African Diaspora Journal of Mathematics
Special Volume in Honor of Prof. Augustin Banyaga
Volume 14, Number 2, pp. 187–196 (2012)

Integration of Conformal Jacobi Fibrations and Prequantization of Poisson Fibrations

Aïssa Wade *
Department of Mathematics, Penn State University,
University Park, PA 16802, USA.

Abstract

We show that integrable conformal Jacobi fibrations are in one-to-one correspondence with source-simply connected fibered conformal contact groupoids. We also prove that prequantizable Poisson fibrations give rise to Jacobi fibrations. In addition, source-simply connected symplectic groupoids associated to prequantizable and integrable Poisson fibrations are also prequantizable.

AMS Subject Classification: 53D17, 53D50, 55R05.

Keywords: Poisson structure, Jacobi structure. groupoid, prequantization, geometric quantization.

1 Introduction

In this note, our purpose is two-fold. Firstly, we introduce conformal Jacobi fibrations (which unify Poisson and contact fiber bundles) and discuss their integration problem. Secondly, we examine applications of this theory to the prequantization of Poisson fibrations. Recall that a Jacobi structure on a smooth manifold is given by a pair (Λ, R) consisting of a bivector field Λ and a vector field R (called the Reeb vector field) that satisfy some geometric properties (see below). Jacobi manifolds encompass Poisson manifolds, contact manifolds as well as locally conformal symplectic manifolds. They were first introduced by Lichnerowicz (see [13]). Here, by a *conformal Jacobi fibration*, we mean a locally trivial fiber bundle $p: M \to B$ whose fiber type F is a Jacobi manifold together with a collection of trivializations whose transition maps preserve the conformal class of the Jacobi structure on F. This notion naturally extends the concept of a Poisson fibration that was considered in [2, 19].

Jacobi manifolds can be viewed as the infinitesimal counterpart of contact Lie groupoids, in the sense that, the global geometric objects that integrate Jacobi manifolds are contact Lie groupoids [6]. Lie groupoids which are generalizations of Lie groups, have been intensively studied in differential geometry in recent years. They provide a fruitful framework

*E-mail address: wade@math.psu.edu

for the study of many geometric objects (see [14] and references therein). Their corresponding infinitesimal geometric objects are Lie algebroids which, include both Lie algebras and foliations. The classical Lie's third theorem says that there is a one-to-one correspondence between finite dimensional Lie algebras and finite dimensional connected and simply connected Lie groups. It is known that Lie's third theorem cannot be directly extended to Lie algebroids and Lie groupoids, nonetheless, the obstructions to its extension are controllable as shown in [5].

Our first main result lies in this context: we show that the global objects associated with conformal Jacobi fibrations are fibered conformal contact groupoids (see Theorem 3.5). Another interesting observation is that the prequantization of a Poisson fibration is a Jacobi fibration. In the context of Lie groupoids, the notion of prequantization of symplectic groupoids was introduced in [20]. Here, we show that the source-simply connected symplectic groupoid associated to a prequantizable and integrable Poisson fibration is also prequantizable (see Theorem 4.3).

The paper is organized as follows: In Section 2, we review known facts about Jacobi manifolds and we introduce the notion of a conformal Jacobi fibration. We also brush up groupoids, in particular, contact groupoids. Section 3 deals with the integration problem for conformal Jacobi fibrations. In Section 4, we develop the prequantization of Poisson fibrations.

2 Definitions and Basic Results

All manifolds are assumed to be paracompact, Hausdorff, smooth and connected. We also assume that all maps between manifolds are smooth.

2.1 Locally conformal symplectic and Jacobi structures

A *locally conformal symplectic* (lcs for short) form on a manifold M is a non-degenerate 2-form Ω for which, there exists an open cover $\mathcal{U} = (U_i)$ of M and smooth positive functions $f_i : U_i \to \mathbb{R}$ such that $\Omega_i = f_i \Omega_{|U_i}$ is a symplectic 2-form on U_i.

Obviously, if $f_i = 1$ for all i then Ω is a symplectic form on M. In [11], Lee noticed that, in the general case, the 1-forms $d(\ln f_i)$ fit together into a closed 1-form ω on M such that:

$$d\Omega = -\omega \wedge \Omega. \tag{2.1}$$

More precisely, the above definition of a lcs form is equivalent to the existence of a closed 1-form ω satisfying Equation (2.1). The 1-form ω is called the Lee form of Ω. It is known that [11] that if the dimension of M is greater than 2 then the Lee 1-form ω is uniquely determined by its corresponding lcs form Ω.

Two lcs forms Ω and Ω' are said to be conformally equivalent if there exists some positive function f such that

$$\Omega' = f\Omega.$$

A locally conformal symplectic structure is an equivalence class of lcs forms for this relation. Notice that the de Rham cohomology class of the Lee form is an invariant of the lcs structure since a conformal rescaling of Ω changes its Lee form ω by adding an exact form. Observe that any lcs structure that contains a symplectic representative is globally conformal symplectic. This happens when the Lee form is exact.

Example [1]. Consider a co-oriented contact structure on a manifold B, let α be a corresponding contact 1-form and let $p : M \to B$ be a flat principal $U(1)$-bundle over B. Then any connection 1-form ω on M induces a canonical lcs structure on M determined by the conformal class of the pair $(d\theta + \omega \wedge \theta, \omega,)$ where $\theta = p^*\alpha$.

To any lcs form Ω is associated a pair (Λ, R) of tensors defined by:

$$\iota_R \Omega = -\omega \quad \text{and} \quad \Lambda = \Omega^{-1},$$

where ω is the corresponding Lee 1-form. In fact, locally conformal symplectic manifolds are special cases of Jacobi manifolds. Recall that a Jacobi structure on M is defined by a pair (Λ, R) where Λ is a bivector field and R is a vector field (called the Reeb vector field) such that :

$$[\Lambda, \Lambda] = -2R \wedge \Lambda \quad \text{and} \quad [R, \Lambda] = 0. \tag{2.2}$$

Here, the bracket $[\cdot, \cdot]$ stands for the Schouten-Nijenhuis bracket on multivector fields. Poisson structures are Jacobi structures for which $R = 0$. Jacobi structures were introduced and studied by Lichnerowicz [13].

Given a Jacobi manifold (M, Λ, R), let $\Lambda^\sharp : T^*M \to TM$ be the bundle map defined by:

$$\langle \Lambda^\sharp(\alpha), \beta \rangle = \Lambda(\alpha, \beta).$$

Then the distribution generated by $\mathrm{Im}(\Lambda^\sharp)$ and R is called the *characteristic distribution* of the Jacobi structure. When the distribution coincides with TM we say that its corresponding Jacobi structure is *transitive*. Locally conformal symplectic structures are odd-dimensional transitive Jacobi structures. While even-dimensional transitive Jacobi structures are co-orientable contact structures. Recall that a *contact structure* on a $(2n+1)$-dimensional manifold is given by a hyperplane field $\xi \subset TM$ which can be written locally as the kernel of a 1-form α and such that $\alpha \wedge (d\alpha)^n$ is a volume form. Co-orientable contact structures are those for which the normal bundle $\xi^\perp$ is trivial. In other words, these are contact structures globally defined by a 1-form.

Let (Λ, R) be a Jacobi structure on M and let f be a smooth nowhere vanishing function on M. The conformal transformation of (Λ, R) by f is a new Jacobi structure given by the tensors:

$$\Lambda_f = f\Lambda, \quad R_f = fR + \Lambda^\sharp(df). \tag{2.3}$$

We say that two Jacobi structures are *conformally equivalent* if they are related by a conformal transformation. Such an equivalence class $\mathcal{J}$ of Jacobi structures is called a *conformal Jacobi structure* and the pair $(M, \mathcal{J})$ is said to be a conformal Jacobi manifold. A conformal Jacobi structure is just a locally conformal Jacobi structure with an orientable line bundle. When M is simply connected, all locally conformal Jacobi structures on M are globally conformal.

It is well-known that the poissonization of a Jacobi structure (Λ, R) on M is a Poisson structure on the manifold $M \times \mathbb{R}$ with corresponding bivector:

$$\widetilde{\Lambda} = e^{-t}\left(\Lambda + \frac{\partial}{\partial t} \wedge R\right), \tag{2.4}$$

where t is the coordinate on $\mathbb{R}$. In fact (Λ, R) defines a Jacobi structure if and only if $\widetilde{\Lambda}$ is a Poisson tensor.

2.2 Conformal Jacobi fibrations

Let $(F, \mathcal{J}_F)$ be a conformal Jacobi manifold. A *conformal Jacobi fibration* is a locally trivial fiber bundle $F \hookrightarrow M \xrightarrow{p} B$ whose structure group preserves the conformal Jacobi structure on F. In other words, there is an open cover $\mathcal{U} = (U_i)$ of B and diffeomorphisms $\phi_i : p^{-1}(U_i) \to U_i \times F$ satisfying the properties:

1. The following diagram commutes

$$\begin{array}{ccccc} p^{-1}(U_i) & & \xrightarrow{\phi_i} & & U_i \times F \\ & p \searrow & & \swarrow \mathrm{pr} & \\ & & U_i & & \end{array}$$

2. If $b \in U_i \cap U_j$ then the transition map $\phi_{ij}(b) = \phi_i(b) \circ \phi_j(b)^{-1}$ preserves the conformal class $\mathcal{J}_F$ of Jacobi structures on F.

In particular, one obtains a *lcs fibration* if the fiber type F is a lcs manifold. When F is a co-oriented contact manifold, this definition coincide with Lerman's definition of a *contact fiber bundle* [10]. Given a conformal Jacobi fibration, we consider the vertical sub-bundle

$$\mathrm{Vert} = Ker(Tp) \subset TM.$$

There is a vertical bivector field Λ and a vertical vector field $R \in \Gamma(\mathrm{Vert}(M))$ such that the conformal class of the pair (Λ, R) defines a conformal Jacobi structure which coincide with the conformal Jacobi structure along the fibers.

2.3 Lie groupoids

A *Lie groupoid* over a manifold M is given by a smooth manifold $\mathcal{G}$ together with two surjective submersions $\alpha, \beta : \mathcal{G} \to M$ (called the source map and the target map), a smooth associative multiplication $m : \mathcal{G}_2 \to \mathcal{G}$, a unit section $\epsilon : M \to \mathcal{G}$ and an inversion map $i : \mathcal{G} \to \mathcal{G}$, where $\mathcal{G}_2 = \{(g,h) \in \mathcal{G} \times \mathcal{G} \mid \beta(g) = \alpha(h)\}$ is the set of composable pairs and the following properties are satisfied:

1. $\alpha(m(g,h)) = \alpha(h)$ and $\beta(m(g,h)) = \beta(g)$, $\forall\, (g,h) \in \mathcal{G}_2$,
2. $m(g, m(h,k)) = m(m(g,h),k)$, $\forall\, g,h,k \in \mathcal{G}$ such that $\alpha(g) = \beta(h)$ and $\alpha(h) = \beta(k)$,
3. $\alpha(\epsilon(x)) = x$ and $\beta(\epsilon(x)) = x$, $\forall x \in M$,

4. $m(g, \epsilon(\alpha(g))) = g$ and $m(\epsilon(\beta(g)), g) = g$, $\forall\, g \in \mathcal{G}$,
5. $m(g, \iota(g)) = \epsilon(\beta(g))$ and $m(\iota(g), g) = \epsilon(\alpha(g))$, $\forall g \in \mathcal{G}$.

Here, the base manifold M, the α-fibers and the β-fibers are supposed to be Hausdorff but $\mathcal{G}$ is not necessarily Hausdorff. We will often identify M with $\epsilon(M)$.

A real-valued function r defined on a Lie groupoid $\mathcal{G}$ is *multiplicative* if:

$$r \circ m(g_1, g_2) = r(g_1) + r(g_2), \quad \forall\, (g_1, g_2) \in \mathcal{G}_2.$$

Given a Lie groupoid $\mathcal{G}$ and a multiplicative function r on $\mathcal{G}$, we define

$$\mathcal{G} \times_r \mathbb{R} = \mathcal{G} \times \mathbb{R} \rightrightarrows M \times \mathbb{R},$$

together with its source α, target β and multiplication given by:

$$\alpha(g, s) = (\alpha(g), s), \quad \beta(g, s) = (\beta(g), s - r(g)), \quad (g_1, s_1)(g_2, s_2) = (g_1 g_2, s_2). \tag{2.5}$$

Then $\mathcal{G} \times_r \mathbb{R}$ is a Lie groupoid if and only if r is multiplicative.

2.4 Contact groupoids

A *contact groupoid* [7, 8] is a Lie groupoid $\mathcal{G} \overset{\alpha}{\underset{\beta}{\rightrightarrows}} M$ together with a pair (θ, r) consisting of a contact form θ, and a multiplicative function r on $\mathcal{G}$ satisfying the following property:

$$m^*\theta = pr_2^*(e^{-r}) \cdot pr_1^*\theta + pr_2^*\theta. \tag{2.6}$$

where m is the multiplication and the pr_i are the projections. The function r called the *Reeb function*, or the *Reeb cocycle* of $\mathcal{G}$.

Let $\mathcal{G} \rightrightarrows M$ be a contact groupoid with a contact structure (θ, r). Given a smooth nowhere vanishing function τ on M, we set

$$\theta_\tau = \alpha^*(\tau)\theta, \quad r_\tau = r + \ln(\frac{\alpha^*\tau}{\beta^*\tau})$$

Then the pair (θ_τ, r_τ) defines a new contact structure on $\mathcal{G}$. We say that these two contact structures are conformally equivalent. A *conformal contact groupoid* is a Lie groupoid $\mathcal{G} \rightrightarrows M$ endowed with such a equivalence class of pairs (θ, r).

2.5 Prequantization

Prequantization is the first step in the geometric quantization procedure. Geometric quantization for symplectic manifolds was independently developed by Kostant [9] and Souriau [16]. The existence of a prequantization bundle for a symplectic manifold is guaranteed if the symplectic form is integral. In fact, given a symplectic manifold (V, Ω) , where Ω is integral, (i.e. its de Rham cohomology class $[\Omega]$ is integral), then its prequantization consists of the choice of a Hermitian line bundle K over V endowed with a connection ∇ on K with curvature $2\pi i\Omega$. This is equivalent to a principal $U(1)$-bundle $\tau : \widetilde{V} \to V$ together with a connection 1-form θ on $\widetilde{V}$ satisfying $\tau^*\Omega = d\theta$.

Geometric quantization has been extended to various geometric settings. Vaisman studied the geometric quantization for Poisson manifolds [17]. The case of Jacobi manifolds was investigated by de León, Marrero and Padrón [12]. Recently, Weinstein and Zambon considered a generalization to Dirac structures and quantizability conditions of a Dirac structure. In the framework of generalized complex, generalized para-complex and generalized tangent geometry, prequantization spaces are discussed in [18]. The connection between integrability and prequantization was studied by Crainic [4].

The prequantization condition for a Poisson manifold (P, Λ_P) [17] is that the class $[\Lambda_P]$ relative to the Lichnerowicz-Poisson cohomology is the image of some integral de Rham class under the cochain map obtained by extending of the canonical bundle map $\Lambda_P^\sharp : T^*P \to TP$, that is,

$$\Lambda_P^\sharp(\Omega_P) = \Lambda_P + [\Lambda_P, Z],$$

for some integral closed 2-form Ω_P and for some vector field $Z \in \mathfrak{X}(P)$.

Suppose (P, Λ_P) is prequantizable. Let $\tau : \widetilde{P} \to P$ be a $U(1)$-bundle with first Chern class $[\Omega_P]$ and let θ be a connection 1-form θ on $\widetilde{P}$ with $\tau^*\Omega_P = d\theta$. It was shown in [3] that if Λ_P^H and Z^H denote the horizontal lifts of Λ_P and Z, respectively, then $(\Lambda_P^H + R \wedge Z^H, R)$ is a Jacobi structure on $\widetilde{P}$, where R is the generator of the $U(1)$-action on $\widetilde{P}$.

A *prequantization of the symplectic groupoid* $(\mathcal{G}, \Omega)$ is a Lie groupoid extension of $\mathcal{G}$ by the trivial bundle of Lie groups $U(1)$:

$$1 \to U(1) \to \tilde{\mathcal{G}} \xrightarrow{p} \mathcal{G} \to 1$$

together with a a connection 1-form θ which is multiplicative and such that $p^*\Omega = d\theta$.

3 Integration of Jacobi fibrations

Following [2], we call **Fib** the category of locally trivial fiber bundle $p : M \to B$ over a fixed base manifold B whose morphisms are the fiber preserving maps over the identity:

$$\begin{array}{ccc} M_1 & \xrightarrow{\Phi} & M_2 \\ & p_1 \searrow \quad \swarrow p_2 & \\ & B & \end{array}$$

A *fibered groupoid* $\mathcal{G} \rightrightarrows M$ is an internal groupoid in **Fib**, that is, an internal category where every morphism is an isomorphism.

Thus, a fibered groupoid can be viewed as a fiber bundle with fiber type a groupoid where the structure group acts by groupoid automorphisms. Notice that both the total space $\mathcal{G}$ and the base M of a fibered groupoid are fibrations over B and all structure maps are fiber preserving maps.

A *fibered conformal contact groupoid* is a fibered Lie groupoid $\mathcal{G} \rightrightarrows M$ whose fiber type is a conformal contact groupoid $\mathcal{F} \rightrightarrows F$.

Given a fibered contact groupoid $\mathcal{G} \rightrightarrows M$, its fiber type $\mathcal{F}$ is the total space of a Lie groupoid $\mathcal{F} \rightrightarrows F$ over the fiber type F of M. It is well known that the base manifold of a contact groupoid admits a Jacobi structure [6, 7, 8]. We have the following analogous result:

Theorem 3.1. *The base M of a fibered contact groupoid $\mathcal{G} \rightrightarrows M$ has a natural structure of a Jacobi fibration.*

To prove the above theorem, we need the following result which was established in [6]:

Proposition 3.2. [6] *There is a one-to-one correspondence between conformal contact groupoids $\mathcal{G}$ over M which are source-simply connected and integrable conformal Jacobi structures on the base manifold M.*

Proof of Theorem 3.1. From Proposition 3.2, one deduces that each fiber of the base M of the fibered contact groupoid $\mathcal{G} \rightrightarrows M$ naturally admits a conformal Jacobi structure. Moreover, any local trivialization of the fibered contact groupoid $\mathcal{G}$ covers a local trivialization of M. In addition, the transition maps preserve the conformal Jacobi structure on the fiber type F. There follows Theorem 3.1. ∎

Definition 3.3. *A conformal Jacobi fibration is said to be integrable if it is the base of some fibered conformal contact groupoid $\mathcal{G} \rightrightarrows M$.*

We have:

Proposition 3.4. *Any conformal Jacobi fibration whose fiber type is integrable is also integrable.*

Proof: Suppose the fiber F of a conformal Jacobi fibration $p : M \to B$ is integrable. Then the induced vertical conformal Jacobi structure on M is also integrable. Let $\mathcal{G} \rightrightarrows M$ be the corresponding source-simply connected contact groupoid that integrates this vertical conformal Jacobi structure. Since the conformal Jacobi structure on F is integrable, it determines a contact groupoid $\mathcal{F}$ with base F which is the fiber type of the fibered Lie groupoid structure on $\mathcal{G} \rightrightarrows M$. There follows Proposition 3.4. ∎

As an immmediate consequence of Theorem 3.1 and Proposition 3.4, we get the following:

Theorem 3.5. *There is a one-to-one correspondence between source-simply connected fibered conformal contact groupoids and integrable conformal Jacobi fibrations.*

4 Applications to the prequantization of symplectic fibered groupoids

Recall that a *Poisson fibration* is a locally trivial fibration whose fiber type is a Poisson manifold together with a collection of local trivializations whose transition functions are Poisson diffeomorphisms. The global object associated to any Poisson fibration is called a *fibered symplectic groupoid*, that is, a fiber bundle with fiber type a groupoid where the structure group acts by groupoid automorphisms.

Definition 4.1. *A Poisson fibration $\pi : N \to B$ is said to be prequantizable if its fiber type (P, Λ_P) is prequantizable as a Poisson manifold.*

From now on, we assume that the fiber type of the considered Poisson fibrations are compact.

Theorem 4.2. *Prequantizable Poisson fibrations admit Jacobi fibrations as their prequantization bundles.*

Proof: Let $P \hookrightarrow N \xrightarrow{\pi} B$ be a prequantizable Poisson fibration. Since the fiber P is prequantizable, we can choose a prequantization bundle $Q \to P$ so that there exists a Jacobi structure on Q, a 1-form θ with curvature form $\Omega_P \in H^2(P, \mathbb{Z})$ and a vector field Z on P such that:

$$\Lambda_P^\sharp(\Omega_P) = \Lambda_P + [\Lambda_P, Z].$$

The vertical Poisson structure on N is also prequantizable. We can pick a prequantization bundle $M \to N$ so that M is the total space of a Jacobi fibration over B with fiber Q, endowed with a connection 1-form $\widetilde{\theta}$ compatible with the θ, (e.g. the pullback connection). Then this determines a natural Jacobi fibration $\widetilde{\pi} : M \to B$ with fiber type Q. We have the commutatif diagram:

$$\begin{array}{ccc} M & \xrightarrow{\overline{p}} & N \\ \widetilde{\pi}\downarrow & & \pi\downarrow \\ B & \xrightarrow{id} & B \end{array}$$

Theorem 4.3. *Let $P \hookrightarrow N \xrightarrow{\pi} B$ be an integrable Poisson fibration which is prequantizable. Then the symplectic groupoid $\mathcal{G}_s(N)$that integrates N is also prequantizable.*

Proof: Suppose the Poisson fibration $P \hookrightarrow N \to B$ is integrable and prequantizable. Denote by $\mathcal{G}_s(P) \rightrightarrows P$ and $\mathcal{G}_s(N) \rightrightarrows N$ the source 1-connected symplectic groupoids that integrate Poisson structure on the fiber P and the induced vertical Poisson structure on N, respectively. They are related to their corresponding contact topological groupoids denoted by $\mathcal{G}_c(P)$ and $\mathcal{G}_c(N)$, respectively. More precisely, we have the short exact sequences of topological groupoids [6]:

$$1 \to \Sigma_P \to \mathcal{G}_c(P) \to \mathcal{G}_s(P) \to 1$$

and

$$1 \to \Sigma_N \to \mathcal{G}_c(N) \to \mathcal{G}_s(N) \to 1$$

where Σ_P (resp. Σ_N) is the quotient of the trivial groupoid $P \times \mathbb{R}$ (resp. $N \times \mathbb{R}$) by a group bundle over P (resp. N) whose fiber at $x \in P$ (resp. $n \in N$) is the group of periods of the restriction of the symplectic form of $\mathcal{G}_s(P)$ (resp.$\mathcal{G}_s(N)$) to the source-fiber $\alpha^{-1}(x)$(resp, $\alpha^{-1}(n)$). Because the vertical Poisson structure on N is prequantizable, it follows that its symplectic groupoid $\mathcal{G}_s(N)$ is automatically prequantizable [22].

References

[1] A. Banyaga, Some properties of locally conformal symplectic structures. *Comment. Math. Helv.* **77** (2002), 383–398.

[2] O. Brahic and R. L. Fernandes, Poisson fibrations and fibered symplectic groupoids. *Poisson geometry in mathematics and physics*, 41-59, Contemp. Math., **450** Amer. Math. Soc., Providence, RI, 2008

[3] D. Chinea, J. C. Marrero, and M. de León. Prequantizable Poisson manifolds and Jacobi structures. *J. Phys. A*, **29** (1996) 6313–6324.

[4] M. Crainic, Prequantization and Lie brackets. *Journal of Symplectic Geometry,* **2** (2005) 579–602.

[5] M. Crainic and R. L. Fernandes, Integrability of Lie brackets. *Ann. of Math.* **157** (2003) 575–620.

[6] M. Crainic and C. Zhu, Integrability of Jacobi structure, *Ann. Inst. Fourier,* **57** (2007), no. 4, 1181–1216

[7] P. Dazord Intégration d'algèbres de Lie locales et groupoïdes de contact *C. R. Acad. Sci. Paris Sér. I Math.* **320** (1995), no. 8, 959–964.

[8] Y. Kerbrat and Z. Souici-Benhammadi, Variétés de Jacobi et groupoïdes de Lie. *C. R. Acad. Sci. Paris Sér. I Math.* **317** (1993), 81–86.

[9] B. Kostant, Quantization and unitary representations. I. Prequantization. In *Lectures in modern analysis and applications, III*, pages 87–208. Lecture Notes in Math., Vol. 170. Springer, Berlin, 1970.

[10] E. Lerman, Contact fiber bundles, *J. Geom. Phys.* **49** (2004), no. 1, 52–66.

[11] H. C. Lee A kind of even-dimensional differential geometry and its application to exterieur calculus *Amer. J. Math.* **65** (1943). 433–438.

[12] M. de Leon, J. C. Marrero, E. Padrón, *On the geometric quantization of Jacobi manifolds*, J. Math. Phys. **38** (1997), 6185–6213.

[13] A. Lichnerowicz Les variétés de Jacobi et leurs algèbres de Lie associées, *J. Math. Pures Appl.* (9) **57** (1978), 453-488.

[14] K. C. H. Mackenzie, *General theory of Lie groupoids and Lie algebroids*, volume 213 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2005.

[15] K. Mikami and A. Weinstein, Moments and reduction for symplectic groupoids. *Publ. Res. Inst. Math. Sci.*, **24** (2008), 121–140.

[16] J.-M. Souriau, Quantification géométrique, in *Physique quantique et géométrie (Paris, 1986)*, *Travaux en Cours*, Vol. **32** (1988), 141–193. Hermann, Paris.

[17] I. Vaisman, On the geometric quantization of Poisson manifolds. *J. Math. Phys.*, **32** (1991), 3339–3345.

[18] I Vaisman, *On some quantizable generalized structures*, An. Univ. Vest Timi. Ser. Mat.-Inform. **48** (2010), 275–284.

[19] A. Wade. Poisson fiber bundles and coupling Dirac structures *Ann. Global Anal. Geom.*, **33** (2008), 207-217.

[20] A. Weinstein and P. Xu. Extensions of symplectic groupoids and quantization. *J. Reine Angew. Math.*, **417** (1991), 159–189.

[21] A. Weinstein and M. Zambon, Variations on prequantization. In *Travaux mathématiques. Fasc. XVI*, Trav. Math., **XVI**, pages 187–219. Univ. Luxemb., Luxembourg, 2005.

[22] M. Zambon and C. Zhu, On the geometry of prequantization spaces, *J. Geom. Phys.* **57** (2007), 2372–2397.