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PERIODICITY IN THE α-NORM FOR SOME PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

KHALIL EZZINBI*
Université Cadi Ayyad, Faculté des Sciences Semlalia
Département de Mathématiques
B.P. 2390, Marrakech, Morocco

BILA ADOLPHE KYELEM†
Université de Ouagadougou, Unité de Formation et de Recherche
en Sciences Exactes et Appliquées
Département de Mathématiques
B.P.7021 Ouagadougou 03, Burkina Faso

STANISLAS OUARO‡
Université de Ouagadougou, Unité de Formation et de Recherche
en Sciences Exactes et Appliquées
Département de Mathématiques
B.P.7021 Ouagadougou 03, Burkina Faso

Abstract

The aim of this work is to study the existence of periodic solutions in the α-norm for some partial differential equations with infinite delay. A linear part of equations is assumed to generate an analytic semigroup. The delayed part is assumed to be continuous with respect to the fractional norm of the linear part and σ-periodic with respect to the first argument. Using Massera's approach we prove the existence of periodic solutions in the linear case. In the nonlinear case, a fixed point theorem for multivalued mapping is used to prove the existence of periodic solutions. We use also Horn's fixed point theorem to get the existence of periodic solutions when solutions are ultimate bounded. For illustration an example is provided for some reaction-diffusion equation involving infinite delay.

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*E-mail address: ezzinbi@ucam.ac.ma
†E-mail address: kyeleadoc@yahoo.fr
‡E-mail address: souaro@univ-ouaga.bf

1 Introduction

In this work, we study the existence of periodic solutions in the α-norm for the following partial functional differential equations

$$\frac{du(t)}{dt} = -Au(t) + f(t, u_t) \text{ for } t \in \mathbb{R}, \tag{1.1}$$

where $f : \mathbb{R} \times \mathscr{B}_\alpha \to X$ is a continuous function, σ-periodic in its first argument and $A : D(A) \subseteq X \to X$ is a closed linear operator. $\mathscr{B}_\alpha$ is the space of functions mapping $(-\infty, 0]$ into X which will be defined later.
We denote by u_t for $t \in \mathbb{R}$ as usual, the historic function defined on $(-\infty; 0]$ by

$$u_t(\theta) = u(t + \theta) \text{ for } \theta \leq 0,$$

where u is a function from $\mathbb{R}$ into X.

For this purpose, we consider the following Cauchy problem

$$\begin{cases} \dfrac{du(t)}{dt} = -Au(t) + f(t, u_t) \text{ for } \quad t \geq 0, \\ \\ u_0(\theta) = \phi(\theta) \text{ for } \theta \leq 0, \end{cases} \tag{1.2}$$

where $-A$ generates an analytic semigroup $\{T(t)\}_{t \geq 0}$ on the Banach space X. The initial function ϕ belongs to a Banach space $\mathscr{B}$ of functions mapping $(-\infty, 0]$ to X and satisfying some axioms to be introduced later. For $0 < \alpha < 1$, A^α denotes the fractional power of A.
The function $f : \mathbb{R}^+ \times \mathscr{B} \to X$ is continuous and σ-periodic in t i.e $f(\sigma + t, \phi) = f(t, \phi)$ for $t \in \mathbb{R}$ and $\phi \in \mathscr{B}$.

In [21], the authors studied equation (1.2) for A generating an analytic semigroup on X and f a continuous function in the finite delayed case.
The theory and applications of partial functional differential equations were studied in [2, 22] and where extensively treated by Wu [23]. In [9], the authors studied the existence of periodic solutions for partial differential equations with infinite delay of the following form:

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t) + f(t, u_t) \text{ for } t \geq 0, \\ \\ u(s) = \phi(s) \quad \text{ for } s \leq 0, \end{cases} \tag{1.3}$$

where A is an unbounded nondensely defined linear operator on a Banach space X and satisfies Hille-Yosida's condition. The function f is continuous, σ-periodic with respect to the first argument and lipschitz function with respect to the second argument.

In [1, 6], the authors studied the stability of solutions in the α-norm for some partial differential equations with finite and infinite delays. The theory of semigroups of linear operators [7, 19] is an important working tools to study partial functional differential equations with

delays.
In this work, we deal with the existence of periodic solutions of equation (1.1) following the works done in [16, 17, 18, 24], in the case where A is densely defined linear operator, $-A$ generates an analytic semigroup and f is not necessarily a lipschitz function. The problem of finding periodic solutions is an important subject in the qualitative study of functional differential equations. The Massera's approach [18] on periodic partial functional differential equations is used in [9] to explain the relationship between the boundedness of solutions and periodic solutions.
In many of those studies, the most important feature is to show that the Poincaré's mapping

$$P_\sigma(\phi) = u_\sigma(.,\phi)$$

where σ is the period of the system and u the unique mild solution determined by ϕ is condensing. Then, a fixed point theorem can be used to derive periodic solutions.

In [4, 5, 8, 9, 13, 15, 16, 17, 18], the authors proved the existence of periodic solutions in the linear case or in the case of lipschitz function f by using the boundedness and ultimate boundedness of solutions of some partial functional differential equations and Hale's fixed point theorem [10]. In many of those works, the authors showed that the Poincaré's mapping is condensing. Note also that the Sadovskii's fixed point theorem is used in [20] to prove the existence of σ-periodic solutions.
In [13], the author used the phase space $\mathscr{B}$ that is a space of linear and continuous functions from $(-\infty,0]$ to X endowed with the norm denoted by $\|.\|_{\mathscr{B}}$ constructed for the first time by Hale and Kato (see [11]). They showed the existence of periodic solutions of the partial functional differential equations of the form:

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t) + F(t,u_t) \text{ for } t \geq 0, \\ \\ u_0 = \phi \in \mathscr{B}, \end{cases} \tag{1.4}$$

where the function $F : \mathbb{R} \times \mathscr{B} \to X$ is continuous and X a Banach space.

The organization of this work is as follows: in section **2**, we recall some preliminary results on the fractional powers of linear operators generating analytic semigroups and the existence and uniqueness of solutions of partial functional differential equations. We develop in section **3**, the existence of periodic solutions of equation in the nonhomogeneous linear case. In section **4**, we prove the relationship between the existence of periodic solutions and the boundedness and ultimate boundedness of solutions of equation (1.1). We study in section **5**, the existence of periodic solutions in the nonlinear case using the multivalued theory for equation (1.1). Finally, we apply our theoretical results to some examples reaction-diffusion system involving infinite delay.

2 Analytic semigroup, fractional power of its generator and partial functional differential equations

Troughout this work we assume the following:

($\mathbf{H}_1$) $(-A)$ is the infinitesimal generator of an analytic semigroup of linear operators $\{T(t)\}_{t\geq 0}$ on a Banach space $(X, |.|)$.

Without loss of generality, we suppose that $0 \in \rho(A)$; otherwise instead of A, we take $A - \delta I$ where δ is chosen such that $0 \in \rho(A - \delta I)$ where $\rho(A)$ is the resolvent set of A.

It is well known that $|T(t)x| \leq Me^{\omega t}|x|$ for all $t \geq 0$, $x \in X$ where $M \geq 1$ and $\omega \in \mathbb{R}$.
For all $0 < \alpha < 1$, we define (see [19]) the operator $A^{-\alpha}$ by

$$A^{-\alpha}x = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} T(t)x dt \text{ for all } x \in X,$$

where $\Gamma(\alpha)$ denotes the well-known gamma function. The operator $A^{-\alpha}$ is bijective and the operator A^{α} is defined by

$$A^{\alpha} = (A^{-\alpha})^{-1}.$$

We denote by $D(A^{\alpha})$, the domain of the operator A^{α}. Then, $D(A^{\alpha})$ endowed with the norm $|x|_{\alpha} = |A^{\alpha}x|$ for all $x \in D(A^{\alpha})$ is a Banach space (more details can be found in [19]). In the sequel, we denote $(D(A^{\alpha}), |.|_{\alpha})$ by X_{α}. Moreover, we recall the following known results.

Theorem 2.1. *([19], p.69-75) Let $0 < \alpha < 1$ and assume that ($\boldsymbol{H}_1$) holds. Then,*
(i) $T(t) : X \to D(A^{\alpha})$ for each $t > 0$ and $\alpha \geq 0$;
(ii) For all $x \in D(A^{\alpha})$, one has $T(t)A^{\alpha}x = A^{\alpha}T(t)x$;
(iii) For each $t > 0$, the linear operator $A^{\alpha}T(t)$ is bounded and $|A^{\alpha}T(t)x| \leq M_{\alpha}t^{-\alpha}e^{\omega t}|x|$, where M_{α} is a positive real constant;
(iv) For $0 < \alpha \leq 1$ and $x \in D(A^{\alpha})$, one has $|T(t)x - x| \leq N_{\alpha}t^{\alpha}|A^{\alpha}x|$, for $t > 0$, where N_{α} is a positive real constant;
(v) For $0 < \alpha < \beta < 1$, $X_{\beta} \hookrightarrow X_{\alpha}$.

From now on, we use an axiomatic definition of the phase space $\mathscr{B}$ which was firstly introduced by Hale and Kato in [11]. We assume that $\mathscr{B}$ is the normed space of functions mapping $(-\infty, 0]$ into X and satisfying the following axioms:

(**A**) there exist a positive constant N, a locally bounded function $M(.)$ on $[0, +\infty)$ and a continuous function $K(.)$ on $[0, +\infty)$, such that if $u : (-\infty, a] \to X$ is continuous on $[\xi, a]$ with $u_{\xi} \in \mathscr{B}$ for some $\xi < a$ where $0 < a$, then for all $t \in [\xi, a]$,

(i) $u_t \in \mathscr{B}$,
(ii) $t \to u_t$ is continuous on $[\xi, a]$,
(iii) $N|u(t)| \leq |u_t|_{\mathscr{B}} \leq K(t-\xi) \sup_{\xi \leq s \leq t} |u(s)| + M(t-\xi)|u_{\xi}|_{\mathscr{B}}$.
(**B**) $\mathscr{B}$ is a Banach space.

Lemma 2.2. *[6] Let C_{00} be the space of continuous functions mapping $(-\infty,0]$ into X with compact supports and C_{00}^a be the subspace of functions with supports included in $[-a,0]$ endowed with the uniform norm topology. Then $C_{00}^a \hookrightarrow \mathscr{B}$.*

Let
$$\mathscr{B}_\alpha = \{\phi \in \mathscr{B} : \ \phi(\theta) \in D(A^\alpha) \text{ for } \theta \le 0 \text{ and } A^\alpha\phi \in \mathscr{B}\}$$
and provide $\mathscr{B}_\alpha$ with the following norm
$$|\phi|_{\mathscr{B}_\alpha} = |A^\alpha \phi|_{\mathscr{B}} \text{ for } \phi \in \mathscr{B}_\alpha.$$

We assume also that

(H$_2$) $A^{-\alpha}\phi \in \mathscr{B}$ for all $\phi \in \mathscr{B}$, where the function $A^{-\alpha}\phi$ is defined by
$$(A^{-\alpha}\phi)(\theta) = A^{-\alpha}(\phi(\theta)) \text{ for } \theta \le 0.$$

Lemma 2.3. *[6] Assume that ($\boldsymbol{H}_1$) and ($\boldsymbol{H}_2$) hold. Then $\mathscr{B}_\alpha$ is a Banach space.*

Let us now give the notions of solutions which will be studied in our work.

Definition 2.4. *A continuous function $u : \mathbb{R} \to X_\alpha$ satisfying for all $\tau, t \in \mathbb{R}$ with $\tau \le t$*
$$u(t) = T(t-\tau)u(\tau) + \int_\tau^t T(t-s)f(s,u_s)ds$$
is called a mild solution of equation (1.1) on $\mathbb{R}$.

Definition 2.5. *A function $u : (-\infty,+\infty) \to X_\alpha$ satisfying*
$$\begin{cases} u(t) = T(t)\phi(0) + \int_0^t T(t-s)f(s,u_s)ds \ \ for \ \ t \ge 0, \\ \\ u_0 = \phi, \end{cases}$$
is called a mild solution of equation (1.2) corresponding to the initial data $\phi \in \mathscr{B}_\alpha$.

We make the following additional assumption on f:

(H$_3$) $|f(t,\phi) - f(t,\psi)| \le k|\phi - \psi|_{\mathscr{B}_\alpha}$ for every ϕ, $\psi \in \mathscr{B}_\alpha$, $t \ge 0$, where k is a positive constant.

Theorem 2.6. *[6] Assume that conditions ($\boldsymbol{H}_1$), ($\boldsymbol{H}_2$) and ($\boldsymbol{H}_3$) hold. Then, for each $\phi \in \mathscr{B}_\alpha$, there exists a unique mild solution of equation (1.2) defined for all $t \ge 0$.*

Let $u(.,\phi)$ be the unique mild solution of equation (1.2) associated to $\phi \in \mathscr{B}_\alpha$. We define the operator $U(t)$ on $\mathscr{B}_\alpha$ for each $t \ge 0$ by
$$U(t)(\phi) = u_t(.,\phi).$$

We have the following important results.

Theorem 2.7. *[6] Suppose that ($\boldsymbol{H}_1$), ($\boldsymbol{H}_2$) and ($\boldsymbol{H}_3$) hold and let ϕ and ψ be in $\mathscr{B}_\alpha$. Then, there exists a positive function $l \in l^\infty_{loc}(\mathbb{R}^+, \mathbb{R}^+)$ such that*

$$|U(t)\phi - U(t)\psi|_{\mathscr{B}_\alpha} \leq l(a)|\phi - \psi|_{\mathscr{B}_\alpha} \text{ for all } t \in [0,a] \text{ and for all } a > 0.$$

Assume now that
($\mathbf{H}_4$) The semigroup $(T(t))_{t\geq 0}$ is compact for $t > 0$.

Theorem 2.8. *[6] Assume that ($\boldsymbol{H}_1$), ($\boldsymbol{H}_2$), ($\boldsymbol{H}_3$) and ($\boldsymbol{H}_4$) hold. Then, the solution $u(.,\phi)$ of equation (1.2) is decomposed as follows*

$$u_t(.,\phi) = \mathscr{U}(t)\phi + \mathscr{W}(t)\phi \text{ for } t \geq 0,$$

where $\mathscr{W}(t)$ is a compact operator on $\mathscr{B}_\alpha$, for each $t > 0$ and $\mathscr{U}(t)$ is the semigroup solution of the following equation

$$\begin{cases} \dfrac{du(t)}{dt} = -Au(t) \quad \text{for } t \geq 0, \\ \\ u_0 = \phi \in \mathscr{B}_\alpha. \end{cases} \tag{2.1}$$

Let $K : \mathcal{D}(K)Y \to Y$ be a closed linear oprerator with dense domain $D(K)$ in a Banach space Y. We denote by $\sigma(K)$ the spectrum of K.

Definition 2.9. *The essential spectrum $\sigma_{ress}(K)$ of K is the set of all $\lambda \in \mathbb{C}$ such that at least one of the following holds:*
(i) the range $Im(\lambda I - K)$ is not closed;
(ii) the generalized eigenspace $M_\lambda(K) = \cup_{n\geq 0} ker(\lambda I - K)$ of λ is an infinite dimensional space;
(iii) λ is a limit of $\sigma(K)$, that is $\lambda \in \overline{\sigma(K) - \{\lambda\}}$.

The essential radius denoted by $r_{ess}(K)$ is given by

$$r_{ess}(K) = \sup\{|\lambda| : \lambda \in \sigma_{ess}(K)\}.$$

Definition 2.10. *The spectral bound of the linear operateur A denoted by $s(A)$ is defined as follows*

$$s(A) = \sup\{Re\lambda : \lambda \in \sigma(A)\}.$$

Definition 2.11. *The type of the semigroup $\{T(t)\}_{t\geq 0}$ is defined by*

$$\omega_0(T) = \inf\left\{\omega \in \mathbb{R} : \sup_{t\geq 0}\left\{e^{-\omega t}|T(t)|\right\} < \infty\right\}$$

In the sequel, we recall the χ measure of noncompactness, which will be used in the next to study the existence of periodic solutions via a fixed point theorem for condensing operators. The χ measure of noncompactness for a bounded set H of a Banach space Y with the norm $|.|_Y$ is defined by

$$\chi(H) = \inf\{\epsilon > 0 : H \text{ has a finite cover of diameter} < \epsilon\}.$$

Some fondamental properties on χ measure of noncompactness are given below.

Lemma 2.12. *[14] Let A_1 and A_2 be bounded sets of a Banach space Y. Then*
(i) $\chi(A_1) \le dia(A_1)$, where $dia(A_1) = \sup_{x,y\in A_1} |x-y|_Y$.
(ii) $\chi(A_1) = 0$ if and only if A_1 is relatively compact in Y.
(iii) $\chi(A_1 \bigcup A_2) = max\{\chi(A_1), \chi(A_2)\}$.
(iv) $\chi(\lambda A_1) = |\lambda|\chi(A_1)$, $\lambda \in \mathbb{R}$ where $\lambda A_1 = \{\lambda x : x \in A_1\}$.
(v) $\chi(A_1 + A_2) \le \chi(A_1) + \chi(A_2)$ where $A_1 + A_2 = \{x + y : x \in A_1, y \in A_2\}$.
(vi) $\chi(A_1) \le \chi(A_2)$ if $A_1 \subseteq A_2$.

Definition 2.13. *The essential norm of a bounded linear operator K on Y is defined by*

$$|K|_{ess} = \inf\{M \ge 0 : \chi(K(B)) \le M\chi(B) \text{ for any bounded set } B \text{ in } Y\}.$$

Let $V = \{V(t)\}_{t\ge 0}$ be a c_0-semigroup on a Banach space Y.

Definition 2.14. *The essential growth $\omega_{ess}(V)$ of $\{V(t)\}_{t\ge 0}$ is defined by*

$$\omega_{ess}(V) = \inf\left\{\omega \in \mathbb{R} : \sup_{t\ge 0} e^{-\omega t}|V(t)|_{ess} < \infty\right\}.$$

Theorem 2.15. *[6] The essential growth bound of $\{V(t\}_{t\ge 0}$ is computed by the following formulas*

$$\omega_{ess}(V) = \lim_{t\to+\infty} \frac{1}{t} \log|V(t)|_{ess} = \inf_{t>0} \frac{1}{t} \log|V(t)|_{ess}. \tag{2.2}$$

Moreover

$$r_{ess}(V(t)) = exp(t\omega_{ess}(V)) \text{ for } t \ge 0. \tag{2.3}$$

3 Existence of periodic solutions in the α-norm for nonhomogeneous linear equations

In this section, we study the following nonhomogeneous linear equation

$$\frac{du(t)}{dt} = -Au(t) + L(t, u_t) + f(t) \text{ for } t \in \mathbb{R}, \tag{3.1}$$

where $L : \mathbb{R} \times \mathscr{B}_\alpha \to X$ is a continuous function, linear with respect to the second argument, σ-periodic in t, $f : \mathbb{R} \to X$ is continuous and σ-periodic.

We consider the following Cauchy problem associated to (3.1):

$$\begin{cases} \dfrac{du(t)}{dt} = -Au(t) + L(t, u_t) + f(t) \quad \text{for } t \ge 0, \\ u_0 = \phi \in \mathscr{B}_\alpha. \end{cases} \tag{3.2}$$

Recall that for equation (3.2), the existence of a mild solution is true, since $F(t,\phi) = L(t,\phi) + f(t)$ is continuous and lipschitzian with respect to ϕ.

For $\phi \in \mathscr{B}$, $t \geq 0$ and $\theta \leq 0$, we define the following.

$$[S(t)\phi](\theta) = \begin{cases} \phi(0) \text{ if } t+\theta \geq 0 \\ \\ \phi(t+\theta) \text{ if } t+\theta < 0. \end{cases} \tag{3.3}$$

Then $\{S(t)\}_{t\geq 0}$ is a strongly continuous semigroup on $\mathscr{B}$. We set

$$S_0(t) = S(t)/\mathscr{B}_0, \text{ where } \mathscr{B}_0 = \{\phi \in \mathscr{B} : \ \phi(0) = 0\}.$$

Definition 3.1. *[6] We say that $\mathscr{B}$ is a uniform fading memory space if the following conditions hold:*
(i) *if an uniformly bounded sequence $(\phi_n)_n$ in C_{00} converges to a function ϕ compactly on $(-\infty, 0]$, then ϕ is in $\mathscr{B}$ and $|\phi_n - \phi|_{\mathscr{B}} \to 0$ as $n \to +\infty$,*
(ii) *$|S_0(t)|_{\mathscr{B}} \to 0$ as $t \to +\infty$.*

Lemma 3.2. *[6] If $\mathscr{B}$ is an uniform fading memory space, then K and M can be chosen such that K is bounded on $\mathbb{R}^+$ and $M(t) \to 0$ as $t \to +\infty$.*

If y is a bounded solution of equation (3.2) on $\mathbb{R}^+$ in the sens that

$$\sup_{t\in\mathbb{R}^+} |y(t)|_\alpha < \infty,$$

then, using (iii) in Axiom (**A**) we obtain that there exists a positive constant N_1 such that

$$\sup_{t\in\mathbb{R}^+} |y_t|_{\mathscr{B}_\alpha} < N_1. \tag{3.4}$$

Theorem 3.3. *[6] Assume that ($\boldsymbol{H}_1$), ($\boldsymbol{H}_2$), ($\boldsymbol{H}_3$) and ($\boldsymbol{H}_4$) hold. Then, the solution $u(.,\phi)$ of equation (3.2) is decomposed as follows*

$$u_t(.,\phi) = \mathscr{U}_1(t)\phi + \mathscr{U}_2(t)\phi \text{ for } t \geq 0,$$

where $\mathscr{U}_2(t)$ is a compact operator on $\mathscr{B}_\alpha$, for each $t > 0$ and $\mathscr{U}_1(t)$ is the semigroup solution of the following equation

$$\begin{cases} \dfrac{du(t)}{dt} = -Au(t) \quad \text{for } t \geq 0, \\ \\ u_0 = \phi \in \mathscr{B}_\alpha. \end{cases}$$

Moreover, for all $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\chi(\mathscr{U}_1(t)) \leq C_\epsilon M(t-\epsilon) \text{ for } t > \epsilon.$$

Theorem 3.4. *[6] Assume that ($\boldsymbol{H}_1$), ($\boldsymbol{H}_2$), ($\boldsymbol{H}_3$) and ($\boldsymbol{H}_4$) hold. If $\mathscr{B}$ is a uniform fading memory space, then $\omega_{ess}(\mathscr{U}_1) < 0$.*

For the existence of a periodic solution, we use the following important theorem due to Hale and Chow which is valid for an affine map.

Theorem 3.5. *[10] Let Y be a Banach space, $\Phi : Y \to Y$ a bounded linear and continuous operator, $y_0 \in Y$ given. Let $\Theta : Y \to Y$ be defined by*

$$\Theta x = \Phi x + y_0.$$

Suppose that $Im(I - \Phi)$ is closed and there exists $x_0 \in Y$ such that $\{\Theta^n x_0 : n \in \mathbb{N}\}$ is bounded in Y, then Θ has at least one fixed point in Y.

We have now, the following existence result.

Theorem 3.6. *Assume that $(\boldsymbol{H}_1)$, $(\boldsymbol{H}_2)$, $(\boldsymbol{H}_3)$ and $(\boldsymbol{H}_4)$ hold and $\mathscr{B}$ is a uniform fading memory space. Moreover, suppose that $f : \mathbb{R} \to X$ is continuous and σ-periodic and that equation* (3.2) *has a bounded mild solution v on $\mathbb{R}^+$ in the sense that $\sup_{t\geq 0} |v(t)|_\alpha < +\infty$, for some $\phi \in \mathscr{B}_\alpha$. Then, equation* (3.1) *has a σ-periodic solution defined on $\mathbb{R}$.*

Proof. Consider the Poincaré's map $P_\sigma : \mathscr{B}_\alpha \to \mathscr{B}_\alpha$ which is defined by

$$P_\sigma(\phi) = u_\sigma(., \phi, f),$$

where $u(., \phi, f)$ is the unique mild solution with respect to the initial data $\phi \in \mathscr{B}_\alpha$ for equation (3.2). Then we can write $P_\sigma(\phi) = u_\sigma(., \phi, 0) + u_\sigma(., 0, f)$ where $u(., 0, f)$ is the mild solution of equation (3.2) when $\phi = 0$ and $u(., \phi, 0)$ is the mild solution of equation (3.2) when $f = 0$. Then, P_σ is an affine map.
Recall that $u_\sigma(., \phi, 0)$ is decomposed as follows

$$u_\sigma(., \phi, 0) = \mathscr{U}_1(\sigma)\phi + \mathscr{U}_2(\sigma)\phi,$$

where $\mathscr{U}_2(\sigma)$ is a compact operator on $\mathscr{B}_\alpha$ and $\{\mathscr{U}_1(t)\}_{t\geq 0}$ is the semigroup solution of equation (2.1).
Using Theorem 3.4, we obtain that $\omega_{ess}(\mathscr{U}_1(\sigma)) < 0$ which gives that $r_{ess}(\mathscr{U}_1(\sigma)) < 1$. Then, $1 \notin \sigma_{ess}(\mathscr{U}_1(\sigma))$ which gives that $Im(I - u_\sigma(., \phi, 0))$ is closed in $\mathscr{B}_\alpha$. Note also that Proposition 2.7 gives the continuity of $u_\sigma(., \phi, 0)$ on $\mathscr{B}_\alpha$.

Moreover by virtue of the boundedness of v and the relation (3.4), one can see that

$$\{P_\sigma^n \phi : n \in \mathbb{N}\} = \{v_{n\sigma} : n \in \mathbb{N}\}$$

is bounded in $\mathscr{B}_\alpha$.
We obtain that all conditions of Theorem **3.5** are satisfied. Therefore, P_σ has a fixed point ψ in $\mathscr{B}_\alpha$ and we conclude that equation (3.2) has a σ-periodic solution. Finally, equation (3.1) has a σ-periodic solution on $\mathbb{R}$. In fact, Let $v(.) = v(., \psi)$ be a bounded mild solution of equation (3.2) such that $v_\sigma(., \psi) = \psi$. Let us define a function w such that $w(.) = v(. + \sigma)$. Then, for all $t \geq 0$, we obtain

$$
\begin{aligned}
w(t) &= T(t+\sigma)\psi(0) + \int_0^{t+\sigma} T(t+\sigma-s)[L(s,v_s)+f(s)]ds \\
&= T(t)\Big[T(\sigma)\psi(0) + \int_0^{\sigma} T(\sigma-s)[L(s,v_s)+f(s)]ds\Big] + \int_{\sigma}^{t+\sigma} T(t+\sigma-s)[L(s,v_s)+f(s)]ds \\
&= T(t)\Big[T(\sigma)\psi(0) + \int_0^{\sigma} T(\sigma-s)[L(s,v_s)+f(s)]ds\Big] + \int_0^{t} T(t-s)[L(s+\sigma,v_{s+\sigma})+f(s+\sigma)]ds \\
&= T(t)v(\sigma) + \int_0^{t} T(t-s)[L(s,w_s)+f(s)]ds \\
&= T(t)w(0) + \int_0^{t} T(t-s)[L(s,w_s)+f(s)]ds.
\end{aligned}
$$

Since $w_0 = v_\sigma = \psi$, for the uniqueness of the mild solution $v(.,\psi)$ associated to the initial data ϕ, we have $w(t) = v(t)$ for all $t \geq 0$. Thus $v(t+\sigma) = v(t)$ for all $t \geq 0$. □

4 Boundedness, ultimate boundedness and existence of periodic solutions in the α-norm in the nonlinear case

Now, we will study the existence of σ-periodic solutions of equation (1.1).
Throughout this section, the mild solutions of equation (1.2) are denoted by $u(.,\phi)$ and will be called also solutions of equation (1.2).
We study the existence of periodic solutions of equation (1.2) by using boundedness and ultimate boundedness of solutions. To do it, we make the following additional assumption.

($\mathbf{H}_5$) The locally bounded function M in axiom (**A**) is strictly decreasing and satisfies M(0)=1.
According to ($\mathbf{H}_5$), there exists a positive real $\eta \in (0,1)$ such that

$$M(s) \leq \eta, \text{ for all } s \in (0,\sigma].$$

Since K is locally bounded then, there exists K_1 (the bound of the function K in the definition of uniform fading memory space) such that

$$K(s) \leq K_1 \text{ for all } s \in [0,\sigma].$$

Let $M_0 = \sup_{s\in[0,\sigma]} |T(s)|$.
As $\eta \in (0,1)$ and since $M(t) \leq 1$ for all $t \geq 0$, we can find an integer N_0 such that

$$\eta^{N_0-1}[K_1 M_0 \frac{1}{N} + 1] < 1 \tag{4.1}$$

and

$$0 < \omega_0 = \frac{\sigma}{N_0}. \tag{4.2}$$

Using the same technic as in [15], we define the following sets and we obtain important results for the existence of σ-periodic solutions.
For $D \subset \mathscr{B}_\alpha$ and $u(\phi)$ the unique mild solution of equation (1.2) with respect to ϕ, we define the sets

$$W_l(D) = \{u_l(.,\phi) : \ \phi \in D\} \ \text{and} \ W_{[h,r]}(D) = \{u_{[h,r]}(\phi) : \ \phi \in D\},$$

where $u_{[h,r]}$ means the restriction of u on the interval $[h,r]$ and h, r the positive constants with $0 < h < r$.

Proposition 4.1. *Let $D \subset \mathscr{B}_\alpha$ be a bounded subset. Then*

$$\chi(W_{[h,r]}(D)) = 0, \tag{4.3}$$

for any $0 < h < r \leq \sigma$.

Proof. Let $0 < h < r \leq \sigma$, $\phi \in D$ and $t \in [h,r]$ with $t > \epsilon > 0$. Then

$$\begin{aligned}
u(t,\phi) &= T(t)\phi(0) + \int_0^t T(t-s)f(s,u_s(.,\phi))ds \\
&= T(\epsilon)T(t-\epsilon)\phi(0) + T(\epsilon)\int_0^{t-\epsilon} T(t-\epsilon-s)f(s,u_s(.,\phi))ds + \int_{t-\epsilon}^t T(t-s)f(s,u_s(.,\phi))ds \\
&= T(\epsilon)\Big[T(t-\epsilon)\phi(0) + \int_0^{t-\epsilon} T(t-\epsilon-s)f(s,u_s(.,\phi))ds\Big] + \int_{t-\epsilon}^t T(t-s)f(s,u_s(.,\phi))ds \\
&= T(\epsilon)u(t-\epsilon,\phi) + \int_{t-\epsilon}^t T(t-s)f(s,u_s(.,\phi))ds.
\end{aligned}$$

The set $\{u(t-\epsilon,\phi) : \ \phi \in D\}$ is bounded in X_α for some fixed $t \in [h,r]$. Using the compactness of $T(\epsilon)$, it follows that

$$\{T(\epsilon)u(t-\epsilon,\phi) : \ \phi \in D\} \, \text{is compact in} \, X_\alpha.$$

Also, $f(s,u_s(.,\phi))$ is bounded for $s \in [0,\sigma]$ since f is σ-periodic with respect to the first argument and u is locally bounded. Indeed, for $s \in [0,\sigma]$ we have

$$\begin{aligned}
|f(s,u_s(.,\phi))| &= |f(s,u_s(,\phi)) - f(s,0) + f(s,0)| \\
&\leq |f(s,u_s(.,\phi)) - f(s,0)| + |f(s,0)| \\
&\leq k|u_s(.,\phi)|_{\mathscr{B}_\alpha} + |f(s,0)| \\
&= N_2 \\
&< \infty.
\end{aligned}$$

Moreover, for $0 < \beta < \alpha < 1$ and each $\phi \in D$, we have

$$|A^\beta \int_{t-\epsilon}^t T(t-s)f(s,u_s(.,\phi))ds| \leq M_\beta N_2 \int_0^\epsilon e^{\omega s}s^{-\beta}ds.$$

Thus,
$\left\{A^\beta \int_{t-\epsilon}^t T(t-s)f(s,u_s(.,\phi))ds\right\}$ is bounded in X. Using the compactness of

$$A^{-\beta} : X \to X_\alpha,$$

we conclude that $\overline{\left\{\int_{t-\epsilon}^{t} T(t-s)f(s,u_s(.,\phi))ds : \ \phi \in D\right\}}$ is compact in X_α.

Consequently, $\overline{\{u(t,\phi) : \ \phi \in D\}}$ is compact in X_α for every fixed $t \in [h,r]$.
Now, let $\phi \in D$ be fixed and $0 < h \le t_0 \le r$. Take $\epsilon > 0$ be small enough. Then

$$\begin{aligned}
u(t_0+\epsilon,\phi) - u(t_0,\phi) &= T(t_0+\epsilon)\phi(0) + \int_0^{t_0+\epsilon} T(t_0+\epsilon-s)f(s,u_s(.,\phi))ds \\
&- \Big(T(t_0)\phi(0) + \int_0^{t_0} T(t_0-s)f(s,u_s(.,\phi))ds\Big) \\
&= T(t_0+\epsilon)\phi(0) + \int_0^{t_0} T(t_0+\epsilon-s)f(s,u_s(.,\phi))ds \\
&+ \int_{t_0}^{t_0+\epsilon} T(t_0+\epsilon-s)f(s,u_s(.,\phi))ds - T(t_0)\phi(0) - \int_0^{t_0} T(t_0-s)f(s,u_s(.,\phi))ds \\
&= (T(\epsilon)-I)u(t_0,\phi) + \int_{t_0}^{t_0+\epsilon} T(t_0+\epsilon-s)f(s,u_s(.,\phi))ds.
\end{aligned}$$

Since $\overline{\{u(t,\phi) : \ \phi \in D\}}$ is compact in X_α, by Banach-Steinhaus's theorem, we obtain that

$$\lim_{\epsilon\to 0} |(T(\epsilon)-I)u(t_0,\phi)|_\alpha = 0 \quad \text{uniformly with respect to} \quad \phi \in D.$$

Moreover,

$$|A^\alpha \int_{t_0}^{t_0+\epsilon} T(t_0+\epsilon-s)f(s,u_s(.,\phi))ds| \le M_\alpha N_2 \int_0^{\epsilon} e^{\omega s} s^{-\alpha} ds.$$

Letting ϵ goes to 0, we obtain

$$\lim_{\epsilon\to 0} |u(t_0+\epsilon,\phi) - u(t_0,\phi)|_\alpha = 0 \quad \text{uniformly with respect to} \quad \phi \in D.$$

Using the same argument, we get for $\epsilon > 0$ with $t_0 - \epsilon > 0$,

$$\lim_{\epsilon\to 0} |u(t_0-\epsilon,\phi) - u(t_0,\phi)|_\alpha = 0 \quad \text{uniformly with respect to} \quad \phi \in D.$$

Thus, the family $\{u_{[h,r]}(\phi) : \phi \in D\}$ is equicontinuous in $C([h,r], X_\alpha)$.
Using Ascoli-Arzelà's theorem, we conclude that $\overline{\{u_{[h,r]}(\phi) : \phi \in D\}}$ is compact in $C([h,r], X_\alpha)$.
Then, we get

$$\chi(\{u_{[h,r]}(\phi) : \phi \in D\}) = 0. \qquad \square$$

Now, we give the definitions of a condensing operator.

Definition 4.2. *A continuous mapping $P : Y \to Y$ is said to be χ-contraction if P maps bounded sets into bounded sets and there exists a constant $k \in (0,1)$ such that*

$$\chi(P(B)) \le k\chi(B),$$

for every bounded subset B in Y such that $\chi(B) > 0$.

Definition 4.3. *A continuous mapping $P : Y \to Y$ is said to be χ-condensing map of Y if P maps bounded sets into bounded sets and*

$$\chi(P(B)) < \chi(B),$$

for every bounded subset B in Y such that $\chi(B) > 0$.

Definition 4.4. *The solutions u of equation* (1.2) *are said to be bounded if for each $B_1 > 0$, there exists a constant $\overline{B_1} > 0$, such that $|\phi|_{\mathscr{B}_\alpha} \le B_1$ implies that $|u(t,\phi)|_\alpha \le \overline{B_1}$, for $t \ge 0$.*

Definition 4.5. *The solutions u of equation* (1.2) *are said to be ultimate bounded if there is a bound $B > 0$ such that for each $B_2 > 0$, there exists a constant $k > 0$ such that $|\phi|_{\mathscr{B}_\alpha} \le B_2$ and $t \ge k$ imply that $|u(t,\phi)|_\alpha \le B$.*

The relationship between the local boundedness, the boundedness and ultimate boundedness is given below.

Theorem 4.6. *The local boundedness and ultimate boundedness of solutions of equation* (1.2) *imply the boundedness of solutions.*

Proof. Let $B > 0$ be given by the ultimate boundedness, then for any $B_1 > 0$, there exists a constant $k > 0$ such that $|\phi|_{\mathscr{B}_\alpha} \le B_1$ and $t \ge 0$ imply that $|u(t,\phi)|_\alpha \le B$. The local boundedness of solutions gives that there exists a constant $B_2 > B$ such that $|\phi|_{\mathscr{B}_\alpha} \le B_1$ implies that $|u(t,\phi)|_\alpha < B_2$, for $t \in [0,k]$. It follows that for any positive constant B_1, there exists a constant $B_2 > B$ such that $|\phi|_{\mathscr{B}_\alpha} \le B_1$ implies that $|u(t,\phi)|_\alpha < B_2$ for all $t \ge 0$. □

Proposition 4.7. *Under the assumptions ($\boldsymbol{H}_1$), ($\boldsymbol{H}_2$), ($\boldsymbol{H}_3$), ($\boldsymbol{H}_4$) and ($\boldsymbol{H}_5$), the Poincaré's map*

$$P_\sigma \phi = u_\sigma(.,\phi),$$

is χ-condensing on $\mathscr{B}_\alpha$.

For the proof of Proposition 4.7, we need the following lemmas.

Lemma 4.8. *For each bounded set D in $\mathscr{B}_\alpha$ such that $\chi(D) > 0$, we have*

$$\chi(W_t(D)) \le K(t-\xi)\chi(W_{[\xi,t]}(D)) + M(t-\xi)\chi(W_\xi(D)),\ \xi < t \text{ and } t-\xi \ge \omega_0. \tag{4.4}$$

Proof. Since the space $\mathscr{B}_\alpha$ satisfies the Axiom (**A**), then Lemma 4.8 follows as in [13, 15].

Lemma 4.9. *For each bounded set D in $\mathscr{B}_\alpha$ such that $\chi(D) > 0$, we have*

$$\chi(W_{[0,\omega_0]}(D)) \le M_0 \frac{1}{N}\chi(D).$$

Proof. According to [15], since $\mathscr{B}_\alpha$ satisfies Axiom (**A**), then Lemma 4.9 follows.

Proof of Proposition 4.7. By Proposition 2.7, P_σ is a continuous function from the Banach space $\mathscr{B}_\alpha$ into itself.

Then, using Lemma **4.8** and relation (4.3) one can write

$$\begin{aligned}
\chi(P_\sigma(D)) &= \chi(W_\sigma(D)) \\
&\le K(\sigma-(\sigma-\omega_0))\chi(W_{[\sigma-\omega_0,\sigma]}(D)) + M(\sigma-(\sigma-\omega_0))\chi(W_{\sigma-\omega_0}(D)) \\
&= K(\omega_0)\chi(W_{[\sigma-\omega_0,\sigma]}(D)) + M(\omega_0)\chi(W_{\sigma-\omega_0}(D)) \\
&= M(\omega_0)\chi(W_{\sigma-\omega_0}(D)) \\
&\le \eta\chi(W_{\sigma-\omega_0}(D)) \\
&\le \eta[K(\omega_0)\chi(W_{[\sigma-2\omega_0,\sigma-\omega_0]}(D)) + M(\omega_0)\chi(W_{\sigma-2\omega_0}(D))] \\
&= \eta M(\omega_0)\chi(W_{\sigma-2\omega_0}(D)) \\
&\le \eta^2\chi(W_{\sigma-2\omega_0}(D)) \\
&. \\
&. \\
&. \\
&\le \eta^{N_0-1}\chi(W_{\omega_0}(D)) \\
&\le \eta^{N_0-1}[K(\omega_0)\chi(W_{[0,\omega_0]}(D)) + M(\omega_0)\chi(D)].
\end{aligned}$$

From (4.1) and according to Lemma 4.9, we obtain that

$$\begin{aligned}
\chi(P_\sigma(D)) &\le \eta^{N_0-1}[K(\omega_0)\chi(W_{[0,\omega_0]}(D)) + M(\omega_0)(D)] \\
&\le \eta^{N_0-1}[K_1M_0\frac{1}{N}\chi(D) + \chi(D)] \\
&\le \eta^{N_0-1}[K_1M_0\frac{1}{N} + 1]\chi(D) \\
&< \chi(D).
\end{aligned}$$

This implies that P_σ is χ-condensing on $\mathscr{B}_\alpha$. □

The following result is usefull for the boundedness of solutions of equation (1.2).

Theorem 4.10. *Assume that there exists a constant $C > 0$ such that for any bounded set D in $\mathscr{B}_\alpha$ we have*

$$\overline{\lim_{t\to+\infty}} \sup_{\phi\in D} |u(t,\phi)|_\alpha < C.$$

Then, the solutions of equation (1.2) *are ultimate bounded.*

Proof. Let us choose $D_0 = C + 1 > 0$. Then, for each $D_1 > 0$, we define the bounded subset D_{D_1} of $\mathscr{B}_\alpha$ by

$$D_{D_1} = \{\phi \in \mathscr{B}_\alpha : |\phi| \le D_1\}.$$

Since $\overline{\lim_{t\to+\infty}} \sup_{\phi\in D_{D_1}} |u(t,\phi)|_\alpha < C$, then for all $\phi \in D_{D_1}$ there exists $k > 0$ such that $t \ge k$ imply $|u(t,\phi)|_\alpha < C + 1 = D_0$ for all $\phi \in D$. Thus, the solutions of (1.2) are ultimate bounded. □

Theorem 4.11. *Assume that* ($\boldsymbol{H}_1$)*,* ($\boldsymbol{H}_2$)*,* ($\boldsymbol{H}_3$)*,* ($\boldsymbol{H}_4$) *and* ($\boldsymbol{H}_5$) *hold. Moreover suppose that the function f is continuous, σ-periodic with respect to the first argument and that the solutions of equation* (1.2) *are ultimate bounded. Then, equation* (1.1) *has a σ-periodic solution on $\mathbb{R}$.*

For the proof, we use the following Hale-Lunel's fixed point theorem which is an extension of the well known Horn's fixed point theorem for condensing maps [12].

Theorem 4.12. *Suppose that $S_0 \subseteq S_1 \subseteq S_2$ are convex bounded subsets of a Banach space Y, such that S_0, S_2 are closed and S_1 is open in S_2. Let P be a condensing map on Y such that $P^j(S_1) \subseteq S_2$ for $j \geq 0$ and there exists a number $N(S_1)$ such that $P^k(S_1) \subseteq S_0$, for $k \geq N(S_1)$. Then P has a fixed point.*

Proof of Theorem 4.11. Using the ultimate boundedness and locally boundedness, we obtain that the solutions of equation (1.2) are bounded and ultimate bounded. Let B be the bound in the definition of ultimate boundedness. By the boundedness of solutions, there exists a constant $B_1 > (K_1+1)B > 0$ such that for $|\phi|_{\mathscr{B}_\alpha} \leq (K_1+1)B$ and $t \geq 0$, one has $|u(t,\phi)|_\alpha \leq B_1$ where K_1 comes from the definition of the uniform fading memory space. Moreover, there exists a constant $B_2 > B_1$ such that for $|\phi|_{\mathscr{B}_\alpha} \leq B_1$ and $t \geq 0$, one has $|u(t,\phi)|_\alpha \leq B_2$. As the space is a uniform fading memory space, then there exists M_1 such that $M(.) \leq M_1$. Now let $\tilde{B}_2 > \max\{B_1, K_1B_2 + M_1B_1\}$ and let

$$S_2 = \left\{\phi \in \mathscr{B}_\alpha : \ |\phi|_{\mathscr{B}_\alpha} \leq \tilde{B}_2\right\},$$

$$S_1 = \{\phi \in \mathscr{B}_\alpha : \ |\phi|_{\mathscr{B}_\alpha} < B_1\},$$

$$S_0 = \{\phi \in \mathscr{B}_\alpha : \ |\phi|_{\mathscr{B}_\alpha} \leq (K_1+1)B\}.$$

Therefore, $S_0 \subseteq S_1 \subseteq S_2$ are convex bounded subsets of the Banach space $\mathscr{B}_\alpha$, S_0 and S_2 are closed and $S_1 = S_1 \bigcap S_2$ is an open of S_2.
It is known that the existence of fixed point of Poincaré's map gives rise to the existence of σ-periodic solution. In fact, let v be such that $v_\sigma(.,\psi) = \psi$. Define $y(t) = v(t+\sigma)$. Then, for $t \geq 0$, we obtain

$$\begin{aligned}
y(t) &= T(t+\sigma)\psi(0) + \int_0^{t+\sigma} T(t+\sigma-s)f(s,v_s)ds \\
&= T(t)\Big[T(\sigma)\psi(0) + \int_0^{\sigma} T(\sigma-s)f(s,v_s)ds\Big] + \int_\sigma^{t+\sigma} T(t+\sigma-s)f(s,v_s)ds \\
&= T(t)\Big[T(\sigma)\psi(0) + \int_0^{\sigma} T(\sigma-s)f(s,v_s)ds\Big] + \int_0^{t} T(t-s)f(s+\sigma,v_{s+\sigma})ds \\
&= T(t)v(\sigma) + \int_0^t T(t-s)f(s,y_s)ds \\
&= T(t)y(0) + \int_0^t T(t-s)f(s,y_s)ds.
\end{aligned}$$

This implies that $y(t) = v(t+\sigma)$ is also a mild solution of equation (1.2) with respect to ψ. Then the uniqueness of mild solution associated to ψ implies that $y = v$. Hence, v is σ-periodic. Note also that $P_\sigma^j\psi = v_{j\sigma}(.,\psi)$ for all $j \geq 0$.

Next, for $\phi \in S_1$ and $j \geq 1$,

$$\begin{aligned}
|P_\sigma^j \phi|_{\mathcal{B}_\alpha} &= |u_{j\sigma}(.,\phi)|_{\mathcal{B}_\alpha} \\
&\leq K(j\sigma) \sup_{0\leq s\leq j\sigma} |u(s)|_\alpha + M(j\sigma)|u_0|_{\mathcal{B}_\alpha} \\
&\leq K(j\sigma) \sup_{0\leq s\leq j\sigma} |u(s)|_\alpha + M(j\sigma)|\phi|_{\mathcal{B}_\alpha} \\
&\leq K_1 B_2 + M_1 B_1 \\
&\leq \tilde{B}_2,
\end{aligned}$$

which implies $P_\sigma^k(S_1) \subseteq S_2$, for all $k \geq 0$.
Next, we prove that there is a number $N(S_1)$ such that $P_\sigma^k(S_1) \subseteq S_0$ for $k \geq N(S_1)$. To this end, we note that using ultimate boundedness of solutions of equation (1.2), there is a positive number $m = m(B_1)$ such that for $|\phi|_{\mathcal{B}_\alpha} \leq B_1$ and $t \geq m\sigma$, we have $|u(t,\phi)|_\alpha \leq B$. Now, let $k \geq m$, one can write

$$\begin{aligned}
|P_\sigma^k \phi|_{\mathcal{B}_\alpha} &= |u_{k\sigma}(.,\phi)|_{\mathcal{B}_\alpha} \\
&\leq K(k\sigma - m\sigma) \sup_{m\sigma\leq s\leq k\sigma} |u(s)|_\alpha + M(k\sigma - m\sigma)|u_{m\sigma}|_{\mathcal{B}_\alpha} \\
&\leq K_1 B + M(k\sigma - m\sigma)\Big[K(m\sigma) \sup_{0\leq s\leq m\sigma} |u(s)|_\alpha + M(m\sigma)|\phi|_{\mathcal{B}_\alpha}\Big] \\
&\leq K_1 B + M(k\sigma - m\sigma)[K_1 B_2 + M_1 B_1] \\
&\leq K_1 B + M(k\sigma - m\sigma)\tilde{B}_2.
\end{aligned}$$

Thus, as $M(t) \to 0$ when $t \to 0$ for the uniform fading memory space, we can find an integer $N(S_1)$ such that $k\sigma - m\sigma > 0$ and $M(k\sigma - m\sigma)\tilde{B}_2 < B$ for $k \geq N(S_1)$. Therefore, for $k \geq N(S_1)$, we have

$$\begin{aligned}
|P_\sigma^j \phi|_{\mathcal{B}_\alpha} &\leq K_1 B + M(k\sigma - m\sigma)B_2 \\
&\leq K_1 B + B \\
&= (K_1 + 1)B,
\end{aligned}$$

which implies that $P_\sigma^k(S_1) \subseteq S_0$ for all $k \geq N(S_1)$.
Using Proposition **4.7**, we deduce that P_σ is a χ-condensing map on $\mathcal{B}_\alpha$. Consequently, the Hale-Lunel's fixed point theorem implies that the Poincaré's map P_σ has at least one fixed point on $\mathcal{B}_\alpha$ which leads to the existence of a σ-periodic solution of equation (1.2) on $\mathbb{R}^+$. Thus, equation (1.1) has a σ-periodic solution on $\mathbb{R}$. □

5 Nonlinear partial functional differential equation and periodic solutions using multivalued fixed point theory

We study in this part, the existence of periodic solutions of the following partial functional differential equation

$$\frac{d}{dt}u(t) = -Au(t) + L(t,u_t) + G(t,u_t) \text{ for } t \in \mathbb{R}, \tag{5.1}$$

where L is continuous, linear with respect to its second variable, σ-periodic in its first variable and $G : \mathbb{R}\times\mathscr{B}_\alpha \to X$ is continuous.

Definition 5.1. *We say that $u : \mathbb{R} \to X_\alpha$ is a mild solution of equation* (5.1) *if for any $t,\tau \in \mathbb{R}$ such that $t \ge \tau$, one has*

$$u(t) = T(t-\tau)u(\tau) + \int_\tau^t T(t-s)[L(s,u_s) + G(s,u_s)]ds.$$

We assume in addition that

($\mathbf{H}_6$) G is σ-periodic in its first variable and takes every bounded set into a bounded set.

Let B_σ be the space of all continuous σ-periodic functions with values in X_α endowed with the uniform norm topology which is for $u \in B_\sigma$: $|u|_\sigma = \sup_{s\in[0,\sigma]} |u(s)|_\alpha$.

Definition 5.2. *[3] Let $g : M \to 2^M$ be a multivalued map, where M is a subset of a Banach space Y and 2^M is the power set of M.*
(i) For $D \subset M$, the inverse image $g^{-1}(D)$ is the set of all $x \in M$ such that $g(x)\cap D \neq \emptyset$.
(ii) The map g is called upper semi-continuous if $g^{-1}(D)$ is closed for all closed set D in M.

We have the following result.

Theorem 5.3. *[3] Let $g : M \to 2^M$ be a multivalued map, where M is a nonempty convex set in the Banach space Y such that*
(i) the set $g(x)$ is nonempty, closed and convex for all $x \in M$,
(ii) the set $g(M)$ is relatively compact,
(iii) the map $g : M \to 2^M$ is upper semi-continuous.
Then, g has a fixed point in the sense that there exists $x \in M$ such that $x \in g(x)$.

We can now announce the main result of this section.

Theorem 5.4. *Suppose the hypothesis ($\boldsymbol{H}_1$), ($\boldsymbol{H}_2$), ($\boldsymbol{H}_4$) and ($\boldsymbol{H}_6$) hold. If there exists a positive constant ρ such that for every $y \in S_\rho = \{v \in B_\sigma : |v|_\sigma \le \rho\}$, the equation*

$$\frac{d}{dt}u(t) = -Au(t) + L(t,u_t) + G(t,y_t) \quad for \quad t \in \mathbb{R} \tag{5.2}$$

has a σ-periodic solution in S_ρ. Then the equation (5.1) *has at least one σ-periodic solution.*

The following lemma is needed to prove Theorem 5.4.

Lemma 5.5. *The linear mapping $L : \mathbb{R}\times\mathscr{B}_\alpha \to X$ satisfies $\sup_{s\ge 0}|L(s,.)| < \infty$.*

Proof. Let us define the linear operator $T_s : \mathscr{B}_\alpha \to X$ by

$$T_s(\phi) = L(s,\phi).$$

Since L is continuous and σ-periodic on $\mathbb{R}$, for every $\phi \in \mathscr{B}_\alpha$,
$\sup_{s\geq 0}|T_s(\phi)| = \sup_{s\geq 0}|L(s,\phi)| = \sup_{s\in[0,\sigma]} |L(s,\phi)| < \infty$. Then, using the Banach-Steinhaus's Theorem, we conclude that

$$\sup_{s\in[0,\sigma]} |L(s,.)| < \infty. \quad \square$$

Proof of Theorem 5.4. Let us define the multivalued map $g : S_\rho \to 2^{S_\rho}$ for $y \in S_\rho$ by

$$g(y) = \left\{ u \in S_\rho : \ u(t) = T(t)u(0) + \int_0^t T(t-s)[L(s,u_s) + G(s,y_s)]ds \text{ for } t \in [0,\sigma] \right\}.$$

Let us show that the mapping g satisfies the conditions (i)-(iii) of Theorem 5.3.
(i) For each $y \in S_\rho$, $g(y)$ is nonempty. Let u_1, u_2 in $g(y)$ and $\lambda \in [0,1]$. We have $\lambda u_1 + (1-\lambda)u_2 \in g(y)$, which implies that $g(y)$ is convex. Moreover for every $y \in S_\rho$, let $(u^n)_{n\geq 0}$ be a sequence in $g(y)$ such that $u^n \to u$ as $n \to +\infty$. Then, we can write

$$u^n(t) = T(t)u^n(0) + \int_0^t T(t-s)[L(s,u^n_s) + G(s,y_s)]ds \text{ for } t \in [0,\sigma].$$

We have $u^n \to u$ in B_σ. Then,

$$\sup_{s\in[0,\sigma]} |u^n(s) - u(s)|_\alpha \to 0 \text{ as } n \to +\infty.$$

Since

$$u^n(t) = T(t)u^n(0) + \int_0^t T(t-s)[L(s,u^n_s) + G(s,y_s)]ds \text{ for } t \in [0,\sigma],$$

then

$$L(s,u^n_s) \to L(s,u_s).$$

We know that

$$|L(s,u^n_s)| \leq |L(s,.)|\, |u^n_s|_{\mathscr{B}_\alpha}.$$

Using Lemma 5.5, we deduce that

$$\sup_{s\in[0,\sigma]} |L(s,.)| < \infty. \tag{5.3}$$

It follows that for all $t \in [0,\sigma]$,

$$\begin{aligned} \int_0^t |A^\alpha T(t-s)[L(s,u^n_s) + G(s,y_s)]|ds &\leq M_\alpha \int_0^t e^{\omega(t-s)}(t-s)^{-\alpha}[|L(s,u^n_s)| + |G(s,y_s)|]ds \\ &< \infty. \end{aligned}$$

Therefore,

$$u(t) = T(t)u(0) + \int_0^t T(t-s)[L(s,u_s) + G(s,y_s)]ds, \text{ for all } t \in [0,\sigma].$$

Consequently, $u \in g(y)$ and $g(y)$ is closed.

(ii) Note that $g(S_\rho) = \bigcup_{y\in S_\rho} g(y)$.

Let $u \in g(S_\rho)$. Then there exists $y \in S_\rho$ such that

$$u(t) = T(t)u(0) + \int_0^t T(t-s)[L(s,u_s) + G(s,y_s)]ds, \text{ for all } t \geq 0.$$

Let us prove the relative compactness in X_α of $\left\{u(t) : u \in g(S_\rho)\right\}$. We take $t \in [0,\sigma]$ and $\epsilon > 0$ such that $t - \epsilon > 0$. Then,

$$\begin{aligned} u(t) &= T(\epsilon)[T(t-\epsilon)u(0) + \int_0^{t-\epsilon} T(t-\epsilon-s)[L(s,u_s)+G(s,y_s)]ds] \\ &+ \int_{t-\epsilon}^t T(t-s)[L(s,u_s)+G(s,y_s)]ds \\ &= T(\epsilon)u(t-\epsilon) + \int_{t-\epsilon}^t T(t-s)[L(s,u_s)+G(s,y_s)]ds. \end{aligned}$$

The boundedness of $u \in S_\rho$ and the compactness of the semigroup $(T(t))_{t\geq 0}$ for every $t > 0$ give the relative compactness in X of the family $\{T(\epsilon)A^\alpha u(t-\epsilon) : t > \epsilon\}$ since $\{A^\alpha u(t-\epsilon) : t > \epsilon\}$ is bounded in X.

Moreover, let $0 < \alpha < \beta < 1$. Using (5.3) and ($\mathbf{H}_6$), we get that for all $y \in S_\rho$,

$$\sup_{s\in\mathbb{R}} |L(s,u_s) + G(s,y_s)| = N_3 < \infty. \tag{5.4}$$

Next, we have

$$\begin{aligned} |A^\beta \int_{t-\epsilon}^t T(t-s)[L(s,u_s)+G(s,y_s)]ds| &\leq \int_{t-\epsilon}^t |A^\beta T(t-s)||L(s,u_s)+G(s,y_s)|ds \\ &\leq M_\beta N_3 \int_0^\epsilon e^{\omega s} s^{-\beta} ds. \end{aligned}$$

It follows that $\left\{A^\beta \int_{t-\epsilon}^t T(t-s)[L(s,u_s)+G(s,y_s)]ds,\ \ u \in g(S_\rho)\right\}$ for every $t \in [\epsilon,\sigma]$, is totally bounded in X. The pre-compactness of

$$\left\{\int_{t-\epsilon}^t T(t-s)[L(s,u_s)+G(s,y_s)]ds,\ \ u \in g(S_\rho)\right\}$$

for every $t \in [\epsilon,\sigma]$ in X_α follows from the compactness of the map

$$A^{-\beta} : X \to X_\alpha.$$

Therefore,

$$\left\{u(t) : u \in g(S_\rho)\right\}$$

for all $t \in [0,\sigma]$ is relatively compact in X_α.

Now, let us show the equicontinuity of $\left\{u : \; u \in g(S_\rho)\right\}$.
Let $0 \leq t_2 < t_1 \leq \sigma$ and $u \in g(S_\rho)$. Then,

$$\begin{aligned}
u(t_1) - u(t_2) &= T(t_1)u(0) + \int_0^{t_1} T(t_1 - s)[L(s,u_s) + G(s,y_s)]ds - T(t_2)u(0) \\
&- \int_0^{t_2} T(t_2 - s)[L(s,u_s) + G(s,y_s)]ds \\
&= T(t_1 - t_2)[T(t_2)u(0) + \int_0^{t_2} T(t_2 - s)[L(s,u_s) + G(s,y_s)]ds] \\
&+ \int_{t_2}^{t_1} T(t_1 - s)[L(s,u_s) + G(s,y_s)]ds \\
&- T(t_2)u(0) - \int_0^{t_2} T(t_2 - s)[L(s,u_s) + G(s,y_s)]ds \\
&= (T(t_1 - t_2) - I)u(t_2) + \int_{t_2}^{t_1} T(t_1 - s)[L(s,u_s) + G(s,y_s)]ds.
\end{aligned}$$

Then

$$|u(t_1) - u(t_2)|_\alpha \leq |(T(t_1 - t_2) - I)u(t_2)|_\alpha + \int_{t_2}^{t_1} |A^\alpha T(t_1 - s)||L(s,u_s) + G(s,y_s)|ds.$$

It is well known that $\lim_{h \to 0}(T(h) - I)\zeta = 0$, uniformly in $\zeta \in K$ for any compact set K.
It follows that

$$\lim_{\substack{t_2 \to t_1 \\ t_2 < t_1}} |(T(t_1 - t_2) - I)u(t_2)|_\alpha = 0.$$

Moreover by (5.4), we have

$$\begin{aligned}
|A^\alpha \int_{t_2}^{t_1} T(t_1 - s)[L(s,u_s) + G(s,y_s)]ds| &\leq \int_{t_2}^{t_1} |A^\alpha T(t_1 - s)||L(s,u_s) + G(s,y_s)|ds \\
&\leq N_3 M_\alpha \int_0^{t_1 - t_2} e^{\omega s} s^{-\alpha} ds.
\end{aligned}$$

Finally,

$$\lim_{\substack{t_2 \to t_1 \\ t_2 < t_1}} |(u(t_1) - u(t_2))|_\alpha = 0 \text{ for all } u \in g(S_\rho).$$

Arguing as above, one can also show that

$$\lim_{\substack{t_2 \to t_1 \\ t_2 > t_1}} |(u(t_1) - u(t_2))|_\alpha = 0 \text{ for all } u \in g(S_\rho).$$

Thus, $\left\{u : \; u \in g(S_\rho)\right\}$ is an equicontinuous family. Note also that, $\left\{u(t) : \; u \in g(S_\rho)\right\}$ for each $t \in [0,\sigma]$ is relatively compact in X_α.
Therefore, $g(S_\rho)$ is a family of uniformly bounded and equicontinuous σ-periodic functions.
Using Ascoli-Arzela's theorem, we deduce that $g(S_\rho)$ is relatively compact in B_σ.

(iii) Let us show that the mapping g is upper semi-continuous. For that, let $(y^n)_{n\geq 0}$ and $(z^n)_{n\geq 0}$ be sequences respectively in S_ρ and $g(S_\rho)$ such that

$$y^n \to y,\, z^n \to z \text{ as } n \to +\infty \text{ and } z^n \in g(y^n), \text{ for all } n \geq 0.$$

Then,

$$z^n(t) = T(t)z^n(0) + \int_0^t T(t-s)[L(s,z_s^n) + G(s,y_s^n)]ds \text{ for } t \geq 0.$$

We have by virtue of $(\mathbf{H}_6)$ that

$$\sup_{s\in[0,\sigma]} |G(s,y_s^n)| < \infty. \tag{5.5}$$

Using the relations (5.3) and (5.5), we deduce that

$$\begin{aligned} |\int_0^t T(t-s)[L(s,z_s^n) + G(s,y_s^n)]|_\alpha ds &\leq \int_0^t |A^\alpha T(t-s)||[L(s,z_s^n) + G(s,y_s^n)]|ds \\ &\leq M_\alpha \int_0^t e^{\omega(t-s)}(t-s)^{-\alpha}[|L(s,z_s^n)| + |G(s,y_s^n)|]ds \\ &< \infty. \end{aligned}$$

It follows that

$$z(t) = T(t)z(0) + \int_0^t T(t-s)[L(s,z_s) + G(s,y_s)]ds \text{ for } t \geq 0.$$

Thus, $z \in g(y)$ and g is closed. From the relative compactness of $g(S_\rho)$, we deduce that g is upper semi-continuous. Using Theorem 5.3, we obtain the existence of $u \in S_\rho$ such that $u \in g(S_\rho)$. Therefore, u is a σ-periodic solution of the equation (5.1) on $\mathbb{R}^+$. □

6 Application

To apply the theoritical results of this work, we consider the following reaction-diffusion system with infinite delay

$$\begin{cases} \dfrac{\partial}{\partial t}v(t,x) = \dfrac{\partial^2}{\partial x^2}v(t,x) + a(t)\displaystyle\int_{-\infty}^0 g(\theta)h(\frac{\partial}{\partial x}v(t+\theta))d\theta \quad \text{for } t \geq 0 \text{ and } x \in [0,\pi] \\ v(t,0) = v(t,\pi) = 0 \text{ for } t \geq 0 \\ v(\theta,x) = \psi(\theta,x) \text{ for } \theta \in (-\infty,0] \text{ and } x \in [0,\pi], \end{cases} \tag{6.1}$$

where $h:\mathbb{R}\to\mathbb{R}$ is Lipschitz continuous, $a:\mathbb{R}\to\mathbb{R}$ is continuous, σ-periodic and $g:\mathbb{R}^-\to\mathbb{R}$ is continuous. The initial data ψ will be precised in the next.

In order to write the system (6.1) in an abstract form, we introduce the space $X = L^2([0,\pi];\mathbb{R})$. Let A be the operator defined on X by

$$\begin{cases} D(A) = H^2((0,\pi);\mathbb{R}) \cap H_0^1((0,\pi);\mathbb{R}), \\ Ay = -y'' \text{ for } y \in D(A). \end{cases}$$

Then, $-A$ generates an analytic semigroup $(T(t))_{t\geq 0}$ on X. Moreover, $T(t)$ is compact on X for every $t > 0$. The spectrum $\sigma(-A)$ is equal to the point spectrum $P\sigma(-A)$ and is given by $\sigma(-A) = \left\{-n^2 : n \geq 1\right\}$ and the associated eigenfunctions $(\phi_n)_{n\geq 1}$ are given by $\phi_n = \sqrt{\frac{2}{\pi}}\sin(nx)$ for $x \in [0,\pi]$; the associated analytic semigroup is explicitly given by

$$T(t)y = \sum_{n=1}^{\infty} e^{-n^2 t}(y,\phi_n)\phi_n \text{ for } t \geq 0 \text{ and } y \in X,$$

where $(.,.)$ is an inner product on X.
We have the following Lemma (see [21]).

Lemma 6.1. *[21] If $\alpha = \frac{1}{2}$, then*

$$Ay = \sum_{n=1}^{+\infty} n^2(y,\phi_n)\phi_n \textit{ for } y \in D(A),$$

$$A^{\frac{1}{2}}y = \sum_{n=1}^{+\infty} n(y,\phi_n)\phi_n \textit{ for } y \in X$$

$$A^{\frac{1}{2}}T(t)y = \sum_{n=1}^{+\infty} ne^{-n^2 t}(y,\phi_n)\phi_n \textit{ for } y \in X,$$

$$A^{-\frac{1}{2}}y = \sum_{n=1}^{+\infty} \left(\frac{1}{n}\right)(y,\phi_n)\phi_n \textit{ for } y \in X$$

and

$$A^{-\frac{1}{2}}T(t)y = \sum_{n=1}^{+\infty} \left(\frac{1}{n}\right)e^{-n^2 t}(y,\phi_n)\phi_n \textit{ for } y \in X.$$

There exists $M \geq 1$ (see [21]) such that for $t \geq 0$, $|T(t)| \leq Me^{\omega t}$ for some $-1 < \omega < 0$.

Note also that (see [21]) there exists $M_{\frac{1}{2}} \geq 0$ such that

$$|A^{\frac{1}{2}}T(t)| \leq M_{\frac{1}{2}} t^{-\frac{1}{2}} e^{\omega t} \text{ for each } t > 0.$$

Therefore, hypothesis $(\mathbf{H}_1)$ and $(\mathbf{H}_4)$ are satisfied.
According to [6], we have the following.

Lemma 6.2. *[6] If $m \in D(A^{\frac{1}{2}})$, then m is absolutely continuous, $\frac{\partial}{\partial x}m \in X$. Moreover, there exist positive constants K_0 and K_1 such that*

$$K_0|A^{\frac{1}{2}}m|_X \leq |\frac{\partial}{\partial x}m|_X \leq K_1|A^{\frac{1}{2}}m|_X.$$

Let $\gamma > 0$. We consider the following phase space

$$\mathscr{B} = C_\gamma = \left\{\phi \in C((-\infty,0];X) : \lim_{\theta\to-\infty} e^{\gamma\theta}|\phi(\theta)| \text{ exists in } X\right\}$$

provided with the following norm

$$|\phi|_{C_\gamma} = \sup_{\theta\leq 0} e^{\gamma\theta}|\phi(\theta)| \text{ for } \phi \in C_\gamma.$$

According to [6], $\mathscr{B}$ satisfies axioms (**A**), (**B**) and is a uniform fading memory space. Moreover, it is well known that $K(t) = 1$ for every $t \in \mathbb{R}^+$ and $M(t) = e^{-\gamma t}$ for $t \in \mathbb{R}^+$. ($\mathbf{H}_5$) is then satisfied. Therefore, the norm in $\mathscr{B}_{\frac{1}{2}}$ is given (see [6]) by

$$|\phi|_{\mathscr{B}_{\frac{1}{2}}} = \sup_{\theta\leq 0} e^{\gamma\theta}|A^{\frac{1}{2}}\phi(\theta)|.$$

Next, we assume the following.

($\mathbf{H}_7$) $\quad e^{-2\gamma\cdot}g \in L^2(\mathbb{R}^-)$.

Let f be defined on $\mathbb{R}^+ \times \mathscr{B}_{\frac{1}{2}}$ by

$$f(t,\phi)(x) = a(t)\int_{-\infty}^{0} g(\theta)h(\frac{\partial}{\partial x}\phi(\theta)(x))d\theta \text{ for } x \in [0,\pi] \text{ and } t \geq 0.$$

Proposition 6.3. *For each $\phi \in \mathscr{B}_{\frac{1}{2}}$ and $t \in [0,+\infty)$, $f(t,\phi) \in L^2([0,\pi];\mathbb{R})$ and f is continuous on $\mathbb{R}^+ \times \mathscr{B}_{\frac{1}{2}}$.*

Proof. Let $(t,\phi) \in \mathbb{R}^+ \times \mathscr{B}_{\frac{1}{2}}$ and taking $|a|_\infty = \sup_{s\geq 0}|a(s)|$. Since $h : \mathbb{R} \to \mathbb{R}$ is lipschitzian, there exists a positive constant K such that

$$|h(x)| \leq K|x| + |h(0)| \text{ for all } x \in \mathbb{R}.$$

Then, for all $x \in [0,\pi]$

$$\begin{aligned}
|f(t,\phi)(x)| &\leq |a(t)|K\int_{-\infty}^{0}|g(\theta)||\frac{\partial}{\partial x}\phi(\theta)(x)|d\theta + |a(t)|\int_{-\infty}^{0}|g(\theta)h(0)|d\theta \\
&\leq |a|_\infty K\int_{-\infty}^{0}|g(\theta)||\frac{\partial}{\partial x}\phi(\theta)(x)|d\theta + |a|_\infty|h(0)|\int_{-\infty}^{0}|g(\theta)|d\theta.
\end{aligned}$$

Using Hölder's inequality, we get

$$\begin{aligned}
\int_{-\infty}^{0}|g(\theta)|d\theta &= \int_{-\infty}^{0} e^{-2\gamma\theta}|g(\theta)|e^{2\gamma\theta}d\theta \\
&\leq \left(\int_{-\infty}^{0}|g(\theta)e^{-2\gamma\theta}|^2 d\theta\right)^{\frac{1}{2}}\left(\int_{-\infty}^{0} e^{4\gamma\theta}d\theta\right)^{\frac{1}{2}} \\
&< \infty.
\end{aligned}$$

We put

$$|a|_\infty \int_{-\infty}^{0} |g(\theta)||h(0)|d\theta = C_1$$

to obtain

$$\int_0^\pi \Big||a|_\infty \int_{-\infty}^{0} |g(\theta)||h(0)|d\theta\Big|^2 dx = (C_1)^2\pi.$$

Also, let us set

$$B(x) = \int_{-\infty}^{0} |g(\theta)||\frac{\partial}{\partial x}\phi(\theta)(x)|d\theta \text{ for } x \in [0,\pi].$$

Using again Hölder's inequality, one can write

$$\begin{aligned} B(x) &= \int_{-\infty}^{0} e^{-2\gamma\theta}|g(\theta)||\frac{\partial}{\partial x}\phi(\theta)(x)|e^{2\gamma\theta}d\theta \\ &\leq \left(\int_{-\infty}^{0} |e^{-2\gamma\theta}g(\theta)|^2 d\theta\right)^{\frac{1}{2}} \left(\int_{-\infty}^{0} |\frac{\partial}{\partial x}\phi(\theta)(x)e^{2\gamma\theta}|^2 d\theta\right)^{\frac{1}{2}}. \end{aligned}$$

Then

$$\begin{aligned} \int_0^\pi |B(x)|^2 dx &\leq \int_0^\pi \left(\Big(\int_{-\infty}^{0} |e^{-2\gamma\theta}g(\theta)|^2 d\theta\Big)\Big(\int_{-\infty}^{0} |\frac{\partial}{\partial x}\phi(\theta)(x)e^{2\gamma\theta}|^2 d\theta\Big)\right)dx \\ &= \int_0^\pi \left(|e^{-2\gamma\cdot}g|^2_{L^2(\mathbb{R}^-)} \int_{-\infty}^{0} |\frac{\partial}{\partial x}\phi(\theta)(x)e^{2\gamma\theta}|^2 d\theta\right)dx \\ &\leq |e^{-2\gamma\cdot}g|^2_{L^2(\mathbb{R}^-)} \left(\int_{-\infty}^{0} e^{2\gamma\theta}\left(e^{2\gamma\theta}\int_0^\pi |\frac{\partial}{\partial x}\phi(\theta)(x)|^2 dx\right)d\theta\right) \\ &= |e^{-2\gamma\cdot}g|^2_{L^2(\mathbb{R}^-)} \left(\int_{-\infty}^{0} e^{2\gamma\theta}\left(e^{2\gamma\theta}|\frac{\partial}{\partial x}\phi(\theta)|^2_{L^2([0,\pi],\mathbb{R})}\right)d\theta\right) \\ &\leq |e^{-2\gamma\cdot}g|^2_{L^2(\mathbb{R}^-)} \left(\int_{-\infty}^{0} e^{2\gamma\theta}\left(\sup_{\theta\leq 0} e^{2\gamma\theta}|\frac{\partial}{\partial x}\phi(\theta)|^2_{L^2([0,\pi],\mathbb{R})}\right)d\theta\right) \\ &\leq |e^{-2\gamma\cdot}g|^2_{L^2(\mathbb{R}^-)} \left(\int_{-\infty}^{0} e^{2\gamma\theta}\Big(\sup_{\theta\leq 0} e^{2\gamma\theta}K_1^2|A^{\frac{1}{2}}\phi(\theta)|^2_{L^2([0,\pi],\mathbb{R})}\Big)d\theta\right) \\ &\leq K_1^2|e^{-2\gamma\cdot}g|^2_{L^2(\mathbb{R}^-)} \left(\int_{-\infty}^{0} e^{2\gamma\theta}|A^{\frac{1}{2}}\phi|^2_{C_\gamma} d\theta\right) \\ &< \infty. \end{aligned}$$

We conclude that $f(t,\phi) \in L^2([0,\pi],\mathbb{R})$ for all $(t,\phi) \in \mathbb{R}^+ \times \mathscr{B}_{\frac{1}{2}}$.

Let us show that f is continuous. For this purpose, let $(t_n,\phi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^+ \times \mathscr{B}_{\frac{1}{2}}$ and $(t,\phi) \in \mathbb{R}^+ \times \mathscr{B}_{\frac{1}{2}}$ such that $(t_n,\phi_n) \to (t,\phi)$ in $\mathbb{R}^+ \times \mathscr{B}_{\frac{1}{2}}$ as $n \to +\infty$. Then

$$\begin{aligned}
\Big(f(t_n,\phi_n)-f(t,\phi)\Big)(x) &= a(t_n)\int_{-\infty}^{0} g(\theta)h(\frac{\partial}{\partial x}\phi_n(\theta)(x))d\theta - a(t)\int_{-\infty}^{0} g(\theta)h(\frac{\partial}{\partial x}\phi(\theta)(x))d\theta \\
&= a(t_n)\int_{-\infty}^{0} g(\theta)\Big[h(\frac{\partial}{\partial x}\phi_n(\theta)(x)) - h(\frac{\partial}{\partial x}\phi(\theta)(x))\Big]d\theta \\
&+ (a(t_n)-a(t))\int_{-\infty}^{0} g(\theta)h(\frac{\partial}{\partial x}\phi(\theta)(x))d\theta
\end{aligned}$$

and we obtain that

$$\begin{aligned}
|(f(t_n,\phi_n)-f(t,\phi))(x)| &\le |a(t_n)|\int_{-\infty}^{0} |g(\theta)|\,|h(\frac{\partial}{\partial x}\phi_n(\theta)(x)) - h(\frac{\partial}{\partial x}\phi(\theta)(x))|d\theta \\
&+ |a(t_n)-a(t)|\int_{-\infty}^{0} |g(\theta)h(\frac{\partial}{\partial x}\phi(\theta)(x))|d\theta.
\end{aligned}$$

Let us set for all $x\in[0,\pi]$

$$J_n(x) = |a(t_n)|\int_{-\infty}^{0} |g(\theta)|\Big|h(\frac{\partial}{\partial x}\phi_n(\theta)(x)) - h(\frac{\partial}{\partial x}\phi(\theta)(x))\Big|d\theta$$

and

$$I_n(x) = |a(t_n)-a(t)|\int_{-\infty}^{0} |g(\theta)h(\frac{\partial}{\partial x}\phi(\theta)(x))|d\theta.$$

Then

$$\begin{aligned}
|J_n(x)| &\le |a|_\infty K\int_{-\infty}^{0} e^{-2\gamma\theta}|g(\theta)||\frac{\partial}{\partial x}\phi_n(\theta)(x) - \frac{\partial}{\partial x}\phi(\theta)(x)|e^{2\gamma\theta}d\theta \\
&\le |a|_\infty K\left(\int_{-\infty}^{0}\Big|e^{-2\gamma\theta}g(\theta)\Big|^2 d\theta\right)^{\frac{1}{2}}\left(\int_{-\infty}^{0}\Big|\Big(\frac{\partial}{\partial x}\phi_n(\theta)(x) - \frac{\partial}{\partial x}\phi(\theta)(x)\Big)e^{2\gamma\theta}\Big|^2 d\theta\right)^{\frac{1}{2}},
\end{aligned}$$

which leads to

$$\begin{aligned}
\int_0^\pi |J_n(x)|^2dx &\le |a|_\infty^2K^2|e^{-2\gamma\cdot}g|^2_{L^2(\mathbb{R}^-)}\int_{-\infty}^{0}\left(e^{2\gamma\theta}e^{2\gamma\theta}\int_0^\pi\Big|\frac{\partial}{\partial x}\phi_n(\theta)(x) - \frac{\partial}{\partial x}\phi(\theta)(x)\Big|^2dx\right)d\theta \\
&\le |a|_\infty^2K^2|e^{-2\gamma\cdot}g|^2_{L^2(\mathbb{R}^-)}\int_{-\infty}^{0} e^{2\gamma\theta}\left(\sup_{\theta\le 0} e^{2\gamma\theta}\int_0^\pi\Big|\frac{\partial}{\partial x}\phi_n(\theta)(x) - \frac{\partial}{\partial x}\phi(\theta)(x)\Big|^2dx\right)d\theta \\
&\le |a|_\infty^2K^2|e^{-2\gamma\cdot}g|^2_{L^2(\mathbb{R}^-)}\int_{-\infty}^{0} e^{2\gamma\theta}K_1^2(\sup_{\theta\le 0} e^{2\gamma\theta}\Big|A^{\frac{1}{2}}(\phi_n(\theta)-\phi(\theta))\Big|^2_{L^2([0,\pi],\mathbb{R})})d\theta \\
&\le |a|_\infty^2K^2\Big|e^{-2\gamma\cdot}g\Big|^2_{L^2(\mathbb{R}^-)}K_1^2\Big|A^{\frac{1}{2}}(\phi_n-\phi)\Big|^2_{C_\gamma}\int_{-\infty}^{0} e^{2\gamma\theta}d\theta.
\end{aligned}$$

Since $\phi_n\to\phi$ in $\mathscr{B}_{\frac{1}{2}}$, then $\int_0^\pi |J_n(x)|^2dx \to 0$ as $n\to+\infty$.

Moreover, since $a:\mathbb{R}\to\mathbb{R}$ is continuous and $\int_0^{\pi}\Big|\int_{-\infty}^{0}|g(\theta)h(\frac{\partial}{\partial x}\phi(\theta)(x))|d\theta\Big|^2dx<\infty$, then

$$I_n(x)=|a(t_n)-a(t)|\int_{-\infty}^{0}|g(\theta)h(\frac{\partial}{\partial x}\phi(\theta)(x))|d\theta\to 0 \text{ in } L^2([0,\pi],\mathbb{R}) \text{ as } n\to 0.$$

Hence, $f(t_n,\phi_n)\to f(t,\phi)$ in $L^2([0,\pi],\mathbb{R})$ as $n\to+\infty$ and the proof is complete. □

Let

$$\begin{cases} u(t)(x)=v(t,x) & \text{for } t\geq 0 \text{ and } x\in[0,\pi],\\ u_0(\theta)(x)=\psi(\theta,x) & \text{for } \theta\in(-\infty,0] \text{ and } x\in[0,\pi]. \end{cases}$$

We need the following result to prove that $(\mathbf{H}_3)$ is satisfied.

Proposition 6.4. *Assume that $(\boldsymbol{H}_7)$ holds. Then, f is lipschitzian with respect to the second argument and σ-periodic with respect to the first argument.*

Proof. Let ϕ and ψ in $\mathscr{B}_{\frac{1}{2}}$. Then, for $x\in[0,\pi]$, one has

$$(f(t,\phi)-f(t,\psi))(x)=a(t)\int_{-\infty}^{0}g(\theta)\Big[h(\frac{\partial}{\partial x}\phi(\theta)(x))-h(\frac{\partial}{\partial x}\psi(\theta)(x))\Big]d\theta \text{ for } x\in[0,\pi] \text{ and } t\geq 0.$$

As a is σ-periodic and continuous, then f is σ-periodic with respect to the first argument. Using now the Hölder inequality, we get that

$$\begin{aligned} |(f(t,\phi)-f(t,\psi))(x)| &\leq |a|_\infty\int_{-\infty}^{0}|g(\theta)|\Big|h(\frac{\partial}{\partial x}\phi(\theta)(x))-h(\frac{\partial}{\partial x}\psi(\theta)(x))\Big|d\theta\\ &\leq K|a|_\infty\int_{-\infty}^{0}|g(\theta)|\Big|\frac{\partial}{\partial x}\phi(\theta)(x)-\frac{\partial}{\partial x}\psi(\theta)(x)\Big|d\theta\\ &= K|a|_\infty\int_{-\infty}^{0}e^{-2\gamma\theta}|g(\theta)|\,e^{2\gamma\theta}\Big|\frac{\partial}{\partial x}\phi(\theta)(x)-\frac{\partial}{\partial x}\psi(\theta)(x)\Big|d\theta\\ &\leq K|a|_\infty\Big(\int_{-\infty}^{0}|e^{-2\gamma\theta}g(\theta)|^2d\theta\Big)^{\frac{1}{2}}\Big(\int_{-\infty}^{0}e^{4\gamma\theta}\Big|\frac{\partial}{\partial x}\phi(\theta)(x)-\frac{\partial}{\partial x}\psi(\theta)(x)\Big|^2d\theta\Big)^{\frac{1}{2}}. \end{aligned}$$

Therefore,

$$|(f(t,\phi)(x)-f(t,\psi)(x)|^2\leq(K|a|_\infty)^2\Big(\int_{-\infty}^{0}|e^{-2\gamma\theta}g(\theta)|^2d\theta\Big)\Big(\int_{-\infty}^{0}e^{4\gamma\theta}\Big|\frac{\partial}{\partial x}\phi(\theta)(x)-\frac{\partial}{\partial x}\psi(\theta)(x)\Big|^2d\theta\Big)$$

$$
\begin{aligned}
\int_0^{\pi} |f(t,\phi)(x)-f(t,\psi)(x)|^2 dx &\leq (K|a|_\infty)^2\Big(\int_{-\infty}^0 |e^{-2\gamma\theta}g(\theta)|^2 d\theta\Big)\\
&\times \int_{-\infty}^0 e^{4\gamma\theta}\left(\int_0^{\pi}\left|h(\frac{\partial}{\partial x}\phi(\theta)(x))-h(\frac{\partial}{\partial x}\psi(\theta)(x))\right|^2 dx\right)d\theta\\
&\leq (K|a|_\infty)^2\Big(\int_{-\infty}^0 |e^{-2\gamma\theta}g(\theta)|^2 d\theta\Big)\\
&\times \int_{-\infty}^0 e^{2\gamma\theta}\left(\sup_{\theta\leq 0} e^{2\gamma\theta}\int_0^{\pi}\left|(\frac{\partial}{\partial x}\phi(\theta)(x))-\frac{\partial}{\partial x}\psi(\theta)(x)\right|^2 dx\right)d\theta\\
&\leq (K|a|_\infty)^2\Big(\int_{-\infty}^0 |e^{-2\gamma\theta}g(\theta)|^2 d\theta\Big)\\
&\times \int_{-\infty}^0 e^{2\gamma\theta}\left(\sup_{\theta\leq 0} e^{\gamma\theta}\sqrt{\int_0^{\pi}\left|(\frac{\partial}{\partial x}\phi(\theta)(x))-\frac{\partial}{\partial x}\psi(\theta)(x)\right|^2 dx}\right)^2 d\theta\\
&\leq (K|a|_\infty)^2\Big(\int_{-\infty}^0 |e^{-2\gamma\theta}g(\theta)|^2 d\theta\Big)\\
&\times \int_{-\infty}^0 e^{2\gamma\theta}\left(\sup_{\theta\leq 0} e^{\gamma\theta}K_1|A^{\frac{1}{2}}(\phi(\theta)-\psi(\theta))|_{L^2([0,\pi],\mathbb{R})}\right)^2 d\theta\\
&\leq (K_1K|a|_\infty)^2\Big(\int_{-\infty}^0 |e^{-2\gamma\theta}g(\theta)|^2 d\theta\Big)\int_{-\infty}^0 e^{2\gamma\theta}\left|A^{\frac{1}{2}}(\phi-\psi)\right|_{C_\gamma}^2 d\theta\\
&\leq \frac{(K_1K|a|_\infty)^2}{2\gamma}\left|e^{-2\gamma\cdot}g\right|^2_{L^2(\mathbb{R}^-)}\left|\phi-\psi\right|^2_{\mathscr{B}_{\frac{1}{2}}}.
\end{aligned}
$$

Finally, we obtain that

$$|f(t,\phi)-f(t,\psi)|_{L^2([0,\pi],\mathbb{R})} \leq K_2\,|\phi-\psi|_{\mathscr{B}_{\frac{1}{2}}}\,,\ \text{for } \phi,\psi\in\mathscr{B}_{\frac{1}{2}} \text{ and } t\in\mathbb{R},$$

where

$$K_2=\frac{K_1K|a|_\infty}{\sqrt{2\gamma}}\left(\int_{-\infty}^0 |e^{-2\gamma\theta}g(\theta)|^2 d\theta\right)^{\frac{1}{2}}.$$

Therefore, f is lipschitzian with respect to its second variable. □

Let φ be defined by $\varphi(\theta)(x)=\psi(\theta,x)$ for all $\theta\in(-\infty,0]$ and $x\in[0,\pi]$. We make the following additional assumption:

($\mathbf{H}_8$) $\varphi(\theta)\in D(A^{\frac{1}{2}})$ for all $\theta\leq 0$, with

$$\sup_{\theta\leq 0} e^{\gamma\theta}\sqrt{\int_0^{\pi}\left(\frac{\partial}{\partial x}\psi(\theta,x)\right)^2 dx}<\infty$$

and

$$\lim_{\theta\to\theta_0}\int_0^{\pi}\left(\frac{\partial}{\partial x}\psi(\theta,x)-\frac{\partial}{\partial x}\psi(\theta_0,x)\right)^2 dx=0 \text{ for all } \theta_0\leq 0.$$

Remark that $(\mathbf{H}_8)$ implies $\varphi \in \mathscr{B}_{\frac{1}{2}}$. Then, equation (6.1) can be written as follows

$$\begin{cases} \dfrac{d}{dt}u(t) = -Au(t) + f(t, u_t) \text{ for } t \geq 0, \\ \\ u_0 = \varphi. \end{cases} \tag{6.2}$$

Moreover, for each function $\psi \in \mathscr{B}_{\frac{1}{2}}$, one has $A^{-\frac{1}{2}}\psi \in \mathscr{B}_{\frac{1}{2}}$. In fact, let $\psi \in \mathscr{B}_{\frac{1}{2}}$ then, $\psi \in C_\gamma$ and

$$|\psi|_{C_\gamma} = \sup_{\theta \leq 0} e^{\gamma\theta}|\psi(\theta)|$$

Since $A^{-\frac{1}{2}}$ is a bounded operator,

$$\begin{aligned} |A^{-\frac{1}{2}}\psi|_{C_\gamma} &= \sup_{\theta \leq 0} e^{\gamma\theta}|A^{-\frac{1}{2}}\psi(\theta)|_{C_\gamma} \\ &\leq |A^{-\frac{1}{2}}|\sup_{\theta \leq 0} e^{\gamma\theta}|\psi(\theta)|_{C_\gamma}. \end{aligned}$$

Therefore $(\mathbf{H}_2)$ is satisfied.

Now, we have the following result of existence of periodic solutions.

Theorem 6.5. *Assume that* $(\boldsymbol{H}_7)$ *and* $(\boldsymbol{H}_8)$ *hold and there exists a positive constant* N_4 *such that*

$$|f(t, \phi)| \leq N_4 \text{ for } t \in [0, +\infty) \text{ and } \phi \in \mathscr{B}_{\frac{1}{2}}.$$

Then, equation (6.2) *has at least one* σ*-periodic solution on* $\mathbb{R}^+$.

Proof. We know that $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)$, $(\mathbf{H}_4)$ and $(\mathbf{H}_5)$ hold. It remains to show that the solutions of equation (6.2) are ultimate bounded. In fact, we have

$$\begin{aligned} |u(t,\phi)|_{\frac{1}{2}} &\leq |A^{\frac{1}{2}}T(t)\phi(0)| + \int_0^t |A^{\frac{1}{2}}T(t-s)f(s, u_s(\phi))|ds \\ &\leq Me^{\omega t}|A^{\frac{1}{2}}\phi(0)| + \int_0^t M_{\frac{1}{2}}e^{\omega(t-s)}(t-s)^{-\frac{1}{2}}|f(s, u_s(\phi))|ds \\ &\leq Me^{\omega t}|A^{\frac{1}{2}}\phi(0)| + N_4 M_{\frac{1}{2}}\int_0^t e^{\omega(t-s)}(t-s)^{-\frac{1}{2}}ds \\ &= Me^{\omega t}|\phi(0)|_{\frac{1}{2}} + N_4 M_{\frac{1}{2}}\int_0^t e^{\omega s}s^{-\frac{1}{2}}ds \\ &\leq Me^{\omega t}|\phi(0)|_{\frac{1}{2}} + N_4 M_{\frac{1}{2}}\int_0^{+\infty} e^{\omega s}s^{-\frac{1}{2}}ds \\ &= Me^{\omega t}|\phi(0)|_{\frac{1}{2}} + N_4 M_{\frac{1}{2}}\Gamma(\frac{1}{2})\Big(\frac{1}{-\omega}\Big)^{\frac{1}{2}}. \end{aligned}$$

Since $-1 < \omega < 0$, it follows that there exists a positive constant $C > 0$ such that

$$\overline{\lim_{t\to+\infty}}\sup_{\phi \in B}|u(t,\phi)|_{\frac{1}{2}} < C \text{ for any bounded subset } B \subset \mathscr{B}_{\frac{1}{2}},$$

which implies that the solutions of equation (6.2) are ultimate bounded.
Using now Theorem 4.11, we conclude that equation (6.2) has at least one σ-periodic solution. □

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