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Observability of Linear Difference Equations in Hilbert Spaces and Applications

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Abstract

Here we present a necessary and sufficient conditions for the exact and approximate observability of the following linear difference equation

$$\begin{cases} z(n+1) = A(n)z(n), \quad n \in \mathbb{N}^*, \quad z(0) = z_0 \in Z, \\ y(n) = Cz(n), \end{cases}$$

where $A \in l^\infty(\mathbb{N}, L(Z))$, $C \in L(Z,U)$, Z, U are Hilbert spaces and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$. We apply these results to a flow-discretization of the wave equation and the heat equation.

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1 Introduction

Observability is an important property of a control system, and this governs the existence of an optimal control solution. Roughly speaking, observability means that it is possible to determine the internal states of a system by measuring only the external outputs. Hence it is useful in solving the problem of reconstructing unmeasurable state variables from measurable ones. Formally, a system is said to be observable if, for any possible sequence of state and control vectors, the current state can be determined in finite time using only the outputs.

The observability problem has been studied by many authors, one can see, [2], [4], [5], but in their works they only deal with continuous systems. With respect to discrete systems, there are a few works where the study of the observability is considered for systems like

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(1.1), we can cite [3], [6], [12], [13], but some of these works are in finite dimension spaces and only the exact observability is characterized. In others words, the approximate observability is not studied in those works, and their techniques are based on the concept of detectability or admissibility.

In this work we present necessary and sufficient conditions for the exact and approximate observability of the following linear difference equation

$$\begin{cases} z(n+1) = A(n)z(n), \quad n \in \mathbb{N}^*, \quad z(0) = z_0 \in Z, \\ y(n) = Cz(n). \end{cases} \tag{1.1}$$

where *the state* $z(n) \in Z$, Z is a Hilbert space, $\mathbb{N}^* = \mathbb{N} \cup \{0\}$, $A \in l^\infty(\mathbb{N}, L(Z))$, $C \in L(Z,U)$, where U is another Hilbert space.
Consider the set $\Delta = \{(m,n) \in \mathbb{N} \times \mathbb{N} : m \geq n\}$ and let $\Phi = \{\Phi(m,n)\}_{(m,n)\in\Delta}$ be the evolution operator associated to A, i.e.,

$$\Phi(m,n) = \begin{cases} A(m-1)\cdots A(n), & m > n, \\ I, & m = n, \end{cases}$$

where I is the identity operator in the Hilbert space Z.

Then the state $z(\cdot)$ of (1.1) is given by

$$z(n) = \Phi(n,0)z(0), \quad n \in \mathbb{N}, \tag{1.2}$$

and the *output* $y(\cdot)$ given by (1.1) takes the form

$$y(n) = C\Phi(n,0)z(0).$$

Here we will employ the notation used in [2] and [10].

In this paper we exhibit results that characterize both the exact and approximate observability of (1.1). In Section 2, present some results needed to characterize both the exact and approximate observability of system (1.1), showing its duality with the controlled system

$$z(n+1) = A^*(n)z(n) + C^*u(n), \qquad z(0) = z_0, \tag{1.3}$$

where the *input* $u(n) \in U$. In order to reach our goals we use the concepts of exact and approximate controllability and follow the techniques used in [10]. In Section 3, we present new characterizations of the exact and approximate observability different than used in [3], [6], [12] and [13]. Finally, in Section 4, we will apply these results to a discrete version of the wave and heat equations.

2 Preliminaries

In this section we give the definition of observability for the system (1.1) and present some results needed to characterize both the exact and approximate observability in the next section.

Definition 2.1. (see [2]) For the system (1.1) we define the following concepts:

a) The **observability map** (for $n \in \mathbb{N}$) is define as follows $C^n : Z \longrightarrow l^2(\mathbb{N}, U)$ by

$$C^n z = \begin{cases} C\Phi(n,k)z, & k \leq n, \\ 0, & k > n. \end{cases} \tag{2.1}$$

b) The **observability grammian map** (for $n \in \mathbb{N}$) is define by $L_{C^n} = C^{n*}C^n$.

c) The system (1.1) is **exactly observable** if for each $z_0 \in Z$, there exists $n_0 \in \mathbb{N}$ such that $z(0) = z_0$ can be uniquely and continuously constructed from the knowledge of the outputs (observations) $y(0), y(1), \dots, y(n_0 - 1)$, i.e., if C^{n_0} is injective and its inverse is bounded on the range of C^{n_0}.

d) The system (1.1) is **approximately observable** if for each $z_0 \in Z$, there exists $n_0 \in \mathbb{N}$ such that the knowledge of the outputs $y(0), y(1), \dots, y(n_0 - 1)$ determine the initial state $z(0) = z_0$ uniquely, i.e., if $Ker(C^{n_0}) = \{0\}$.

Proposition 2.2. *The adjoint C^{n_0*} of the operator C^{n_0} is given by*
$C^{n_0*} : l^2(\mathbb{N}, U) \longrightarrow Z$

$$(C^{n_0*}u)(k-1) = \sum_{k=1}^{n_0} \Phi^*(n_0, k)C^*u. \tag{2.2}$$

and

$$L_{C^{n_0}} z = \sum_{k=1}^{n_0} \Phi^*(n_0, k)C^*C\Phi(n_0, k)z, \quad z \in Z. \tag{2.3}$$

Proof

$$\begin{aligned}
\langle C^{n_0} z, u\rangle_{l^2,\ l^2} &= \sum_{k=1}^{\infty} \langle C\Phi(n_0,k)z, u(k-1)\rangle_{U,U} \\
&= \sum_{k=1}^{n_0} \langle C\Phi(n_0,k)z, u(k-1)\rangle_{U,U} + \sum_{k=n+1}^{\infty} \langle 0, u(k-1)\rangle_{U,U} \\
&= \sum_{k=1}^{n_0} \langle C\Phi(n_0,k)z, u(k-1)\rangle_{U,U} \\
&= \sum_{k=1}^{n_0} \langle z, \Phi^*(n_0,k)C^*u(k-1)\rangle_{Z,Z} \\
&= \left\langle z, \sum_{k=1}^{n_0} \Phi^*(n_0,k)C^*u(k-1) \right\rangle_{Z,Z} \\
&= \langle z, C^{n_0*}u\rangle_{Z,Z}
\end{aligned}$$

which prove (2.2). Clearly, (2.3) follows immediately from definition 2.1 *b*) and (2.2).

□

Consider the dual control system of (1.1).

$$z(n+1) = A^*(n)z(n) + C^*u(n), \qquad z(0) = z_0, \tag{2.4}$$

where the *inputs* $u(n) \in U$.

Then, for this control system we have the usual definitions of exact and approximate controllability.

Definition 2.3. The system (2.4) is said to be **exactly controlable** if there is $n_0 \in \mathbb{N}$ such that for every z_0, $z_1 \in Z$ there exists $u \in l^2(\mathbb{N}, U)$ such that $z(0) = z_0$ and $z(n_0) = z_1$.

Definition 2.4. The system (2.4) is said to be **approximately controlable** if there is $n_0 \in \mathbb{N}$ such that for every z_0, $z_1 \in Z$, $\varepsilon > 0$ there exists $u \in l^2(\mathbb{N}, U)$ such that $z(0) = z_0$ and $\|z(n_0) - z_1\| < \varepsilon$.

Definition 2.5. (see [2], [10]) For the system (2.4) we introduce the following concepts:

a) The **controllability map**, $\mathcal{B}^n : l^2(\mathbb{N}, U) \longrightarrow Z$ (for $n \in \mathbb{N}$), is define as follows by

$$\mathcal{B}^n u = \sum_{k=1}^{n} \Phi^*(n,k) B u(k-1), \tag{2.5}$$

where $B = C^*$.

b) The **grammian map** (for $n \in \mathbb{N}$) is define by $L_{\mathcal{B}^n} = \mathcal{B}^n \mathcal{B}^{n*}$.

The following theorem is a discrete version of Theorem 4.1.7 of [2] and its proof may be seen in [10].

Theorem 2.6. *(a) The system (2.4) is exactly controllable for some $n \in \mathbb{N}$ if, and only if, one of the following statements holds:*

(i) $\text{Range}(\mathcal{B}^n) = Z$

(ii) There exists $\gamma > 0$ such that

$$\langle L_{\mathcal{B}^n} z, z\rangle \geq \gamma \|z\|_Z^2, \quad \forall z \in Z,$$

(iii) There exists $\gamma > 0$ such that

$$\|\mathcal{B}^* z\|^2_{l^2(\mathbb{N},U)} \geq \gamma \|z\|_Z^2, \quad \forall z \in Z.$$

(b) The system (2.4) is approximately controllable for some $n \in \mathbb{N}$ if, and only if, one of the following statements holds:

(i) $\text{Ker}(\mathcal{B}^{n*}) = \{0\}$ *and $\mathcal{B}^{n*}$ has close range.*

(ii) $\langle L_{\mathcal{B}^n} z, z\rangle > 0$, *$z \neq 0$ in Z.*

(iii) $B^* \Phi^*(n,k) z = 0 \Rightarrow z = 0$, *$k \geq n$.*

(iv) $\overline{\text{Range}(\mathcal{B}^n)} = Z$.

The following lemma establishes a duality between controllability and observability.

Lemma 2.7. *For the system (1.1) we have the following duality results:*

(a) The system (1.1) is approximately observable in $n \in \mathbb{N}$ if, and only if, the dual system (2.4) is approximately controllable in $n \in \mathbb{N}$.

(b) The system (1.1) is exactly observable in $n \in \mathbb{N}$ if, and only if, the dual system (2.4) is exactly controllable in $n \in \mathbb{N}$.

Proof Let us denote the controllability map of (2.4) by $\mathcal{B}^n$ (see [10]), then, from definition of the controllability map, we know that $\mathcal{C}^{n*} = \mathcal{B}^n$ and $\mathcal{C}^n = \mathcal{B}^{n*}$.

(a) (1.1) is approximately observable, iff $\mathrm{Ker}(\mathcal{C}^n) = \{0\}$, iff $\mathrm{Ker}(\mathcal{B}^{n*}) = \{0\}$, iff (2.4) is approximately controllable (see Theorem 2.6).

(b) Let us suppose that (1.1) is exactly observable. Then there exists $(\mathcal{C}^n)^{-1}$ on Range$(\mathcal{C}^n)$ and it is bounded. Thus, $(\mathcal{C}^n)^{-1}\mathcal{C}^n z = z$, $\forall z \in Z$ and

$$||(\mathcal{C}^n)^{-1}y|| \leq M||y||, \quad \forall y \in \mathrm{Range}(\mathcal{C}^n).$$

Then we have that

$$||z|| = ||(\mathcal{C}^n)^{-1}\mathcal{C}^n z|| \leq M||\mathcal{C}^n z|| = M||\mathcal{B}^{n*}z||.$$

i.e.,

$$||\mathcal{B}^{n*}z|| \geq \frac{1}{M}||z||.$$

Therefore, by Theorem 2.6, we have that (2.4) is exactly controllable.

Now, let us suppose that (2.4) is exactly controllable, then, by Theorem 2.6, we have that $\mathcal{B}^{n*}$ is injective and has closed range. In consequence, $\mathcal{C}^n$ is injective and has closed range, which implies that (1.1) is exactly observable. □

The following corollary is immediately consequence of Theorem 2.6 and Lemma 2.7.

Corollary 2.8. *(a) The system (1.1) is exactly observable, for some $n \in \mathbb{N}$, if, and only if, one of the following statements holds:*

(i) $\mathrm{Ker}(\mathcal{C}^n) = \{0\}$ *and* $\mathcal{C}^n$ *has close range.*

(ii) There exists $\gamma > 0$ *such that*

$$\langle L_{\mathcal{C}^n} z, z\rangle \geq \gamma ||z||_Z^2, \quad \forall z \in Z.$$

(iii) There exists $\gamma > 0$ *such that*

$$||\mathcal{C}^n z||^2 \geq \gamma ||z||^2, \quad \forall z \in Z.$$

(b) The system (1.1) is approximately observable for some $n \in \mathbb{N}$ if, and only if, one of the following statements holds:

(i) $\mathrm{Ker}(\mathcal{C}^n) = \{0\}$.

(ii) $\langle L_{\mathcal{C}^n} z, z\rangle > 0$, $z \neq 0$ *in* Z.

(iii) $C\Phi(n,k)z = 0 \Rightarrow z = 0$, $k \geq n$.

3 Characterizations of the Observability

In this section we present new characterizations of the exact and approximate observability different than the one mentioned in the foregoing sections.

Lemma 3.1. *The system (1.1) is exactly observable for some* $n_0 \in \mathbb{N}$ *if, and only if,* $L_{C^{n_0}}$ *es invertible.*

Proof Suppose that the system (1.1) is exactly observable. Then, from Corollary 2.8 $(a)-(iii)$, there exists $\gamma > 0$ such that $||C^{n_0}z|| \geq \gamma ||z||$, for all $z \in Z$, i.e.,

$$||C^{n_0}z||^2 \geq \gamma^2 ||z||^2, \quad z \in Z,$$

equivalently,

$$\langle C^{n_0*}C^{n_0}z, z\rangle \geq \gamma^2 ||z||^2, \quad z \in Z,$$

and,

$$\langle L_{C^{n_0}}z, z\rangle \geq \gamma^2 ||z||^2, \quad z \in Z. \tag{3.1}$$

This implies that $L_{C^{n_0}}$ is injective. Now, we probe that $L_{C^{n_0}}$ es surjective. That is,

$$\mathcal{R}(L_{C^{n_0}}) = \text{Range}(L_{C^{n_0}}) = Z.$$

For the purpose of contradiction, suppose that $\mathcal{R}(L_{C^{n_0}}) \subsetneq Z$. On the other hand, using the Cauchy-Schwarz inequality and (3.1) we obtain

$$||L_{C^{n_0}}z||_{l^2} \geq \gamma^2 ||z||, z \in Z,$$

which implies that $\mathcal{R}(L_{C^{n_0}})$ is closed. From here, applying the Hahn Banach Theorem, we can prove that $\text{Range}(L_{C^{n_0}}) = Z$. In consequence, $L_{C^{n_0}}$ is a bijection and from Open Mapping Theorem, $L^{-1}_{C^{n_0}}$ is a bounded linear operator.
Now we suppose that $L_{C^{n_0}} = C^{n_0*}C^{n_0}$ is invertible. Then, from Theorem 2.6 and Lemma 2.7 we have that the system (1.1) is exactly observable. □

Lemma 3.2. *The system (1.1) is exactly observable for some* $n_0 \in \mathbb{N}$ *if, and only if,*

$$\sup_{\alpha \in (0,1]} ||(\alpha I + L_{C^{n_0}})^{-1}|| < \infty. \tag{3.2}$$

Proof Suppose that (1.1) is exactly observable. Then, from Corollary 2.8 $(a)-(ii)$, there exists $\gamma > 0$ such that

$$\langle L_{C^{n_0}}z, z\rangle \geq \gamma ||z||^2_Z, \quad \forall z \in Z.$$

Then, for all $z \in Z$ and $\alpha \geq 0$, we have

$$\langle z, (\alpha I + L_{C^{n_0}})z\rangle = \langle z, \alpha z\rangle + \langle z, L_{C^{n_0}}z\rangle = \alpha ||z||^2 + \langle z, L_{C^{n_0}}z\rangle \geq (\alpha+\gamma)||z||^2,$$

i.e.,

$$\langle z, (\alpha I + L_{C^{n_0}})z\rangle \geq (\alpha+\gamma)||z||^2.$$

Using the Cauchy-Schwarz inequality, we obtain

$$\|(\alpha I + L_{C^{n_0}})z\| \geq (\alpha+\gamma)\|z\|.$$

So,

$$(\alpha+\gamma)\|(\alpha I + L_{C^{n_0}})^{-1}y\| \leq \|y\|,$$

in consequence, for all $\alpha \geq 0$,

$$\|(\alpha I + L_{C^{n_0}})^{-1}\| \leq \frac{1}{\alpha+\gamma} \leq \frac{1}{\gamma}.$$

Therefore, $\|(\alpha I + L_{C^{n_0}})^{-1}\|$ is bounded as function of $\alpha \geq 0$ and we have (3.2).

Reciprocally, suppose that (3.2) is true. This implies that there exists

$$\lim_{\alpha\to 0^+}(\alpha I + L_{C^{n_0}})^{-1}$$

and it is finite.
In fact, we know that $(\alpha I + L_{C^{n_0}})^{-1} = R(\alpha I, -L_{C^{n_0}})$ the resolvent of $-L_{C^{n_0}}$, and the identity for the resolvent,

$$R(\alpha I, -L_{C^{n_0}}) - R(\beta I, -L_{C^{n_0}}) = (\beta-\alpha)R(\alpha I, -L_{C^{n_0}})R(\beta I, -L_{C^{n_0}}),$$

together with (3.2), show that $\{R(\alpha I, -L_{C^{n_0}})\}$ is a Cauchy sequence of bounded linear operators. Therefore,

$$S = \lim_{\alpha\to 0^+} R(\alpha I, -L_{C^{n_0}}) = \lim_{\alpha\to 0^+}(\alpha I + L_{C^{n_0}})^{-1}.$$

Then,

$$L_{C^{n_0}}(\lim_{\alpha\to 0^+}(\alpha I + L_{C^{n_0}})^{-1}) = L_{C^{n_0}}S.$$

So,

$$\lim_{\alpha\to 0^+}(\alpha I + L_{C^{n_0}} - \alpha I)(\alpha I + L_{C^{n_0}})^{-1} = L_{C^{n_0}}S,$$

i.e.,

$$I - \lim_{\alpha\to 0^+}\alpha(\alpha I + L_{C^{n_0}})^{-1} = L_{C^{n_0}}S.$$

But the condition (3.2) implies that

$$\lim_{\alpha\to 0^+}\alpha(\alpha I + L_{C^{n_0}})^{-1} = 0.$$

Therefore, for all $z \in Z$, we have that

$$z = L_{C^{n_0}}Sz = C^{n_0 *}C^{n_0}Sz.$$

So, C^{n_0} is injective and the proof follows from Corollary 2.8. $\square$

With respect to approximate observability of the system (1.1), we have the following characterizations.

Lemma 3.3. *The system (1.1) is approximately observable for some $n_0 \in \mathbb{N}$ if, and only if,* $\overline{\text{Range}(L_{C^{n_0}})} = Z$.

Proof Suppose the system (1.1) is approximately observable for some $n_0 \in \mathbb{N}^*$. Then, from Corollary 2.8 $(b)-(ii)$ we have that

$$\langle L_{C^{n_0}} z, z\rangle > 0, \quad \forall z \in Z, \quad z \neq 0. \tag{3.3}$$

For the purpose of contradiction, let us assume that

$$\overline{\text{Range}(L_{C^{n_0}})} \subset Z.$$

Then, from Hanh-Banach's Theorem there exists $z_0 \neq 0$ such that

$$\langle L_{C^{n_0}} z, z_0\rangle = 0, \quad \forall z \in Z.$$

In particular, if we put $z = z_0$, then $\langle L_{C^{n_0}} z_0, z_0\rangle = 0$, which contradicts (3.3).
Now, suppose that $\overline{\text{Range}(L_{C^{n_0}})} = Z$, i.e., $\overline{\text{Range}(C^{n_0 *}C^{n_0})} = Z$, then $\overline{\text{Range}(C^{n_0 *})} = Z$. Then, from Theorem 2.6 and Lemma 2.7 we have that (1.1) is approximately observable. □

Lemma 3.4. *The system (1.1) is approximately observable for some* $n_0 \in \mathbb{N}$ *if, and only if, for each* $z \in Z$,

$$\lim_{\alpha \to 0^+} \alpha(\alpha I + L_{C^{n_0}})^{-1} z = 0. \tag{3.4}$$

Proof Suppose that the system (1.1) is approximately observable for some $n_0 \in \mathbb{N}$. Then, from Corollary 2.8 $(b)-(ii)$, we have that, for $z \neq 0$ in Z

$$\langle L_{C^{n_0}} z, z\rangle > 0. \tag{3.5}$$

Suppose that there exists $z_0 \in Z$ such that

$$\lim_{\alpha \to 0^+} \alpha(\alpha I + L_{C^{n_0}})^{-1} z_0 = y_0 \neq 0.$$

Then,

$$\lim_{\alpha \to 0^+} \alpha L_{C^{n_0}}(\alpha I + L_{C^{n_0}})^{-1} z_0 = L_{C^{n_0}} y_0,$$

and

$$\lim_{\alpha \to 0^+} \alpha z_0 - \alpha[\alpha(\alpha I + L_{C^{n_0}})^{-1} z_0] = L_{C^{n_0}} y_0.$$

That is, $L_{C^{n_0}} y_0 = 0$, and this contradicts (3.5). Therefore, (3.4) is true.

Reciprocally, suppose that

$$\lim_{\alpha \to 0^+} \alpha(\alpha I + L_{C^{n_0}})^{-1} z = 0, \forall z \in Z.$$

We want to probe that $\overline{\text{Range}(L_{C^{n_0}})} = Z$.

For all $z \in Z$, let us define

$$u_\alpha = (\alpha I + L_{C^{n_0}})^{-1} z,$$

then

$$\begin{aligned} L_{C^{n_0}} u_\alpha &= (\alpha I + L_{C^{n_0}} - \alpha I)(\alpha I + L_{C^{n_0}})^{-1} z \\ &= z - \alpha(\alpha I + L_{C^{n_0}})^{-1} z. \end{aligned}$$

From that and (3.4) it follows that

$$\lim_{\alpha \to 0^+} L_{C^{n_0}} u_\alpha = z.$$

In consequence, the system (1.1) is approximately observable. □

4 Applications

Now, as an application of the main results of this research we shall consider two important examples, a flow-discretization of the wave equation and the heat equation.

Example 4.1. Wave Equation

Consider the wave equation

$$\begin{cases} w_{tt} = w_x + u(t,x), \\ w(t,0) = w(t,1) = 0, \\ w(0,x) = w_0, w_t(0,x) = w_1(x), \end{cases} \tag{4.1}$$

with observation in derivative, i.e., $y(t,x) = w_t(t,x)$.

The system (4.1) can be written as an abstract second order equation in the Hilbert space $X = L^2[0,1]$ as follows:

$$\begin{cases} w'' = -Aw + u(t), \\ w(0) = w_0, w'(0) = w_1, \\ y = w', \end{cases} \tag{4.2}$$

where the operator A is given by $A\phi = -\phi_{xx}$ with domain $D(A) = H^2 \cap H_0^1$, and has the following spectral decomposition.

For all $x \in D(A)$ we have

$$Ax = \sum_{j=1}^{\infty} \lambda_j \langle x, \phi_j \rangle \phi_j = \sum_{j=1}^{\infty} \lambda_j E_j x$$

where $\lambda_j = j^2\pi^2$, $\phi_j(x) = \sqrt{2}\sin(j\pi x)$, $\langle \cdot, \cdot \rangle$ is the inner product in X and $E_j x = \langle x, \phi_j \rangle \phi_j$.

So, $\{E_j\}$ is a family of complete orthogonal projections in X and $x = \sum_{j=1}^{\infty} E_j x$, $x \in X$.

Using the change of variables $y' = v$, the second order equation (4.2) can be written as a first order system of ordinary differential equations in the Hilbert space $Z = X^{1/2} \times X$ as

$$\begin{cases} z' = \mathcal{A}z + Bu(t), \quad z(0) = z_0, \quad z \in Z, \\ y = Cz, \end{cases} \tag{4.3}$$

where

$$z = \begin{bmatrix} w \\ v \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix}, \quad C = B^* = [0 \quad I], \tag{4.4}$$

$\mathcal{A}$ is an unbounded linear operator with domain $D(\mathcal{A}) = D(A) \times X$ and $u \in L^2(0,\tau,X) = U$. The proof of the following theorem follows from Theorem 3.1 (see [9]) by putting $c = 0$ and $d = 1$.

Theorem 4.2. *The operator $\mathcal{A}$ given by (4.4), is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\in\mathbb{R}}$ given by*

$$T(t)z = \sum_{j=1}^{\infty} e^{A_j t} P_j z, \quad z \in Z, \quad t \geq 0, \tag{4.5}$$

where $\{P_j\}_{j\geq 1}$ *is a complete family of orthogonal projections in the Hilbert space* Z *given by*

$$P_j = diag[E_j, E_j], j \geq 1, \tag{4.6}$$

and

$$A_j = \widetilde{B}_j P_j, \qquad \widetilde{B}_j = \begin{bmatrix} 0 & 1 \\ -\lambda_j & 0 \end{bmatrix}, j \geq 1. \tag{4.7}$$

Now, the discretization of (4.3) on flow is given by

$$\begin{cases} z(n+1) = T(n)z(n), \quad z(0) = z_0, \quad z \in Z, \\ y(n) = Cz(n). \end{cases} \tag{4.8}$$

Proposition 4.3. *The system (4.8) is approximately observable for any* $n_0 \in \mathbb{N}$.

Proof Consider the operator

$$C^{n_0} : Z \longrightarrow l^2(\mathbb{N}, U), \quad C^{n_0} z = \begin{cases} CT(\Theta(n_0,k))z, & k \leq n_0, \\ 0, & k > n_0. \end{cases}$$

Then

$$C^{n_0 *} u = \sum_{k=1}^{n_0} T^*(\Theta(n_0,k))C^* u,$$

and

$$L_{C^{n_0}} : Z \longrightarrow Z, \quad L_{C^{n_0}} = C^{n_0 *} C^{n_0}.$$

Since

$$C^* C = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix},$$

we obtain that

$$E_j C^* C = C^* C E_j, \quad j = 1, 2, 3, \ldots. \tag{4.9}$$

On the other hand, we have that $T^*(t) = T(-t)$. Then

$$\begin{aligned} L_{C^{n_0}} z &= \sum_{k=1}^{n_0} T^*(\Theta(n_0,k))C^* C T(\Theta(n_0,k))z \\ &= \sum_{k=1}^{n_0} \sum_{j=1}^{\infty} e^{-A_j \Theta(n_0,k)} P_j C^* C \sum_{i=1}^{\infty} e^{A_i \Theta(n_0,k)} P_i z \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^{n_0} e^{-A_j \Theta(n_0,k)} C^* C e^{A_j \Theta(n_0,k)} P_j z \\ &= \sum_{j=1}^{\infty} L_{C_j^{n_0}} P_j z, \end{aligned}$$

where $L_{C_j^{n_0}} y = C_j^{n_0 *} C_j^{n_0} y = \sum_{k=1}^{n_0} e^{-A_j \Theta(n_0,k)} C^* C e^{A_j \Theta(n_0,k)} y, \quad y \in \mathcal{R}(P_j).$

Hence, $L_{C^{n_0}} = \sum_{j=1}^{\infty} L_{C_j^{n_0}}$.

Let $z = [z_1, z_2]^T$ in Z. Since

$$e^{\widetilde{B}_j s} = \left[cos(\sqrt{\lambda_j} s)\right] I + \frac{sen(\sqrt{\lambda_j} s)}{\sqrt{\lambda_j}} \widetilde{B}_j, \; j \geq 1,$$

we can see that

$$\begin{aligned} e^{-A_j \Theta(n_0,k)} C^* C e^{A_j \Theta(n_0,k)} P_j z &= e^{\widetilde{B}_j^* \Theta(n_0,k)} C^* C e^{\widetilde{B}_j \Theta(n_0,k)} P_j z \\ &= [0, E_j z_2]^T, \quad j \geq 1. \end{aligned}$$

So,

$$L_{C_j^{n_0}} P_j z = \sum_{k=1}^{n_0} [0, E_j z_2]^T = n_0 [0, E_j z_2]^T.$$

Then

$$\langle L_{C_j^{n_0}} P_j z, P_j z \rangle = \langle n_0 [0, E_j z_2]^T, [E_j z_1, E_j z_2]^T \rangle = n_0 \|E_j z_2\|^2 > 0, \;\; \forall j.$$

Hence, using (4.9), we get for $z \neq 0$ in Z that

$$\begin{aligned} \langle L_{C^{n_0}} z, z \rangle &= \langle \sum_{j=1}^{\infty} L_{C_j^{n_0}} P_j z, \sum_{j=1}^{\infty} P_j z \rangle \\ &= \sum_{j=1}^{\infty} \langle L_{C_j^{n_0}} P_j z, P_j z \rangle = n_0 \sum_{j=1}^{\infty} \|E_j z_2\|^2 = n_0 \|z_2\|^2 > 0. \end{aligned}$$

In consequence, by Corollary 2.8 part $(b)-(ii)$, the equation (4.8) is approximately observable. □

Example 4.4. Heat Equation

Consider the heat equation

$$\begin{cases} y_t = y_{xx} + u(t,x), \\ y(0,x) = y_0(x), \\ y_x(t,0) = y_x(t,1) = 0. \end{cases} \tag{4.10}$$

The system (4.10) can be written as an abstract equation in the Hilbert space $Z = L^2[0,1]$

$$\begin{cases} z' = -Az + Bu(t), \;\; z \in Z, \\ z(0) = z_0, \end{cases} \tag{4.11}$$

where $B = I$, the control function u belong to $L^2[0, r, Z]$ and the operator A is given by $A\phi = -\phi_{xx}$ with domain $D(A) = H^2 \cap H_0^1$, and has the following spectral decomposition.

a) For all $z \in D(A)$ we have

$$Az = \sum_{j=1}^{\infty} j^2 \pi^2 \langle z, \phi_j \rangle \phi_j,$$

where $\phi_j(x) = \sqrt{2} \sin(j\pi x)$.

b) $-A$ is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ given by

$$T(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j z, \quad z \in Z, \quad t \geq 0, \tag{4.12}$$

where $E_j z = \langle \phi_j, z \rangle$ and $\lambda_j = j^2\pi^2$.

So, $\{E_j\}$ is a family of complete orthogonal projections in Z and

$$z = \sum_{j=1}^{\infty} E_j z, \quad z \in Z.$$

Now, the discretization of (4.11) on flow is given by

$$\begin{cases} z(n+1) = T(n)z(n) + B(n)u(n), & z \in Z, \\ z(0) = z_0. \end{cases} \tag{4.13}$$

In this case, $T^*(t) = T(t)$ and $B = I$.

We proved in [10] the following result.

Proposition 4.5. *The system (4.13) is exactly controllable for $n \in \mathbb{N}$.*

Therefore, if we consider the dual system with observation

$$\begin{cases} z(n+1) = T(n)z(n), \\ y(n+1) = z(n), \end{cases} \tag{4.14}$$

with $C = B$, we have that (4.14) is exactly observable. □

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