

## TWO FAMILIES OF AFFINE OSSERMAN CONNECTIONS ON 3-DIMENSIONAL MANIFOLDS \*

ABDOUL SALAM DIALLO<sup>†</sup>

African Institute for Mathematical Sciences  
AIMS-Sénégal, Km2, Route de Joal (Centre IRD de Mbour),  
B.P. 14 18, Mbour, Sénégal

MOUHAMADOU HASSIROU<sup>‡</sup>

Université Abdou Moumouni,  
Faculté des Sciences, Département de Mathématiques et Informatique,  
B.P. 10 662, Niamey, Niger

### Abstract

The aim of this note is to study the Osserman condition on two families affine connections. As applications, examples of affine Osserman connections which are Ricci flat but not flat are given.

**AMS Subject Classification:** 53B05, 53B15, 53C50.

**Keywords:** affine connection, Jacobi operator, Osserman manifold, Riemann extension.

## 1 Introduction

Curvature is a fundamental notion in pseudo-Riemannian geometry. It has always been a pursuit of great interest to understand to what extent the sectional curvatures of a pseudo-Riemannian manifold can provide information about the curvature and metrics tensors. The Jacobi operator  $J_{\mathcal{R}}(X)$  is an important tool for studying the curvature. The geodesic deviation is measured by this part of the curvature tensor and because of its fundamental role in the Jacobi equation, many geometrics properties can be derived from the knowledge of the Jacobi operator. Since for each vector  $X$ , the Jacobi operator is a self-adjoint operator, the study of its eigenvalues is very important. For examples, in Riemannian case, the eigenvalues indicate the extreme values of the sectional curvature and, in Lorentzian geometry, the eigenvalues play a fundamental role in the construction of mathematical models in general relativity.

---

\* Acknowledgements: Professor A. Banyaga visited AIMS-Sénégal, the 12th, 13th, and 14th of November 2012. All students, tutors and staff members of AIMS-Sénégal are grateful toward him for his visit and for sharing his knowledges and experiences with us.

<sup>†</sup>E-mail address: [abdoul@aims-senegal.org](mailto:abdoul@aims-senegal.org)

<sup>‡</sup>E-mail address: [hassirou@refer.ne](mailto:hassirou@refer.ne)

A pseudo-Riemannian manifold  $(M, g)$  is said to be Osserman if the eigenvalues of the Jacobi operators are constant on the unit pseudo-sphere bundle  $S^\pm(M, g)$ . Any two-point homogeneous space is Osserman and the inverse is true in Riemannian ( $\dim M \neq 16$ ) and Lorentzian setting [7]. However there exists many non symmetric Osserman pseudo-Riemannian metrics in other signature [6] and symmetric Osserman which are not of rank one. The investigation of Osserman manifolds has been extremely attractive and fruitful over the recent years; we refer to [1, 7, 9] for further details.

In the paper [6] the authors introduced so-called *affine Osserman connections*. The concept of affine Osserman connection originated from the effort to build up examples of pseudo-Riemannian Osserman manifolds (see [4], [5],[6]) via the construction called the *Riemannian extension*. This construction assigns to every  $m$ -dimensional manifold  $M$  with a torsion-free affine connection  $\nabla$  a pseudo-Riemannian metric  $g_\nabla$  of signature  $(m, m)$  on the cotangent bundle  $T^*M$ . (See [11], for more details). The authors in [6] pay attention to dimension  $m = 2$ . They prove that  $\nabla$  is affine Osserman if and only if the Ricci tensor of  $\nabla$  is skew-symmetric on  $M$ . Recently, the first author in [4] gave an explicit form of affine Osserman connection on 2-dimensional manifolds. For dimension  $m = 3$ , making a description seems to be a hard problem. Partial results was published in [5]. The aim of the present note is to give the explicit form of two families of affine connections which are *affine Osserman*.

Our paper is organized as follows. Section 1 introduce this topic. In section 2 we recall some basics definitions and results about affine Osserman. In section 3, we study the Osserman condition on two particular affine connections (cf. Proposition 3.1) and (cf. Proposition 3.7). As applications, examples (cf. Examples 3.5 and 3.11) of affine Osserman connections which are Ricci flat but not flat are given.

## 2 Preliminaries

### 2.1 Affine connections

Let  $M$  be a 3-dimensional and  $\nabla$  a smooth affine connection. We choose a fixed coordinate domain  $\mathcal{U}(u_1, u_2, u_3) \subset M$ . In  $\mathcal{U}$ , the connection is given by

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k,$$

where we denote  $\partial_i = (\frac{\partial}{\partial u_i})$  and the functions  $\Gamma_{ij}^k(i, j, k = 1, 2, 3)$  are called the *Christoffel symbols* for the affine connection relative to the local coordinate system. We define a few tensors fields associated to a given affine connection  $\nabla$ . The *torsion tensor field*  $T^\nabla$ , which is of type  $(1, 2)$ , is defined by

$$T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

The components of the torsion tensor  $T^\nabla$  in local coordinates are

$$T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k.$$

If the torsion tensor of a given affine connection  $\nabla$  is 0, we say that  $\nabla$  is torsion-free.

The curvature tensor field  $\mathcal{R}^\nabla$ , which is of type (1, 3), is defined by

$$\mathcal{R}^\nabla(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z.$$

The components in local coordinates are

$$\mathcal{R}^\nabla(\partial_k, \partial_l)\partial_j = \sum_i R_{jkl}^i \partial_i$$

We shall assume that  $\nabla$  is torsion-free. If  $\mathcal{R}^\nabla = 0$  on  $M$ , we say that  $\nabla$  is *flat affine connection*. It is known that  $\nabla$  is flat if and only if around point there exist a local coordinates system such that  $\Gamma_{ij}^k = 0$  for all  $i, j$  and  $k$ .

We define the *Ricci tensor*  $Ric^\nabla$ , of type (0, 2) by

$$Ric^\nabla(Y, Z) = \text{trace}\{X \mapsto \mathcal{R}^\nabla(X, Y)Z\}.$$

The components in local coordinates are given by

$$Ric^\nabla(\partial_j, \partial_k) = \sum_i R_{kij}^i.$$

It is known in Riemannian geometry that the Levi-Civita connection of a Riemannian metric has symmetric Ricci tensor, that is,  $Ric(Y, Z) = Ric(Z, Y)$ . But this property is not true for an arbitrary affine connection with torsion-free.

We will write  $T^\nabla, \mathcal{R}^\nabla$  and  $Ric^\nabla$  when it is necessary to distinguish the role of the connection.

## 2.2 Affine Osserman manifolds

Let  $(M, \nabla)$  be a  $m$ -dimensional affine manifold, i.e.,  $\nabla$  is a torsion free connection on the tangent bundle of a smooth manifold  $M$  of dimension  $m$ . Let  $\mathcal{R}^\nabla(X, Y)$  be the associated curvature operator. We define the *affine Jacobi operator*  $J_{\mathcal{R}^\nabla}(X) : T_p M \longrightarrow T_p M$  with respect to a vector  $X \in T_p M$  by

$$J_{\mathcal{R}^\nabla}(X)Y := \mathcal{R}^\nabla(Y, X)X.$$

**Definition 2.1.** [7] Let  $(M, \nabla)$  be a  $m$ -dimensional affine manifold. Then  $(M, \nabla)$  is called affine Osserman at  $p \in M$  if the characteristic polynomial of  $J_{\mathcal{R}^\nabla}(X)$  is independent of  $X \in T_p M$ . Also  $(M, \nabla)$  is called affine Osserman if  $(M, \nabla)$  is affine Osserman at each  $p \in M$ .

**Theorem 2.2.** [7] Let  $(M, \nabla)$  be a  $m$ -dimensional affine manifold. Then  $(M, \nabla)$  is called affine Osserman at  $p \in M$  if and only if the characteristic polynomial of  $J_{\mathcal{R}^\nabla}(X)$  is  $P_\lambda[J_{\mathcal{R}^\nabla}(X)] = \lambda^m$  for every  $X \in T_p M$ .

**Corollary 2.3.** We say that  $(M, \nabla)$  is affine Osserman if the affine Jacobi operators are nilpotent, i.e., 0 is the only eigenvalue of  $J_{\mathcal{R}^\nabla}(\cdot)$  on the tangent bundle  $TM$ .

**Corollary 2.4.** If  $(M, \nabla)$  is affine Osserman at  $p \in M$  then the Ricci tensor is skew-symmetric at  $p \in M$ .

Affine Osserman connections are well-understood in dimension two, due to the fact that an affine connection is Osserman if and only if its Ricci tensor is skew-symmetric. The situation is however more involved in higher dimensions where the skew-symmetric is a necessary (but not sufficient) condition for an affine connection to be Osserman.

Affine Osserman connections are of interest not only in affine geometry, but also in the study of pseudo-Riemannian Osserman metrics since they provide some nice examples without Riemannian analogue by means of the Riemannian extensions. Here it is worth to emphasize that some recent modifications of the usual Riemann extensions allowed some new applications [2, 3, 10]

Let  $X = \sum_{i=1}^3 \alpha_i \partial_i$  is a vector on a 3-dimensional affine manifold  $M$ , then the affine Jacobi operator is given by

$$\begin{aligned} J_{\mathcal{R}^\nabla}(X) &= \alpha_1^2 \mathcal{R}^\nabla(\cdot, \partial_1) \partial_1 + \alpha_1 \alpha_2 \mathcal{R}^\nabla(\cdot, \partial_1) \partial_2 + \alpha_1 \alpha_3 \mathcal{R}^\nabla(\cdot, \partial_1) \partial_3 \\ &+ \alpha_1 \alpha_2 \mathcal{R}^\nabla(\cdot, \partial_2) \partial_1 + \alpha_2^2 \mathcal{R}^\nabla(\cdot, \partial_2) \partial_2 + \alpha_2 \alpha_3 \mathcal{R}^\nabla(\cdot, \partial_2) \partial_3 \\ &+ \alpha_1 \alpha_3 \mathcal{R}^\nabla(\cdot, \partial_3) \partial_1 + \alpha_2 \alpha_3 \mathcal{R}^\nabla(\cdot, \partial_3) \partial_2 + \alpha_3^2 \mathcal{R}^\nabla(\cdot, \partial_3) \partial_3. \end{aligned}$$

### 2.3 Riemannian extension construction

Let  $(M, \nabla)$  be a 3-dimensional affine manifold. Let  $(u_1, u_2, u_3)$  be the local coordinates on  $M$ . We expand  $\nabla_{\partial_i} \partial_j = \sum_k \Gamma_{ij}^k \partial_k$  for  $i, j, k = 1, 2, 3$  to define the Christoffel symbols of  $\nabla$ . Let  $\omega = u_4 du_1 + u_5 du_2 + u_6 du_3 \in T^*M : (u_4, u_5, u_6)$  are the dual fiber coordinates. The *Riemannian extension* is the pseudo-Riemannian metric  $\bar{g}$  on the cotangent bundle  $T^*M$  of neutral signature  $(3, 3)$  defined by setting

$$\begin{aligned} \bar{g}(\partial_1, \partial_4) &= \bar{g}(\partial_2, \partial_5) = \bar{g}(\partial_3, \partial_6) = 1, \\ \bar{g}(\partial_1, \partial_1) &= -2u_4 \Gamma_{11}^1 - 2u_5 \Gamma_{11}^2 - 2u_6 \Gamma_{11}^3, \\ \bar{g}(\partial_1, \partial_2) &= -2u_4 \Gamma_{12}^1 - 2u_5 \Gamma_{12}^2 - 2u_6 \Gamma_{12}^3, \\ \bar{g}(\partial_1, \partial_3) &= -2u_4 \Gamma_{13}^1 - 2u_5 \Gamma_{13}^2 - 2u_6 \Gamma_{13}^3, \\ \bar{g}(\partial_2, \partial_2) &= -2u_4 \Gamma_{22}^1 - 2u_5 \Gamma_{22}^2 - 2u_6 \Gamma_{22}^3, \\ \bar{g}(\partial_2, \partial_3) &= -2u_4 \Gamma_{23}^1 - 2u_5 \Gamma_{23}^2 - 2u_6 \Gamma_{23}^3, \\ \bar{g}(\partial_3, \partial_3) &= -2u_4 \Gamma_{33}^1 - 2u_5 \Gamma_{33}^2 - 2u_6 \Gamma_{33}^3. \end{aligned}$$

We refer to [11] for details. We have the following result:

**Theorem 2.5.** ( [6] ) *Let  $(T^*M, \bar{g})$  be the cotangent bundle of an affine manifold  $(M, \nabla)$  equipped with the Riemannian extension of the torsion free connection  $\nabla$ . Then  $(T^*M, \bar{g})$  is a pseudo-Riemannian globally Osserman manifold if and only if  $(M, \nabla)$  is an affine Osserman manifold.*

The Riemann extension  $\bar{g}$  was used in [8] to construct nonsymmetric Osserman metrics of signature  $(2, 2)$ .

### 3 Affine Osserman connections on 3-dimensional manifolds

In this section we study two families of affine connections and we give the conditions for them to be Osserman.

#### Family I

**Proposition 3.1.** *Let  $M$  be a 3-dimensional manifold with torsion free connection given by*

$$\begin{cases} \nabla_{\partial_1}\partial_1 &= f_1(u_1, u_2, u_3)\partial_1; \\ \nabla_{\partial_2}\partial_2 &= f_2(u_1, u_2, u_3)\partial_1; \\ \nabla_{\partial_3}\partial_3 &= f_3(u_1, u_2, u_3)\partial_1. \end{cases} \quad (3.1)$$

*Then  $(M, \nabla)$  is affine Osserman if and only if the Christoffel symbols of the connection (3.1) satisfy:*

$$f_1(u_1, u_2, u_3) = f_1(u_1), \quad \partial_1 f_2 + f_1(u_1)f_2 = 0, \quad \text{and} \quad \partial_1 f_3 + f_1(u_1)f_3 = 0.$$

Proposition 3.1 follows from the three Lemmas below.

**Lemma 3.2.** *The components of the curvature operator of the connection (3.1) are given by*

$$\begin{aligned} \mathcal{R}^\nabla(\partial_1, \partial_2)\partial_1 &= -\partial_2 f_1 \partial_1; & \mathcal{R}^\nabla(\partial_1, \partial_2)\partial_2 &= (\partial_1 f_2 + f_1 f_2)\partial_1; \\ \mathcal{R}^\nabla(\partial_1, \partial_3)\partial_1 &= -\partial_3 f_1 \partial_1; & \mathcal{R}^\nabla(\partial_1, \partial_3)\partial_3 &= (\partial_1 f_3 + f_1 f_3)\partial_1; \\ \mathcal{R}^\nabla(\partial_2, \partial_3)\partial_2 &= -\partial_3 f_2 \partial_1, & \mathcal{R}^\nabla(\partial_2, \partial_3)\partial_3 &= \partial_2 f_3 \partial_1. \end{aligned}$$

**Lemma 3.3.** *The nonzero components of the Ricci tensor of the connection (3.1) is given by*

$$\begin{aligned} Ric^\nabla(\partial_2, \partial_1) &= -\partial_2 f_1; & Ric^\nabla(\partial_2, \partial_2) &= \partial_1 f_2 + f_1 f_2; \\ Ric^\nabla(\partial_3, \partial_1) &= -\partial_3 f_1; & Ric^\nabla(\partial_3, \partial_3) &= \partial_1 f_3 + f_1 f_3. \end{aligned}$$

Now, since the Ricci tensor of any affine Osserman connection is skew-symmetric, it follows from Lemma 3.3 that we have the following conditions for the connection to be Osserman

$$\partial_2 f_1 = 0, \quad \partial_3 f_1 = 0, \quad \partial_1 f_2 + f_1 f_2 = 0 \quad \text{and} \quad \partial_1 f_3 + f_1 f_3 = 0; \quad (3.2)$$

which implies that the connection is indeed Ricci flat, but not flat.

**Lemma 3.4.** *Let  $X = \sum_1^3 \alpha_i \partial_i$  be a vector on  $M$ , then the affine Jacobi operator is given by*

$$J_{\mathcal{R}^\nabla}(X)\partial_2 = \alpha_3(-\alpha_2\partial_3 f_2 + \alpha_3\partial_2 f_3)\partial_1 \quad J_{\mathcal{R}^\nabla}(X)\partial_3 = \alpha_2(\alpha_2\partial_3 f_2 - \alpha_3\partial_2 f_3)\partial_1.$$

The matrix associated to  $J_{\mathcal{R}^\nabla}(X)$  with respect to the basis  $\{\partial_1, \partial_2, \partial_3\}$  is given by

$$(J_{\mathcal{R}^\nabla}(X)) = \begin{pmatrix} 0 & 0 & 0 \\ a_2 & 0 & 0 \\ a_3 & 0 & 0 \end{pmatrix},$$

with

$$a_2 = \alpha_3(-\alpha_2\partial_3f_2 + \alpha_3\partial_2f_3) \quad a_3 = \alpha_2(\alpha_2\partial_3f_2 - \alpha_3\partial_2f_3).$$

It follows from the matrix associated to  $J_{\mathcal{R}^\nabla}(X)$ , that its characteristic polynomial can be written as follows:

$$P_\lambda[J_{\mathcal{R}^\nabla}(X)] = -\lambda^3,$$

which has zero eigenvalues.

*Proof of Proposition 3.1.* Systems (3.2) shows that the connection given by Equation (3.1) is affine Osserman if and only if the Christoffel symbols given by the functions  $f_1, f_2$  and  $f_3$  satisfy:

$$f_1(u_1, u_2, u_3) = f_1(u_1) \quad \partial_1f_2 + f_1(u_1)f_2 = 0 \quad \text{and} \quad \partial_1f_3 + f_1(u_1)f_3 = 0. \quad \square$$

**Example 3.5.** In the Euclidean 3-dimensional space  $\mathbb{R}^3$ , consider the connection  $\nabla$  defined by

$$\nabla_{\partial_1}\partial_1 = u_1\partial_1, \quad \nabla_{\partial_2}\partial_2 = u_2u_3e^{-\frac{1}{2}u_1^2}\partial_1, \quad \nabla_{\partial_3}\partial_3 = (u_2 + u_3)e^{-\frac{1}{2}u_1^2}\partial_1. \quad (3.3)$$

Then the nonvanishing components of the curvature tensor are

$$\mathcal{R}^\nabla(\partial_2, \partial_3)\partial_2 = -u_2e^{-\frac{1}{2}u_1^2}\partial_1 \quad \text{and} \quad \mathcal{R}^\nabla(\partial_2, \partial_3)\partial_3 = e^{-\frac{1}{2}u_1^2}\partial_1,$$

from which it follows that  $\mathbb{R}^3$  with torsion free affine connection (3.3) is a nonflat affine Osserman manifold.

We have this observation

**Corollary 3.6.** *The connection given by Equation (3.1) is affine Osserman flat if and only if*

$$\partial_2f_3(u_1, u_2, u_3) = 0 \quad \partial_3f_2(u_1, u_2, u_3) = 0.$$

**Family II**

**Proposition 3.7.** *Let  $M$  be a 3-dimensional manifold with torsion free connection given by*

$$\begin{cases} \nabla_{\partial_1}\partial_1 = f_1(u_1, u_2, u_3)\partial_1; \\ \nabla_{\partial_2}\partial_2 = f_2(u_1, u_2, u_3)\partial_2; \\ \nabla_{\partial_3}\partial_3 = f_3(u_1, u_2, u_3)\partial_3. \end{cases} \quad (3.4)$$

*Then  $(M, \nabla)$  is affine Osserman, if and only the Christoffel symbols given by the the functions  $f_1, f_2$  and  $f_3$  satisfy:*

$$\begin{cases} \partial_2 f_1 + \partial_1 f_2 = 0; & \partial_3 f_1 + \partial_1 f_3 = 0; & \partial_3 f_2 + \partial_2 f_3 = 0; \\ (\partial_1 f_2)(\partial_3 f_1) = 0; & (\partial_2 f_1)(\partial_3 f_2) = 0; & (\partial_3 f_2)(\partial_1 f_3) = 0. \end{cases}$$

Proposition 3.7 follows from the three Lemmas below.

**Lemma 3.8.** *The components of the curvature operator of the connection (3.4) are given by*

$$\begin{aligned} \mathcal{R}^\nabla(\partial_1, \partial_2)\partial_1 &= -\partial_2 f_1 \partial_1; & \mathcal{R}^\nabla(\partial_1, \partial_2)\partial_2 &= \partial_1 f_2 \partial_2; & \mathcal{R}^\nabla(\partial_1, \partial_3)\partial_1 &= -\partial_3 f_1 \partial_1; \\ \mathcal{R}^\nabla(\partial_1, \partial_3)\partial_3 &= \partial_1 f_3 \partial_3; & \mathcal{R}^\nabla(\partial_2, \partial_3)\partial_2 &= -\partial_3 f_2 \partial_2; & \mathcal{R}^\nabla(\partial_2, \partial_3)\partial_3 &= \partial_2 f_3 \partial_3. \end{aligned}$$

**Lemma 3.9.** *The nonzero components of the Ricci tensor of the connection (3.4) is given by*

$$\begin{aligned} Ric^\nabla(\partial_1, \partial_2) &= -\partial_1 f_2; & Ric^\nabla(\partial_1, \partial_3) &= -\partial_1 f_3; \\ Ric^\nabla(\partial_2, \partial_1) &= -\partial_2 f_1; & Ric^\nabla(\partial_2, \partial_3) &= -\partial_2 f_3; \\ Ric^\nabla(\partial_3, \partial_1) &= -\partial_3 f_1; & Ric^\nabla(\partial_3, \partial_2) &= -\partial_3 f_2; \end{aligned}$$

Now, since the Ricci tensor of any affine Osserman connection is skew-symmetric, it follows from Lemma 3.9 that for the connection (3.4) to be Osserman, we have the following conditions

$$\partial_1 f_2 = -\partial_2 f_1, \quad \partial_1 f_3 = -\partial_3 f_1, \quad \partial_2 f_3 = -\partial_3 f_2. \quad (3.5)$$

**Lemma 3.10.** *If  $X = \sum_1^3 \alpha_i \partial_i$  is a vector on  $M$ , then the affine Jacobi operator is given by*

$$\begin{aligned} J_{\mathcal{R}^\nabla}(X)\partial_1 &= b_1 \partial_1 + c_1 \partial_2 + d_1 \partial_3, \\ J_{\mathcal{R}^\nabla}(X)\partial_2 &= b_2 \partial_1 + c_2 \partial_2 + d_2 \partial_3, \\ J_{\mathcal{R}^\nabla}(X)\partial_3 &= b_3 \partial_1 + c_3 \partial_2 + d_3 \partial_3; \end{aligned}$$

where

$$\begin{aligned} b_1 &= -\alpha_1 \alpha_2 \partial_2 f_1 - \alpha_1 \alpha_3 \partial_3 f_1, & b_2 &= \alpha_1^2 \partial_2 f_1, & b_3 &= \alpha_1^2 \partial_3 f_1; \\ c_1 &= \alpha_2^2 \partial_1 f_2, & c_2 &= -\alpha_1 \alpha_2 \partial_1 f_2 - \alpha_2 \alpha_3 \partial_3 f_2, & c_3 &= \alpha_2^2 \partial_3 f_2; \\ d_1 &= \alpha_3^2 \partial_1 f_3, & d_2 &= \alpha_3^2 \partial_2 f_3, & d_3 &= -\alpha_1 \alpha_3 \partial_1 f_3 - \alpha_2 \alpha_3 \partial_2 f_3. \end{aligned}$$

The matrix associated to  $J_{\mathcal{R}^\nabla}(X)$  with respect to the basis  $\{\partial_1, \partial_2, \partial_3\}$  is given by

$$(J_{\mathcal{R}^\nabla}(X)) = \begin{pmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{pmatrix}.$$

It follows from the matrix associated to  $J_{\mathcal{R}^\nabla}(X)$ , that its characteristic polynomial can be written as follows:

$$\begin{aligned} P_\lambda[J_{\mathcal{R}^\nabla}(X)] &= [b_1(c_2d_3 - c_3d_2) + b_2(c_3d_1 - c_1d_3) + b_3(c_1d_2 - c_2d_1)] \\ &\quad - \lambda[(b_1c_2 - b_2c_1) + (b_1d_3 - b_3d_1) + (c_2d_3 - c_3d_2)] \\ &\quad + \lambda^2(b_1 + c_2 + d_3) - \lambda^3. \end{aligned}$$

*Proof of Proposition 3.7.* From the characteristic polynomial we have:

$$\begin{cases} b_1 + c_2 + d_3 & = 0; \\ (b_1c_2 - b_2c_1) + (b_1d_3 - b_3d_1) + (c_2d_3 - c_3d_2) & = 0; \\ b_1(c_2d_3 - c_3d_2) + b_2(c_3d_1 - c_1d_3) + b_3(c_1d_2 - c_2d_1) & = 0. \end{cases}$$

By straightforward calculation of this previous system, one obtains the following:

$$\begin{cases} \partial_2 f_1 + \partial_1 f_2 = 0; & \partial_3 f_1 + \partial_1 f_3 = 0; & \partial_3 f_2 + \partial_2 f_3 & = 0; \\ (\partial_1 f_2)(\partial_3 f_1) = 0; & (\partial_2 f_1)(\partial_3 f_2) = 0; & (\partial_3 f_2)(\partial_1 f_3) & = 0. \end{cases} \tag{3.6}$$

**Example 3.11.** In the Euclidean 3-dimensional space  $\mathbb{R}^3$ , consider the connection  $\nabla$  defined by

$$\nabla_{\partial_1} \partial_1 = \frac{1}{2}u_1u_2^2\partial_1, \quad \nabla_{\partial_2} \partial_2 = -\frac{1}{2}u_1^2u_2\partial_2, \quad \nabla_{\partial_3} \partial_3 = u_3\partial_3. \tag{3.7}$$

Then the nonvanishing components of the curvature tensor are

$$\mathcal{R}^\nabla(\partial_1, \partial_2)\partial_1 = -u_1u_2\partial_1 \quad \text{and} \quad \mathcal{R}^\nabla(\partial_1, \partial_2)\partial_2 = -u_1u_2\partial_2,$$

from which it follows that  $\mathbb{R}^3$  with torsion free affine connection (3.7) is a nonflat affine Osserman manifold.

**Acknowledgements** The authors wish to express their thanks the referee for helpful comments and suggestions. The first author would like to thank Nomenjanahary A. Raharinirina for many useful discussions.

## References

[1] M. Brozos-Vázquez, E. García-Río, P. Gilkey, S. Nikčević and R. Vázquez-Lorenzo, *The geometry of Walker manifolds*. Synthesis Lectures on Mathematics and Statistics, 5. Morgan & Claypool Publishers, Williston, VT, 2009.

- 
- [2] E. Calviño, E. García-Rio, P. Gilkey and R. Vázquez-Lorenzo, The geometry of modified Riemannian extensions. *Proc. R. Soc. A* **465** (2009), 2023-2040.
- [3] E. Calviño, E. García-Rio, P. Gilkey and R. Vázquez-Lorenzo, Higher-dimensional Osserman metrics with non-nilpotent Jacobi operators. *Geom. Dedicata* **156** (2012), 151-163.
- [4] A. S. Diallo, Affine Osserman connections on 2-dimensional manifolds. *Afric. Dispora J. Math.* **11** (1) (2011), 103-109.
- [5] A. S. Diallo and M. Hassirou, Examples of Osserman metrics of (3,3)-signature. *J. Math. Sci. Adv. Appl.* **7** (2) (2011), 95-103.
- [6] E. García-Rio, D. N. Kupeli, M. E. Vázquez-Abal and R. Vázquez-Lorenzo, Affine Osserman connections and their Riemannian extensions. *Differential Geom. Appl.* **11** (1999), 145-153.
- [7] E. García-Rio, D. N. Kupeli and R. Vázquez-Lorenzo, *Osserman Manifolds in Semi-Riemannian Geometry*, Lectures Notes in Mathematics 1777, Springer-Verlag, Berlin 2002.
- [8] E. García-Rio, M. E. Vázquez-Abal and R. Vázquez-Lorenzo, Nonsymmetric Osserman pseudo-Riemannian manifolds. *Proc. Amer. Math. Soc.* **126** (1998), 2771-2778.
- [9] P. Gilkey, *Geometric Properties of Natural Operators defined by the Riemann Curvature Tensor*, World Scientific Press Singapoure 2001.
- [10] O. Kowalski and M. Sekizawa, Almost Osserman structures on natural Riemannian extensions. *Differential Geom. Appl.* **to appear**.
- [11] K. Yano and S. Ishihara, *Tangent and Cotangent Bundles*, Marcel Dekker, New York, 1973.