African Diaspora Journal of Mathematics
Special Volume in Honor of Prof. Gaston M. N'Guérékata
Volume 15, Number 1, pp. 25–42 (2013)

Periodic Solutions of Non-densely Non-autonomous Differential Equations with Delay

Thami Akrid [*], **Lahcen Maniar** [†], **Aziz Ouhinou** [‡]
Laboratory of Mathematics and Population Dynamics
UMMISCO (IRD-UPMC),
Cadi Ayyad University, Faculty of Sciences Semlalia,
Marrakesh 40000, Morocco

Abstract

In this paper we study the Massera problem for the existence of a periodic mild solution of a class of non-densely non-autonomous semilinear differential equations with delay. We assume that the linear part satisfies the conditions introduced by Tanaka. First, we prove that the existence of a periodic solution for non-autonomous inhomogeneous linear differential equations with delay is equivalent to the existence of a bounded solution on the right half real line. Next, we undertake the analysis of the existence of periodic solutions in the semilinear case. To this end, we use a fixed point Theorem concerning set-valued maps. To illustrate the obtained results, we consider a periodic reaction diffusion equation with delay.

AMS Subject Classification: 35B10, 47D06, 34G20.

Keywords : Evolution family, stability conditions, variation of constants formula, Massera problem, Poincaré map, fixed point Theorem.

1 Introduction

This work is concerned with the non-autonomous differential equation with delay

$$\begin{cases} \dfrac{du}{dt} & = A(t)u(t) + L(t)u_t + F(t,u_t), \quad t \geq s \\ u(t) & = \varphi(t-s), \quad s-r \leq t \leq s, \end{cases} \tag{1.1}$$

in a Banach space $(X, \|\cdot\|)$. The family of closed linear operators $A(\cdot)$ is τ-periodic satisfying the *hyperbolic conditions* (**A1**)-(**A3**) in Section **2** and with non necessarily dense domain. We take r to be a non negative real constant. We denote $C_r := C([-r,0],\ X)$ the space of continuous functions from $[-r,\ 0]$ to X endowed with the sup-norm $\|\varphi\| = \max_{\theta\in[-r,\ 0]} \|\varphi(\theta)\|$.

[*]E-mail address: akridthami@gmail.com
[†]E-mail address: maniar@uca.ma
[‡]E-mail address: ouhaziz@yahoo.fr

The history function u_t is defined by $u_t(\theta) := u(t+\theta)$ for $\theta \in [-r,0]$. $L(\cdot)$ is a family of bounded linear operators in $\mathfrak{L}(C_r, X)$ which is τ-periodic and strongly continuous. F is a τ-periodic continuous function from $\mathbb{R} \times C_r$ into X with respect to the first variable. In the autonomous case where $A(t) = A$, Equation (1.1) has been the subject of various quantitative and qualitative studies, among others, we cite [1, 4, 5, 6, 12, 14, 16, 32]. The present paper is devoted to deal with the well-posedness and the existence of periodic mild solutions to the non-autonomous equation (1.1). This work is a continuation of the works done in [2, 6, 29].

In [17] the author initiated a study on the evolution family solution of *hyperbolic* linear evolution equations of the form

$$\begin{cases} \dfrac{du}{dt} &= A(t)u(t), \quad t \geq s \\ u(s) &= u_s, \end{cases} \tag{1.2}$$

in a Banach space X. Some fundamental and basic results about the well posedness and dynamical behavior of equation (1.2) were established under the so called *stability condition*, (**(A2)** in Section 2), introduced by Kato in [17]. This has been followed by various attempts to extend the Kato's stability condition to a more generic context. Tanaka, in [29], proposed an explicit stability condition with finite difference approximations in a non-dense domain case. Other attempts, see [6, 22, 23, 25, 28, 29], have analysed the well posedness of non-autonomous evolution equations in different contexts. When $F = 0$, [6, 21] investigated the well-posedness of the linear part of equation (1.1). Moreover, a variation of constants formula is established for densely inhomogeneous linear delayed differential equations of the form

$$\begin{cases} \dfrac{du}{dt} &= A(t)u(t) + L(t)u_t + f(t), \quad t \geq s \\ u(t) &= \varphi(t-s), \quad s-r \leq t \leq s. \end{cases} \tag{1.3}$$

Among the important questions in the qualitative study of dynamical systems are on the existence of bounded solutions and the existence of periodic solutions. The periodicity in evolution equations has a great theoretical and practical significance, see e.g [2, 4, 7, 12, 13, 15, 19, 31, 32]. Under the exponential dichotomy hypothesis and using a variation of constants formula, the authors in [6, 21] proved the existence and uniqueness of bounded (respectively periodic) solution provided that f is bounded (respectively periodic). When the exponential dichotomy assumption fails, the problem becomes complicated. To remedy this defect, we consider other alternatives such as the so called *Massera problem* [20]. Initially, it consists to prove the existence of periodic solutions of the ordinary differential equation $\dfrac{du}{dt} = Qu(t) + f(t)$, provided that it has a bounded solution. Later on, *Massera problem* has been widely investigated in general cases and in various directions. For instance, in [4, 8, 9, 10, 12, 13, 14, 18, 19, 24], the authors proved the existence of periodic solutions for different evolution equations through the existence of a bounded solution. The method used in these works is based on the existence of a fixed point of the associated Poincaré map. In [12], by using Horn's fixed point Theorem, the authors derived the existence of periodic solutions in the case of the existence of bounded and the ultimate bounded solutions.

In this paper, we will extend the Massera problem for equation (1.3) and then for the equation (1.1). We assume that the common domain D of $A(t)$ is not necessarily dense in X. Using similar techniques as in [22, 29], we give a variation constant formula for Equation (1.1). Next, we construct the Poincaré map P for equation (1.3) given by

$$P\varphi := u_\tau(\cdot, \varphi, f),$$

where $u(\cdot, \varphi, f)$ is the unique mild solution of (1.3) through the initial condition φ. Using Chow-Hale fixed point Theorem, we prove that P has a fixed point in the phase space C_r which generates a τ-periodic mild solution of (1.3). Afterwards, basing on the established results and using a fixed point Theorem for set-valued maps, we analyze the existence of a τ-periodic mild solution for the semilinear equation (1.1). At the end, we discuss an example as an illustration to the given theoretical results. The obtained results extend the results in [4, 8, 12, 14, 18, 20].

2 Preliminary results

Let $\big(A(t), D(A(t))\big)_{0\leq t\leq T}$ be a family of linear operators on a Banach space X satisfying the following assumptions :

(A1) There exists a Banach space $D := D(A(t))$ independent of t and there are positive constants c_1 and c_2 such that

$$c_1\|x\|_D \leq \|x\| + \|A(t)x\| \leq c_2\|x\|_D, \quad x \in D,\ 0 \leq t \leq T.$$

(A2) The family $(A(t))_{0\leq t\leq T}$ is stable that means there are constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A(t))$ for $t \in [0, T]$ and

$$\|\prod_{i=1}^{k} R(\lambda, A(t_j))\| \leq M(\lambda - \omega)^{-k}$$

for $\lambda > \omega$ and every finite sequence $\{t_j\}_{j=1}^{k}$ with $0 \leq t_1 \leq t_2 \leq ... \leq t_k \leq T$ and $k = 1, 2, \cdots$.

(A3) The mapping $t \longmapsto A(t)x$ is continuously differentiable in X for all $x \in D$.

There are several conditions implying the well-posedness of the evolution equation (1.2). In most cases, the authors assumed $\bigcap_{0\leq t\leq T} D(A(t))$ to be dense in X. Among others, we recall here a "classical" result, which is referred to Kato [17] and Tanabe [30].

Theorem 2.1. *Let $\big(A(t), D(A(t))\big)_{0\leq t\leq T}$ be a family of linear operators on a Banach space X satisfying* **(A1)** – **(A3)** *such that D is dense in X. Then, the equation* (1.2) *is well-posed and the family of operators $A(\cdot)$ generates an evolution family $(U(t, s))_{0\leq s\leq t\leq T}$. Moreover, for $x \in D$, the map $t \mapsto U(t, s)x$ is the unique continuous function which solves* (1.2).

In [27], Tanaka showed that the density of the domain D is not needed for the well-posedness of the equation (1.2).

Theorem 2.2. *[27] Let $\big(A(t), D(A(t))\big)_{0\le t\le T}$ be a family of linear operators on a Banach space X satisfying the conditions* (**A1**)-(**A3**). *Then, there exists an evolution family $(U(t,s))_{0\le s\le t\le T}$ on $\overline{D}$ satisfying :*

a) $U(t,s)D(s)\subset D(t)$ *for all* $0\le s\le t\le T$, *where* $D(r)$ *is defined by*

$$D(t) := \{x\in D : A(t)x\in \overline{D}\},$$

b) *for all* $x\in D(s)$ *and* $t\ge s$, *the function* $t\mapsto U(t,s)x$ *is continuously differentiable,*

$$\frac{d}{dt}U(t,s)x = A(t)U(t,s)x,$$

and

$$\frac{d^+}{ds}U(t,s)x = -U(t,s)A(s)x.$$

Assuming the three conditions (**A1**)-(**A3**), Da Prato et al. [26] and Tanaka [29] have studied the well-posedness of the following equation

$$\begin{cases} \dfrac{du}{dt} &= A(t)u(t)+f(t), \quad t\ge 0\\ u(0) &= u_0. \end{cases} \tag{2.1}$$

They gave an explicit formula for the evolution family $(U(t,s))_{0\le s\le t\le T}$ defined on $\overline{D}$. Indeed, they introduced the following family of operators defined on X by

$$U_\lambda(t,s) = \prod_{i=[\frac{s}{\lambda}]+1}^{[\frac{t}{\lambda}]} (I-\lambda A(i\lambda))^{-1}$$

for $(t,s)\in\Omega := \{(t,s) : 0\le s\le t\le T\}$.

Lemma 2.3. *[22, 29] Under the notation above, the following properties hold.*

i) *For* $x\in\overline{D}, \lambda>0$ *and* $0\le s\le r\le t\le T$, *one has*

$$U_\lambda(t,t)x = x \text{ and } U_\lambda(t,s)x = U_\lambda(t,r)U_\lambda(r,s)x.$$

ii) *For* $x\in\overline{D}$, *the limit*

$$U(t,s)x = \lim_{\lambda\to 0^+} U_\lambda(t,s)x$$

exists in X *uniformly for* $(t,s)\in\Omega$.

iii) *For* $x\in\overline{D}$ *and* $0\le s\le r\le t\le T$, *one has*

$$U(t,t)x = x \text{ and } U(t,s)x = U(t,r)U(r,s)x.$$

iv) *For every* $x\in\overline{D}$, *the mapping* $(t,s)\longmapsto U(t,s)x$ *is continuous from* Ω *into* X.

vi) There exist constants $M \geq 1$, $\omega \in \mathbb{R}$ such that for $x \in \overline{D}$ and $(t,s) \in \Omega$, one has $\|U(t,s)x\| \leq Me^{\omega(t-s)}\|x\|$.

In [29], the mild solutions of the evolution equation (2.1) is given by a "generalized" variation of constants formula.

Theorem 2.4. *[29] For every $f \in L^1([0,T],\ X)$, the limit*

$$v(t) := \lim_{\lambda \longrightarrow 0^+} \int_0^t U_\lambda(t,\sigma) f(\sigma) d\sigma$$

exists uniformly for $t \in [0,T]$ and v is a continuous function on $[0,T]$. Furthermore, for every $u_0 \in \overline{D}$ the function given by the following formula

$$u(t) := U(t,0)u_0 + \lim_{\lambda \longrightarrow 0^+} \int_0^t U_\lambda(t,\sigma) f(\sigma) d\sigma \tag{2.2}$$

is well defined, and is known as the mild solution of equation (2.1) *on* $[0,T]$.

3 Well-posedness of the non-densely non-autonomous differential equation with delay

The "generalized" variation of constants formula (2.2) enables us to extend the results of [29] to the equation (1.1). For this, we adopt the following definition.

Definition 3.1. A continuous function $u : [-r,+\infty) \longrightarrow X$ is called a mild solution of the equation (1.1) if it satisfies the following

$$u(t) := \begin{cases} U(t,0)\varphi(0) + \lim\limits_{\lambda \longrightarrow 0^+} \displaystyle\int_0^t U_\lambda(t,\sigma)[L(\sigma)u_\sigma + F(\sigma,u_\sigma)]d\sigma, & t \geq 0, \\ \varphi(t)\ , \quad -r \leq t \leq 0. & \end{cases} \tag{3.1}$$

For the existence of mild solutions of equation (1.1), we impose the Lipschitz continuous assumption:

(**A4**) there exists $\widetilde{M} > 0$ such that for $\varphi, \phi \in C_r$ and $t \geq 0$ we have

$$\|F(t,\varphi) - F(t,\phi)\| \leq \widetilde{M}\|\varphi - \phi\|.$$

By using a well known extension of the Banach contraction principle, we show the following result.

Theorem 3.2. *Assume that (**A1**)-(**A4**) hold and $\varphi \in C_r$ such that $\varphi(0) \in \overline{D}$. Then there exists a unique mild solution $u(\cdot,\varphi,f)$ on $\mathbb{R}^+$ for the equation* (1.1)*. Moreover, the mild solution depends continuously on the initial data φ.*

Proof. Let $T > 0$ and the space $S_\varphi = \{u \in C([-r,T],\ X) : u_0 = \varphi\}$ endowed with the sup-norm $\|\cdot\|$. It is clear that S_φ is closed in $C([-r,T],\ X)$.
Let the mapping $\mathbb{K}$ defined on S_φ by

$$(\mathbb{K}u)(t) := \begin{cases} U(t,0)\varphi(0) + \lim\limits_{\lambda \longrightarrow 0^+} \displaystyle\int_0^t U_\lambda(t,\sigma)[L(\sigma)u_\sigma + F(\sigma,u_\sigma)]d\sigma, & 0 \leq t \leq T, \\ \varphi(t)\ , \quad -r \leq t \leq 0. & \end{cases} \tag{3.2}$$

We have, $\mathbb{K}u \in C([-r,T],\ X)$. For the simplicity, we set $H(\cdot,\phi) := L(\cdot)\phi + F(\cdot,\phi)$. Hence, for $-r \le t \le T$, $c = M\widetilde{M}e^{\omega T}$ and $u,v \in S_{\varphi}$

$$\begin{aligned}
\|(\mathbb{K}u)(t)-(\mathbb{K}v)(t)\| &\le \lim_{\lambda\longrightarrow 0^+}\int_0^t \|U_\lambda(t,\sigma)\|\|H(\sigma,u_\sigma)-H(\sigma,v_\sigma)\|d\sigma \\
&\le c\int_0^t \|u_\sigma - v_\sigma\|d\sigma \\
&\le c\ t\|u-v\|.
\end{aligned}$$

Hence, it follows that

$$\|\mathbb{K}^n u - \mathbb{K}^n v\| \le \frac{c^n\ T^n}{n!}\|u-v\|. \tag{3.3}$$

For n large enough $(\frac{c^n\ T^n}{n!}) < 1$ and by a well known extension of the Banach contraction principle, $\mathbb{K}$ has a unique fixed point $u \in C([-r,T],\ X)$. This fixed point is a mild solution of the equation (1.1).

The uniqueness of the solution and its dependance continuity on the initial data are deduced from the following argument. Let $u(\cdot,\psi)$ be a mild solution of (1.1) with the initial data ψ. Then, for $0 \le t \le T$ and $M_1 = Me^{\omega T}$, one has

$$\begin{aligned}
\|u(t,\varphi)-u(t,\psi)\| &\le \|U(t,0)\varphi(0)-U(t,0)\psi(0)\| \\
&\quad + \lim_{\lambda\longrightarrow 0^+}\int_0^t \|U_\lambda(t,\sigma)\|\|H(\sigma,u_\sigma(\cdot,\varphi))-H(\sigma,u_\sigma(\cdot,\psi))\|d\sigma \\
&\le M_1\|\varphi-\psi\| + c\int_0^t \|u_\sigma(\cdot,\varphi)-u_\sigma(\cdot.\psi)\|d\sigma,
\end{aligned}$$

Hence,

$$\|u(t+\theta,\varphi)-u(t+\theta,\psi)\| \le \begin{cases} \|\varphi(t+\theta)-\psi(t+\theta)\|, & -r \le t+\theta \le 0, \\ M_1\|\varphi-\psi\| + c\displaystyle\int_0^t \|u_\sigma(\cdot,\varphi)-u_\sigma(\cdot,\psi)\|d\sigma, & 0 \le t+\theta \le T. \end{cases}$$

Thus, by the Gronwall's inequality we deduce that

$$\|u_t(\cdot,\varphi)-u_t(\cdot,\psi)\| \le \max(1, M_1 e^{cT})\|\varphi-\psi\| \tag{3.4}$$

which yields both the uniqueness of u and the continuity of the map $\varphi \mapsto u_t(\cdot,\varphi)$ uniformly for $t \in [0,T]$. □

4 Periodic solutions of the inhomogeneous linear equation

In what follows, we assume that $A(\cdot), L(\cdot)$ and $f(\cdot)$ are τ-periodic, and the assumptions (**A1**)-(**A3**) hold for $T = \tau$. As an immediate consequence, $U(t,s)$ is a τ-periodic evolution family defined for $t \ge s$ in $\mathbb{R}$. A function u defined on $\mathbb{R}$ is said to be a mild solution of equation

$$\begin{cases} \dfrac{du}{dt} & = A(t)u(t) + L(t)u_t + f(t), \quad t \ge 0 \\ u(t) & = \varphi(t), \quad -r \le t \le 0. \end{cases} \tag{4.1}$$

if it satisfies the following formula

$$u(t) = U(t,s)u(s) + \lim_{\lambda \longrightarrow 0^+} \int_s^t U_\lambda(t,\sigma)\left[L(\sigma)u_\sigma + f(\sigma)\right] d\sigma, \quad t \geq s.$$

We state now the following result which connects bounded and periodic solutions for the equation (4.1).

Theorem 4.1. *Assume that* (**A1**)-(**A3**) *hold and* $U(t,s)$ *is a compact operator on* $\overline{D}$ *for* $t > s \geq 0$. *Then, the following assertions are equivalent:*

i) The equation (4.1) *has a bounded mild solution on* $\mathbb{R}^+$.

ii) The equation (4.1) *has a* τ*-periodic mild solution.*

For the proof of Theorem 4.1, we use Chow-Hale fixed point Theorem for linear affine maps, presented in the following lemma.

Lemma 4.2. *[9] Let* X *be a Banach space,* $P_0 : X \longrightarrow X$ *be a continuous linear operator and* $y \in X$. *Consider the operator* $P : X \longrightarrow X$ *defined by* $Px = P_0 x + y$.
Suppose that there exists $x_0 \in X$ *such that* $\{P^n x_0,\ n \in \mathbb{N}\}$ *is a relatively compact set in* X. *Then* P *has a fixed point in* X.

We need first to prove the following lemma.

Lemma 4.3. *Let* v *be a bounded mild solution of equation* (4.1). *Then,* v *is a uniformly continuous function with relatively compact range* $\{v(t),\ t \geq 0\}$ *in* X. *Furthermore, the set* $\{v_t,\ t \geq 0\}$ *is relatively compact in* C_r.

Proof. Let $\varepsilon > 0$, one has

$$\{v(t),\ t \geq 0\} = \{v(t),\ 0 \leq t \leq \varepsilon\} \cup \{v(t),\ t \geq \varepsilon\}.$$

Since v is continuous, then the first set on the right hand side is relatively compact in X.
For simplicity, we set $G(t,\varphi) = L(t)\varphi + f(t)$. For $t > \varepsilon$, we have

$$\begin{aligned}
v(t) &= U(t,0)v(0) + \lim_{\lambda \longrightarrow 0^+} \int_0^t U_\lambda(t,s)G(s,v_s)ds \\
&= U(t,0)v(0) + \lim_{\lambda \longrightarrow 0^+} \int_0^{t-\varepsilon} U_\lambda(t,s)G(s,v_s)ds + \lim_{\lambda \longrightarrow 0^+} \int_{t-\varepsilon}^t U_\lambda(t,s)G(s,v_s)ds \\
&= U(t,t-\varepsilon)\left[U(t-\varepsilon,0)v(0) + \lim_{\lambda \longrightarrow 0^+} \int_0^{t-\varepsilon} U_\lambda(t-\varepsilon,s)G(s,v_s)ds\right] \\
&\quad + \lim_{\lambda \longrightarrow 0^+} \int_{t-\varepsilon}^t U_\lambda(t,s)G(s,v_s)ds \\
&= U(t,t-\varepsilon)v(t-\varepsilon) + \lim_{\lambda \longrightarrow 0^+} \int_{t-\varepsilon}^t U_\lambda(t,s)G(s,v_s)ds.
\end{aligned}$$

Firstly, we show that the set $K := \{U(t,t-\varepsilon)v(t-\varepsilon),\ t > \varepsilon\}$ is relatively compact in X. To this end, let $(y_n)_n$ be a sequence in K, then there exists a sequence $(t_n)_n$ with values $t_n > \varepsilon$ such that

$$y_n = U(t_n, t_n - \varepsilon)v(t_n - \varepsilon).$$

So, there exits a unique $q_n \in \mathbb{N}$ such that : $t_n = q_n\tau + r_n + \varepsilon$ with $0 \le r_n < \tau$. Using the τ-periodicity of $(U(t,s))_{t\ge s}$, we get

$$y_n = U(r_n+\varepsilon, r_n)v(q_n\tau + r_n).$$

Since $(r_n)_n$ is a bounded sequence in $[0,\tau]$, then there exists a subsequence $(r_{nk})_k$ which converges to $r_0 \in [0,\tau]$. Since the evolution family $(U(t,s))_{t>s\ge 0}$ is compact, we get that $(U(t,s))_{t\ge s}$ is continuous with respect to the operator norm. Hence,

$$\lim_{k\longrightarrow\infty} \|U(r_{nk}+\varepsilon, r_{nk}) - U(r_0+\varepsilon, r_0)\| = 0. \tag{4.2}$$

Since v is a bounded mild solution of (4.1) and $U(r_0+\varepsilon, r_0)$ is compact, then there exists a subsequence of $(r_{n_k})_k$ which we denote similarly $(r_{n_k})_k$ such that the sequence $(U(r_0+\varepsilon, r_0)v(q_{nk}\tau + r_{nk}))_k$ converges to y^* in X, and we write

$$\lim_{k\longrightarrow\infty} U(r_0+\varepsilon, r_0)v(q_{nk}\tau + r_{nk}) = y^*. \tag{4.3}$$

Now, we show that

$$\lim_{k\longrightarrow\infty} U(r_{nk}+\varepsilon, r_{nk})v(q_{nk}\tau + r_{nk}) = y^*.$$

We have,

$$\begin{aligned}
\|U(r_{nk}+\varepsilon, r_{nk})v(q_{nk}\tau + r_{nk}) - y^*\| &\le \|U(r_{nk}+\varepsilon, r_{nk})v(q_{nk}\tau + r_{nk}) - U(r_0+\varepsilon, r_0)v(q_{nk}\tau + r_{nk})\| \\
&\quad + \|U(r_0+\varepsilon, r_0)v(q_{nk}\tau + r_{nk}) - y^*\| \\
&\le \|U(r_{nk}+\varepsilon, r_{nk}) - U(r_0+\varepsilon, r_0)\|\|v(q_{nk}\tau + r_{nk})\| \\
&\quad + \|U(r_0+\varepsilon, r_0)v(q_{nk}\tau + r_{nk}) - y^*\|.
\end{aligned}$$

From (4.2), (4.3) and the boundedness of the mild solution v, we conclude that the set $\{U(t, t-\varepsilon)v(t-\varepsilon),\ t > \varepsilon\}$ is relatively compact in X. Secondly, from the boundedness of G there exists a positive constant α such that

$$\left\| \lim_{\lambda\longrightarrow 0^+} \int_{t-\varepsilon}^{t} U_\lambda(t,s)G(s, v_s)ds \right\| \le \alpha\varepsilon.$$

Hence $\{v(t),\ t > \varepsilon\}$ is relatively compact in X. Consequently, the set $\{v(t),\ t \ge 0\}$ is relatively compact in X. To show the uniform continuity of the mild solution v. Let $t > s \ge 0$, one has

$$\begin{aligned}
v(t) - v(s) &= (U(t,0) - U(s,0))v(0) + \lim_{\lambda\longrightarrow 0^+} \int_0^t U_\lambda(t,\sigma)G(\sigma, v_\sigma)d\sigma \\
&\quad - \lim_{\lambda\longrightarrow 0^+} \int_0^s U_\lambda(s,\sigma)G(\sigma, v_\sigma)d\sigma \\
&= (U(t,s) - I)U(s,0)v(0) + (U(t,s) - I) \lim_{\lambda\longrightarrow 0^+} \int_0^s U_\lambda(s,\sigma)G(\sigma, v_\sigma)d\sigma \\
&\quad + \lim_{\lambda\longrightarrow 0^+} \int_s^t U_\lambda(t,\sigma)G(\sigma, v_\sigma)d\sigma \\
&= (U(t,s) - I)v(s) + \lim_{\lambda\longrightarrow 0^+} \int_s^t U_\lambda(t,\sigma)G(\sigma, v_\sigma)d\sigma.
\end{aligned}$$

Since the set $\{v(t),\ t \geq 0\}$ is relatively compact in X, then

$$\lim_{\substack{t-s\to 0\\ t>s}} \|(U(t,s)-I)v(s)\| = 0.$$

Using the boundedness of v and the boundedness of G, there exists a positive constant $c > 0$ such that

$$\left\| \lim_{\lambda\longrightarrow 0^+} \int_s^t U_\lambda(t,\sigma)G(\sigma, v_\sigma)d\sigma \right\| \leq c(t-s).$$

Then,

$$\lim_{\substack{t-s\to 0\\ t>s}} \| \lim_{\lambda\longrightarrow 0^+} \int_s^t U_\lambda(t,\sigma)F(\sigma, v_\sigma)d\sigma\| = 0.$$

Therefore,

$$\lim_{\substack{t-s\to 0\\ t>s}} \|v(t)-v(s)\| = 0.$$

Similarly, we show

$$\lim_{\substack{t-s\to 0\\ t<s}} \|v(t)-v(s)\| = 0.$$

Thus, v is uniformly continuous. In particular, the set of history function $\{v_t,\ t \geq 0\}$ is equicontinuous. Together with the relative compactness of the range of v, by Arzelà-Ascoli Theorem, we obtain that $\{v_t,\ t \geq 0\}$ is relatively compact in C_r. □

Proof of Theorem 4.1. We define the Poincaré map on the phase space $C_0 := \{\varphi \in C_r : \varphi(0) \in \overline{D}\}$ by

$$P(\varphi) = u_\tau(\cdot, \varphi, f),$$

where $u(\cdot,\varphi, f)$ refers to the mild solution of (4.1) through φ. The variation of constants formula (3.1) allows to decompose P map as

$$P\varphi = u_\tau(\cdot,\varphi,0) + u_\tau(\cdot,0,f) := P_0\varphi + \psi,$$

where $u_\tau(\cdot,\varphi,0)$ is the mild solution of (4.1) with $f = 0$ and $u_\tau(\cdot,0,f)$ is the mild solution of (4.1) with $\varphi = 0$. Let v be a bounded solution of (4.1) on $\mathbb{R}^+$ with $v_0 = \widetilde{\varphi}$. Then the uniqueness of the solution of (4.1) implies that

$$P^n\widetilde{\varphi} = v_{n\tau}(\cdot, \widetilde{\varphi}, f) \quad \text{for} \quad n \in \mathbb{N}.$$

By Lemma 4.3, the set

$$\{P^n\widetilde{\varphi},\ n \in \mathbb{N}\} = \{v_{n\tau}(\cdot, \widetilde{\varphi}, f),\ n \in \mathbb{N}\}$$

is relatively compact in C_r. From Lemma 4.2, we conclude that the mapping P has a fixed point in C_r. Hence, the equation (4.1) has a τ-periodic mild solution. □

5 Periodic solutions of the semilinear equation

We consider the semilinear equation with delay

$$\frac{du}{dt} = A(t)u(t) + L(t)u_t + F(t,u_t), \quad t \geq 0. \tag{5.1}$$

We denote by $\mathbf{B}_\tau$ the space of τ-periodic continuous functions from $\mathbb{R}$ into X endowed with the uniform norm topology. To get the aim of this section, we give some definitions and a fixed point Theorem for set-valued maps.

Definition 5.1. [33] Let $\Gamma : M \longrightarrow 2^M$ be a multivalued map, where M is a subset of a Banach space E and 2^M is the power set of M.

i) For $D \subset M$, the inverse $\Gamma^{-1}(D)$ is the set of all $x \in M$ such that $\Gamma(x) \cap D \neq \emptyset$.

ii) The map Γ is called upper semi continuous if $\Gamma^{-1}(D)$ is closed for all closed $D \subset M$.

Theorem 5.2. *[33] Let $\Gamma : M \longrightarrow 2^M$ be a multivalued map, where M is a nonempty convex set in a Banach space E such that:*

i) the set $\Gamma(x)$ is nonempty, closed and convex for all $x \in M$,

ii) the set $\Gamma(M)$ is relatively compact in E,

iii) the map Γ is upper semi continuous.

Then Γ has a fixed point in the sense that there exists $x \in M$ such that $x \in \Gamma(x)$.

Then, we come to the aim of this section,

Theorem 5.3. *Assume that* (**A1**)-(**A4**) *hold, $U(t,s)$ is a compact operator on $\overline{D}$ for $t > s \geq 0$ and there exists a positive constant ρ such that for every function $y \in S_\rho := \{v \in \mathbf{B}_\tau : \|v\| \leq \rho\}$, the equation*

$$\frac{du}{dt} = A(t)u(t) + L(t)u_t + F(t,y_t), \quad t \geq 0 \tag{5.2}$$

has a τ-periodic mild solution in S_ρ. Then, the equation (5.1) *has a τ-periodic mild solution.*

Proof. Let $\Gamma : S_\rho \longrightarrow 2^{S_\rho}$ be the mapping defined for $y \in S_\rho$ by

$$\Gamma(y) = \left\{ u \in S_\rho : u(t) = U(t,0)u(0) + \lim_{\lambda \longrightarrow 0^+} \int_0^t U_\lambda(t,\sigma)[L(\sigma)u_\sigma + F(\sigma,y_\sigma)]d\sigma,\ t \geq 0 \right\}.$$

We show that the mapping Γ satisfies the conditions i), ii) and iii) of Theorem 5.2.
i) By assumption, $\Gamma(y)$ is nonempty for all $y \in S_\rho$. Let $y \in S_\rho, u_1,\ u_2 \in \Gamma(y)$ and $\alpha \in [0,1]$. Then $\alpha u_1 + (1-\alpha)u_2 \in \Gamma(y)$, which implies that $\Gamma(y)$ is convex. From the boundedness of the linear operator L and (**A4**) we deduce that $\Gamma(y)$ is a closed set.
ii) Using Arzelà-Ascoli Theorem, we show that $\Gamma(S_\rho)$ is relatively compact in $\mathbf{B}_\tau$. To this end, consider the functions in $\Gamma(S_\rho)$ on $[0,\tau]$ with length the period τ. Let $0 < t \leq \tau$ and

$\varepsilon > 0$ such that $t > \varepsilon$. We prove that the set $\left\{u(t),\ u \in \Gamma(S_\rho)\right\}$ is relatively compact in X. For $u \in \Gamma(S_\rho)$, there exists $y \in S_\rho$ such that

$$\begin{aligned}
u(t) = U(t,0)u(0) + \lim_{\lambda \longrightarrow 0^+} \int_0^{t-\varepsilon} U_\lambda(t,\sigma)[L(\sigma)u_\sigma + F(\sigma, y_\sigma)d\sigma] \\
+ \lim_{\lambda \longrightarrow 0^+} \int_{t-\varepsilon}^{t} U_\lambda(t,\sigma)[L(\sigma)u_\sigma + F(\sigma, y_\sigma)]d\sigma \\
= U(t,0)u(0) + U(t,t-\varepsilon) \lim_{\lambda \longrightarrow 0^+} \int_0^{t-\varepsilon} U_\lambda(t-\varepsilon,\sigma)[L(\sigma)u_\sigma + F(\sigma, y_\sigma)]d\sigma \\
+ \lim_{\lambda \longrightarrow 0^+} \int_{t-\varepsilon}^{t} U_\lambda(t,\sigma)[L(\sigma)u_\sigma + F(\sigma, y_\sigma)]d\sigma.
\end{aligned}$$

From the compactness of the operator $U(t,0)$ for $0 < t \le \tau$, we deduce that the set $\{U(t,0)u(0),\ u \in \Gamma(S_\rho)\}$ is relatively compact in X. Moreover, since $U(t,t-\varepsilon)$ is a compact operator, L is a bounded linear operator and the operator F satisfies (**A4**), in particular F transforms bounded sets into bounded sets, then we deduce that the subset

$$\left\{U(t,t-\varepsilon)\left[\lim_{\lambda \longrightarrow 0^+} \int_0^{t-\varepsilon} U_\lambda(t-\varepsilon,\sigma)[L(\sigma)u_\sigma + F(\sigma, y_\sigma)]d\sigma\right],\ u \in \Gamma(S_\rho)\right\}$$

is relatively compact in X. On the other hand, there exists a positive constant α such that

$$\sup_{u \in \Gamma(S_\rho)} \left\|\lim_{\lambda \longrightarrow 0^+} \int_{t-\varepsilon}^{t} U_\lambda(t,\sigma)[L(\sigma)u_\sigma + F(\sigma, y_\sigma)]d\sigma\right\| \le \alpha\varepsilon.$$

Thus $\left\{u(t),\ u \in \Gamma(S_\rho)\right\}$ is relatively compact in X for $t \in (0,\tau]$. Now, from the periodicity of $u(\cdot)$, the subset $\left\{u(0),\ u \in \Gamma(S_\rho)\right\}$ is also relatively compact in X. Consequently, the set $\left\{u(t),\ u \in \Gamma(S_\rho)\right\}$ is relatively compact in X for $t \in [0,\tau]$. To conclude, we prove the equicontinuity of $\Gamma(S_\rho)$. Let $t_0 \in [0,\tau)$ and $t_0 < t \le \tau$. Then,

$$\begin{aligned}
\|u(t) - u(t_0)\| \le \left\|(U(t,t_0) - I)\left[U(t_0,0)u(0) + \lim_{\lambda \longrightarrow 0^+} \int_0^{t_0} U_\lambda(t_0,\sigma)[L(\sigma)u_\sigma + F(\sigma, y_\sigma)]d\sigma\right]\right\| \\
+ \left\|\lim_{\lambda \longrightarrow 0^+} \int_{t_0}^{t} U_\lambda(t,\sigma)[L(\sigma)u_\sigma + F(\sigma, y_\sigma)]d\sigma\right\| \\
\le \|(U(t,t_0) - I)u(t_0)\| + \left\|\lim_{\lambda \longrightarrow 0^+} \int_{t_0}^{t} U_\lambda(t,\sigma)[L(\sigma)u_\sigma + F(\sigma, y_\sigma)]d\sigma\right\|.
\end{aligned}$$

Since $\{u(t_0),\ u \in \Gamma(S_\rho)\}$ is relatively compact in X, then

$$\lim_{\substack{t \longrightarrow t_0 \\ t > t_0}} \sup_{u \in \Gamma(S_\rho)} \|(U(t,t_0) - I)u(t_0)\| = 0.$$

Using the boundedness of $u(t_0)$ in $\Gamma(y)$ independently from $y \in S_\rho$, the assumption (**A4**) and the boundedness of L, there exists a positive constant c such that

$$\left\|\lim_{\lambda \longrightarrow 0^+} \int_{t_0}^{t} U_\lambda(t,\sigma)[L(\sigma)u_\sigma + F(\sigma, y_\sigma)]d\sigma\right\| \le c(t - t_0) \text{ uniformly on } y \in S_\rho.$$

So,

$$\lim_{\substack{t\to t_0\\ t>t_0}} \sup_{u\in\Gamma(S_\rho)} \|u(t)-u(t_0)\| = 0.$$

Similarly, for $t_0 \in (0,\tau]$, we can prove that

$$\lim_{\substack{t\to t_0\\ t<t_0}} \sup_{u\in\Gamma(S_\rho)} \|u(t)-u(t_0)\| = 0.$$

We deduce then that $\Gamma(S_\rho)$ is relatively compact in $\mathbf{B}_\tau$.
iii) To prove that Γ is upper semi continuous, we show firstly that Γ is closed.
Let $(y^n)_{n\geq 0}$ and $(z^n)_{n\geq 0}$ be sequences respectively in S_ρ and $\Gamma(S_\rho)$ such that for $n \geq 0$, $z^n \in \Gamma(y^n)$ with $y^n \longrightarrow y$ and $z^n \longrightarrow z$ as $n \to +\infty$. Then,

$$z^n(t) = U(t,0)z^n(0) + \lim_{\lambda\longrightarrow 0^+}\int_0^t U_\lambda(t,\sigma)[L(\sigma)z_\sigma + F(\sigma,y^n_\sigma)]d\sigma,\ t\geq 0.$$

Using the Lipschitz assumption of F, the boundedness of $L(\cdot)$ and the boundedness of $(U_\lambda(t,s))$ independently of λ, we obtain by letting $n \longrightarrow +\infty$, that

$$z(t) = U(t,0)z(0) + \lim_{\lambda\longrightarrow 0^+}\int_0^t U_\lambda(t,\sigma)[L(\sigma)z_\sigma + F(\sigma,y_\sigma)]d\sigma,\ t\geq 0.$$

Hence $z \in \Gamma(y)$ which implies that Γ is closed. Now, let D be a closed set in S_ρ, and take a sequence $(u_n)_{n\geq 0}$ in $\Gamma^{-1}(D)$ such that $u_n \longrightarrow u$, as $n \to +\infty$. Then there exists $y_n \in D$ such that $y_n \in \Gamma(u_n)$. Since $\Gamma(S_\rho)$ is relatively compact, thus there exists a subsequence $(y_{n_k})_k$ of $(y_n)_n$ such that $y_{n_k} \longrightarrow y$, as $k \longrightarrow +\infty$. Since Γ is closed, then $y \in \Gamma(u)$ and $u \in \Gamma^{-1}(D)$. Consequently, Γ is upper semi continuous.
All assumptions of Theorem 5.2 are satisfied. Hence, there exists $u \in S_\rho$ such that $u \in \Gamma(u)$. Which implies the existence of a τ-periodic mild solution of equation (5.1). From Theorem 4.1, to prove that (5.2) has a τ-periodic mild solution in S_ρ it suffices to show that (5.2) has a mild solution which is bounded by ρ.

Corollary 5.4. *Assume that* (**A1**)-(**A4**) *hold and* $U(t,s)$ *is a compact operator on* $\overline{D}$ *for* $t > s \geq 0$. *If there exists a positive constant* ρ *such that for any* $y \in S_\rho = \{v \in \mathbf{B}_\tau : \|v\| \leq \rho\}$, *the equation* (5.2) *has a mild solution that is bounded by* ρ *then the equation* (5.1) *has a* τ*-periodic mild solution on* $\mathbb{R}^+$.

Proof. Let $y \in S_\rho$ and v be a bounded mild solution of (5.2) with $v_0 = \widetilde{\varphi}$. Following the proof of [15, *Theorem* 2.5], the Poincaré map P has a fixed point which belongs to $\overline{Co}\{P^n\widetilde{\varphi},\ n\geq 0\}$, where $\overline{Co}$ denotes the closure of the convex hull. Let ψ be the fixed point of P and $v(\cdot,\psi,F(\cdot,y_\cdot))$ be the associated mild solution of (5.2) through ψ. By virtue of the linearity of $(U(t,s))_{t\geq s\geq 0}$ and $(L(t))_{t\geq 0}$ together with the continuity dependance on the initial data, we deduce that the mild solution $v(\cdot,\psi,F(\cdot,y_\cdot))$ is also bounded by ρ.

6 Application

In order to illustrate the previous results, we consider the non-autonomous partial differential equation with delay

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) = \delta(t)\dfrac{\partial^2 v}{\partial x^2}(t,x) + \beta(t)v(t,x) + b_1(t)v(t-r,x) + b_2(t)h(v(t-r,x)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad +g(t,x),\ t \geq 0, x \in [0,\pi], \\ v(t,0) = v(t,\pi) = 0, \quad t \geq 0, \\ v(\theta,x) = \phi(\theta,x), \theta \in [-r,0], x \in [0,\pi], \end{cases} \tag{6.1}$$

with $\delta(\cdot)$ is a τ-periodic and C^1-positive function in $\mathbb{R}^+$, with $\delta_0 := \inf_{t\in\mathbb{R}^+} \delta(t) > 0$, the functions $\beta, b_1, b_2 : \mathbb{R}^+ \to \mathbb{R}$ are τ- periodic continuous, $h : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous such that

$$|h(x)| \leq k|x|,\ x \in \mathbb{R}. \tag{6.2}$$

$g : \mathbb{R}^+ \times [0,\pi] \longrightarrow \mathbb{R}$ is a continuous function, τ-periodic in t. The function $\phi : [-r,0] \times [0,\pi] \longrightarrow \mathbb{R}$ is continuous. Let $X := C([0,\pi],\mathbb{R})$ and Δ be the Laplacien operator on $[0,\pi]$ with domain

$$D := \{z \in C^2([0,\pi],\mathbb{R}) : z(0) = z(\pi) = 0\}.$$

By [25], Δ satisfies the following conditions :

$$(0,+\infty) \subset \rho(\Delta),\ ||R(\lambda,\Delta)|| \leq \frac{1}{\lambda}, \quad \lambda > 0. \tag{6.3}$$

Let $(A(t))_{t\geq 0}$ be the family of operators defined by $A(t)z = \delta(t)\Delta$ with domain $D(A(t)) = D$. It is known that

$$R(\lambda, A(t)) = \frac{1}{\delta(t)} R(\frac{\lambda}{\delta(t)}, \Delta).$$

Using (6.3), we deduce that for every $\lambda > 0$, $\lambda \in \rho(A(t))$ and $||R(\lambda, A(t))|| \leq \frac{1}{\lambda}$. Then,

$$\left\| \prod_{i=1}^{n} R(\lambda, A(t_i)) \right\| \leq \frac{1}{\lambda^n}\ ,\ t_1 \leq t_2 \leq ... \leq t_n.$$

Hence, the operator $A(\cdot)$ satisfies the assumptions (**A1**)-(**A3**).
Moreover,

$$\overline{D} = \{z \in C([0,\pi],\mathbb{R}) : z(0) = z(\pi) = 0\} \neq X.$$

By [3] the part Δ_0 of Δ in $\overline{D}$ given by

$$\begin{cases} D(\Delta_0) = \{z \in D : \Delta z \in \overline{D}\} \\ \Delta_0 z = \Delta z, \end{cases} \tag{6.4}$$

generates an immediately compact semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D}$ such that

$$||T_0(t)|| \leq e^{-t},\ t \geq 0. \tag{6.5}$$

Thus, the part $A_0(\cdot)$ of $A(\cdot)$ in $\overline{D}$ generates an evolution family $(U(t,s))_{t\geq s\geq 0}$ on $\overline{D}$ which is given by

$$U(t,s) = T_0(\int_s^t \delta(\tau)d\tau),\ t \geq s \geq 0,$$

Therefore, the evolution family $(U(t,s))_{t\geq s}$ is immediately compact. By (6.5), one has

$$||U(t,s)|| \leq e^{-\delta_0(t-s)},\ t \geq s \geq 0. \tag{6.6}$$

Let $L,\ F : \mathbb{R}\times C([-r,0],X) \longrightarrow X$ be defined, for $t \in \mathbb{R}^+$ and $\psi \in C([-r,0],X)$, by

$$(L(t)\psi)(x) = \beta(t)\psi(0)(x) + b_1(t)\psi(-r)(x),\ x \in [0,\pi] \tag{6.7}$$

$$(F(t,\psi))(x) = b_2(t)h(\psi(-r)(x)) + g(t,x),\ x \in [0,\pi], \tag{6.8}$$

$$\varphi(t)(x) = \phi(t,x),\ x \in [0,\pi]. \tag{6.9}$$

$(L(\cdot))$ is a family of bounded linear operators from $C([-r,0],X)$ to X, and F satisfies the assumption (**A4**). Using the above notations and set $u(t)(x) := v(t,x)$, the equation (6.1) takes the abstract form

$$\begin{cases} \dfrac{du}{dt} & = A(t)u(t) + L(t)u_t + F(t,u_t), \quad t \geq 0 \\ u(t) & = \varphi(t), \quad -r \leq t \leq 0. \end{cases} \tag{6.10}$$

Proposition 6.1. *Assume that there exists $d \in (\max\{0, 1-\delta_0\}, 1)$ such that*

$$|\beta(t)| + |b_1(t)| + |b_2(t)|k \leq 1-d,\ \textit{for } t \in [0,\tau]. \tag{6.11}$$

Then, (6.10) has a τ-periodic mild solution.

Proof. Let $m := \max\{|g(t,x)|;\ x \in [0,\pi],\ t \in [0,\tau]\}$ and $\rho := \dfrac{m+d}{\delta_0 - 1 + d}$. Hence,

$$m - \rho(d-1) = \rho\delta_0 - d. \tag{6.12}$$

We claim that if y is a τ-periodic continuous function such that $||y|| \leq \rho$, then for all φ with $||\varphi|| \leq \rho$, the solution u of

$$\begin{cases} \dfrac{du}{dt} & = A(t)u(t) + L(t)u_t + F(t,y_t), \quad t \geq 0 \\ u(t) & = \varphi(t), \quad -r \leq t \leq 0, \end{cases} \tag{6.13}$$

satisfies $||u(t)|| \leq \rho$, for all $t \geq 0$. Indeed, let

$$t_0 = \inf\{t > 0 : ||u(t)|| > \rho\}. \tag{6.14}$$

If $t_0 < \infty$, by continuity, we get $||u(t_0)|| = \rho$. By using the generalized variation of constants formula, we have

$$u(t_0) = U(t_0,0)\varphi(0) + \lim_{\lambda\longrightarrow 0^+} \int_0^{t_0} U_\lambda(t_0,s)\,[L(s)u_s + F(s,y_s)]\,ds, \quad t \geq 0.$$

Using (6.5) and the integral representation of the resolvent [11], we deduce that

$$(-1,+\infty)\subset\rho(\Delta_0),\ \|R(\lambda,\Delta_0)\|\le\frac{1}{\lambda+1},\quad \lambda>-1, \tag{6.15}$$

and

$$\|R(\lambda,A_0(t))\|\le\frac{1}{\lambda+\delta_0},\ t\ge 0. \tag{6.16}$$

For $\lambda>0$ and $x\in\overline{D}$, we have

$$U_\lambda(t_0,s)x=\prod_{i=[\frac{s}{\lambda}]+1}^{[\frac{t_0}{\lambda}]}(I-\lambda A(i\lambda))^{-1}x=\prod_{i=[\frac{s}{\lambda}]+1}^{[\frac{t_0}{\lambda}]}\lambda^{-1}(\frac{1}{\lambda}-A_0(i\lambda))^{-1}x.$$

Using (6.16), one has

$$\begin{aligned}\|U_\lambda(t_0,s)\|&=\|\prod_{i=[\frac{s}{\lambda}]+1}^{[\frac{t_0}{\lambda}]}\lambda^{-1}(\frac{1}{\lambda}-A_0(i\lambda))\|\\&\le\left(\frac{1}{1+\lambda\delta_0}\right)^{[\frac{t_0}{\lambda}]-[\frac{s}{\lambda}]}\\&\le\left(\frac{1}{1+\lambda\delta_0}\right)^{\frac{t_0-s}{\lambda}+1}\\&\le\frac{1}{1+\lambda\delta_0}\exp[-\delta_0(t_0-s)\frac{\ln(1+\delta_0\lambda)}{\delta_0\lambda}].\end{aligned}$$

Hence,

$$\begin{aligned}\|\lim_{\lambda\longrightarrow0^+}\int_0^{t_0}U_\lambda(t_0,s)ds\|&\le\lim_{\lambda\longrightarrow0^+}\int_0^{t_0}\|U_\lambda(t_0,s)\|ds\\&\le\lim_{\lambda\longrightarrow0^+}\int_0^{t_0}\frac{1}{1+\lambda\delta_0}\exp[-\delta_0(t_0-s)\frac{\ln(1+\delta_0\lambda)}{\delta_0\lambda}]ds.\end{aligned}$$

Thus, by Lebesgue's Theorem we deduce that

$$\lim_{\lambda\longrightarrow0^+}\int_0^{t_0}\|U_\lambda(t_0,s)\|ds\le\frac{1}{\delta_0}(1-e^{-\delta_0t_0}) \tag{6.17}$$

By (6.6)-(6.8) and (6.17), we get that

$$\|u(t_0)\|\le\rho e^{-\delta_0t_0}+\frac{1}{\delta_0}((|\beta|+|b_1|+|b_2|k)\rho+m)(1-e^{-\delta_0t_0}).$$

From (6.11) and (6.12), we obtain

$$\begin{aligned}\|u(t_0)\|&\le\rho e^{-\delta_0t_0}+\frac{1}{\delta_0}[(1-d)\rho+m](1-e^{-\delta_0t_0})\\&\le\rho e^{-\delta_0t_0}+\frac{1}{\delta_0}(\rho\delta_0-d)(1-e^{-\delta_0t_0})\\&\le\rho-\frac{d}{\delta_0}(1-e^{-\delta_0t_0}),\end{aligned}$$

which gives that $\|u(t_0)\| < \rho$. This contradicts the fact that $t_0 < \infty$. Consequently, $\|u(t)\| \leq \rho$ for all $t \geq 0$, and by Corollary 5.4 the equation (6.10) has a τ-periodic mild solution in S_ρ. We deduce that Equation (6.1) has a τ-periodic mild solution. □

References

[1] B. Amir and L. Maniar, Asymptotic behavior of hyperbolic Cauchy problems for Hille-Yosida operators with an application to retarded differential equations, *Q*uaest. Math. **23** (2000), 343-357.

[2] T. Akrid, L. Maniar and A. Ouhinou , Periodic solutions for some non-autonomous boundary evolutions equations, *Nonlinear Analysis: Theory, Methods and Applications* **71** (2009), 1000-1011.

[3] W. Arendt, A. Grabosch, G. Greiner, U. Groh, H. P. Lotz, U. Moustakas, R. Nagel, B. Neubrander and U. Schlotterbeck, One-Parameter Semigroup of Positive Operators, *Lecture Notes in Mathematics, Springer-Verlag, Berlin*, **1184**, 1984.

[4] R. Benkhalti and K. Ezzinbi, Periodic solutions for some partial functional differential equations, *Journal of Applied Mathmatics and Stochastic Analysis* **1** (2004), 9-18.

[5] R. Benkhalti and K. Ezzinbi, A Massera type criterion for some partial functional differential equations, Dynamic Systems and Applications **9** (2000), 221-228.

[6] S. Boulite, L. Maniar and M. Moussi, Non-autonomous retarded differential equations: variation of constants formula and asymptotic behavior, *Electronic Journal of Differential Equations* **62** (2003), 1-15.

[7] T. Burton, Stability and periodic solutions of ordinary differential equations and functional differential equations, *Academic Press,* New York (1985), 197-308.

[8] S. N. Chow, Remarks on one-dimensional delay differential equations, *Journal of Mathematical Analysis and Applications* **41** (1973), 426-429.

[9] S. N. Chow and J. K. Hale, Strongly limit-compact maps, *Funkcialaj Ekvacioj* **17** (1974), 31-38.

[10] Y. Li, F. cong, Z. Lin, Z. Li and W. Liu, Periodic solutions for evolution equations, *Nonlinear Analysis: Theory, Methods and Applications* **36** (1999), 275-293.

[11] K. J. Engel and R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, *Springer-Verlag, New York,* 2000.

[12] K. Ezzinbi and J. H. Liu, Periodic solutions of non-densely defined delay evolutions equations, *Journal of Applied Mathematics and Stochastic Analysis* **15** (2002), 113-123.

[13] K. Ezzinbi, T. Naito, N. Minh and J. H. Liu, Periodic solutions of evolutions equations, dynamics of continuous, *discrete and impulsive systems, series A: mathematical analysis* **11** (2004), 601-613.

[14] G. Gühring and F. Räbiger, Asymptptic properties of non-autonomous evolution equations with applications to retarted differential equations, *Abstract and Applied Analysis* **4** (1999), 169-194.

[15] J. K. Hale, Asymptotic Behavior of Dissipative Systems, *Mathematical Surveys and Monographs, American Mathematical Society, Rhode Island* **25**, 1988.

[16] Y. Hino, T. Naito, N. V. Minh and J. S. Shin, Almost Periodic Solutions of Differential Equations in Banach Spaces, *Stability and Control : Theory, Methods and Applications*, 2002.

[17] T. Kato, Linear evolution equations of "hyperbolic" type, Journal of the Faculty of Science, *University of Tokyo. Section 1A, Mathematics* **17** (1970), 241-258.

[18] J. H. Liu, Bounded and periodic solutions of finite delay evolutions equations, *Nonlinear Analysis: Theory, Methods and Applications* **34** (1998), 101-111.

[19] J. H. Liu, Bounded and periodic solutions of differential equations in Banach space, *Journal of Applied Mathematics and Computing* **65** (1994), 141-150.

[20] J. L. Massera, The existence of periodic solutions of systems of differential equations, *Duke Mathematical Journal* **17** (1950), 457-475.

[21] M. Moussi, Well-posedness and Asymptotic Behavior of Non-Autonomous Boundary Cauchy Problems, *Ph.D. Thesis, Oujda*, Morrocco, 2003.

[22] H. Oka and N. Tanaka, Evolution operators generated by non-densely defined operators, *Mathematische Nachrichten* **278** (2005), 1285-1296.

[23] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, *Springer Verlag, New york*, 1983.

[24] J. Prüss, Periodic solutions of semilinear evolution equations, *Nonlinear Analysis: Theory, Methods and Applications* **3** (1979), 221-235.

[25] G. Da Prato and E. Sinestrari, Differential operators with non-dense domain, *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze* **14** (1987), 285-344.

[26] G. Da Prato and E. Sinestrari, Non-autonomous evolution operators of hyperbolic type, *Semigoup Forum* **45** (1992), 302-321.

[27] N. Tanaka, Quasilinear evolution equations with non-densely defined operators, *Differential and Integral Equations* **5** (1996), 1067-1106.

[28] N. Tanaka, Generation of linear evolution equations, *Proceedings of the American Mathematical Society* **128** (1999), 2007-2015.

[29] N. Tanaka, Semilinear equations in the "hyperbolic" case, *Nonlinear Analysis: Theory, Methods and Applications* **24** (1995), 773-78.

[30] H. Tanabe, Equations of Evolutions, Pitman, London, 1979.

[31] G. F. Webb, Theory of Nonlinear Age-Dependent Population Dynamics, Marcel Dekker, 1985.

[32] J. Wu, Theory and Applications of Partial Functional Differential Equations, *Springer-Verlag, New York*, 1996.

[33] E. Zeidler, Nonlinear Functioanl Analysis and its Applications. I. Fixed-point Theorems, *Springer-Verlag, New York*, 1986.