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On the Existence of Almost Automorphic Solutions of Nonlinear Stochastic Volterra Difference Equations

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Abstract

In this paper, we introduce a concept of almost automorphy for random sequences. Using the Banach contraction principle, we establish the existence and uniqueness of an almost automorphic solution to some Volterra stochastic difference equation in a Banach space. Our main results extend some known ones in the sense of mean almost automorphy. As an application, almost automorphic solution to a concrete stochastic difference equation is analyzed to illustrate our abstract results.

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1 Introduction

The paper deals with the existence and uniqueness of almost automorphic solution of some nonlinear stochastic Volterra difference equations of convolution type

$$X(\omega,n+1)=\sum_{j=-\infty}^{n} a(n-j)TX(\omega,j)+f(n,X(\omega,n))\xi(\omega,n+1),\ \omega\in\Omega,\ n\in\mathbb{Z} \tag{1.1}$$

on a Banach space $\mathbb{B}$, where T is a bounded linear operator on $\mathbb{B}$, $a:\mathbb{N}\to\mathbb{C}$ is summable, f is an appropriate function to be specified later. Each $\xi(n)$ is a real-valued random variable.

The concept of almost automorphy is an important generalization of the classical almost periodicity. It was introduced by Bochner [4], for more details about this topics we refer the reader to [12, 13]. In recent years, the existence of almost periodic and almost automorphic solutions on different kinds of differential equations has been considerably investigated in lots of publications [2, 5, 9] because of its significance and applications in physics, mechanics, and mathematical biology. Difference equations have received less studies about them. During the last few years, Volterra difference equations have emerged vigorously in several applied fields. Volterra systems mainly arise to model many real phenomena and describe

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processes whose current state is determined by their entire pre-history. They also play a key role in the study of competitive species in the population dynamics.

For a study of almost periodic or almost automorphic sequences we refer the reader to Araya *et al* [1], Bezandry and Diagana [2], Bezandry *et al.* [3], Corduneanu [5], Cuevas *et al.* [6], Diagana [7], Diagana *et al.* [8], Han and Hong [10], Hong and Nunez [11] and references therein.

Motivated by the work of Araya *et al* [1] and Cuevas *et al.* [6], the main purpose of this paper is to introduce the notion of almost automorphy for random sequences and apply the concept to investigate the existence of almost automorphic solutions to Eq.(1.1).

The rest of this paper is organized as follows. In Section 2, we introduce the notion of almost automorphic random sequences and study some of their basic properties. In Section 3, we prove the existence and uniqueness of almost automorhic solutions to some linear and nonlinear stochastic difference equations, respectively. We illustrate our main result by providing an example in Section 4.

2 Preliminaries

In this section we establish a basic theory for almost automorphic random sequences. To facilitate our task, we first introduce the notations needed in the sequel.

Let $(\mathbb{B}, \|\cdot\|)$ be a Banach space and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Throughout the rest of the paper, $\mathbb{Z}_+$ denotes the set of all nonnegative integers. Define $L^1(\Omega;\mathbb{B})$ to be the space of all $\mathbb{B}$-valued random variables V such that

$$\mathbf{E}\|V\| := \Big(\int_\Omega \|V(\omega)\| d\mathbb{P}(\omega)\Big) < \infty. \tag{2.1}$$

It is then routine to check that $L^1(\Omega;\mathbb{B})$ is a Banach space when it is equipped with its natural norm $\|\cdot\|_1$ defined by, $\|V\|_1 := \mathbf{E}\|V\|$ for each $V \in L^1(\Omega,\mathbb{B})$.

Let $X = \{X_n\}_{n\in\mathbb{Z}}$ be a sequence of $\mathbb{B}$-valued random variables satisfying $\mathbf{E}\|X_n\| < \infty$ for each $n \in \mathbb{Z}$. Thus, interchangeably we can, and do, speak of such a sequence as a function, which goes from $\mathbb{Z}$ into $L^1(\Omega;\mathbb{B})$.

This setting requires the following preliminary definitions.

Definition 2.1. An $L^1(\Omega;\mathbb{B})$-valued random sequence $X = \{X(n)\}_{n\in\mathbb{Z}}$ is said to be Bohr almost periodic in mean if for each $\varepsilon > 0$ there exists $N_0(\varepsilon) > 0$ such that among any N_0 consecutive integers there exists at least an integer p for which

$$\mathbf{E}\|X(n+p) - X(n)\| < \varepsilon, \ \forall\, n \in \mathbb{Z}.$$

An integer p with the above-mentioned property is called an ε-almost period for X. The collection of all $\mathbb{B}$-valued random sequences $X = \{X(n)\}_{n\in\mathbb{Z}}$ which are Bohr almost periodic in mean is then denoted by $AP(\mathbb{Z}; L^1(\Omega;\mathbb{B}))$.

Similarly, one defines the Bochner almost periodicity in mean as follows:

Definition 2.2. An $L^1(\Omega;\mathbb{B})$-valued random sequence $X = \{X(n)\}_{n\in\mathbb{Z}}$ is called mean Bochner almost periodic if for every sequence $\{m_k\}_{k\in\mathbb{Z}} \subset \mathbb{Z}$ there exists a subsequence $\{m'_k\}_{k\in\mathbb{Z}}$ such that $\{X(n+m'_k))\}_{k\in\mathbb{Z}}$ converges (in the mean) uniformly in $n \in \mathbb{Z}$.

Following along the same arguments as in the proof of [Diagana *et al.* [8], Theorem 2.4, p. 241], one can show that those two notions of almost periodicity coincide.

Theorem 2.3. *An $L^1(\Omega;\mathbb{B})$-valued random sequence $X = \{X(n)\}_{n\in\mathbb{Z}}$ is Bochner almost periodic in mean if and only if it is Bohr almost periodic in mean.*

The above characterization as well as the definition of automorphic functions in the continuous case will motivate the following definition.

Definition 2.4. An $L^1(\Omega;\mathbb{B})$-valued random sequence $X = \{X(n)\}_{n\in\mathbb{Z}}$ is said to be almost automorphic in mean if for every sequence $\{k'_n\} \subset \mathbb{Z}$ there exists a subsequence $\{k_n\}$ such that

$$\lim_{n\to\infty} X(k+k_n) =: \bar{X}(k) \text{ in } L^1(\Omega;\mathbb{B}) \tag{2.2}$$

is well defined for each $k \in \mathbb{Z}$ and

$$\lim_{n\to\infty} \bar{X}(k-k_n) = X(k) \text{ in } L^1(\Omega;\mathbb{B}) \tag{2.3}$$

for each $k \in \mathbb{Z}$

The collection of all $\mathbb{B}$-valued random sequences $X = \{X(n)\}_{n\in\mathbb{Z}}$ which are almost automorphic in mean is then denoted by $AA(\mathbb{Z}; L^1(\Omega;\mathbb{B}))$.

Remark 2.5. Note that if the convergence in Definition 2.4 is uniform on $\mathbb{Z}$, then we obtain almost automorphy in mean.

Example 2.6. Consider the random mapping $X : \mathbb{Z} \to L^1(\Omega,\mathbb{R})$ defined by

$$X(k) = \frac{\gamma}{2+\cos(k)+\cos(\sqrt{2}k)},$$

where γ is a random variable with $\mathbf{E}|\gamma| < \infty$.
It can be shown that X is almost automorphic in mean.

Almost automorphic random sequences have the following fundamental properties.

Theorem 2.7. *Let X, Y be almost automorphic random sequences. Then the following assertions hold:*

(i) *$X + Y$ is almost automorphic in mean;*

(ii) *cX is almost automorphic in mean for every scalar c;*

(iii) *For each fixed $l \in \mathbb{Z}$, $X_l : \mathbb{Z} \to L^1(\Omega;\mathbb{B})$ defined by $X_l(k) := X(k+l)$ is almost automorphic in mean;*

(iv) *The mapping $\hat{X} : \mathbb{Z} \to L^1(\Omega;\mathbb{B})$ defined by $\hat{X}(k) = X(-k)$ is almost automorphic in mean;*

(v) *X is bounded in $L^1(\Omega;\mathbb{B})$. That is, $\sup_{k\in\mathbb{Z}} \mathbf{E}\|X(k)\| < \infty$.*

(vi) $\sup_{k\in\mathbb{Z}} \mathbf{E}\|\overline{X}(k)\| \le \sup_{k\in\mathbb{Z}} \mathbf{E}\|X(k)\|$ *where* $\overline{X}$ *is defined in (2.2) and (2.3).*

The proof of all statements of Theorem 2.7 follows the same lines as in the determinsitic continuous case [12] and therefore is omitted.

As a consequence of the above theorem, the space $AA(\mathbb{Z}; L^1(\Omega;\mathbb{B}))$ of almost automorphic random sequences equipped with the norm

$$\|X\|_\infty = \sup_{k\in\mathbb{Z}} \mathbf{E}\|X(k)\|,$$

becomes a Banach space.

Theorem 2.8. *If A is a bounded linear operator on $L^1(\Omega;\mathbb{B})$ and $X : \mathbb{Z} \to L^1(\Omega;\mathbb{B})$ is almost automorphic random sequence, then $AX(k)$, $k \in \mathbb{Z}$ is also almost automorphic in mean.*

Proof. Let (k'_n) be a sequence in $\mathbb{Z}$. By assumption, we can then choose a subsequence (k_n) of (k'_n) such that

$$\lim_{n\to\infty} \|X(k+k_n) - \overline{X}(k)\| = 0 \text{ and } \lim_{n\to\infty} \|\overline{X}(k-k_n) - X(k)\| = 0,$$

for each $k \in \mathbb{Z}$.
We then have

$$\mathbf{E}\|AX(k+k_n) - A\overline{X}(k)\| \le \|A\|\mathbf{E}\|X(k+k_n) - \overline{X}(k)\|.$$

Hence, $\lim_{n\to\infty} \mathbf{E}\|AX(k+k_n) - A\overline{X}(k)\| = 0$ for each $k \in \mathbb{Z}$.

In a similar way, we can also prove

$$\lim_{n\to\infty} \mathbf{E}\|A\overline{X}(k-k_n) - AX(k)\| = 0$$

for each $k \in \mathbb{Z}$, and therefore AX is almost automorphic in mean. □

Theorem 2.9. *Let $\alpha : \mathbb{Z} \to \mathbb{C}$ be an almost automorphic deterministic sequence and $X : \mathbb{Z} \to L^1(\Omega;\mathbb{B})$ be an almost automorphic random sequence. Then $\alpha X : \mathbb{Z} \to L^1(\Omega;\mathbb{B})$ defined by $(\alpha X)(k) = \alpha(k)X(k)$, $k \in \mathbb{Z}$ is almost automorphic in mean.*

Proof. Let (k'_n) be a sequence in $\mathbb{Z}$. By assumption, we can then choose a subsequence (k_n) of (k'_n) such that

$$1)\ \lim_{n\to\infty} |\alpha(k+k_n) - \overline{\alpha}(k)| = 0 \text{ and } \lim_{n\to\infty} |\overline{\alpha}(k-k_n) - \alpha(k)| = 0$$

and

$$2)\ \lim_{n\to\infty} \|X(k+k_n) - \overline{X}(k)\| = 0 \text{ and } \lim_{n\to\infty} \|\overline{X}(k-k_n) - X(k)\| = 0,$$

for each $k \in \mathbb{Z}$.
Also, since α and $\overline{X}$ are bounded, there exist $M_1 > 0$ and $M_2 > 0$ such that $\sup_{n\in\mathbb{Z}} |\alpha(n)| \le M_1$ and $\sup_{n\in\mathbb{Z}} \mathbf{E}\|\overline{X}(n)\| \le M_2$.

We then have

$$\begin{aligned}
&\mathbf{E}\|\alpha(k+k_n)X(k+k_n)-\overline{\alpha}(k)\overline{X}(k)\| \\
&\le \mathbf{E}\|\alpha(k+k_n)[X(k+k_n)-\overline{X}(k)]\|+\mathbf{E}\|[\alpha(k+k_n)-\overline{\alpha}(k)]\overline{X}(k)\| \\
&\le |\alpha(k+k_n)|\mathbf{E}\|X(k+k_n)-\overline{X}(k)\|+|\alpha(k+k_n)-\overline{\alpha}(k)|\mathbf{E}\|\overline{X}(k)\| \\
&\le M_1\mathbf{E}\|X(k+k_n)-\overline{X}(k)\|+M_2|\alpha(k+k_n)-\overline{\alpha}(k)|\,.
\end{aligned}$$

Hence, $\lim_{n\to\infty}\mathbf{E}\|\alpha(k+k_n)X(k+k_n)-\overline{\alpha}(k)\overline{X}(k)\|=0$ for each $k\in\mathbb{Z}$.

In a similar way, we can also prove

$$\lim_{n\to\infty}\mathbf{E}\|\overline{\alpha}(k-k_n)\overline{X}(k-k_n)-\alpha(k)X(k)\|=0$$

for each $k\in\mathbb{Z}$, and therefore αX is almost automorphic in mean. □

More generally, we have

Theorem 2.10. *Let $X,Y:\mathbb{Z}\to L^1(\Omega;\mathbb{B})$ be almost automorphic random sequences. Assume that X and Y are independent. Then $XY:\mathbb{Z}\to L^1(\Omega;\mathbb{B})$ defined by $(XY)(k)=X(k)Y(k)$, $k\in\mathbb{Z}$ is almost automorphic in mean.*

Proof. Let (k'_n) be a sequence in $\mathbb{Z}$. By assumption, we can then choose a subsequence (k_n) of (k'_n) such that

$$1)\ \lim_{n\to\infty}\mathbf{E}\|X(k+k_n)-\overline{X}(k)|=0 \text{ and } \lim_{n\to\infty}|\overline{X}(k-k_n)-X(k)|=0$$

and

$$2)\ \lim_{n\to\infty}\|Y(k+k_n)-\overline{Y}(k)\|=0 \text{ and } \lim_{n\to\infty}\|\overline{Y}(k-k_n)-Y(k)\|=0,$$

for each $k\in\mathbb{Z}$.
Also, since X and $\overline{Y}$ are bounded, there exist $M_1>0$ and $M_2>0$ such that $\sup_{n\in\mathbb{Z}}|\alpha(n)|\le M_1$ and $\sup_{n\in\mathbb{Z}}\mathbf{E}\|\overline{X}(n)\|\le M_2$.
Using that fact that X and Y are independent, we then have

$$\begin{aligned}
&\mathbf{E}\|X(k+k_n)Y(k+k_n)-\overline{X}(k)\overline{Y}(k)\| \\
&\le \mathbf{E}\|X(k+k_n)[Y(k+k_n)-\overline{Y}(k)]\|+\mathbf{E}\|[X(k+k_n)-\overline{X}(k)]\overline{Y}(k)\| \\
&\le \mathbf{E}\|X(k+k_n)\|\mathbf{E}\|Y(k+k_n)-\overline{Y}(k)\|+\mathbf{E}\|X(k+k_n)-\overline{X}(k)\|\mathbf{E}\|\overline{Y}(k)\| \\
&\le M_1\mathbf{E}\|Y(k+k_n)-\overline{Y}(k)\|+M_2\mathbf{E}\|X(k+k_n)-\overline{X}(k)\|\,.
\end{aligned}$$

Hence, $\lim_{n\to\infty}\mathbf{E}\|X(k+k_n)Y(k+k_n)-\overline{X}(k)\overline{Y}(k)\|=0$ for each $k\in\mathbb{Z}$.

In a similar way, we can also prove

$$\lim_{n\to\infty}\mathbf{E}\|\overline{X}(k-k_n)\overline{Y}(k-k_n)-X(k)Y(k)\|=0$$

for each $k\in\mathbb{Z}$, and therefore XY is almost automorphic in mean. □

For applications to nonlinear stochastic difference equations the following concept of almost automorphic random sequence depending on parameters will be useful.

Let $(\mathbb{B}_1,\|\cdot\|_1)$ and $(\mathbb{B}_2,\|\cdot\|_2)$ be Banach spaces and let $L^1(\Omega;\mathbb{B}_1)$ and $L^1(\Omega;\mathbb{B}_2)$ be their corresponding L^1-spaces, respectively.

Definition 2.11. A function $F : \mathbb{Z} \times L^1(\Omega;\mathbb{B}_1) \mapsto L^1(\Omega;\mathbb{B}_2)$, $(n,U) \mapsto F(n,U)$ is said to be almost automorphic in mean in $n \in \mathbb{Z}$ uniformly in $U \in L^1(\Omega;\mathbb{B}_1)$, if for every sequence $\{k'_n\} \subset \mathbb{Z}$ there exists a subsequence $\{k_n\}$ such that

$$\lim_{n\to\infty} F(k+k_n,U) =: \bar{F}(k,U) \text{ in } L^1(\Omega;\mathbb{B}_2) \tag{2.4}$$

is well defined for each $k \in \mathbb{Z}$, $U \in L^1(\Omega;\mathbb{B}_1)$, and

$$\lim_{n\to\infty} \bar{F}(k-k_n,U) = F(k,U) \text{ in } L^1(\Omega;\mathbb{B}_2) \tag{2.5}$$

for each $k \in \mathbb{Z}$ and $U \in L^1(\Omega;\mathbb{B}_1)$.

The proofs of the following results follows the same lines as in the deterministic continuous case (see [12] Theorem 2.1.3).

Theorem 2.12. *Let F, $G : \mathbb{Z} \times L^1(\Omega;\mathbb{B}_1) \to L^1(\Omega;\mathbb{B}_2)$ be almost automorphic random sequences in $k \in \mathbb{Z}$ uniformly in $U \in L^1(\Omega;\mathbb{B}_1)$. Then the following assertions hold:*

(i) *$F+G$ is almost automorphic in mean;*

(ii) *cF is almost automorphic in mean for every scalar c;*

(iii) $\sup_{k\in\mathbb{Z}} \mathbf{E}\|F,U(k)\| = M_U < \infty$ *for each $U \in L^1(\Omega;\mathbb{B}_1)$;*

(vi) $\sup_{k\in\mathbb{Z}} \mathbf{E}\|\bar{F}(k,U)\| = N_U < \infty$ *for each $U \in L^1(\Omega;\mathbb{B}_1)$, where $\bar{F}$ is defined in (2.4) and (2.5).*

We now state the following composition result.

Theorem 2.13. *Let $F : \mathbb{Z}_+ \times L^1(\Omega;\mathbb{B}_1) \to L^1(\Omega;\mathbb{B}_2)$, $(n,U) \mapsto F(n,U)$ be almost automorphic in mean in $n \in \mathbb{Z}_+$ uniformly in $U \in L^1(\Omega;\mathbb{B}_1)$. If in addition, F is Lipschitz in U in the following sense: there exists $L > 0$ such that*

$$\mathbf{E}\|F(t,U) - F(t,V)\|_2 \le L\,\mathbf{E}\|U - V\|_1 \ \ \forall\, U,V \in L^1(\Omega;\mathbb{B}_1),\, n \in \mathbb{Z}_+$$

then for any almost automorphic random sequence $X = \{X(n)\}_{n\in\mathbb{Z}}$, then the $L^1(\Omega;\mathbb{B}_1)$-valued random sequence $Y(n) = F(n,X(n))$ is almost automorphic in mean.

The following result will play a key role in the study of almost automorphic solutions of linear and nonlinear stochastic Volterra difference equations.

Theorem 2.14. *Let $b : \mathbb{Z}_+ \to \mathbb{C}$ be a summable sequence, i.e $\sum_{l=0}^{\infty} |b(l)| < \infty$. Then for any almost automorphic random sequence $X : \mathbb{Z} \to L^1(\Omega;\mathbb{B})$, the random sequence $W(\cdot)$ defined by*

$$W(k) = \sum_{l=-\infty}^{k} b(k-l)X(l),\ k \in \mathbb{Z}$$

is also almost automorphic in mean.

Proof. Let (k_n') be an arbitrary sequence of integers. Since X is almost automorphic in mean, there exists a subsequence (k_n) of (k_n') such that

$$\lim_{n\to\infty} \mathbf{E}\|X(k+k_n)-\bar{X}(k)\| = 0$$

for each $k \in \mathbb{Z}$ and

$$\lim_{n\to\infty} \mathbf{E}\|\bar{X}(k-k_n)-X(k)\| = 0$$

for each $k \in \mathbb{Z}$.

Define $\bar{W}(k) = \sum_{l=-\infty}^{k} b(k-l)\bar{X}(l)$. We then have

$$\begin{aligned}
\mathbf{E}\|W(k+k_n)-\bar{W}(k)\| &= \mathbf{E}\Big\|\sum_{k=-\infty}^{k+k_n} b(k+k_n-l)X(l) - \sum_{l=-\infty}^{k} b(k-l)\bar{X}(l)\Big\| \\
&= \mathbf{E}\Big\|\sum_{l=-\infty}^{k} b(k-l)[X(l+k_n)-\bar{X}(l)]\Big\| \\
&\le \sum_{l=-\infty}^{k} |b(k-l)|\mathbf{E}\|X(l+k_n)-\bar{X}(l)\|\,.
\end{aligned}$$

Note that

$$\mathbf{E}\|W(k)\| \le \sum_{l=-\infty}^{k} |b(k-l)|\mathbf{E}\|X(l)\| \le \|X\|_\infty \sum_{l=-\infty}^{k} |b(k-l)| < \infty\,.$$

Thus, by Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \mathbf{E}\|W(k+k_n)-\bar{W}(k)\| \le \sum_{l=-\infty}^{k} |b(k-l)| \lim_{n\to\infty} \mathbf{E}\|X(l+k_n)-\bar{X}(l)\| = 0\,.$$

Hence,

$$\lim_{n\to\infty} \mathbf{E}\|W(k+k_n)-\bar{W}(k)\| = 0$$

for each $k \in \mathbb{Z}$.

In a similar way, we can also prove

$$\lim_{n\to\infty} \mathbf{E}\|\bar{W}(k-k_n)-W(k)\| = 0$$

for each $k \in \mathbb{Z}$, and therefore W is almost automorphic in mean. □

3 Almost Automorphic Solutions of Stochastic Volterra Difference Equations

3.1 Linear case

In this subsection we study the existence of almost automorphic solutions for linear stochastic Volterra difference equation of type

$$X(\omega,n+1) = \sum_{j=-\infty}^{n} a(n-j)TX(\omega,j) + f(n)\xi(\omega,n+1)\,,\ \omega\in\Omega,\ n\in\mathbb{Z} \tag{3.1}$$

where T is a bounded linear operator on $\mathbb{B}$, $a : \mathbb{N} \to \mathbb{C}$ is summable, f is an almost automorphic function. Each $\xi(n)$ is a real-valued random variable.

We make the following assumption about ξ through the paper:

(i) $\xi = \{\xi(n), n \in \mathbb{Z}\}$ is a sequence of independent real-valued random variables;

(ii) ξ is an almost automorphic random sequence.

Let $\mathcal{L}(\mathbb{B})$ be the space of all bounded linear operator on $\mathbb{B}$. For $T \in \mathcal{L}(\mathbb{B})$, define $s(T,k) \in \mathcal{L}(\mathbb{B})$ as a solution of the difference equation

$$\begin{aligned} s(T,k+1) &= \sum_{k=0}^{k} Ta(k-j)s(T,j), \; k = 0,1,2,\cdots \\ s(T,0) &= I \end{aligned}$$

We have the following theorem.

Theorem 3.1. *Assume that $s(T,\cdot)$ is a summable function and that $f : \mathbb{Z} \to \mathbb{B}$ is an almost automorphic function. Then, Eq.(3.1) has an almost automorphic solution given by*

$$X(n+1) = \sum_{k=-\infty}^{n} s(T,n-k)f(k)\xi(k+1)\,. \tag{3.2}$$

Proof. Let X be the random sequence given in (3.2). Using the fact that T is linear and that the function $s(T,\cdot)$ is summable, we then have

$$\begin{aligned} &\sum_{j=-\infty}^{n} a(n-j)TX(j) \\ &= \sum_{j=-\infty}^{n} a(n-j)T\Big(\sum_{\tau=-\infty}^{j-1} s(T,j-1-\tau)f(\tau)\xi(\tau+1)\Big) \\ &= T\sum_{j=-\infty}^{n-1}\sum_{\tau=-\infty}^{j} a(n-1-j)s(T,j-\tau)f(\tau)\xi(\tau+1) \\ &= T\sum_{\tau=-\infty}^{n-1}\sum_{j=\tau}^{n-1} a(n-1-j)s(T,j-\tau)f(\tau)\xi(\tau+1) \\ &= \sum_{\tau=-\infty}^{n-1}\Big(\sum_{j=0}^{n-1-\tau} Ta(n-1-\tau-j)s(T,j)\Big)f(\tau)\xi(\tau+1) \\ &= \sum_{\tau=-\infty}^{n-1} s(T,n-\tau)f(\tau)\xi(\tau+1) \\ &= \sum_{\tau=-\infty}^{n} s(T,n-\tau)f(\tau)\xi(\tau+1) - s(T,0)f(n)\xi(n+1) \end{aligned}$$

$$= X(n+1) - f(n)\xi(n+1)$$

which proves that X is the solution of Eq.(3.1). Applying Theorems 2.7 and 2.9, we conclude that X is almost automorphic in mean. □

3.2 Nonlinear case

To analyze Eq.(1.1), our strategy consists of studying the existence of almost automorphic solutions to the corresponding stochastic Volterra difference of the form:

$$X(n+1) = \sum_{j=-\infty}^{n} a(n-j)TX(j) + f(n, X(n))\xi(n+1)\,,\; n \in \mathbb{Z} \tag{3.3}$$

where T is a bounded linear operator on $L^1(\Omega, \mathbb{B})$, $a : \mathbb{N} \to \mathbb{C}$ is summable, f is an almost automorphic random function in $n \in \mathbb{Z}$ uniformly in the second variable. Each $\xi(n)$ is a real-valued random variable.

In addition to Assumptions (i) and (ii) on ξ, we assume that

(iii) ξ is independent of $X = \{X(n), n \in \mathbb{Z}\}$, the solution of Eq. (3.3).

For $T \in \mathcal{L}(\mathbb{B})$, define $N_T = \sum_{j=0}^{\infty} \|s(T, j)\|$.

We now state our main result.

Theorem 3.2. *Let $f : \mathbb{Z} \times L^1(\Omega; \mathbb{B}) \to L^1(\Omega; \mathbb{B})$ be an almost automorphic random function in $k \in \mathbb{Z}$ for each $U \in L^1(\Omega; \mathbb{B})$. Suppose that f satisfies Lipschitz condition: there exists an $L > 0$ such that*

$$\mathbf{E}\|f(k,U) - f(k,V)\| \le L\mathbf{E}\|U - V\|\,, \tag{3.4}$$

for all $U, V \in L^1(\Omega; \mathbb{B})$ and $k \in \mathbb{Z}$. Then, Eq.(3.3) has a unique almost automorphic solution X defined by

$$X(n+1) = \sum_{j=-\infty}^{n} s(T, n-k) f(k, X(k))\xi(k+1)$$

provided that $MLN_T < 1$.

Proof. We define the nonlinear operator $\Gamma : AA(\mathbb{Z}; L^1(\Omega; \mathbb{B})) \to AA(\mathbb{Z}; L^1(\Omega; \mathbb{B}))$ by

$$F(U)(n) = \sum_{j=-\infty}^{n-1} s(T, n-1-k) f(k, U(k))\xi(k+1)\,.$$

Since $U \in AA(\mathbb{Z}; L^1(\Omega; \mathbb{B}))$ and f satisfies (3.4), it follows from Theorem 2.13 that $f(\cdot, X(\cdot))$ is in $AA(\mathbb{Z}; L^1(\Omega; \mathbb{B}))$. We deduce from Theorems 2.10 and 2.14 that Γ is well defined.

Now, let U and $V \in AA(\mathbb{Z}; L^1(\Omega;\mathbb{B}))$ chosen independently of ξ. We then have

$$\begin{aligned}
&\mathbf{E}\|\Gamma(U)(n)-\Gamma(V)(n)\| \\
&\le \sum_{k=-\infty}^{n-1} |s(T,n-1-k)|\mathbf{E}\left[\|f(k,U(k))-f(k,V(k))\|\,|\xi(k+1)|\right] \\
&= \sum_{k=-\infty}^{n-1} |s(T,n-1-k)|\mathbf{E}\|f(k,U(k))-f(k,V(k))\|\,\mathbf{E}|\xi(k+1)| \\
&\le ML \sum_{k=-\infty}^{n-1} |s(T,n-1-k)|\mathbf{E}\|U(k)-V(k)\| \\
&\le MLN_T\|U-V\|_\infty \sum_{k=-\infty}^{n-1} |s(T,n-1-k)| \\
&\le MLN_T\|U-V\|_\infty
\end{aligned}$$

for any $n \in \mathbb{Z}$.

Thus,

$$\|\Gamma(U)-\Gamma(V)\|_\infty \le MLN_T\|U-V\|_\infty .$$

Hence, Γ is a contraction provided that $MLN_T < 1$. Using the Banach fixed point theorem, we obtain that Γ has a unique fixed point $\overline{U}$, which is the unique almost automorphic solution of Eq.(3.3). □

4 Application

For a given $\lambda \in \mathbb{C}$, define $s(\lambda,k) \in \mathbb{C}$ as a solution of the difference equation

$$s(\lambda,k+1) = \lambda \sum_{k=0}^{k} p^{n-j} s(\lambda,j),\ k = 0,1,2,\cdots \tag{4.1}$$

$$s(\lambda,0) = 1 \tag{4.2}$$

where $|p| < 1$.

Define

$$C_s = \left\{\lambda \in \mathbb{C} : \sum_{k=0}^{\infty} |s(\lambda,k)| < \infty\right\}.$$

Using (4.1)-(4.2), it is not hard to show that $s(\lambda,k) = \lambda(\lambda+p)^{k-1}$, $k \ge 1$, and hence

$$\mathbb{D}(-p,1) := \{z \in \mathbb{C} : |z+p| < 1\} \subset C_s .$$

Let $|p| < 1$ be fixed and take $\lambda \in \mathbb{D}(-p,1)$. Consider the following stochastic difference equation

$$X(n+1) = \lambda \sum_{j=-\infty}^{n} p^{n-j} X(j) + \sin\Big(\frac{1}{2-\sin(n)-\sin(\sqrt{2}n)}\Big) X(n)\xi(n+1),\ n \in \mathbb{Z} \tag{4.3}$$

where $\xi = \{\xi(n), n \in \mathbb{Z}\}$ is almost automorphic random sequence.
By Theorem 3.2, Equation (4.3) has an almost automorphic solution X given by

$$X(n+1) = \lambda \sum_{k=-\infty}^{n} (p+\lambda)^{n-k} \sin\Big(\frac{1}{2 - \sin(k) - \sin(\sqrt{2}k)}\Big) X(k)\xi(k+1)\,.$$

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