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An Extension of Fuglede-Putnam Theorem for w-Hyponormal Operators

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Abstract

In this paper, we prove the following: assume that either (i) T^* is w-hyponormal and S is w-hyponormal such that $\ker(T^*) \subset \ker(T)$ and $\ker(S) \subset \ker(S^*)$ or (ii) T^* is p-hyponormal or log-hyponormal and S is w-hyponormal such that $\ker(S) \subset \ker(S^*)$ or (iii) T^* is an injective w-hyponormal and S is a dominant holds. Then the pair (T,S) satisfy Fuglede-Putnam theorem. Also, other related results are given.

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1 Introduction

For complex infinite dimensional Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, $\mathscr{L}(\mathscr{H})$, $\mathscr{L}(\mathscr{K})$ and $\mathscr{L}(\mathscr{H},\mathscr{K})$ denote the set of bounded linear operators on $\mathscr{H}$, the set of bounded linear operators on $\mathscr{K}$ and the set of bounded linear operators from $\mathscr{H}$ to $\mathscr{K}$, respectively. An operator $T \in \mathscr{L}(\mathscr{H})$ is called positive (in symbol $T \geq 0$) if $\langle Tx,x\rangle \geq 0$ for all $x \in \mathscr{H}$. An operator $T \in \mathscr{L}(\mathscr{H})$ is called normal if $T^*T = TT^*$. Following [24, 28], an operator $T \in \mathscr{L}(\mathscr{H})$ is called dominant if

$$\Re(T-\lambda) \subset \Re(T-\lambda)^* \qquad \text{for all } \lambda \in \mathbb{C}.$$

This condition is equivalent to the existence of a positive constant M_λ for each $\lambda \in \mathbb{C}$ such that

$$(T-\lambda)(T-\lambda)^* \leq M_\lambda (T-\lambda)^*(T-\lambda).$$

If there exists a constant M such that $M_\lambda \leq M$ for all $\lambda \in \mathbb{C}$, then T is called M-hyponormal, and if $M = 1$, T is hyponormal. Hence the following inclusion relations hold:

$$\{\text{Normal}\} \subset \{\text{Hyponormal}\} \subset \{M\text{-hyponormal}\} \subset \{\text{Dominant}\}.$$

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According to [1, 3, 11], an operator $T \in \mathscr{L}(\mathscr{H})$ is called p-hyponormal for $p \in (0,1]$ if $|T|^{2p} \geq |T^*|^{2p}$, when $p = 1$, T is called hyponormal, when $p = \frac{1}{2}$, T is called semi-hyponormal. An operator $T \in \mathscr{L}(\mathscr{H})$ is called log-hyponormal if T is invertible and $\log(T^*T) \geq \log(TT^*)$. And $T \in \mathscr{L}(\mathscr{H})$ is called paranormal if $\left\|T^2x\right\| \geq \|Tx\|^2$ for every unit vector $x \in \mathscr{H}$.

In order to discuss the relations between paranormal and p-hyponormal and log-hyponormal operators, Furuta el al. [12] introduced a class A defined by $|T^2| \geq |T|^2$ and they showed that class A is a subclass of paranormal and contains p-hyponormal and log-hyponormal operators. Class A operators have been studied by many researchers, for example [12, 10]. Fujii et al. [10] introduced a new class $A(t,s)$ of operators: For $t > 0$ and $s > 0$ an operator T belongs to class $A(s,t)$ if it satisfies an operator inequality

$$\left(|T^*|^t|T|^{2s}|T^*|^t\right)^{\frac{t}{t+s}} \geq |T^*|^{2t}.$$

Recall from [2] that an operator $T \in \mathscr{L}(\mathscr{H})$ is called w-hyponormal if $|\widetilde{T}| \geq |T| \geq |\widetilde{T}^*|$, where $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is the Aluthge transformation. As a generalization of w-hyponormal and class $A(s,t)$, Ito [15] introduced a class of operators called $wA(s,t)$: For $t > 0$ and $s > 0$ an operator T belongs to class $wA(s,t)$ if it satisfies an operator inequality

$$\left(|T^*|^t|T|^{2s}|T|^t\right)^{\frac{t}{t+s}} \geq |T^*|^{2t}.$$

and

$$|T|^{2s} \geq \left(|T|^s|T^*|^{2t}|T|^s\right)^{\frac{s}{s+t}}.$$

In [14], they showed that class w-hyponormal coincides with class $wA(\frac{1}{2},\frac{1}{2})$, class A coincides with class $wA(1,1)$ and class $A(s,t)$ coincides with class $wA(s,t)$ for each $s > 0$, and $t > 0$. Inclusion relations among these classes are known as follows:

$$\begin{aligned}
\{\text{hyponormal operators}\} &\subset \{p\text{-hyponormal operators for } 0 < p \leq 1\} \\
&\subset \{\text{class } A(s,t) \text{ operators for } s,t \in [0,1]\ \} \\
&= \{\text{class } wA(s,t) \text{ operators for } s,t \in [0,1]\ \} \\
&\subset \{\text{class } A \text{ operators}\} \\
&\subset \{\text{paranormal operators}\}.
\end{aligned}$$

A pair (T,S) is said to have the Fuglede-Putnam property if $T^*X = XS^*$ whenever $TX = XS$ for every $X \in \mathscr{L}(\mathscr{K},\mathscr{H})$. The Fuglede-Putnam theorem is well-known in the operator theory. It asserts that for any normal operators T and S, the pair (T,S) has the Fuglede-Putnam property. There exist many generalization of this theorem which most of them go into relaxing the normality of T and S, see [4, 5, 8, 9, 19, 20, 21, 22, 24, 25, 27, 28, 29, 30, 31] and references therein. The two next lemmas are concerned with the Fuglede-Putnam theorem and we need them in the future.

Lemma 1.1. ([30]) Let $T \in \mathscr{L}(\mathscr{H})$ and $S \in \mathscr{L}(\mathscr{K})$. Then the following assertions are equivalent.

(i) The pair (T,S) has the Fuglede-Putnam property.

(ii) If $TX = SX$, then $\overline{\Re(X)}$ reduces T, $\ker(X)^{\perp}$ reduces S, and $T|_{\overline{\Re(X)}}$, $S|_{\ker(X)^{\perp}}$ are unitarily equivalent normal operators.

Lemma 1.2. ([16]) Let $T \in \mathscr{L}(\mathscr{H})$ and $S^* \in \mathscr{L}(\mathscr{K})$ be either log-hyponormal or p-hyponormal operators. Then the pair (T, S) has the Fuglede-Putnam property.

2 Complementary Results

In this section, we present some results that will be needed in the section which follows.

Lemma 2.1. *([13]) If $A, B \in \mathscr{L}(\mathscr{H})$ satisfy $A \geq 0$ and $\|B\| \leq 1$, then*

$$(B^*AB)^{\alpha} \geq B^*A^{\alpha}B \qquad \text{for all } \alpha \in (0,1].$$

Lemma 2.2. *Let A, B and C be positive operators. Then*

$$\left(B^{\frac{1}{2}}AB^{\frac{1}{2}}\right)^{\alpha} \geq B \text{ and } B \geq C \Longrightarrow \left(C^{\frac{1}{2}}AC^{\frac{1}{2}}\right)^{\alpha} \geq C, \text{ for all } 0 < \alpha \leq 1.$$

Proof. There exists an operator X such that

$$C^{\frac{1}{2}} = B^{\frac{1}{2}}X = X^*B^{\frac{1}{2}} \quad \text{and} \quad \|X\| \leq 1$$

by Douglas theorem [7]. Then with $C^{\frac{1}{2}} = B^{\frac{1}{2}}X$ we have

$$\begin{aligned}\left(C^{\frac{1}{2}}AC^{\frac{1}{2}}\right)^{\alpha} &= \left(X^*B^{\frac{1}{2}}AB^{\frac{1}{2}}X\right)^{\alpha} \\ &\geq X^*\left(B^{\frac{1}{2}}AB^{\frac{1}{2}}\right)^{\alpha}X \geq X^*BX = C\end{aligned}$$

by Lemma 2.1. □

Theorem 2.3. *Let $0 < s, t \leq 1$. Let $T \in \mathscr{L}(\mathscr{H})$ be a class $A(s,t)$ operator and $\mathscr{M}$ be its invariant subspace. Then the restriction $T|_{\mathscr{M}}$ of T to $\mathscr{M}$ is also class $A(s,t)$ operator.*

Proof. Let P be the projection onto $\mathscr{M}$, and $T_1 = TP$. Then

$$|T_1|^{2s} = (P|T|^2P)^s \geq P|T|^{2s}P$$

by Lemma 2.1, so that $|T_1^*|^t|T_1|^{2s}|T_1^*|^t \geq |T_1^*|^t|T|^{2s}|T_1^*|^t$. And also,

$$|T_1^*|^{2t} = (TPT^*)^t \leq (TT^*)^t = |T^*|^{2t}$$

by Löwner-Heinz theorem [23]. Since T belongs to class $A(s,t)$, we have

$$\left(|T^*|^t|T|^{2s}|T|^t\right)^{\frac{t}{t+s}} \geq |T^*|^{2t},$$

it follows from Lemma 2.2 that

$$\left(|T_1^*|^t|T|^{2s}|T_1^*|^t\right)^{\frac{t}{t+s}} \geq |T_1^*|^{2t},$$

and so

$$\left(|T_1^*|^t|T_1|^{2s}|T_1^*|^t\right)^{\frac{t}{t+s}} \geq |T_1^*|^{2t}, \tag{2.1}$$

by Löwner-Heinz theorem. That is, the restriction $T|_{\mathscr{M}}$ of T to $\mathscr{M}$ is class $A(s,t)$ operator. □

Since class $A(s,t)$ operators coincides with class $wA(s,t)$ for each $s>0$ and $t>0$, we have the following corollary.

Corollary 2.4. *Let $0<s,t\leq 1$. Let $T\in\mathcal{L}(\mathcal{H})$ be a class $wA(s,t)$ operator and $\mathcal{M}$ be its invariant subspace. Then the restriction $T|_{\mathcal{M}}$ of T to $\mathcal{M}$ is also class $wA(s,t)$ operator.*

Since class $wA(\frac{1}{2},\frac{1}{2})$ operators coincides with class w-hyponormal operators, we have the following corollary.

Corollary 2.5. *Let $T\in\mathcal{L}(\mathcal{H})$. If T is w-hyponormal operator and $\mathcal{M}$ be its invariant subspace. Then the restriction $T|_{\mathcal{M}}$ of T to $\mathcal{M}$ is also w-hyponormal operator.*

Lemma 2.6. *Let $0<s,t\leq 1$. Let $T\in\mathcal{L}(\mathcal{H})$ belongs to class $wA(s,t)$ and $T=U|T|$ be the polar decomposition of T. If $\mathcal{M}$ is an invariant subspace of T and $T|_{\mathcal{M}}$ is an injective normal operator, then the generalized Aluthge transformation has the form $\widetilde{T}_{s,t}=N\oplus R$ on $\mathcal{H}=\mathcal{M}\oplus\mathcal{M}^{\perp}$, where N is a normal operator on $\mathcal{M}$.*

Proof. First, we show that if T is a class $wA(s,t)$, then the generalized Aluthge transformation $\widetilde{T}_{s,t}$ has the form $\widetilde{T}_{s,t}=N\oplus R$. Since T is a class $wA(s,t)$, it follows from [15] that $\widetilde{T}_{s,t}$ is a p-hyponormal operator, where $p=\dfrac{\min\{s,t\}}{s+t}$. By Lemma 5 and Lemma 11 of [31], $\widetilde{T}_{s,t}$ has the form $\begin{pmatrix} N & S\\ 0 & R\end{pmatrix}$ on $\mathcal{H}=\mathcal{M}\oplus\mathcal{M}^{\perp}$, where N is normal and $\mathfrak{R}(S)\subset\ker(N)$. Then

$$\widetilde{T}^*_{s,t}\widetilde{T}_{s,t}=\begin{pmatrix}|N|^2 & 0\\ 0 & |S|^2+|R|^2\end{pmatrix}$$
$$\widetilde{T}_{s,t}\widetilde{T}^*_{s,t}=\begin{pmatrix}|N|^2+|S^*|^2 & SR^*\\ RS^* & |R^*|^2\end{pmatrix}$$

Put $(\widetilde{T}_{s,t}\widetilde{T}^*_{s,t})^p=\begin{pmatrix} X & Y\\ Y^* & Z\end{pmatrix}$. Then the p-hyponormality of $\widetilde{T}_{s,t}$ implies that

$$(\widetilde{T}^*_{s,t}\widetilde{T}_{s,t})^p=\begin{pmatrix}|N|^{2p} & 0\\ 0 & (|S|^2+|R|^2)^p\end{pmatrix}\geq\begin{pmatrix} X & Y\\ Y^* & Z\end{pmatrix}=(\widetilde{T}_{s,t}\widetilde{T}^*_{s,t})^p.$$

We have $\mathfrak{R}(Y)\subset\mathfrak{R}(X^{\frac{1}{2}})$ by Lemma 9 of [31] and $\mathfrak{R}(X^{\frac{1}{2}})\subset\mathfrak{R}(|N|^p)$ by Lemma 8 of [31]. Hence we have $\mathfrak{R}(X)\cup\mathfrak{R}(Y)\subset\mathfrak{R}(X^{\frac{1}{2}})\subset\mathfrak{R}(|N|^p)$. Put $(\widetilde{T}_{s,t}\widetilde{T}^*_{s,t})^{1-p}=\begin{pmatrix} A & B\\ B^* & C\end{pmatrix}$. Hence

$$\widetilde{T}_{s,t}\widetilde{T}^*_{s,t}=(\widetilde{T}_{s,t}\widetilde{T}^*_{s,t})^p(\widetilde{T}_{s,t}\widetilde{T}^*_{s,t})^{1-p}=\begin{pmatrix} X & Y\\ Y^* & Z\end{pmatrix}\begin{pmatrix} A & B\\ B^* & C\end{pmatrix}.$$

This implies that $|N|^2+SS^*=XA+YB^*$. Therefore,

$$\mathfrak{R}(SS^*)\subset\mathfrak{R}(|N|^2)+\mathfrak{R}(X)+\mathfrak{R}(Y)\subset\mathfrak{R}(|N|^p)\subset\overline{\mathfrak{R}(N)},$$

while, $\mathfrak{R}(SS^*)\subset\mathfrak{R}(S)\subset\ker(N)$. This shows that $\mathfrak{R}(SS^*)=\{0\}$ and therefore $S=0$. That is, $\widetilde{T}_{s,t}=N\oplus R$.

□

Lemma 2.7. *Let $T \in \mathcal{L}(\mathcal{H})$ be w-hyponormal operator and $T = U|T|$ be the polar decomposition of T. If $\mathcal{M}$ is an invariant subspace of T and $T|_{\mathcal{M}}$ is an injective normal operator, then $\mathcal{M}$ reduces T.*

Proof. Since T is w-hyponormal operator

$$|\widetilde{T}^*| \leq |T| \leq |\widetilde{T}|.$$

Hence we have

$$|N| \oplus |R^*| \leq |T| \leq |N| \oplus |R|,$$

by assumption. This implies that $|T|$ is of the form $|N| \oplus L$ for some positive operator L. Let $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$ be the matrix representation of U with respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$. Then the definition $\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ means that

$$\begin{pmatrix} N & 0 \\ 0 & R \end{pmatrix} = \begin{pmatrix} |N|^{\frac{1}{2}} & 0 \\ 0 & L^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} |N|^{\frac{1}{2}} & 0 \\ 0 & L^{r\frac{1}{2}} \end{pmatrix}.$$

Hence we have

$$N = |N|^{\frac{1}{2}} U_{11} |N|^{\frac{1}{2}}, \tag{2.2}$$

$$|N|^{\frac{1}{2}} U_{12} L^{\frac{1}{2}} = 0, \tag{2.3}$$

$$L^{\frac{1}{2}} U_{21} |N|^{\frac{1}{2}} = 0. \tag{2.4}$$

Since $\ker(U) = \ker(T) = \ker(|T|)$, we have

$$\ker(N) \subset \ker(U_{11}),\ \ker(N) \subset \ker(U_{21}), \tag{2.5}$$

$$\ker(L) \subset \ker(U_{12}),\ \ker(N) \subset \ker(U_{22}). \tag{2.6}$$

Let $N = V|N|$ be the polar decomposition of N. Then $\mathfrak{R}(U_{11} - V) \subset \ker(N)$. Hence for arbitrary $x \in \mathfrak{R}(N)$, we have

$$\begin{aligned} \|x\|^2 &\geq \|Vx\|^2 + \|U_{11} - V\|^2, \quad \text{by Pythagoras's theorem,} \\ &= \|x\|^2 + \|U_{11} - V\|^2, \quad \text{since } V \text{ is unitary on } \overline{\mathfrak{R}(N)}. \end{aligned}$$

Therefore, we obtain $V = U_{11}$. Since

$$\|x\|^2 = \|Ux\|^2 + \|U_{21}x\|^2 = \|x\|^2 + \|U_{21}x\|^2 \text{ for } x \in \mathfrak{R}(N),$$

we have $U_{21} = 0$. Also, we see that $\mathfrak{R}(U_{12}) \subset \ker(N)$ by (2.3) and (2.6). Hence,

$$T = U|T| = \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix} \begin{pmatrix} |N| & 0 \\ 0 & L \end{pmatrix} = \begin{pmatrix} N & U_{12}L \\ 0 & U_{22}L \end{pmatrix}.$$

Since $\mathfrak{R}(U_{12}) \subset \ker(N) = \{0\}$, we have $U_{12} = 0$ and so $T = N \oplus T_1$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$. That is, $\mathcal{M}$ reduces T. □

The following example shows that there exists a w-hyponormal operator T such that $T|_{\mathscr{M}}$ is quasinormal but M does not reduce T.

Example 2.8. Let T be a bilateral shift on $\ell^2(\mathbb{Z})$ defined by $Te_n = e_{n+1}$ and $\mathscr{M} = \vee_{n\geq 0}\mathbb{C}e_n$. Then T is unitary and $T|_{\mathscr{M}}$ is isometry. However, $\mathscr{M}$ does not reduce T.

Lemma 2.9. *Let* $0 < s,t \leq 1$. *Let* $T = \begin{pmatrix} A & S \\ 0 & B \end{pmatrix}$ *be a class* $A(s,t)$ *operator on* $\mathscr{H} = \mathscr{M} \oplus \mathscr{M}^{\perp}$, *where* $\mathscr{M}$ *is a* T*-invariant subspace such that the restriction* $A = T|_{\mathscr{M}}$ *is an injective normal operator. Then* $\mathscr{M}$ *reduces* T.

Proof. Since T belongs to class $A(s,t)$ and $0 < s,t \leq 1$, T belongs to class A. Let P be the orthogonal projection onto $\mathscr{M}$. Then we have

$$\begin{aligned}\begin{pmatrix} A^*A & 0 \\ 0 & 0 \end{pmatrix} &= PT^*TP \leq P|T^2|P \quad \text{(since } T \in \text{class } A)\\ &\leq \begin{pmatrix} (A^{*2}A^2)^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{(by Lemma 2.1)}\\ &= \begin{pmatrix} A^*A & 0 \\ 0 & 0 \end{pmatrix} \quad \text{(since } A \text{ is normal).}\end{aligned}$$

Let $|T^2| = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}$ be the 2×2 matrix representation of $|T^2|$ on $\mathscr{H} = \mathscr{M} \oplus \mathscr{M}^{\perp}$. Then we have $X = A^*A$ by the equality above. Since $|A^2|^2 = T^{*2}T^2$, we have

$$\begin{pmatrix} X^2 + YY^* & XY + YZ \\ ZY^* + Y^*X & Y^*Y + Z^2 \end{pmatrix} = \begin{pmatrix} A^{*2}A^2 & A^{*2}AS \\ S^*A^*A^2 & S^*S + B^{*2}B^2 \end{pmatrix},$$

and hence $X^2 + YY^* = A^{*2}A^2 = (A^*A)^2 = X^2$. This implies that $Y = 0$. Thus we have

$$\begin{aligned}\begin{pmatrix} |A|^4 & 0 \\ 0 & Z^2 \end{pmatrix} &= |T^2|^2 = T^*T^*TT\\ &= \begin{pmatrix} A^*A^*AA & A^*A^*(AS + SB) \\ (S^*A^* + B^*S^*)AA & (AS + SB)^*(AS + SB) + B^*B^*BB \end{pmatrix}\end{aligned}$$

Since A is an injective normal operator, we have $AS + SB = 0$ and $Z = |B^2|$. Now, since T is a class A, we have

$$\begin{aligned}0 &\leq |T^2| - |T|^2\\ &= \begin{pmatrix} 0 & -A^*S \\ -S^*A & -S^*S + (|B^2| - |B|^2) \end{pmatrix}\end{aligned}$$

and hence $A^*S = 0$. Thus the range of S is included in $\ker(A^*) = \ker(A) = \{0\}$. Therefore, $S = 0$ and so $\mathscr{M}$ reduces T. □

An operator $X \in \mathscr{L}(\mathscr{K}, \mathscr{H})$ is called quasiaffinity if X is both injective and has a dense range. For $T \in \mathscr{L}(\mathscr{H})$ and $S \in \mathscr{L}(\mathscr{K})$, if there exist quasiaffinities $X \in \mathscr{L}(\mathscr{K}, \mathscr{H})$ and

$Y \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ such that $TX = XS$ and $YT = SY$, then we say that T and S are quasisimilar.

The operator $T \in \mathscr{L}(\mathscr{H})$ is said to be pure if there exists no non-trivial reducing subspace $\mathscr{M}$ of $\mathscr{H}$ such that the restriction of T to $\mathscr{M}$ is normal and is completely hyponormal if it is pure.

Recall that every operator $T \in \mathscr{L}(\mathscr{H})$ has a direct sum decomposition $T = T_1 \oplus T_2$, where T_1 and T_2 are normal and pure parts, respectively. Of course in the sum decomposition, either T_1 or T_2 may be absent.

The following lemma is due to Williams [33, Lemma 1.1].

Lemma 2.10. *Let $T \in \mathscr{L}(\mathscr{H})$ and $S \in \mathscr{L}(\mathscr{K})$ be normal operators. It there exist injective operators $X \in \mathscr{L}(\mathscr{K}, \mathscr{H})$ and $Y \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ such that $TX = XS$ and $YT = SY$, then T and S are unitarily equivalent.*

Corollary 2.11. *Let $T \in \mathscr{L}(\mathscr{H})$ be w-hyponormal operator. Then $T = T_1 \oplus T_2$ on the space $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$, where T_1 is normal and T_2 is pure and w-hyponormal; i.e., T_2 has no invariant subspace $\mathscr{M}$ such that $T_2|_{\mathscr{M}}$ is normal.*

The next lemma was proved for dominant operators in [28, Theorem 1], for p-hyponormal operators in [17] and for log-hyponormal operators in [16, Lemma 3].

Lemma 2.12. *Let $T \in \mathscr{L}(\mathscr{H})$ be w-hyponormal operator and let $S \in \mathscr{L}(\mathscr{K})$ be a normal operator. If there exists an operator $X \in \mathscr{L}(\mathscr{K}, \mathscr{H})$ with dense range such that $TX = XS$, then T is normal.*

Proof. First, we decompose T into normal and pure parts by $T = T_1 \oplus T_2$ with respect to a decomposition $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$. Let $T_2 = U_2|T_2|$ be the polar decomposition of T_2 and $\widetilde{T}_2 = |T_2|^{\frac{1}{2}} U_2 |T_2|^{\frac{1}{2}}$. Let $\widetilde{T}_2 = V_2|\widetilde{T}_2|$ be the polar decomposition of $\widetilde{T}_2$ and $\widehat{T}_2 = |\widetilde{T}_2|^{\frac{1}{2}} V_2 |\widetilde{T}_2|^{\frac{1}{2}}$. Since T_1 is normal, we have $\widetilde{T} = T_1 \oplus \widetilde{T}_2$ and $\widehat{T} = T_1 \oplus \widehat{T}_2$. Let $W = |\widetilde{T}_2|^{\frac{1}{2}} |T_2|^{\frac{1}{2}}$. Since $\ker(|T_2|) = \ker(T_2) = \{0\}$, by Corollary 2.11, $|T_2|^{\frac{1}{2}}$ is a quasiaffinity. Hence $\widehat{T}_2$ is injective and W is a quasiaffinity such that $\widehat{T}W = WT_2$. Let $Y = I_{\mathscr{H}_1} \oplus W$. Then $\widehat{T}$ is hyponormal and Y is a quasiaffinity such that $\widehat{T}Y = YT$. Thus we have $\widehat{T}(YX) = (YX)S$ and YX has dense range. Hence $\widehat{T}$ is normal, by [28, Theorem 1], and so T is normal by [6, Theorem 1]. □

3 The Fuglede-Putnam Theorem

In this section, we present some results concerning the Fuglede-Putnam theorem.

Theorem 3.1. *Let $T \in \mathscr{L}(\mathscr{H})$ be w-hyponormal such that* $\ker(T) \subset \ker(T^*)$ *and $L \in \mathscr{L}(\mathscr{H})$ be a self-adjoint which satisfies $TL = LT^*$. Then $T^*L = LT$.*

Proof. We first show that if $TL = LT^* = 0$, then $T^*L = LT = 0$. Since $\ker(T) \subset \ker(T^*)$, $\ker(T)$ reduces T by [4], $TL = 0$ implies that $\mathfrak{R}(L) \subseteq \ker(T) \subset \ker(T^*)$ and by taking the orthogonal complement, we obtain $\overline{\mathfrak{R}(T)} \subset \ker(L)$. Hence we have $T^*L = LT = 0$.

Next, we prove the case in which $TL \neq 0$. Since T is w-hyponormal, the Aluthge transform $\widetilde{T}$ of T is semi-hyponormal. Moreover, it satisfies

$$|\widetilde{T}| \geq |T| \geq |\widetilde{T}^*|. \tag{3.1}$$

Put $W = |T|^{\frac{1}{2}} L |T|^{\frac{1}{2}}$. Then W is self-adjoint and satisfies

$$\widetilde{T} W = W \widetilde{T}^*. \tag{3.2}$$

By the argument in the proof of Theorem 2 of [31], we have that the restriction $\widetilde{T}|_{\overline{\Re(W)}}$ of $\widetilde{T}$ to its invariant subspace $\overline{\Re(W)}$ is normal and

$$\widetilde{T}^* W = W \widetilde{T}. \tag{3.3}$$

Hence $\overline{\Re(W)}$ reduces $\widetilde{T}$, by Lemma 2.7, and so $\widetilde{T}$ is of the form $\widetilde{T} = N \oplus S$ on $\overline{\Re(W)} \oplus \ker(W)$, where N is normal. By Corollary 2.5 and Lemma 2.7, $T = N \oplus B$, for some w-hyponormal operator B. Let $W = W_1 \oplus 0$ and $L = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix}$ on $\overline{\Re(W)} \oplus \ker(W)$. Then $L_2 = L_3 = 0$ and $L_4 = 0$ follows from the equality $W = |T|^{\frac{1}{2}} L |T|^{\frac{1}{2}}$. By assumption, $NL_1 = L_1 N^*$, we have $N^* L_1 = L_1 N$ by Fuglede-Putnam theorem and so $T^* L = LT$. □

Example 3.2. Let $\mathscr{H} = \oplus_{n=0}^{\infty} \mathbb{C}^2$ and define an operator R on $\mathscr{H}$ by

$$R(\cdots \oplus x_{-2} \oplus x_{-1} \oplus x_0^{(0)} \oplus x_1 \oplus \cdots) = \cdots \oplus Ax_{-2} \oplus Ax_{-1}^{(0)} \oplus Bx_0 \oplus Bx_1 \oplus \cdots,$$

where $A = \frac{1}{4}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then R is w-hyponormal. Moreover, $\Re(E) = \ker(R)$, E is not a self-adjoint and $\ker(R) \neq \ker(R^*)$, where E is the Riesz idempotent with respect to 0, see [32, Example 13]. Let $T = R$ and $L = P$ be the orthogonal projection onto $\ker(T)$. Then T is w-hyponormal operator and $TL = 0 = LT^*$, but $T^*L \neq LT$. Hence the kernel condition $\ker(T) \subset \ker(T^*)$ is necessary for Theorem 3.1.

Corollary 3.3. *Let $T \in \mathscr{L}(\mathscr{H})$ be w-hyponormal such that* $\ker(T) \subset \ker(T^*)$. *If $X \in \mathscr{L}(\mathscr{H})$ and $TX = XT^*$, then $T^*X = XT$.*

Proof. Let $X = L + iK$ be the cartesian decomposition of X. Then we have $TL = LT^*$ and $TK = KT^*$, by the assumption. By Theorem 3.1, we have $T^*L = LT$ and $T^*K = KT$. This implies that $T^*X = XT$. □

If we use the 2×2 matrix trick, we easily deduce the following result.

Corollary 3.4. *Let $T^* \in \mathscr{L}(\mathscr{H})$ be w-hyponormal and $S \in \mathscr{L}(\mathscr{K})$ be w-hyponormal with* $\ker(T^*) \subset \ker(T)$ *and* $\ker(S) \subset \ker(S^*)$. *If $X \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ and $XT = SX$, then $XT^* = S^*X$.*

Proof. Put $A = \begin{pmatrix} T^* & 0 \\ 0 & S \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix}$ on $\mathscr{H} \oplus \mathscr{K}$. Then A is a w-hyponormal operator on $\mathscr{H} \oplus \mathscr{K}$ that satisfies $BA^* = AB$ and $\ker(A) \subset \ker(A^*)$. Hence we have $BA = A^*B$, by Corollary 3.3, and so $XT^* = S^*X$. □

Example 3.5. Let $S = T^* = R$ as in Example 3.2 and $X = P$ be the orthogonal projection onto $\ker(S)$. Then $SX = 0 = XT$, but $S^*X = XT^*$. Hence the kernel condition is necessary for Corollary 3.4.

Theorem 3.6. *Let $T \in \mathscr{L}(\mathscr{H})$ be such that T^* is p-hyponormal or* log*-hyponormal. Let $S \in \mathscr{L}(\mathscr{K})$ be w-hyponormal with* $\ker(S) \subset \ker(S^*)$. *If $XT = SX$, for some $X \in \mathscr{L}(\mathscr{H}, \mathscr{K})$. Then $XT^* = S^*X$.*

Proof. Let T^* be a p-hyponormal operator for $p \geq \frac{1}{2}$ and let $U|T|$ be the polar decomposition of T. Then the Aluthge transform $\widetilde{T}^*$ of T^* is hyponormal and satisfies

$$|\widetilde{T}^*| \geq |T|^2 \geq |\widetilde{T}|, \tag{3.4}$$

$$X'\widetilde{T} = SX', \tag{3.5}$$

where $X' = XU|T|^{\frac{1}{2}}$. Using the decompositions $\mathscr{H} = \ker(X')^{\perp} \oplus \ker(X')$ and $\mathscr{K} = \overline{\Re(X')} \oplus \Re(X')^{\perp}$, we see that $\widetilde{T}, S$ and X' are of the form

$$\widetilde{T} = \begin{pmatrix} T_1 & 0 \\ T_2 & T_3 \end{pmatrix}, \quad S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}, \quad X' = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$$

where T_1^* is hyponormal, S_1 is w-hyponormal with $\ker(S_1) \subset \ker(S_1^*)$ and X_1 is a one-one operator with dense range. Since $X'\widetilde{T} = SX'$, we have

$$X_1T_1 = S_1X_1. \tag{3.6}$$

Hence T_1 and S_1 are normal by Theorem 3.6 of [4], so that $T_2 = 0$, by Lemma 12 of [31] and $S_2 = 0$ by Lemma 2.7. Then $|T| = |T_1| \oplus P$, for some positive operator P, by (3.4) and $U = \begin{pmatrix} U_1 & U_2 \\ 0 & U_3 \end{pmatrix}$ by Lemma 13 of [31]. Let $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ be a 2×2 matrix representation of X with respect to the decomposition $\mathscr{H} = \ker(X')^{\perp} \oplus \ker(X')$ and $\mathscr{K} = \overline{\Re(X')} \oplus \Re(X')^{\perp}$. Then $X' = XU|T|^{\frac{1}{2}}$ implies that $X_1 = X_{11}U_1|T_1|^{\frac{1}{2}}$ and hence $\ker(T_1) \subset \ker(X_1) = \{0\}$. This shows that T_1 is one-one and hence it has dense range, so that $U_2 = 0$ and $T = T_1 \oplus T_4$ for some hyponormal operator T_4^* by [31, Lemma 13]. Since

$$\begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} = X' = XU|T|^{\frac{1}{2}} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} U_1|T_1|^{\frac{1}{2}} & 0 \\ 0 & U_3|T_4|^{\frac{1}{2}} \end{pmatrix}$$

we deduce the following assertions.

$$X_{12}U_2|T_4|^{\frac{1}{2}} = 0; \text{ hence } X_{12}T_3 = 0 \text{ because } T_4 = U_3|T_4.$$

$$X_{21}U_1|T_1|^{\frac{1}{2}} = 0; \text{ hence } X_{12} = 0 \text{ because } U_1|T_1|^{\frac{1}{2}} \text{ has dense range.}$$

$$X_{22}U_3|T_4|^{\frac{1}{2}} = 0; \text{ hence } X_{22}T_3 = 0.$$

The assumption $XT = SX$ tell us that,

$$\begin{aligned} X_{11}T_1 &= S_1X_{11} \\ X_{12}T_4 &= S_1X_{12} = 0, \\ X_{22}T_4 &= S_3X_{22} = 0. \end{aligned}$$

Since T_1 and S_1 are normal, we have $X_{11}T_1^* = S_1^*X_{11}$, by Fuglede-Putnam theorem. The p-hyponormality of T_4^* shows that $\overline{\Re(T_4^*)} \subset \overline{\Re(T_4)}$. Also, we have $\ker(S_3) \subset \ker(S_3^*)$. Hence, we also have $X_{12}T_4^* = S_1^*X_{12} = 0$ and $X_{22}T_4^* = S_3^*X_22 = 0$. This implies that $XT^* = X_{11}T_1^* \oplus 0 = S_1^*X_{11} \oplus 0 = S^*X$.

Next, we prove the case where T^* is p-hyponormal for $0 < p \leq \frac{1}{2}$. Let X' be as above. Then $\widetilde{T}^*$ is $(p+\frac{1}{2})$-hyponormal and satisfies $X'\widetilde{T} = SX'$. Use the same argument as above. We obtain $\widetilde{T} = T_1 \oplus T_3$ on $\mathscr{H} = \ker(X')^{\perp} \oplus \ker(X')$ and $S = S_1 \oplus S_3$, where T_1 is an injective normal operator and S_1 is also normal. Hence we have $T = T_1 \oplus T_4$ for some p-hyponormal T_4^*, by Lemma 13 of [31]. Again using the same argument as above, we obtain $X_{21} = 0, X_{11}T_1^* = S_1^*X_{11}, X_{12}T_4^* = S_1^*X_{12} = 0$ and $X_{22}T_4^* = S_3^*X_{22} = 0$. hence we have $XT^* = S^*X$.

Finally, we assume that T^* is log-hyponormal. Let $\widetilde{T}$ and X' be as above. Then $X'\widetilde{T} = SX'$ and $\widetilde{T}^*$ is semi-hyponormal and satisfies

$$|\widetilde{T}| \leq |T| \leq |\widetilde{T}^*|.$$

By the same argument as above, we have $\widetilde{T} = T_1 \oplus T_3$ on $\mathscr{H} = \ker(X')^{\perp} \oplus \ker(X')$ and $S = S_1 \oplus S_3$ on $\mathscr{K} = \overline{\Re(X')} \oplus \Re(X')^{\perp}$, where T_1 is an injective normal operator, S_1 is normal, T_3^* is invertible semi-hyponormal and S_3 is w-hyponormal with $\ker(S_3) \subset \ker(S_2^*)$. By Lemma 13 of [31], we have that T is of the form $T = T_1 \oplus T_4$, for some log-hyponormal T_4^*. Let $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$. Then $X' = XU|T|^{\frac{1}{2}}$ implies that $X_{12} = 0, X_{21} = 0$ and $X_{22} = 0$. The assumption $XT = SX$ implies that $X_{11}T_1 = S_1X_{11}$, hence $X_{11}T_1^* = S_1^*X_{11}$ by Fuglede-Putnam theorem. Thus we have $XT^* = X_{11}T_1^* \oplus 0 = S_1^*X_{11} \oplus 0 = S^*X$. Therefore, the proof of the theorem is achieved. □

Example 3.7. Let R be an operator such that $\ker(R)$ does not reduce R and let P be the orthogonal projection onto $\ker(R)$. Then P does not commute with T; otherwise $\Re(R) = \ker(R)$ reduce T. Hence $PR \neq 0 = RP$. It is easy to see that $RP = PR^* = 0$ but $R^*P \neq PR (\neq 0)$ because $\Re(R^*P) \subset \Re(R^*) \subset \ker(R^{\perp}) = I - P$. If we put $T = R$, then the assertion of Theorem 3.1 does not hold for such T. Also, if we put $T = R^*, S = I - P$ and $X = P$, then $XT = PR^* = 0 = (I-P)P = SX$. However, $XT^* = PR \neq 0 = (I-P)P = S^*X$. Hence the assertion of Theorem 3.6 does not hold for such T.

Theorem 3.8. *Let $T \in \mathscr{L}(\mathscr{H})$ be such that T^* is an injective w-hyponormal . Let $S \in \mathscr{L}(\mathscr{K})$ be dominant. If $XT = SX$, for some $X \in \mathscr{L}(\mathscr{H}, \mathscr{K})$. Then $XT^* = S^*X$.*

Proof. Assume that T^* is an injective w-hyponormal and let $U|T|$ be the polar decomposition of T. Let $\widetilde{T}$ be the aluthge transform of T and $X' = XU|T|^{\frac{1}{2}}$. Then $X'\widetilde{T} = SX'$ and $\widetilde{T}^*$ is semi-hyponormal and satisfies

$$|\widetilde{T}| \leq |T| \leq |\widetilde{T}^*|.$$

By the same argument in the proof of Theorem 3.6, we conclude that $\widetilde{T} = T_1 \oplus T_3$ on $\mathscr{H} = \ker(X')^{\perp} \oplus \ker(X')$ and $S = S_1 \oplus S_3$, where T_1 is an injective normal operator and S_1 is also normal, T_3^* is invertible w-hyponormal and S_3 is dominant. Hence by Lemma 2.7, we have that T is of the form $T = T_1 \oplus T_4$ for some w-hyponormal T_4^*. Let

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$$

Then $X' = XU|T|^{\frac{1}{2}}$ implies that $X_{12} = 0, X_{21} = 0$ and $X_{22} = 0$. The assumption $XT = SX$ implies that $X_{11}T_1 = S_1X_{11}$, hence $X_{11}T_1^* = S_1^*X_{11}$ by Fuglede-Putnam theorem. Thus we have $XT^* = X_{11}T_1^* \oplus 0 = S_1^*X_{11} \oplus 0 = S^*X$. Therefore, the proof of the theorem is achieved. □

Example 3.9. Let $T^* = R$ as in Example 3.2. Let $X = P$ be the orthogonal projection onto $\ker(T^*)$ and $S = I - P$. Then $SX = 0 = XT^*$, but $0 = S^*X \neq XT^*$. Hence the injectivity condition is necessary for Theorem 3.8.

Example 3.10. Let $\{e_n\}_{n=-\infty}^{\infty}$ be a complete orthonormal system for $\mathscr{H}$. We denote the orthogonal projection onto $\mathbb{C}e_n$ by P_n. Let W be a weighted shift on $\mathscr{H}$ defined by

$$We_n = \begin{cases} \sqrt{2}e_{n+1}, & \text{if } n \geq 0; \\ e_{n+1}, & \text{if } n < 0. \end{cases}$$

Then $W^*W - WW^* = P_0$. Define an operator T on a Hilbert space $\mathscr{K} = \mathscr{H} \oplus \mathbb{C}e_0$ by

$$T = \begin{pmatrix} W & P_0 \\ 0 & 0 \end{pmatrix}.$$

Then T is class A, see [31, Example 1]. It is easy to see that

$$\ker(T) = \mathbb{C}(-e_{-1} \oplus e_0) \quad \text{and} \quad \ker(T^*) = \{0\} \oplus \mathbb{C}e_0.$$

Hence T does not reduces T and therefore the assertions of Theorems 3.8, 3.6 and Corollary 3.4 are not necessarily true for class A operators.

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