-l')\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}\\&\times\ \|\overline{w}_{(s,t)}-\overline{w^*}_{(s,t)}\|_{\mathcal{B}}dtds\\&\le\ \frac{\ell'}{(1-l')\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}\\&\times\ k\sup_{(s,t)\in[0,x]\times[0,y]}\|\overline{w}(s,t)-\overline{w^*}(s,t)\|_{\mathcal{B}}dtds\\&\le\ \frac{K\ell'}{(1-l')\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}dtds\|\overline{w}-\overline{w^*}\|_{(a,b)}.\end{aligned}$$

Therefore

$$\|P(w) - P(w^*)\|_{(a,b)} \leq \frac{k\ell' a^{r_1} b^{r_2}}{\Gamma(1+r_1+)\Gamma(1+r_2)} \|\overline{w} - \overline{w^*}\|_{(a,b)}.$$

and hence, by (5.1) $P$ is a contraction. Therefore, $P$ has a unique fixed point by Banach's contraction principle.

Now we give an existence result based on the nonlinear alternative of Leray-Schauder type [28].

**Theorem 5.3.** *Assume that the following hypotheses hold:*

$(H_2')$ *There exist* $p', q', d' \in C(J, \mathbb{R}_+)$ *such that*

$$\|f(x,y,u,v)\| \leq p'(x,y) + q'(x,y)\|u\|_{\mathcal{B}} + d'(x,y)\|v\|,$$

*for each* $(x,y) \in J$, $u \in \mathcal{B}$, *and* $v \in \mathbb{R}^n$.

*If*

$$d^{**} + \frac{q^{**} a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1, \tag{5.4}$$

*where* $d^{**} = \sup_{(x,y)\in J} d'(x,y)$ *and* $q^{**} = \sup_{(x,y)\in J} q'(x,y)$, *then the IVP (1.4)-(1.6) has at least one solution on* $(-\infty, a] \times (-\infty, b]$.

*Proof.* Let $P : C_0 \to C_0$ defined as in (5.3). As in Theorem 4.5, we can prove that $P$ is continuous and completely continuous. We now show there exists an open set $U' \subseteq C_0$ with $w \neq \lambda P(w)$, for $\lambda \in (0,1)$ and $w \in \partial U'$. Let $w \in C_0$ and $w = \lambda P(w)$ for some $0 < \lambda < 1$. Thus for each $(x,y) \in J$,

$$w(x,y) = \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} g(s,t)\,dt\,ds.$$

where

$$g(x,y) = f(x,y,u_{(x,y)}, g(x,y));\ (x,y) \in J.$$

By $(H_2')$ and (5.4) we get for each $(x,y) \in J$,

$$\|g(x,y)\| \quad \leq \quad \frac{p^{**} + q^{**}\|\mu\|_\infty}{1 - d^{**} - \frac{q^{**} a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}} := M',$$

where $p^{**} = \sup_{(x,y)\in J} p'(x,y)$. This implies that, for each $(x,y) \in J$, we have

$$\|w(x,y)\| \quad \leq \quad \frac{M' a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} := \widetilde{M}.$$

Hence

$$\|w\|_{(a,b)} \leq \widetilde{M}.$$

Set

$$U' = \{w \in C_0 : \|w\|_{(a,b)} < \widetilde{M} + 1\}.$$

$P : \overline{U'} \to C_0$ is continuous and completely continuous. By our choice of $U'$, there is no $w \in \partial U'$ such that $w = \lambda P(w)$, for $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type (Theorem 4.4), we deduce that $N$ has a fixed point which is a solution to problem (1.4)-(1.6).

# 6 Existence Results with State Dependent Delay

## 6.1 Finite delay case

**Definition 6.1.** A function $u \in C_{(\alpha,\beta)}$ such that $u(x,y)$, $\overline{D}_{0,x}^{r_1}u(x,y)$, $\overline{D}_{0,y}^{r_2}u(x,y), \overline{D}_{\theta}^{r}u(x,y)$ are continuous for $(x,y) \in J$ and $I_{\theta}^{1-r}u(x,y) \in AC(J)$ is said to be a solution of (1.1)-(1.3) if $u$ satisfies equations (1.1),(1.3) on $J$ and the condition (1.2) on $\tilde{J}$.

$$\begin{aligned}\text{Set } \mathcal{R} :=& \mathcal{R}_{(\rho_1^-,\rho_2^-)}\\ =& \{(\rho_1(s,t,u),\rho_2(s,t,u)) : (s,t,u) \in J\times C,\ \rho_i(s,t,u) \le 0;\ i = 1,2\}.\end{aligned}$$

We always assume that $\rho_i : J\times C \to \mathbb{R};\ i = 1,2$ are continuous and the function $(s,t) \longmapsto u_{(s,t)}$ is continuous from $\mathcal{R}$ into $C$.

Further, we present conditions for the existence and uniqueness of a solution of problem (1.7)-(1.9).

**Theorem 6.2.** *Assume that the following hypotheses hold:*

$(H_{01})$ *The $f : J\times C\times\mathbb{R}^n \to \mathbb{R}^n$ is continuous,*

$(H_{02})$ *There exist $\ell_* > 0,\ 0 < l_* < 1$ such that*

$$\|f(x,y,u,v) - f(x,y,\overline{u},\overline{v})\| \le \ell_*\|u-\overline{u}\|_C + l_*\|v-\overline{v}\|;$$

*for any $u,\ \overline{u} \in C,\ v,\ \overline{v} \in \mathbb{R}^n$ and $(x,y) \in J$.*

*If*

$$\frac{\ell_*\, a^{r_1} b^{r_2}}{(1-l_*)\Gamma(1+r_1)\Gamma(1+r_2)} < 1, \tag{6.1}$$

*then there exists a unique solution for IVP (1.7)-(1.9) on $[-\alpha,a]\times[-\beta,b]$.*

*Proof.* Consider the operator $N : C_{(\alpha,\beta)} \to C_{(\alpha,\beta)}$ defined by,

$$(Nu)(x,y) = \begin{cases}\phi(x,y); & (x,y) \in \tilde{J},\\ \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} & \\ \times g(s,t)dtds; & (x,y) \in J,\end{cases} \tag{6.2}$$

where, $g \in C(J)$ such that

$$g(x,y) = f(x,y,u_{(\rho_1(x,y,u_{(x,y)}),\rho_2(x,y,u_{(x,y)}))}, g(x,y)).$$

Let $u,v \in C_{(\alpha,\beta)}$. Then, for $(x,y) \in [-\alpha,a]\times[-\beta,b]$,

$$\begin{aligned}\|(Nu)(x,y)-(Nv)(x,y)\| \le& \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\\ &\times \|g(s,t)-h(s,t)\|dtds,\end{aligned}$$

where, $g,h \in C(J)$ such that

$$g(x,y) = f(x,y,u_{(\rho_1(x,y,u_{(x,y)}),\rho_2(x,y,u_{(x,y)})}, g(x,y))$$

and

$$h(x,y) = f(x,y,v_{(\rho_1(x,y,v_{(x,y)}),\rho_2(x,y,v_{(x,y)})}, h(x,y)).$$

Since

$$\|g-h\|_\infty \le \frac{\ell_*}{1-l_*}\|v-w\|_{C_{(a,b)}},$$

we obtain that

$$\|(Nu)-(Nv)\|_{C_{(a,b)}} \le \frac{\ell_* a^{r_1} b^{r_2}}{(1-l_*)\Gamma(1+r_1)\Gamma(1+r_2)}\|v-w\|_{C_{(a,b)}}.$$

Consequently, by (6.1), $N$ is a contraction, and hence $N$ has a unique fixed point by Banach's contraction principle.

**Theorem 6.3.** *Assume $(H_{01})$ and the following hypothesis holds:*

$(H_{03})$ *There exist $p,q,d \in C(J,\mathbb{R}_+)$ such that*

$$\|f(x,y,u,v)\| \le p(x,y) + q(x,y)\|u\|_C + d(x,y)\|v\|;$$

*for each $u \in C$, $v \in \mathbb{R}^n$ and $(x,y) \in J$.*

*If*

$$d^* + \frac{q^* a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1, \tag{6.3}$$

*where $d^* = \sup_{(x,y)\in J} d(x,y)$ and $q^* = \sup_{(x,y)\in J} q(x,y)$, then the IVP (1.7)-(1.9) has at least one solution on $[-\alpha,a]\times[-\beta,b]$.*

*Proof.* Consider the operator $N$ defined in (6.2). We can easily show that the operator $N$ is continuous and completely continuous.

**A priori bounds.** We now show there exists an open set $U \subseteq C_{(\alpha,\beta)}$ with $u \neq \lambda N(u)$, for $\lambda \in (0,1)$ and $u \in \partial U$. Let $u \in C_{(\alpha,\beta)}$ and $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus for each $(x,y) \in J$,

$$\begin{aligned} u(x,y) &= \lambda\mu(x,y) \\ &+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} g(s,t)\,dt\,ds, \end{aligned}$$

where, $g \in C(J)$ such that

$$g(x,y) = f(x,y,u_{(\rho_1(x,y,u_{(x,y)}),\rho_2(x,y,u_{(x,y)})}, g(x,y)).$$

This implies by $(H_{03})$ and as in step 2 (Theorem 4.5) that, for each $(x,y) \in J$, we have $\|u\|_{C_{(a,b)}} \le M^*$.

$$U = \{u \in C_{(a,b)} : \|u\|_\infty < M^* + 1\}.$$

By our choice of $U$, there is no $u \in \partial U$ such that $u = \lambda N(u)$, for $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type (Theorem 4.4) we deduce that $N$ has a fixed point $u$ in $\overline{U}$ which is a solution to problem (1.7)-(1.9).

## 6.2 Infinite delay case

Let us start in this section by defining what we mean by a solution of the problem (1.10)-(1.12).

**Definition 6.4.** A function $u \in \Omega$ such that $u(x,y)$, $\overline{D}_{0,x}^{r_1}u(x,y)$, $\overline{D}_{0,y}^{r_2}u(x,y)$, $\overline{D}_{\theta}^{r}u(x,y)$ are continuous for $(x,y) \in J$ and $I_{\theta}^{1-r}u(x,y) \in AC(J)$ is said to be a solution of (1.10)-(1.12) if $u$ satisfies equations (1.10),(1.12) on $J$ and the condition (1.11) on $\tilde{J}'$.

Set $\mathcal{R}' := \mathcal{R}'_{(\rho_1^-,\rho_2^-)}$

$$= \{(\rho_1(s,t,u),\rho_2(s,t,u)) : (s,t,u) \in J \times \mathcal{B}\ \rho_i(s,t,u) \le 0;\ i = 1,2\}.$$

We always assume that $\rho_i : J \times \mathcal{B} \to \mathbb{R};\ i = 1,2$ are continuous and the function $(s,t) \longmapsto u_{(s,t)}$ is continuous from $\mathcal{R}'$ into $\mathcal{B}$.

We will need to introduce the following hypothesis:

$(C_\phi)$ There exists a continuous bounded function $L : \mathcal{R}'_{(\rho_1^-,\rho_2^-)} \to (0,\infty)$ such that

$$\|\phi_{(s,t)}\|_{\mathcal{B}} \le L(s,t)\|\phi\|_{\mathcal{B}}, \text{ for any } (s,t) \in \mathcal{R}'.$$

In the sequel we will make use of the following generalization of a consequence of the phase space axioms ([[35], Lemma 2.1]).

**Lemma 6.5.** *If $u \in \Omega$, then*

$$\|u_{(s,t)}\|_{\mathcal{B}} = (M + L')\|\phi\|_{\mathcal{B}} + K \sup_{(\theta,\eta)\in[0,\max\{0,s\}]\times[0,\max\{0,t\}]} \|u(\theta,\eta)\|,$$

*where*

$$L' = \sup_{(s,t)\in\mathcal{R}'} L(s,t).$$

Now, we give (without proof) the existence result for the IVP (1.10)-(1.12)

**Theorem 6.6.** *Assume that the following hypothesis holds:*

$(C_1)$ *there exist $\ell_*'' > 0,\ 0 < l_*'' < 1$ such that*

$$\|f(x,y,u,v) - f(x,y,\overline{u},\overline{v})\| \le \ell_*''\|u - v\|_{\mathcal{B}} + l_*''\|\overline{u} - \overline{v}\|;$$

$$\textit{for any } u,\ v \in \mathcal{B},\ \overline{u},\overline{v} \in \mathbb{R}^n \textit{and } (x,y) \in J.$$

*If*

$$\frac{\ell_*'' K a^{r_1} b^{r_2}}{(1 - l_*'')\Gamma(1 + r_1)\Gamma(1 + r_2)} < 1, \tag{6.4}$$

*then there exists a unique solution for IVP (1.10)-(1.12) on $(-\infty,a] \times (-\infty,b]$.*

**Theorem 6.7.** *Assume $(C_\phi)$ and that the following hypothesis holds:*

$(C_2)$ *There exist* $p,q,d \in C(J,\mathbb{R}_+)$ *such that*

$$\|f(x,y,u,v)\| \le p(x,y) + q(x,y)\|u\|_{\mathcal{B}} + d(x,y)\|v\| \; for \; (x,y) \in J, \; u \in \mathcal{B}, \; v \in \mathbb{R}^n.$$

*If*

$$d^* + \frac{q^* a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1, \tag{6.5}$$

*where* $d^* = \sup_{(x,y)\in J} d(x,y)$ *and* $q^* = \sup_{(x,y)\in J} q(x,y)$, *then the IVP (1.10)-(1.12) has at least one solution on* $(-\infty,a]\times(-\infty,b]$.

# 7 Examples

**Example 1:** Consider the following partial hyperbolic implicit differential equations of the form

$$\overline{D}_\theta^r u(x,y) = \frac{1}{5e^{x+y+2}(1+|u(x-1,y-2)|+|\overline{D}_\theta^r u(x,y)|)}; \; \text{if } (x,y) \in [0,1]\times[0,1], \tag{7.1}$$

$$u(x,y) = x + y^2, \; (x,y) \in [-1,1]\times[-2,1]\backslash(0,1]\times(0,1], \tag{7.2}$$

$$u(x,0) = x, \; u(0,y) = y^2; \; x,y \in [0,1]. \tag{7.3}$$

Set

$$f(x,y,u_{(x,y)},\overline{D}_\theta^r u(x,y)) = \frac{1}{5e^{x+y+2}(1+|u(x-1,y-2)|+|\overline{D}_\theta^r u(x,y)|)}; \; (x,y) \in [0,1]\times[0,1].$$

Clearly, the function $f$ is continuous. For each $u,v \in C$, $\overline{u},\overline{v} \in \mathbb{R}$ and $(x,y) \in [0,1]\times[0,1]$ we have

$$|f(x,y,u(x,y),v(x,y)) - f(x,y,\overline{u}(x,y),\overline{v}(x,y))| \le \frac{1}{5e^2}(\|u-\overline{u}\|_C + \|v-\overline{v}\|).$$

Hence condition $(H_2)$ is satisfied with $\ell = l = \frac{1}{5e^2}$. We shall show that condition (4.1) holds with $a = b = 1$. Indeed

$$\frac{\ell a^{r_1} b^{r_2}}{(1-l)\Gamma(1+r_1)\Gamma(1+r_2)} = \frac{1}{(5e^2-1)\Gamma(1+r_1)\Gamma(1+r_2)} < 1,$$

which is satisfied for each $(r_1,r_2) \in (0,1]\times(0,1]$. Consequently Theorem 4.3 implies that problem (7.1)-(7.3) has a unique solution defined on $[-1,1]\times[-2,1]$.

**Example 2:** Consider now the following partial hyperbolic functional implicit differential equations with infinite delay

$$\overline{D}_\theta^r u(x,y) = \frac{e^{x+y-\gamma(x+y)}\|u_{(x,y)}\|}{2(e^{x+y}+e^{-x-y})(1+c\|u_{(x,y)}\|+|\overline{D}_\theta^r u(x,y)|)}, \; \text{if } (x,y) \in [0,1]\times[0,1], \tag{7.4}$$

$$u(x,y) = x + y^2, \; (x,y) \in (-\infty,1]\times(-\infty,1]\backslash(0,1]\times(0,1], \tag{7.5}$$

$$u(x,0) = x,\ u(0,y) = y^2,\ x,y \in [0,1], \tag{7.6}$$

where $c = \frac{\Gamma(1+r_1)\Gamma(1+r_2)}{2}$, $r = (r_1,r_2) \in (0,1] \times (0,1]$ and $\gamma$ a positive real constant.
Let

$$\mathcal{B}_\gamma = \{u \in C((-\infty,0] \times (-\infty,0],\mathbb{R}) : \lim_{\|(\theta,\eta)\| \to \infty} e^{\gamma(\theta+\eta)} u(\theta,\eta) \text{ exists in } \mathbb{R}\}.$$

The norm of $\mathcal{B}_\gamma$ is given by

$$\|u\|_\gamma = \sup_{(\theta,\eta) \in (-\infty,0] \times (-\infty,0]} e^{\gamma(\theta+\eta)} |u(\theta,\eta)|.$$

Let

$$E := [0,1] \times \{0\} \cup \{0\} \times [0,1],$$

and $u : (-\infty,1] \times (-\infty,1] \to \mathbb{R}$ such that $u_{(x,y)} \in \mathcal{B}_\gamma$ for $(x,y) \in E$, then

$$\lim_{\|(\theta,\eta)\| \to \infty} e^{\gamma(\theta+\eta)} u_{(x,y)}(\theta,\eta) = \lim_{\|(\theta,\eta)\| \to \infty} e^{\gamma(\theta-x+\eta-y)} u(\theta,\eta)$$

$$= e^{-\gamma(x+y)} \lim_{\|(\theta,\eta)\| \to \infty} e^{\gamma(\theta+\eta)} u(\theta,\eta) < \infty.$$

Hence $u_{(x,y)} \in \mathcal{B}_\gamma$. Finally we prove that

$$\|u_{(x,y)}\|_\gamma = K \sup\{|u(s,t)| : (s,t) \in [0,x] \times [0,y]\} + M \sup\{\|u_{(s,t)}\|_\gamma : (s,t) \in E_{(x,y)}\},$$

where $K = M = 1$ and $H = 1$.
If $x+\theta \le 0,\ y+\eta \le 0$ we get

$$\|u_{(x,y)}\|_\gamma = \sup\{|u(s,t)| : (s,t) \in (-\infty,0] \times (-\infty,0]\},$$

and if $x+\theta \ge 0,\ y+\eta \ge 0$ then we have

$$\|u_{(x,y)}\|_\gamma = \sup\{|u(s,t)| : (s,t) \in [0,x] \times [0,y]\}.$$

Thus for all $(x+\theta, y+\eta) \in [0,1] \times [0,1]$, we get

$$\|u_{(x,y)}\|_\gamma = \sup\{|u(s,t)| : (s,t) \in (-\infty,0] \times (-\infty,0]\} + \sup\{|u(s,t)| : (s,t) \in [0,x] \times [0,y]\}.$$

Then

$$\|u_{(x,y)}\|_\gamma = \sup\{\|u_{(s,t)}\|_\gamma : (s,t) \in E\} + \sup\{|u(s,t)| : (s,t) \in [0,x] \times [0,y]\}.$$

$(\mathcal{B}_\gamma, \|.\|_\gamma)$ is a Banach space. We conclude that $\mathcal{B}_\gamma$ is a phase space. Set

$$f(x,y,u_{(x,y)},v) = \frac{e^{x+y-\gamma(x+y)} \|u_{(x,y)}\|}{2(e^{x+y} + e^{-x-y})(1 + c\|u_{(x,y)}\|) + \|v(x,y)\|},\ (x,y) \in [0,1] \times [0,1].$$

For each $u,\ \overline{u} \in \mathcal{B}_\gamma,\ v,\ \overline{v} \in \mathbb{R}$ and $(x,y) \in [0,1] \times [0,1]$ we have

$$\begin{aligned} |f(x,y,u_{(x,y)},v(x,y)) - f(x,y,\overline{u}_{(x,y)},\overline{v}(x,y))| &\le \frac{e^{x+y}}{2(e^{x+y} + e^{-x-y})}(c\|u-\overline{u}\|_B + \|v-\overline{v}\|) \\ &\le \frac{c}{2}\|u-\overline{u}\|_B + \frac{1}{2}\|v-\overline{v}\|. \end{aligned}$$

Hence condition $(H_{1'})$ is satisfied with $\ell' = \frac{c}{2}$, $l' = \frac{1}{2}$. Since $a = b = K = 1$, we get

$$\frac{k\ell' a^{r_1} b^{r_2}}{(1-l')\Gamma(1+r_1)\Gamma(1+r_2)} = \frac{c}{\Gamma(1+r_1)\Gamma(1+r_2)} = \frac{1}{2} < 1.$$

Consequently, Theorem 5.2 implies that problem (7.4)-(7.6) has a unique solution defined on $(-\infty,1]\times(-\infty,1]$.

**Example 3:** Consider wow the following fractional order hyperbolic partial functional differential equations of the form

$$\overline{D}_\theta^r u(x,y) = \frac{1}{10e^{x+y+4}}$$

$$\times\frac{2+|u(x-\sigma_1(u(x,y)),y-\sigma_2(u(x,y)))|}{1+|u(x-\sigma_1(u(x,y)),y-\sigma_2(u(x,y)))|+|\overline{D}_\theta^r u(x,y)|},\ \text{ if } (x,y)\in[0,1]\times[0,1], \tag{7.7}$$

$$u(x,0) = x,\ u(0,y) = y^2;\ x,y\in[0,1], \tag{7.8}$$

$$u(x,y) = x+y^2,\ (x,y)\in[-1,1]\times[-2,1]\backslash[-1,0]\times[-2,0], \tag{7.9}$$

where $\sigma_1\in C(\mathbb{R},[0,1])$, $\sigma_2\in C(\mathbb{R},[0,2])$. Set

$$\rho_1(x,y,\varphi) = x-\sigma_1(\varphi(0,0)),\ \ (x,y,\varphi)\in J\times C([-1,0]\times[-2,0],\mathbb{R}),$$

$$\rho_2(x,y,\varphi) = y-\sigma_2(\varphi(0,0)),\ \ (x,y,\varphi)\in J\times C([-1,0]\times[-2,0],\mathbb{R}),$$

$$f(x,y,\varphi,\psi) = \frac{|\varphi|+2}{(10e^{x+y+4})(1+|\varphi|+|\psi|)},\ (x,y)\in[0,1]\times[0,1],\ \varphi\in C([-1,0]\times[-2,0],\mathbb{R}),\ \psi\in\mathbb{R}.$$

For each $\varphi,\ \overline{\varphi}\in C([-1,0]\times[-2,0],\mathbb{R})$, $\psi,\overline{\psi}\in\mathbb{R}$ and $(x,y)\in[0,1]\times[0,1]$ we have

$$|f(x,y,\varphi,\psi)-f(x,y,\overline{\varphi},\overline{\psi})| \le \frac{1}{10e^4}(\|\varphi-\overline{\varphi}\|_C+\|\psi-\overline{\psi}\|).$$

Hence the condition $(H_{02})$ is satisfied with $\ell_* = l_* = \dfrac{1}{10e^4}$. We shall show that condition (6.1) holds with $a = b = 1$. Indeed

$$\frac{\ell_* a^{r_1} b^{r_2}}{(1-l_*)\Gamma(1+r_1)\Gamma(1+r_2)} = \frac{1}{(10e^4-1)\Gamma(1+r_1)\Gamma(1+r_2)} < 1,$$

which is satisfied for each $(r_1,r_2)\in(0,1]\times(0,1]$. Consequently, Theorem 6.2 implies that problem (7.7)-(7.9) has a unique solution defined on $[-1,1]\times[-2,1]$.

**Example 4:** We consider now the following fractional order partial implicit differential equations with infinite delay of the form

$$\overline{D}_\theta^r u(x,y) = \frac{ce^{x+y-\gamma(x+y)}}{e^{x+y}+e^{-x-y}}$$

$$\times\frac{|u(x-\sigma_1(u(x,y)),y-\sigma_2(u(x,y)))|}{1+|u(x-\sigma_1(u(x,y)),y-\sigma_2(u(x,y)))|+|\overline{D}_\theta^r u(x,y)|};\ \text{ if } (x,y)\in J, \tag{7.10}$$

$$u(x,0)=x,\ u(0,y)=y^2;\ x,y\in[0,1], \tag{7.11}$$

$$u(x,y)=x+y^2,\ (x,y)\in\tilde{J}, \tag{7.12}$$

where $J:=[0,1]\times[0,1]$, $\tilde{J}:=(-\infty,1]\times(-\infty,1]\backslash(0,1]\times(0,1]$,
$c=1+\frac{2}{\Gamma(1+r_1)\Gamma(1+r_2)}$, $\gamma$ a positive real constant and $\sigma_1,\sigma_2\in C(\mathbb{R},[0,\infty))$.
Let the phase space

$$B_\gamma=\{u\in C((-\infty,0]\times(-\infty,0],\mathbb{R}):\lim_{\|(\theta,\eta)\|\to\infty}e^{\gamma(\theta+\eta)}u(\theta,\eta)\text{ exists in }\mathbb{R}\},$$

defined as in Example 2. Set

$$\rho_1(x,y,\varphi)=x-\sigma_1(\varphi(0,0)),\quad (x,y,\varphi)\in J\times B_\gamma,$$

$$\rho_2(x,y,\varphi)=y-\sigma_2(\varphi(0,0)),\quad (x,y,\varphi)\in J\times B_\gamma,$$

$$f(x,y,\varphi,\psi)=\frac{ce^{x+y-\gamma(x+y)}|\varphi|}{(e^{x+y}+e^{-x-y})(1+|\varphi|+|\psi|)},\ (x,y)\in[0,1]\times[0,1],\ \varphi\in B_\gamma.$$

For each $\varphi,\ \overline{\varphi}\in B_\gamma,\ \psi,\ \overline{\psi}\in\mathbb{R}$ and $(x,y)\in[0,1]\times[0,1]$ we have

$$|f(x,y,\varphi,\psi)-f(x,y,\overline{\varphi},\overline{\psi})|\le\frac{1}{c}(\|\varphi-\overline{\varphi}\|_\gamma+\|\psi-\overline{\psi}\|).$$

Hence condition $(C_1)$ is satisfied with $\ell''_*=l''_*=\dfrac{1}{c}$. Since $a=b=K=1$. we get

$$\frac{K\ell''_*a^{r_1}b^{r_2}}{(1-l''_*)\Gamma(1+r_1)\Gamma(r_2+1)}=\frac{1}{(c-1)\Gamma(1+r_1)\Gamma(1+r_2)}=\frac{1}{2}<1,$$

for each $(r_1,r_2)\in(0,1]\times(0,1]$. Consequently Theorem 6.6 implies that problem (7.10)-(7.12) has a unique solution defined on $(-\infty,1]\times(-\infty,1]$.

# References


[1] S. Abbas and M. Benchohra, Darboux problem for perturbed partial differential equations of fractional order with finite delay, *Nonlinear Anal. Hybrid Syst.* **3** (2009), 597-604.

[2] S. Abbas and M. Benchohra, Partial hyperbolic differential equations with finite delay involving the Caputo fractional derivative, *Comm. Math. Anal.* **7** (2009), 62-72.

[3] S. Abbas and M. Benchohra, Darboux problem for partial functional differential equations with infinite delay and Caputo's fractional derivative, *Adv. Dynam. Syst. Appl.* **5** (1) (2010), 1-19.

[4] S. Abbas and M. Benchohra, Fractional order partial hyperbolic differential equations involving Caputo's derivative, *Stud. Univ. Babeş-Bolyai Math.*, **57** (4) (2012), 469-479.

[5] S. Abbas, M. Benchohra and A. Cabada, Partial neutral functional integro-differential equations of fractional order with delay, *Bound. Value Prob.* **Vol. 2012** (2012), 128, 15 pp.

[6] S. Abbas, M. Benchohra and G.M. N'Guérékata, *Topics in Fractional Differential Equations*, Developments in Mathematics, **27**, Springer, New York, 2012.

[7] S. Abbas, M. Benchohra and G.M. N'Guérékata, Asymptotic stability in nonlinear delay differential equations of fractional order, *J. Nonlinear Evolution Equ. Appl.* **2012**, No. 7, 85-96.

[8] S. Abbas and M. Benchohra and J.J. Nieto, Global uniqueness results for fractional order partial hyperbolic functional differential equations. *Adv. Difference Equ.* **2011**, Art. ID 379876, 25 pp.

[9] S. Abbas, M. Benchohra and J.J. Nieto, Global attractivity of solutions for nonlinear fractional order Riemann-Liouville Volterra-Stieltjes partial integral equations, *Electron. J. Qual. Theory Differ. Equ.* **81** (2012), 1-15.

[10] S. Abbas, M. Benchohra and A. N. Vityuk, On fractional order derivatives and Darboux problem for implicit differential equations, *Frac. Calc. Appl. Anal.* **15** (2) (2012), 168-182.

[11] S. Abbas, M. Benchohra and Y. Zhou, Fractional order partial functional differential inclusions with infinite delay, *Proc. A. Razmadze Math. Inst.* **154** (2010), 1-19.

[12] S. Abbas, M. Benchohra and Y. Zhou, Fractional order partial hyperbolic functional differential equations with state-dependent delay, *Int. J. Dynam. Syst. Differ. Equ.* **3** (2011), 459-490.

[13] W.G. Aiello, H.I. Freedman, J. Wu, Analysis of a model representing stage-structured population growth with state-dependent time delay. *SIAM J. Appl. Math.* **52** (1992), 855–869.

[14] O. Arino, K. Boushaba, A. Boussouar, A mathematical model of the dynamics of the phytoplankton-nutrient system. *Nonlinear Anal. RWA.* **1** (2000), 69–87.

[15] J. Belair, Population models with state-dependent delays, *Lect. Notes Pure Appl. Maths., Dekker, New York,* **131** (1990), 156-176.

[16] J. Bélair and M.C. Mackey, Consumer memory and price fluctuations on commodity markets: An intergrodifferential model. *J. Dynam. Differential Equations* **1** (1989), 299-325.

[17] J. Belair, M. C. Mackey, and J. Mahaffy, Age-structured and two delay models for erythropoiesis, *Math. Biosciences* **128** (1995),

[18] Y. Cao, J. Fan, and T.C. Card, The effects of state-dependent time delay on a stage-structured population growth model, *Nonlinear Anal. TMA* **19** (1992), 95-105.

[19] F. Chen, D. Sun, and J. Shi, Periodicity in a food-limited population model with toxicants and state-dependent delays. *J. Math. Anal. Appl.* **288** (2003), 136-146.

[20] C. Colijn , M. C. Mackey, Bifurcation and bistability in a model of hematopoietic regulation, *SIAM J. Appl. Dynam. Syst.* **6** (2007), 378-394.

[21] F. Crauste, Delay model of hematopoietic stem cell dynamics: asymptotic stability and stability switch, *Math. Model. Nat. Phenom.* **4** (2009), 28-47.

[22] T. Czlapinski, On the Darboux problem for partial differential-functional equations with infinite delay at derivatives. *Nonlinear Anal.* **44** (2001), 389–398.

[23] T. Czlapinski, Existence of solutions of the Darboux problem for partial differential-functional equations with infinite delay in a Banach space. *Comment. Math. Prace Mat.* **35** (1995), 111-122.

[24] S. Das, *Functional Fractional Calculus*, Springer-Verlag, Berlin, Heidelberg, 2011.

[25] A. Domoshnitsky, M. Drakhlin and E. Litsyn, On equations with delay depending on solution. *Nonlinear Anal. TMA.* **49** (2002), 689-701.

[26] R.D. Driver, and M.J. Norrisn, Note on uniqueness for a one-dimentional two-body problem of classical electrodynamics. *Ann. Phys.* **42** (1964), 347-351.

[27] W. G. Glockle and T. F. Nonnenmacher, A fractional calculus approach of selfsimilar protein dynamics, *Biophys. J.* **68** (1995), 46-53.

[28] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[29] J. Hale and J. Kato, Phase space for retarded equations with infinite delay, *Funkcial. Ekvac.* **21**, (1978),11-41.

[30] J. K. Hale and S. Verduyn Lunel, *Introduction to Functional -Differential Equations*, Applied Mathematical Sciences, 99, Springer-Verlag, New York, 1993.

[31] F. Hartung, Linearized stability in periodic functional differential equations with state-dependent delays. *J. Comput. Appl. Math.* **174** (2005), 201-211.

[32] F. Hartung, Parameter estimation by quasilinearization in functional differential equations with state-dependent delays: a numerical study. Proceedings of the Third World Congress of Nonlinear Analysis, Part 7 (Catania, 2000) *Nonlinear Anal. TMA.* **47** (2001), 4557–4566.

[33] F. Hartung, T.L. Herdman, J. Turi. Parameter identification in classes of neutral differential equations with state-dependent delays. *Nonlinear Anal. TMA* **39** (2000), 305–325.

[34] F. Hartung, T. Krisztin, H.-O. Walther, J. Wu, Functional differential equations with state-dependent delays: theory and applications. Handbook of differential equations: ordinary differential equations. Vol. III, 435–545, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2006

[35] E. Hernández, A. Prokopczyk and L. Ladeira, A note on partial functional differential equations with state-dependent delay. *Nonlinear Anal. RWA* **7** (2006), 510-519.

[36] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[37] A.A. Kilbas, Hari M. Srivastava, and Juan J. Trujillo, *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[38] V. Kolmanovskii, and A. Myshkis, *Introduction to the Theory and Applications of Functional-Differential Equations*, Kluwer Academic Publishers, Dordrecht, 1999.

[39] F. Metzler, W. Schick, H. G. Kilian and T. F. Nonnenmacher, Relaxation in filled polymers: A fractional calculus approach, *J. Chem. Phys.* **103** (1995), 7180-7186.

[40] M.C. Mackey, Commodity price fluctuations: price dependent delays and nonlinearities as explanatory factors. *J. Econ. Theory* **48** (1989), 479-59.

[41] M.C. Mackey, and J. Milton, Feedback delays and the orign of blood cell dynamics, *Comm. Theor. Biol.* **1** (1990), 299-372.

[42] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[43] M. D. Ortigueira, *Fractional Calculus for Scientists and Engineers*. Lecture Notes in Electrical Engineering, 84. Springer, Dordrecht, 2011.

[44] R. Qesmi, H.-O. Walther, Center-stable manifolds for differential equations with state-dependent delays. *Discrete Contin. Dyn. Syst. 23* (2009), 1009–1033.

[45] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[46] K. Schumacher, Existence and continuous dependence for differential equation with unbounded delay *Arch. Rational Mech. Anal* (1978), 315-355.

[47] V. E. Tarasov, *Fractional Dynamics. Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*. Springer, Heidelberg, 2010.

[48] A. N. Vityuk, Existence of solutions of partial differential inclusions of fractional order, *Izv. Vyssh. Uchebn. , Ser. Mat.* **8** (1997), 13-19.

[49] A. N. Vityuk and A. V. Mykhailenko, On a class of fractional-order differential equation, *Nonlinear Oscil.* **11** (2008), 307-319.

[50] A. N. Vityuk and A. V. Mykhailenko, The Darboux problem for an implicit fractional-order differential equation, *J. Math. Sci.* **175** (2011), 391-401.

[51] H.-O. Walther, Linearized stability for semiflows generated by a class of neutral equations, with applications to state-dependent delays. *J. Dynam. Differential Equations* **22** (2010), 439–462.