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An Elementary Symmetry-Based Derivation of the Heat Kernel on Heisenberg Group

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Abstract

Using symmetry arguments, we propose a simple derivation of a fundamental solution of the operator $\partial_t - \Delta_H$ in which Δ_H is Kohn-Laplace operator on the Heisenberg group H^{2n+1}. Our derivation extends that of Craddock and Lennox [J. Differential Equations 232(2007) 652-674]. Indeed, these authors solved the same problem by employing a symmetry approach in the case $n = 1$. We demonstrate that the case $n = 1$ is quite peculiar from a symmetry standpoint and the extension of symmetry arguments to the case $n > 1$ requires some intermediate results.

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1 Introduction

The main goal of this paper is to construct a solution of the initial-value problem

$$E_t = \Delta_H E, \quad t > 0,\ x = (x_0, x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n+1} \tag{1.1}$$

$$E(0, x) = \delta(x), \tag{1.2}$$

where δ is Dirac measure and Δ_H is given by

$$\Delta_H = \sum_{j=1}^{n} (X_j^2 + X_{j+n}^2), \tag{1.3}$$

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in which

$$X_j = \frac{\partial}{\partial x_j} - x_{n+j} a_j \frac{\partial}{\partial x_0}, \quad 1 \le j \le n, \tag{1.4}$$

$$X_{n+j} = \frac{\partial}{\partial x_{j+n}} + x_j a_j \frac{\partial}{\partial x_0}, \quad 1 \le j \le n, \tag{1.5}$$

where the a_is are positive real numbers. It can be easily verified that the operators $X_0 = \partial/\partial x_0$, $X_1, X_2, \dots, X_{2n}$ constitute a realization of the Heisenberg Lie algebra $\mathfrak{H}^{2n+1}$ which arises in quantum mechanics, harmonic analysis, ergodic theory and classical invariant theory. Several approaches have been adopted in the search of a fundamental solution of the operator $\partial_t - \Delta_H$ i.e a solution of Eqs. (1.1)-(1.2). Historically, an explicit formula for the heat kernel of Δ_H was first obtained by Hulanicki [1] using representation theory. Subsequently, Gaveau [2] and Beals and Greiner [3] obtained a similar formula using probability theory and Laguerre calculus respectively. Arguably, the most unified of these approaches is due to Beals and Gaveau [4] (see also [5]). In their approach, they first look for a fundamental solution of Δ_H i.e. a solution of $\Delta_H G(x) = \delta(x)$ in the form

$$G(x) = \int_{-\infty}^{\infty} \frac{V(\tau)d\tau}{f(x,\tau)^n},$$

where f is associated to a complex Hamiltonian problem and V solves a transport equation. Then, they proved by direct computations that a solution of Eqs (1.1)-(1.2) is

$$E(t,x) = \frac{1}{(2\pi t)^{n+1}} \int_{-\infty}^{\infty} V(\tau) \exp(-f(x,\tau)/t)\, d\tau.$$

In Beals and Gaveau [4], symmetries play a prominent role in the computation of the heat kernel for Δ_H. Indeed the integration of both the Hamilton-Jacobi equation satisfied by f and the transport equation satisfied by V relies on the invariance of the underlying equations (or companion equations) under appropriate scaling symmetries. Therefore we may query whether a solution of the problem (1.1)-(1.2) may be constructed directly using its symmetries. An affirmative answer to this question in the case $n = 1$ was recently given by Craddock and Lennox [6]. However, they did not extend the symmetry argument to the general case. Our objective in this paper is to close this gap. We have organized our work as follows. There are five sections including this introduction. In Section 2 we rewrite the problem (1.1)-(1.2) in appropriate coordinates that exposes some of its symmetries. By using the symmetry principle together with an appropriate Fourier transform, we reduce the search of the heat kernel of Δ_H to that of a simpler operator. In section 3 we perform a Lie symmetry analysis of the reduced problem. It is found that the case $n = 1$ belongs to the most symmetric one which corresponds to $a_1 = a_2 = \ldots = a_n$. If there are at least two distinct a_i, in which case $n > 1$ necessarily, there are not enough Lie symmetries to construct the kernel by Craddock and Lennox method. Fortunately, in Section 4, we demonstrate that the simplified problem possesses a generalized symmetry that allows separation of variables and the reduction of the computations to the case $n = 1$. In the last section, we summarize our findings.

2 Reformulation of the problem (1.1)-(1.2)

Here we introduce transformations that will facilitate the search for a solution of Eqs. (1.1)-(1.2).

The operator Δ_H may be written explicitly as

$$\begin{aligned}\Delta_H &= \sum_{j=1}^{n}(\partial_{x_j}^2+\partial_{x_{j+n}}^2)+2\sum_{j=1}^{n}a_j(x_{j+n}\partial_{x_j}-x_j\partial_{x_{j+n}})\partial_{x_0}\\ &\quad+\sum_{j=1}^{n}a_j^2(x_j^2+x_{j+n}^2)\partial_{x_0}^2.\end{aligned} \tag{2.1}$$

Now, make the change of variables

$$x_j=r_j\cos\theta_j,\quad x_{j+n}=r_j\sin\theta_j,\quad \theta_j\in[0,2\pi),\ r_j\geq 0,\ j=1,2\dots,n. \tag{2.2}$$

In the new variables, Eq. (2.1) becomes

$$\Delta_H=\sum_{j=1}^{n}\left(\partial_{r_j}^2+\frac{1}{r_j}\partial_{r_j}+\frac{1}{r_j^2}\partial_{\theta_j}^2\right)-2\sum_{j=1}^{n}a_j\partial_{x_0}\partial_{\theta_j}+\sum_{j=1}^{n}a_j^2r_j^2\partial_{x_0}^2. \tag{2.3}$$

In the new coordinates (2.2), the problem (1.1)-(1.2) is transformed to

$$\partial_tE=\sum_{j=1}^{n}\left(\partial_{r_j}^2+\frac{1}{r_j}\partial_{r_j}+\frac{1}{r_j^2}\partial_{\theta_j}^2\right)E-2\sum_{j=1}^{n}a_j\partial_{x_0}\partial_{\theta_j}E+\sum_{j=1}^{n}a_j^2r_j^2\partial_{x_0}^2E, \tag{2.4}$$

$$E(0,x_0,r_1,\dots,r_n,\theta_1,\dots,\theta_n)=\delta(x_0)\bigotimes_{j=1}^{n}\frac{\delta(r_j)}{2\pi r_j}, \tag{2.5}$$

where $\bigotimes$ is the direct product of distributions. Note that Eq. (2.4) is invariant under the translations ∂_{θ_j}, $j=1,\dots,n$. The initial condition (2.5) will be also invariant under the same translations if and only if E is independent of θ_1 to θ_n. In such case, E satisfies

$$\partial_tE=\sum_{j=1}^{n}\left(\partial_{r_j}^2+\frac{1}{r_j}\partial_{r_j}\right)E+\sum_{j=1}^{n}a_j^2r_j^2\partial_{x_0}^2E, \tag{2.6}$$

$$E(0,x_0,r_1,\dots,r_n)=\delta(x_0)\bigotimes_{j=1}^{n}\frac{\delta(r_j)}{2\pi r_j}. \tag{2.7}$$

Take the Fourier transform of Eqs. (2.6)-(2.7) with respect to x_0 to obtain

$$\partial_t\hat{E}=\sum_{j=1}^{n}\left(\partial_{r_j}^2+\frac{1}{r_j}\partial_{r_j}\right)\hat{E}-\lambda^2\sum_{j=1}^{n}a_j^2r_j^2\hat{E}, \tag{2.8}$$

$$\hat{E}(0,r_1,\dots,r_n)=\bigotimes_{j=1}^{n}\frac{\delta(r_j)}{2\pi r_j}, \tag{2.9}$$

where the hat stands for Fourier transform with respect to x_0 and λ is the associated Fourier variable. Now make the change of variable

$$u = \hat{E}\exp\left(2|\lambda| t \sum_{j=1}^{n} a_j\right). \tag{2.10}$$

Equations (2.8)-(2.9) become

$$\begin{aligned} u_t &= \sum_{j=1}^{n}(u_{r_j r_j} + r_j^{-1} u_{r_j}) - \left(\lambda^2 \sum_{j=1}^{n} a_j^2 r_j^2 - 2|\lambda| \sum_{j=1}^{n} a_j\right) u, & (2.11)\\ u|_{t=0} &= \bigotimes_{j=1}^{n} \frac{\delta(r_j)}{2\pi r_j}, & (2.12) \end{aligned}$$

where subscripts of the dependent variable u stand for partial differentiations. At this point it is opportune to point that when $n = 1$ and $a_1 = 1$, Eq. (2.11) coincides with Eq. (8.5) of [6]. However we will see in the next section that the case $n = 1$ is quite peculiar from a symmetry standpoint.

3 Lie symmetry Analysis of the transformed problem

This section is dedicated to the computation of Lie symmetries of Eq. (2.11). We assume that the reader is familiar with Lie symmetry algorithm [7, 8, 9].

An operator

$$\mathbf{v} = \tau(t,\overrightarrow{r},u)\partial_t + \xi^j(t,\overrightarrow{r},u)\partial_{r_j} + \eta(t,\overrightarrow{r},u)\partial_u,\ \overrightarrow{r} = (r_1,\ldots,r_n) \tag{3.1}$$

is a Lie symmetry of Eq. (2.11) if

$$\mathbf{v}^{[2]}\left(u_t - \sum_{j=1}^{n}(u_{r_j r_j} + r_j^{-1} u_{r_j}) + \left(\lambda^2 \sum_{j=1}^{n} a_j^2 r_j^2 - 2|\lambda| \sum_{j=1}^{n} a_j\right) u\right)\Bigg|_{(2.11)} = 0, \tag{3.2}$$

where $\mathbf{v}^{[2]}$ is the second prolongation [7, 8, 9] of $\mathbf{v}$. Equation (3.2) is a polynomial equation in the derivatives $u_{r_j r_j}$ and u_{r_j} which results in an over-determined system of linear homogeneous partial differential equations in τ, the ξ^js and η after separation. This determining equations for symmetries simplifies after elementary manipulations to the following equations

$$\begin{aligned} &\tau = \tau(t),\ \xi^j = \xi^j(t,\overrightarrow{r}),\ \eta = \alpha(t,\overrightarrow{r})u + \beta(t,\overrightarrow{r}), & (3.3)\\ &\xi^j_{,k} = 0, \quad j \neq k,\ j,k = 1,2,\ldots,n & (3.4)\\ &r_j^{-1}\tau_{,t} + \xi^j_{,t} = \xi^j_{,jj} - 2\alpha_{,j} + r_j^{-2}\xi^j + r_j^{-1}\xi^j_{,j}, \quad j = 1,2,\ldots,n, & (3.5)\\ &\alpha_{,t} + \tau_{,t}\sum_{k=1}^{n}(\lambda^2 a_k^2 r_k^2 - 2|\lambda| a_k) = \sum_{k=1}^{n}\left(\alpha_{,kk} + r_k^{-1}\alpha_{,k} - 2\lambda^2 a_k^2 r_k \xi^k\right), & (3.6)\\ &\beta_{,t} = \sum_{k=1}^{n}(\beta_{,kk} + r_k^{-1}\beta_{,k}) - \beta\sum_{k=1}^{n}(\lambda^2 a_k^2 r_k^2 - 2|\lambda| a_k), & (3.7) \end{aligned}$$

in which subscripts following the comma in function's notation stand for partial differentiations and subscripts j and k represent differentiations with respect to r_j and r_k respectively.

From Eqs. (3.3)-(3.4), we infer that

$$\xi^j = \frac{\tau_{,t}}{2} r_j + \gamma^j(t), \quad j = 1, \ldots, n, \tag{3.8}$$

where γ^j is an arbitrary function of t.

Substitute Eq. (3.8) in Eq. (3.5) to obtain

$$\alpha_{,j} = \frac{\gamma^j}{2 r_j^2} - \frac{\gamma^j_{,t}}{2} - \frac{\tau_{,tt} r_j}{4}, \; j = 1, \ldots, n. \tag{3.9}$$

Taking the derivative of both sides of Eq. (3.6) with respect to r_j and using Eq. (3.9) in the resulting equation yields

$$\gamma^j_{,tt} + \frac{\tau_{,ttt}}{2} r_j = 8\lambda^2 a_j^2 \tau_{,t} r_j - 3\gamma^j r_j^{-4}, \; j = 1, \ldots, n. \tag{3.10}$$

Since the functions τ and γ^j depend solely on t, we deduce from Eq. (3.10) that

$$\gamma^j = 0, \quad \tau_{,ttt} - 16\lambda^2 a_j^2 \tau_{,t} = 0, \; j = 1, \ldots, n. \tag{3.11}$$

Equation (3.11b) forces the consideration of the following cases.

Case I: $\mathbf{a_1 = a_2 = \ldots = a_n = a > 0}$.

In this case, the solution of Eq. (3.11b) is

$$\tau = C_1 + C_2 e^{4a|\lambda|t} + C_3 e^{-4a|\lambda|t}, \tag{3.12}$$

where C_1, C_2 and C_3 are arbitrary constants. Using Eq. (3.12) in Eq.(3.8) taking into account Eq. (3.11a) produces

$$\xi^j = 2a|\lambda| \left(C_2 e^{4a|\lambda|t} - C_3 e^{-4a|\lambda|t} \right) r_j. \tag{3.13}$$

Substituting Eq. (3.12) into Eq. (3.9) gives

$$\alpha_{,j} = -4a^2\lambda^2 \left(C_2 e^{4a|\lambda|t} + C_3 e^{-4a|\lambda|t} \right) r_j, \; j = 1, \ldots, n. \tag{3.14}$$

Therefore

$$\alpha = -2a^2\lambda^2 \left(C_2 e^{4a|\lambda|t} + C_3 e^{-4a|\lambda|t} \right) r^2 + \Gamma(t), \tag{3.15}$$

where Γ is a function of t and $r^2 = r_1^2 + r_2^2 + \cdots + r_n^2 = \vec{r}.\vec{r}$. Now insert Eqs. (3.12)-(3.13) and (3.15) in Eq. (3.6) to obtain after simplifications

$$\Gamma_{,t} = -16a^2\lambda^2 n C_3 e^{-4a|\lambda|t}. \tag{3.16}$$

Hence

$$\Gamma = 4a|\lambda| n C_3 e^{-4a|\lambda|t} + C_4, \tag{3.17}$$

where C_4 is an arbitrary constant and

$$\begin{aligned}\eta &= -2ar^2\mathrm{e}^{4a|\lambda|t}C_2u - 2a\mathrm{e}^{-4a|\lambda|t}(a\lambda^2r^2 - 2|\lambda|n)C_3u \\ &\quad + C_4u + \beta(t,\overrightarrow{r}),\end{aligned} \tag{3.18}$$

where β solves the linear partial differential equation (3.7) which is in fact our Eq. (2.11).

Case II: there are p and q in $\{1,\ldots,n\}$ such that $a_p \neq a_q$.

Note that in this case we necessarily have $n > 1$. Now Eq. (3.11b) leads to $\tau_{,t} = 0$. That is

$$\tau = K_1, \tag{3.19}$$

where K_1 is an arbitrary constant. From Eqs. (3.8)-(3.9) we infer that

$$\xi^j = 0,\ j = 1,\ldots,n,\ \alpha = K_2, \tag{3.20}$$

where K_2 is an arbitrary constant. Therefore

$$\eta = K_2\,u + \beta(t,\overrightarrow{r}), \tag{3.21}$$

where β solves Eq. (2.11).

In summary we have proved the following theorem.

Theorem 1. *According to whether $a_1 = a_2 = \ldots = a_n = a > 0$, or at least two of the a_j are distinct, the symmetry Lie algebra of Eq. (2.11) is respectively spanned by*

$$\mathbf{v_1} = e^{4a|\lambda|t}\,\partial_t + 2a|\lambda|e^{4a|\lambda|t}r_j\,\partial_j - 2a^2\lambda^2r^2e^{4a|\lambda|t}u\,\partial_u, \tag{3.22}$$

$$\mathbf{v_2} = e^{-4a|\lambda|t}\partial_t - 2a|\lambda|e^{-4a|\lambda|t}r_j\,\partial_j - 2(a^2\lambda^2r^2 - 2a|\lambda|n)e^{-4a|\lambda|t}u\,\partial_u, \tag{3.23}$$

$$\mathbf{v_3} = \partial_t, \quad \mathbf{v_4} = u\,\partial_u \text{ and } \quad \mathbf{v}_\beta = \beta(t,\overrightarrow{r})\,\partial_u, \tag{3.24}$$

or $\mathbf{v_3}$, $\mathbf{v_4}$ and $\mathbf{v}_\beta$, where β is a solution of Eq. (2.11) and summation over repeated indices is used.

It is important to note that when $n = 1$ and $a_1 = 1$, the statement of Proposition 8.1 of [6] is recovered from Theorem 1.

4 Construction of the heat kernel

Our computations in the previous section show that in general Eq. (2.11) does not have enough nontrivial Lie symmetries. Therefore we may not right away apply Craddock and Lennox procedure [6] for the computation of a fundamental solution of Eqs. (2.11)-(2.12). However, the form of the initial condition (2.12) suggests that Eq. (2.11) may admit multiplicative separable solutions. The following lemma confirms our suggestion which will be motivated below using symmetries.

Lemma 4.1. *Let $\varphi_1(t,r_1),\ldots,\varphi_n(t,r_n)$ be functions such that for all $j \in \{1,\ldots,n\}$ we have*

$$\varphi_{j,t} = \varphi_{j,r_jr_j} + r_j^{-1}\varphi_{j,r_j} - \left(\lambda^2a_j^2r_j^2 - 2|\lambda|a_j\right)\varphi_j, \tag{4.1}$$

$$\varphi_j|_{t=0} = \frac{\delta(r_j)}{2\pi r_j}. \tag{4.2}$$

Then $u = \bigotimes_{i=1}^{n}\varphi_i$ solves Eqs.(2.11)-(2.12).

Proof. The proof of this lemma is a straightforward application of Leibnitz differentiation rule.

According to Lemma 4.1, Eq. (2.11) admits a multiplicative separable solution and it is well-known that separable solutions of partial differential equations results from generalized conditional symmetries [10]. It can be verified that in our case the multiplicative separable solutions comes from the generalized conditional symmetry $u\left\{\frac{\partial^n(\ln u)}{\partial r_1 \ldots \partial r_n}\right\}\partial_u$. To see this, make the change of variable $u = \mathrm{e}^v$ and verify that the resulting equation admits the generalized symmetry $v_{r_1\ldots r_n}\partial_v$.

We are left with constructing a solution of the problem (4.1)-(4.2). But this problem has been solved by Craddock and Lennox [6] using symmetry methods. Indeed, by replacing λa_j by λ_j in Eq. (4.1) we recover Eq. (8.5) of [6] up to a change in notations. Starting from the stationary solution (i.e invariant solution under the time-translation $\mathbf{v_3}$) $\psi_j = \mathrm{e}^{-\lambda_j r_j^2/2}$ and using the symmetry $\mathbf{v_2}$, Craddock and Lennox constructed (see [6] for details) the solution

$$\phi_j = \frac{|\lambda_j| e^{2|\lambda_j|t}}{2\pi \sinh(2|\lambda_j|t)} \exp\left\{\frac{-|\lambda_j| r_j^2}{2\tanh(2|\lambda_j|t)}\right\}. \tag{4.3}$$

Note that in [6] The multiplicative factor in Eq. (4.3) is incorrect due to a mistake in setting the initial condition. It can be verified by direct calculations using Lebesgue dominated convergence theorem that

$$\lim_{t\to 0}\int_0^\infty 2\pi r_j \phi_j(r) f(r_j) dr_j = f(0)$$

for any test function $f \in C_0^\infty(\mathbb{R}^+)$. Therefore, the initial condition (2.12) is satisfied. The function u takes the form

$$u = \frac{1}{(2\pi)^n}\prod_{j=1}^n \frac{|\lambda| a_j e^{2|\lambda| a_j t}}{\sinh(2|\lambda| a_j t)} \exp\left\{\frac{-|\lambda| a_j r_j^2}{2\tanh(2|\lambda| a_j t)}\right\}. \tag{4.4}$$

Finally, we obtain

$$\begin{aligned} E &= \frac{1}{(2\pi)^{n+1}}\int_{-\infty}^{\infty} \exp\left\{-2|\lambda| t\sum_{j=1}^n a_j + i\lambda x_0\right\}\times \\ &\qquad \prod_{j=1}^n \frac{|\lambda| a_j e^{2|\lambda| a_j t}}{\sinh(2|\lambda| a_j t)} \exp\left\{\frac{-|\lambda| a_j (x_j^2 + x_{j+n}^2)}{2\tanh(2|\lambda| a_j t)}\right\} d\lambda. \end{aligned} \tag{4.5}$$

It is opportune to stress how symmetries have played a crucial role in the derivation of the fundamental solution (4.5). We have employed the invariance of Eq. (1.1) under the rotations $x_j\partial_{x_{j+n}} - x_{j+n}\partial_{x_j} = \partial_{\theta_j}$ to derive Eq. (2.11). Separation of variables which results from a generalized conditional symmetry has allows us to express the solution of Eq. (2.11) in term of that of Eq. (4.1). Finally a solution of Eq. (4.1) was constructed starting from a stationary solution and using the Lie symmetry $\mathbf{v_2}$.

5 Conclusion

We have derived the heat kernel on Heisenberg group using symmetry arguments. The advantage of the symmetry approach is that it is simple when enough symmetries are available, it relieves us from technicalities of other approaches and it may be pedagogically appealing.

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References

[1] Hulanicki, A. The distribution of energy in Brownian motion in Gaussian field and analytic hypoellipticity of certain subelliptic operators on Heisenberg group. *Studia Mathematicae* **1976**, *56*, 165-173.

[2] Gaveau, B. Principe de moindre action, propagation de la chaleur et estimées sous-elliptics sur certains groupes nilpotents. *Acta Mathematicae* **1977**, *139*, 95-153.

[3] Beals, R.; Greiner, P.C. *Calculus on Heisenberg manifolds*; Princeton University Press: New Jersey, 1988.

[4] Beals, R.; Gaveau, B. The Green funtion of model step two hypoelliptic operators and the analysis of certain Cauchy Riemann complexes, *Advances in Mathematics* **1996**, *121*, 288-345.

[5] Beals, R.; Gaveau, B.; Greinier, P.C. Hamilton-Jacobi theory and the heat kernel on Heisenberg groups. *Journal de Mathématiques Pures et Appliqueés* **2000**, *79*, 633-689.

[6] Craddock, M.; Lennox, K. Lie group symmetries as integral transforms of fundamental solutions. *Journal of Differential Equations* **2007**, *232*, 652-674.

[7] Ovsiannikov, L.V. *Group Analysis of Differential equations*; Academic Press: New York, 1982.

[8] Bluman, G.; Kumei, S. *Symmetries and Differential Equations*; Springer-Verlag: New York, 1989.

[9] Olver, P. *Applications of Lie groups to Differential Equations*; Springer-Verlag: New York, 1993.

[10] Estévez, F.G.; Qu, C.; Zhang, S. Separation of variables of a generalized porous medium equation with nonlinear source. *Journal of Mathematical Analysis and Applications* **2002**, *275*, 44-59.