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An Application of Ultrafilters to the Haar Measure

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Abstract

In this article we will use ultrafilter theory to present a modified proof that a locally compact group with a countable basis has a left invariant and a right invariant Haar measure. To facilitate this result, we shall first show that the topological space consisting of all ultrafilters on a non-empty set X is homeomorphic to the topological space of all nonzero multiplicative functionals in the first dual space $\ell_\infty^\star(X)$.

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1 Introduction

Let $G = ((0,\infty),.)$ be the multiplicative group of the set of positive real numbers with the usual topology. Then G is a locally compact and Hausdorff topological group. Let $C_0(G)$ be the space of all real valued and continuous functions f on G such that f is zero outside some compact set K_f. Then:

$$I(f) = \int_G \frac{f(x)}{x} dx, f \in C_0(G)$$

is an invariant integral or Haar integral on G,i.e.

$$\int_G \frac{f(ax)}{x} dx = \int_G \frac{f(x)}{x} dx, \forall a \in G, f \in C_0(G).$$

Further for each Borel set E in G,

$$\nu(E) = \int_E \frac{1}{x} dx$$

is an invariant measure or Haar measure; i.e. $\nu(aE) = \nu(E)$ for all $a \in G$. Here a Borel set E is a member of the smallest σ-algebra containing all the open sets.

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Invariant integrals for a special class of topological groups has long been known. The first of such integrals was given in 1897 by Hurwitz for $SO(n)$, where $SO(n)$ is the special orthogonal group [3].

Shortly before his death in 1933 Alfred Haar proved the existence of left invariant and right invariant measures on a locally compact group with a countable basis. Haar's construction on invariant measure was reformulated in terms of a linear functional and extended to arbitrary locally compact groups by Andre Weil [3]. The Haar measure has been extended to other spaces including Uniform Spaces [3], [5]. Although Weil's approach is very elegant it obscures the original proof by Haar. Most books, for example [3], [1], [6], and [7] present Weil's and H. Cartan's proof.

In 1999 after nearly 58 years, AMS published the work of J. Von Neumann regarding the Haar measure. The author of this article believes that the original proof by Haar (which was refined and extended by Von Neumann) utilizes the most interesting and illuminating approach.

In his proof of the Haar measure Von Neumann uses the theory of generalized limit developed in the same book [8]. In this article we will present an alternative approach by using ultrafilters in place of Von Neumann's theory of limits. The application of ultrafilters and the basic tools of Banach algebra make this proof more understandable. We will conclude by presenting several examples of the Haar measure.

2 Ultrafilters

Definition 2.1. Let X be a nonempty set. Let $\mathcal{P}$ be a collection of subsets of X, i.e. each member of $\mathcal{P}$ is a subset of X. We call $\mathcal{P}$ an ultrafilter if it satisfies the following properties:

(a) $\mathcal{P} \neq \phi$ and $\phi \notin \mathcal{P}$

(b) $A, B \in \mathcal{P}$ implies $A \cap B \in \mathcal{P}$ so $\mathcal{P}$ is closed under finite intersection.

(c) $A \in \mathcal{P}, B \subseteq X$ and $A \subseteq B$ implies $B \in \mathcal{P}$ hence $\mathcal{P}$ is closed under superset operation.

(d) $A \subseteq X$ implies $A \in \mathcal{P}$ or $X \setminus A \in \mathcal{P}$

Remark 2.2. If $\mathcal{P}$ satisfies only properties (a), (b) and (c) then $\mathcal{P}$ is called a filter. Property (d) is called the maximality property.

Example 2.3. Let $x \in X$ and put $\widehat{x} = \{A \subseteq X : x \in A\}$. Evidently $\widehat{x}$ is an ultrafilter in X. We call $\widehat{x}$ a principal ultrafilter. An ultrafilter Q which is not of this form is called nonprincipal. We shall see shortly that by Zorn's lemma nonprincipal ultrafilter exists if X has infinitely many elements.

Theorem 2.4. *Let X be a nonempty set and let C be a collection of subsets of X with the finite intersection property, i.e. $A_1, \ldots, A_n \in C$ implies $\cap_{i=1}^{n} A_i \neq \phi$. Then there exists an ultrafilter $\mathcal{P}$ in X such that $C \subseteq \mathcal{P}$.*

Proof. Apply Zorn's lemma [4, Theorem 3.8]. □

Remark 2.5. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X with $x_n \neq x_m$ if $n \neq m$. For each natural number n let $A_n = \{x_k : k \geq n\}$. Then $C = \{A_n : n \in N\}$ has the finite intersection property. By Theorem 2.4 C is contained in an ultrafilter P. it is clear that $\mathcal{P}$ is a nonprincipal ultrafilter.

Definition 2.6. Let X be an infinite set. We define the symbol $\beta_1 X$ as:

$$\beta_1 X = \{\mathcal{P} : \mathcal{P} \text{ is an ultrafilter in } X\}.$$

Thus each point of $\beta_1 X$ is an ultrafilter in X.

Definition 2.7. For each $A \subseteq X$ let:

$$\bar{A} = \{\mathcal{P} \in \beta_1 X : A \in \mathcal{P}\}.$$

Then one can show that $\mathcal{B} = \{\bar{A} : A \subseteq X\}$ is a base for a topology on $\beta_1 X$, which we will denote by τ_1.

Theorem 2.8. *The topological space* $(\beta_1 X, \tau_1)$ *is a nonmetrizable compact Hausdorff space. Furthermore* $\hat{X} = \{\hat{x} : x \in X\}$ *is dense in* $\beta_1 X$*;i.e. the set of principal ultrafilters form a dense subset of this space.*

Proof. See [4, Theorems 3.18 and 3.36]. □

3 Bounded Linear Functionals and Ultrafilters

Definition 3.1. Let X be a nonempty set and let C be the field of complex numbers. We define

$$\ell_\infty(X) = \{f : X \to C | f \text{ is a bounded function}\}$$

It is well known that $\ell_\infty(X)$ is a commutative Banach algebra with identity when equipped with the usual addition, scalar multiplication and pointwise multiplication of functions as well as the sup norm. In addition one can show that $\ell_\infty(X)$ is a $B^\star$-algebra where the involution is conjugate of a function.

Definition 3.2. We define the first dual space of $\ell_\infty(X)$ by:

$$\ell_\infty^\star(X) = \{\xi : \ell_\infty(X) \to C | \xi \text{is bounded (continuous)and linear }\}.$$

Definition 3.3. We define $\beta_2(X)$ by the following:

$$\beta_2 X = \{\xi \in \ell_\infty^\star(X) : \xi \text{is nonzero and multiplicative}\}.$$

So each $\xi \in \beta_2(X)$ is a nonzero homomorphism from algebra $\ell_\infty(X)$ in to the algebra of the set of complex numbers C.

Example 3.4. Let $x \in X$ and define $\widetilde{x} : \ell_\infty(X) \to C$ by $\widetilde{x}(f) = f(x)$. Evidently each $\widetilde{x}$ is nonzero multiplicative linear functional. Since each algebra homomorphism is continuous [9, 10.7 (c)] we will have $\widetilde{x} \in \beta_2(X)$. We call each $\widetilde{x}$ an evaluation map. Also we note that if $\xi \in \beta_2(X)$ and f_A is the characteristic function of $A \subseteq X$ then $\xi(f_A)$ is either 1 or zero.

Remark 3.5. Since each algebra homomorphism is continuous we could define $\beta_2 X$ as the set of all nonzero homomorphisms from $\ell_\infty(X)$ into C.

Definition 3.6. We define a topology τ_2 in $\beta_2(X)$ using net convergence.Let $\{\xi_\alpha : \alpha \in D\}$ be a net in $\beta_2(X)$ we say ξ_α converges to ξ, denoted by $\xi_\alpha \to \xi$ if and only if $\xi_\alpha(f) \to \xi(f)$ for each $f \in \ell_\infty(X)$. To see what are the basic open sets in this topology for each $f \in \ell_\infty(X)$, define $\psi_f : \beta_2 X \to C$ by $\psi_f(\xi) = \xi(f)$. Then τ_2 is the smallest topology on $\beta_2 X$ so that each ψ_f is continuous, hence τ_2 is the Gelfand topology. See [9, 11.8].

Theorem 3.7. $(\beta_2 X, \tau_2)$ *is a non metrizable compact Hausdorff space. Furthermore* $\widetilde{X} = \{\widetilde{x} : x \in X\}$ *is dense in* $\beta_2 X$.

Proof. See [9, Theorem 11.9]. □

Theorem 3.8. *The spaces* $(\beta_1 X, \tau_1)$ *and* $(\beta_2 X, \tau_2)$ *are homeomorphic.*

Proof. First we will show that the algebra $\mathcal{A}$ generated by the set of all characteristic functions $\{f_A : A \subseteq X\}$ is dense in $\ell_\infty(X)$. Let $f \in \ell_\infty(X)$ be given. Suppose that f is real valued and that $f(x) \geq 0$ for all $x \in X$. Let $\|f\|_\infty$ be the sup norm of f. For each natural number n and $i \in \{1, \cdots n\}$ put:

$$A_i = f^{-1}[\frac{(i-1)\|f\|_\infty}{n}, \frac{i\|f\|_\infty}{n})$$

and $B = f^{-1}(\{\|f\|_\infty\})$. Evidently $A_1, \cdots, A_n$ and B are pairwise disjoint and their union is X. Now it is easy to see that the sequence:

$$f_n = \|f\|_\infty f_B + 1/n\Sigma_{i=1}^{n}(i-1)\|f\|_\infty f_{A_i}$$

converges in sup norm to f. If f is arbitrary but still real valued we decompose f into its positive and negative parts. So $f = f^+ - f^-$, where both f^+ and f^- are real valued and nonnegative functions in $\ell_\infty(X)$. By application of the first part one can find a sequence f_n in $\ell_\infty(X)$ which converges in sup norm to f. For a complex valued function f we can write $f = g + ih$ where g and h are real valued functions in $\ell_\infty(X)$. An application of the preceding part for g and h shows that the algebra $\mathcal{A}$ is dense in $\ell_\infty(X)$. Now define:

$$\varphi : (\beta_2 X, \tau_2) \to (\beta_1 X, \tau_1)$$

by $\varphi(\xi) = \{A \subseteq X : \xi(f_A) = 1\}$. It is routine to check that $\varphi(\xi)$ is an ultrafilter. Furthermore, $\varphi(\widetilde{x}) = \hat{x}$;i.e. φ sends an evaluation map $\widetilde{x}$ to a principal ultrafilter $\hat{x}$. To prove that φ is continuous it suffices to show that $\varphi^{-1}(\bar{A})$ is open for each $A \subseteq X$. So let $\xi \in \varphi^{-1}(\bar{A})$. Now $O = \{\eta \in \beta_2(X) : |\eta(f_A) - \xi(f_A| < \frac{1}{2}\}$ is an open neighborhood of ξ and lies in $\varphi^{-1}(\bar{A})$,hence φ is continuous. By Theorem 2.8 the set of all principal ultrafilters are dense in $\beta_1 X$ and as we observed the range of φ contains all the principal ultrafilters ,so φ is surjective. To show φ is injective suppose that $\varphi(\xi_1) = \varphi(\xi_2)$. Hence ξ_1 and ξ_2 are equal on algebra $\mathcal{A}$ which is dense in $\ell_\infty(X)$. Thus they are equal on the whole space and φ is a homeomorphism by compactness. □

4 Topological Groups

Definition 4.1. Let G be a group with a topology τ. We say that (G,τ) is a topological group if the mappings : $G\times G\to G$ and : $G\to G$ defined by $(x,y)\to xy$ and $x\to x^{-1}$ respectively are continuous. Also for nonempty subsets A and B of G we define:

$$AB=\{ab : a\in A, b\in B\} \text{ and } A^{-1}=\{a^{-1} : a\in A\}$$

In the sequel we assume G is a locally compact Hausdorff topological group with the identity denoted by e. The following theorem plays a major role in the development of Haar measure.

Theorem 4.2. *Let O be an open neighborhood of e in G and let C and D be disjoint compact sets in G.Then:*

(a) there exists a compact set A in G with $e\in int(A)$ such that:

$$A^{-1}A\subseteq O$$

(b) there exists a compact set B in G with $e\in int(B)$ such that:

$$B^{-1}B\subseteq G\setminus(D^{-1}C).$$

Proof. See [8, 15.2, 15.3.1 and 15.3.2]. □

Lemma 4.3. *Let:*

$$\mathcal{I}=\{A\subseteq G : A \text{ is compact and } e\in int(A)\}$$

Let C and D be disjoint compact subsets of G. Put:

$$J_{(C,D)}=\{A\in\mathcal{I} : A^{-1}A\subseteq G\setminus(D^{-1}C)\}$$

Then:

$$\mathcal{F}=\{J_{(C,D)} : C \text{ and } D \text{ are disjoint compact subsets of } G\}$$

has the finite intersection property.

Proof. Let n be a natural number. Suppose that for each $i\in\{1,\cdots,n\}$, C_i and D_i are disjoint compact subsets of G. By theorem 4.2 there exists for each $i\in\{1,\cdots,n\}$ a compact subset A_i with $e\in int(A_i)$ and $A_i^{-1}A_i\subseteq G\setminus(D_i^{-1}C_i)$. Now let $A=\cap_{i=1}^n A_i$, then $A\in\mathcal{I}$, and for all $i\in\{1,\cdots,n\}$ we have $A^{-1}A\subseteq A_i^{-1}A_i$. Hence $A\in\cap_{i=1}^n J_{(C_i,D_i)}$. Thus $\mathcal{F}$ has the finite intersection property. □

5 Relative Measure of a Compact Set, Haar Number

Lemma 5.1. *Let A and C be compact sets in G with $int(A)\neq\emptyset$. Then there exists $x_1\cdots,x_k\in G$ such that :*

$$C\subseteq\cup_{i=1}^k x_iA.$$

Proof. See [8, 15.1]. □

Definition 5.2. Let A and C be compact sets with $int(A)\neq\emptyset$. We define the left measure of C with respect to A or the Haar number of C with respect to A, denoted $n[\frac{C}{A}]$ by:

$$n[\frac{C}{A}]=\min\{k : \exists x_1,\cdots x_k\in G \text{ such that } C\subseteq\cup_{i=1}^k x_iA\}.$$

Note that $n[\frac{C}{A}]$ is well defined by Lemma 5.1.

Example 5.3. Let $G = (R^2, +), A = \{(x,y) \in R^2 : x^2 + y^2 \leq 1\}$ and $C = \{(x,y) \in R^2 : 2 \leq x \leq 4, 5 \leq y \leq 6\}$. Then $n[\frac{C}{A}] = 2$.

6 Pre-Haar Measure of a Compact Set

Definition 6.1. Let A and E be compact sets in G such that the interiors of A and E are not empty. For each compact set C in G define:

$$\lambda_A(C) = \frac{n[\frac{C}{A}]}{n[\frac{E}{A}]}.$$

We call $\lambda_A(C)$ the left pre-Haar measure of C.

Remark 6.2. Note that $n[\frac{E}{A}]$ is a positive integer. For the remainder of this article we will assume that E denotes a fixed compact set in G with nonempty interior.

Theorem 6.3. *λ_A satisfies the following properties, where C and D are compact sets in G:*

(a) $0 \leq \lambda_A(C) < \infty$

(b) if $C \subseteq D$ then $\lambda_A(C) \leq \lambda_A(D)$

(c) $\lambda_A(C \cup D) \leq \lambda_A(C) + \lambda_A(D)$

(d) if $D^{-1}C \cap A^{-1}A = \emptyset$, then $\lambda_A(C \cup D) = \lambda_A(C) + \lambda_A(D)$

(e) if int$(C) \neq \emptyset$, then $\lambda_A(C) \geq \frac{1}{n[\frac{E}{C}]}$

(f) $\lambda_A(C) \leq n[\frac{C}{E}]$

(g) $\lambda_A(xC) = \lambda_A(C)$ for each $x \in G$

Proof. See [8, 15.1]. □

Remark 6.4. We note that in Theorem 6.3(d) if $D^{-1}C \cap A^{-1}A$ is empty then also $C \cap D = \emptyset$ but the converse is not true. So we cannot show that λ_A is finitely additive on the collection of compact sets.

7 Finitely Additive Left Invariant Measure on Compact Sets

Theorem 7.1. *Let C be the collection of all compact subsets of G. Then there exists a function $\lambda : C \to [0, \infty)$ satisfying the following properties:*
(a) if $C \subseteq D$ then $\lambda(C) \leq \lambda(D)$
(b) $\lambda(C \cup D) \leq \lambda(C) + \lambda(D)$
(c) if C and D are disjoint then $\lambda(C \cup D) = \lambda(C) + \lambda(D)$
(d) if int(C) is not empty then $\lambda(C) > 0$
(e) $\lambda(xC) = \lambda(C)$ for each $x \in G$

Proof. Let $\mathcal{I}$ be the collection of all compact subset A of G such that $e \in int(A)$. For each compact set C in $\mathcal{C}$ define a function:

$$x_C : \mathcal{I} \to R \text{ by } x_C(A) = \lambda_A(C)$$

.

By Theorem 6.3 (f),x_C is bounded, hence x_C is in the commutative Banach algebra $\ell_\infty(\mathcal{I})$. For each pair of disjoint compact sets C and D put:

$$J_{(C,D)} = \{A \in \mathcal{I} : D^{-1}C \cap A^{-1}A = \emptyset\}$$

and

$$\mathcal{F} = \{J_{(C,D)} : C \text{ and } D \text{ in } \mathcal{C} \text{ and } C \cap D = \emptyset\}$$

By Lemma 4.3 $\mathcal{F}$ has the finite intersection property. So Theorem 2.4 implies that there exists an ultrafilter $\mathcal{P}$ in $\mathcal{I}$ such that $\mathcal{F} \subset \mathcal{P}$. By Theorem 3.7 there exists a nonzero multiplicative linear functional $\xi \in \ell_\infty^\star(\mathcal{I})$ such that:

$$\{U \subseteq \mathcal{I} : \xi(f_U) = 1\} = \mathcal{P}$$

where f_U is the characteristic function of U. Now define:

$$\lambda : \mathcal{C} \to [0, \infty) \text{ by } \lambda(C) = \xi(x_C).$$

We will show that λ has the desired properties. First we prove that λ is finitely additive. Let C and D be disjoint where $C, D \in \mathcal{C}$. Hence $J_{(C,D)}$ is a member of $\mathcal{P}$. Since $\mathcal{P} \in cl_{\beta_1 \mathcal{I}} J_{(C,D)}$ there exists a net $\{A_\alpha\}$ in $J_{(C,D)}$ such that $\widehat{A_\alpha} \to \mathcal{P}$ in the topology of $\beta_1\mathcal{I}$. Hence $\widetilde{A_\alpha} \to \xi$ in the topology of $\beta_2\mathcal{I}$ by Theorem 3.2. Therefor:

(i) $\lambda_{A_\alpha}(C) = x_C(A_\alpha) \to \xi(x_C) = \lambda(C)$

(ii) $\lambda_{A_\alpha}(D) = x_D(A_\alpha) \to \xi(x_D) = \lambda(D)$

(iii) $\lambda_{A_\alpha}(C \cup D) = x_{C\cup D}(A_\alpha) \to \xi(x_{C\cup D}) = \lambda(C \cup D)$.

Therefore, by (i), (ii), (iii) and Theorem 6.3(d)

$$\lambda(C \cup D) = \lambda(C) + \lambda(D)$$

So λ satisfies property (c); i.e, λ is finitely additive. Now let $\{B_\alpha\}$ be a net in $\mathcal{I}$ such that $\widetilde{B}_\alpha$ converges to ξ. Hence:

(iv) $\lambda_{B_\alpha}(C) = x_C(B_\alpha) \to \xi(x_C) = \lambda(C)$.

So, $0 \le \lambda(C) \le n[\frac{C}{E}]$ by Theorem 6.3(f). If $int(C) \neq \emptyset$, then Theorem 6.3(e) implies that:

$$\lambda(C) \ge \frac{1}{n[\frac{E}{C}]} > 0$$

so λ also satisfies property (d). A similar argument will show that λ satisfies the remaining properties. □

8 Left Invariant Measure

Definition 8.1. Let λ be a finitely additive measure on compact sets in G as in Theorem 7. We define two new set functions μ and ν by the following:

(a) for each open set O in G, $\mu(O) = \sup\{\lambda(C) : C \subseteq O, C \text{ compact}\}$

(b) for each set M in G, $\nu(M) = inf\{\mu(O) : M \subseteq O, O\ open\}$.

One can show that ν is an outer measure.

Definition 8.2. A set M in G ($M \subseteq G$) is called ν-measurable if for each set K in G:

$$\nu(K) = \nu(K \cap M) + \nu(K \cap (G \setminus M)).$$

Theorem 8.3. *Let $\mathcal{B}$ be the Borel σ-algebra in G; which is the smallest σ-algebra in G containing all the open sets. Then the set of all ν-measurable sets is a σ-algebra Γ which contains the σ-algebra $\mathcal{B}$. Furthermore the restriction of ν to $\mathcal{B}$ satisfies the following:*

(a) if $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{B}$, then $\nu(\cup_{n=1}^{\infty} A_n) \leq \Sigma_{n=1}^{\infty} \nu(A_n)$
(b) if $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{B}$, and $A_n \cap A_m = \emptyset$ whenever $n \neq m$ then
$\nu(\cup_{n=1}^{\infty} A_n) = \Sigma_{n=1}^{\infty} \nu(A_n)$
(c) $\nu(A) \leq \nu(B)$, for A and B in $\mathcal{B}$ with $A \subseteq B$
(d) $\lambda(C) \leq \nu(C)$ for every compact set C
(e) for any open set O we have:
$\nu(O) \leq sup\{\nu(C) : C$ compact and $C \subseteq O\}$
(f) $\nu(A) = inf\{\nu(O) : A \subseteq O, O$ open$\}$
(g) $\nu(int(C)) \leq \lambda(C) \leq \nu(C)$ for each compact set C
(h) $\nu(aB) = \nu(B)$ for each Borel set B and each $a \in G$
(i) $\nu(A) > 0$ if A is any Borel set for which $int(A) \neq \emptyset$
(j) $\nu(A) < \infty$ if A is any Borel set for which $\bar{A}$ is compact
We call ν a left invariant Haar measure on G.

Proof. See [8, 2, 3, and 4]. □

9 Examples of Haar Measure

We will conclude this article by presenting several examples of Haar measure and posing an open question.

Example 9.1. Let c be a positive real number and put $G_c = (-c, c)$, so that G_c is the open interval with end points $-c$ and c.For each x and y in G_c define:

$$x \star y = \frac{x+y}{1+\frac{xy}{c^2}}.$$

To show that $x \star y$ is in G_c first note that for all x and y in G_c we have:

$$(c+x) > 0, (c+y) > 0, (c-x) > 0, (c-y) > 0.$$

So we get (a) $(c+x)(c+y) > 0$ and (b) $(c-x)(c-y) > 0$. Also we have (c) $c^2 + xy > 0$. Now by (a), (b) and (c) $-c < x \star y < c$. The associativity of $\star$ operation follows from the following:

$$(x \star y) \star z = x \star (y \star z) = \frac{x+y+z+\frac{xyz}{c^2}}{1+\frac{xy+yz+zx}{c^2}}.$$

Now it is evident that $(G_c, \star)$ is a locally compact Hausdorff topological group with the usual topology, where the identity is 0 and the inverse of each x is $-x$. To find the Haar measure of G_c we follow the technique developed in [8, 14.2.4]. Let $\mathcal{B}$ be the Borel σ-Algebra of G_c and $h : G_c \to [0, \infty)$ be a measurable function. Define:

(a) $\nu : \mathcal{B} \to [0, \infty)$ by $\nu(E) = \int_E h(t)dt$

where the integral $\int_E h(t)dt$ is the Lebesgue integral. One can show that ν is a measure on $\mathcal{B}$, in particular if $h(t) = 1$ for all t in G_c then ν is the Lebesgue measure on $\mathcal{B}$. Now we find a function h such that the measure ν given in (a) is invariant measure. First note that for each $E \in \mathcal{B}$ and each $a \in G_c$, $a \star E$ is also in $\mathcal{B}$. Now let $E = [\alpha,\beta]$ then for each $a \in G_c$, $a \star E = [a \star \alpha, a \star \beta]$, hence we must have:

$$\nu(E) = \nu(a \star E) = \int_E h(t)dt = \int_{a\star E} h(t)dt,$$

so:

$$\int_\alpha^\beta h(t)dt = \int_{a\star\alpha}^{a\star\beta} h(t)dt.$$

Note that:
$a \star \alpha = \frac{(a+\alpha)c^2}{a\alpha+c^2}$ and $a \star \beta = \frac{(a+\beta)c^2}{a\beta+c^2}$.
Now by change of variable $t = a \star u$ we get:

$$\int_\alpha^\beta h(t)dt = \int_{a\star\alpha}^{a\star\beta} h(t) = \int_\alpha^\beta h(a \star u)\frac{(c^2-a^2)c^2}{(au+c^2)^2}du.$$

Thus for each a and u in G_c:

$$h(u) = \frac{(c^2-a^2)c^2}{(au+c^2)^2}\ h(a \star u),$$

hence $h(-a) = \frac{c^2}{c^2-a^2}h(0)$. Let $h(0) = 1$, so that $h(t){=}\frac{c^2}{c^2-t^2}$. Now one can show that:

$$\nu(E) = \int_E \frac{c^2}{c^2-t^2}dt$$

is a Haar measure on G_c.

Example 9.2. Let $c \in \mathcal{R}, c > 0$. Define:

$$\varphi : \mathcal{R} \to (-c,c),\ \varphi(t) = \frac{2c}{\pi}\tan^{-1}(t).$$

Note that φ is a homeomorphism between $\mathcal{R}$ and $(-c,c)$ where both are equipped with the usual topology. For each x and y in $\mathcal{R}$ define $x \circ y = \varphi^{-1}(\varphi(x) \star \varphi(y))$ where $\star$ is the group operation defined in Example 9.1. It is routine to check that $\mathcal{R}$ with operation $\circ$ is a topological group. Further:

$$x \circ y = \tan\left(\frac{\tan^{-1}x + \tan^{-1}y}{1 + \frac{4}{\pi^2}\tan^{-1}x\tan^{-1}y}\right).$$

In addition for all x, y and z in $\mathcal{R}$ we have:

$$(x \circ y) \circ z = x \circ (y \circ z) = \tan\left(\frac{\tan^{-1}x + \tan^{-1}y + \tan^{-1}z + a\tan^{-1}x\tan^{-1}y\tan^{-1}z}{1 + a\tan^{-1}x\tan^{-1}y + a\tan^{-1}x\tan^{-1}z + a\tan^{-1}y\tan^{-1}z}\right)$$

where $a = \frac{4}{\pi^2}$.

To find Haar measure of $(\mathcal{R}, \circ)$ we will find a function $h = h(t)$ such that $\nu(E) = \int_E h(t)dt$ is an invariant measure. Using a change in variable as in Example 9.1 and letting $h(0) = 1$ we obtain:

$$\nu(E) = \int_E \frac{1}{(1+t^2)(1-(\frac{2}{\pi}\tan^{-1}t)^2)}dt.$$

In particular , if $E = [a,b]$ then:

$$\nu(E) = \frac{\pi}{4} ln \frac{(1 + k\tan^{-1} b)(1 - k\tan^{-1} a)}{(1 - k\tan^{-1} b)(1 + k\tan^{-1} a)}$$

where $k = \frac{2}{\pi}$.

Example 9.3. Let $G = \{(x,y) \in \mathcal{R}^2 : x^2 - y^2 \neq 0\}$. For (x,y) and (a,b) in G define:

$$(x,y) \star (a,b) = (ax + by, bx + ay).$$

It is easy to show that $(G,\star)$ is a commutative topological group where G has the usual topology. Note that G is an open set. To find Haar measure of $(G,\star)$ we find a function $h = h(x,y)$ such that :

$$\nu(E) = \int\int_E h(x,y)d\mu(x,y)$$

is invariant measure on Borel σ-algebra in G , where μ is Lebesgue measure on $\mathcal{R}^2$.

So let (α,β) be a point in G. Hence we must have:

$$(a) \int\int_E h(x,y)d\mu(x,y) = \int\int_{(\alpha,\beta)\star E} h(x,y)d\mu(x,y).$$

Now let $x = \frac{1}{\delta}(\alpha X - \beta Y)$ and $y = \frac{1}{\delta}(-\beta X + \alpha Y)$. By Using change of variables in (a) we get:

$\int\int_E h(x,y)d\mu(x,y) = \int\int_E h(\frac{1}{\delta}(\alpha X - \beta Y), \frac{1}{\delta(-\beta X + \alpha Y)} \frac{1}{|\alpha^2 - \beta^2|} d\mu(X,Y)$

Hence we must have:

$h(x,y) = h(\frac{1}{\delta}(\alpha x - \beta y), \frac{1}{\delta}(-\beta x + \alpha y))\frac{1}{|\alpha^2 - \beta^2|}$ for all (x,y) and all (α,β) in G.

Thus $h(\alpha,\beta) = h(1,0)\frac{1}{|\alpha^2 - \beta^2|}$. Let$h(1,0) = 1$. Then $h(x,y) = \frac{1}{|x^2 - y^2|}$.

Example 9.4. Let $G = \{(x,y,z) \in \mathcal{R}^3 : x^3 + y^3 + z^3 - 3xyz \neq 0\}$. For $(x,y,z),(a,b,c) \in G$ define:

$(x,y,z) \star (a,b,c) = (ax + cy + bz, bx + ay + cz, cx + by + az)$.

Note that G is an open subset of $\mathcal{R}^3$. It is easy to show that $(G,\star)$ is a commutative topological group in fact it is a Lie group. Now using the technique in example 9.3 or the fact that G is a Lie grou one can show that:

$\nu(E) = \int_E \frac{1}{|x^3 + y^3 + z^3 - 3xyz|} d\mu(x,y,z)$ is Haar Measure on G, where E is a Borel set in G.

Example 9.5. Let $G = \{x = (x_1, \cdots x_n) \in \mathcal{R}^n : x_1 \neq 0\}$. For $x, y \in G$ define:

$x \star y = (x_1 y_1, x_1 y_2 + x_2, \cdots x_1 y_k + x_k, \cdots x_1 y_n + x_n)$.

Note that G is an open subset of $\mathcal{R}^n$. Now one can show that $(G,\star)$ is a non commutative topological group, in fact it is a Lie group with dimension n. By [4, 15.17e]:

$$\nu_r(E) = \int_E \frac{1}{|x_1|} dx_1 \cdots dx_n$$

is a right Haar measure and

$$\nu_\ell(E) = \int_E \frac{1}{|x_1|^n} dx_1 \cdots dx_n$$

is left Haar measure on G. In this example the left and right Haar measure on G are different if $n > 1$.

Remark 9.6. Examples 9.1, 9.2, 9.3 and 9.4 are from the author of this article. However in example 9.4 the fact that $(G,\star)$ is a group was mentioned to the author by late Mohsen Hashtroody. One can generalize example 9.4 to dimension n.

Example 9.5 is a generalization of a popular example in [2],

Example 9.7. Let G be the set of all real $n \times n$ upper triangular matrices $A = (a_{ij})_{n\times n}$ such that $a_{ii} = 1$ for $i = 1, \cdots n$. Note that there are $k = \frac{n^2-n}{2}$ elements above the diagonal of each matrix. Identify each matrix A with a point $x = (x_1 \cdots x_k) \in \mathcal{R}^k$. So we may assume $G = \mathcal{R}^k$, hence G is a noncommutative topological group with the usual topology on $\mathcal{R}^k$ with $\star$ operation as matrix multiplication. Note that $(G, +)$ and $(G, \star)$ are Lie groups where one is commutative that is ,$(G, +)$ and the other one $(G, \star)$ is not,when $n > 1$. However one can show that [1, p. 243] the Lebesgue measure on $G = \mathcal{R}^k$ is Haar measure on both topological groups.

In Example 9.6 we mentioned that $G = \mathcal{R}^k$ with $k = \frac{n^2-n}{2}$ has two different Lie group structure. Furthermore the Lebesgue measure on $\mathcal{R}^k$ is Haar measure on both Lie groups. Now we ask the following question:

Question 9.8. *What are the real k dimensional manifolds with at least two different Lie group structures that have the same Haar measure?*

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