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Interior Controllability of the Linear Beam Equation

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Abstract

In this paper we prove the interior controllability of the Linear Beam Equation

$$\begin{cases} u_{tt} - 2\beta\Delta u_t + \Delta^2 u = 1_\omega u(t,x), & \text{in} \quad (0,\tau)\times\Omega, \\ u = \Delta u = 0, & \text{on} \quad (0,\tau)\times\partial\Omega, \end{cases}$$

where $\beta > 1$, Ω is a sufficiently regular bounded domain in $\mathbb{R}^N$ $(N \geq 1)$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω and the distributed control $u \in L^2([0,\tau];L^2(\Omega))$. Specifically, we prove the following statement: For all $\tau > 0$ the system is approximately controllable on $[0,\tau]$. Moreover, we exhibit a sequence of controls steering the system from an initial state to a final state in a prefixed time $\tau > 0$.

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1 Introduction

This paper has been motivated by the works in [1], [3], [4], [5],[7],[8] and [9] where a new technique is used to prove the approximate controllability of some diffusion process.

Following [1] and [4], in this paper we study the interior approximate controllability of the Linear Beam Equation

$$\begin{cases} u_{tt} - 2\beta\Delta u_t + \Delta^2 u = 1_\omega u(t,x), & \text{in} \quad (0,\tau)\times\Omega, \\ u = \Delta u = 0, & \text{on} \quad (0,\tau)\times\partial\Omega, \end{cases} \tag{1.1}$$

where $\beta > 1$, Ω is a sufficiently regular bounded domain in $\mathbb{R}^N$ $(N \geq 1)$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω and the distributed

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control $u \in L^2([0,\tau];L^2(\Omega))$.
The approximate controllability of the following linear beam equation with the controls acting in the whole set Ω follows from [3]

$$\begin{cases} u_{tt} - 2\beta\Delta u_t + \Delta^2 u = u(t,x), & \text{in} \quad (0,\tau)\times\Omega, \\ u = \Delta u = 0, & \text{on} \quad (0,\tau)\times\partial\Omega. \end{cases} \tag{1.2}$$

In this paper,we are interested in the interior approximate controllability of the linear beam equation, which is more interesting problem from the applications point of view since the control is acting only in a subset or part of the plate Ω. Roughly speaking, we prove the following statement(see Theorem 3.4) : For all $\tau > 0$ the system is approximately controllable on $[0,\tau]$. Moreover, we can exhibit a sequence of controls steering the system from an initial state to a final state in a prefixed time (see Theorem 3.4). Equation (1.1) arise in the mathematical study of structural damped nonlinear vibrations of a string or a beam and was considered in [11] and references therein.

2 Abstract Formulation of the Problem.

Let $Z = L^2(\Omega)$ and consider the linear unbounded operator
$A : D(A) \subset Z \to Z$ defined by $A\phi = -\Delta\phi$, where

$$D(A) = H_0^1(\Omega)\cap H^2(\Omega). \tag{2.1}$$

The operator A has the following very well known properties: the spectrum of A consists of only eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty,$$

each one with multiplicity γ_n equal to the dimension of the corresponding eigenspace.
a) There exists a complete orthonormal set $\{\phi_n\}$ of eigenvectors of A.

b) For all $z \in D(A)$ we have

$$Az = \sum_{n=1}^{\infty}\lambda_n \sum_{k=1}^{\gamma_n} < z, \phi_{n,k} > \phi_{n,k} = \sum_{n=1}^{\infty}\lambda_n E_n z, \tag{2.2}$$

where $< \cdot, \cdot >$ is the inner product in X and

$$E_n z = \sum_{k=1}^{\gamma_n} < z, \phi_{n,k} > \phi_{n,k}. \tag{2.3}$$

So, $\{E_n\}$ is a family of complete orthogonal projections in z and

$$z = \sum_{n=1}^{\infty} E_n z, \quad z \in Z. \tag{2.4}$$

c) $-A$ generates an analytic semigroup $\{T(t)\}_{t\geq 0}$ given by

$$T(t)z = \sum_{n=1}^{\infty} e^{-\lambda_n t} E_n z. \tag{2.5}$$

d) The fractional powered spaces Z^r are given by:

$$Z^r = D(A^r) = \{z \in Z : \sum_{j=1}^{\infty} \lambda_j^{2r} \|E_j z\|^2 < \infty\}, \quad r \geq 0,$$

with the norm

$$\|z\|_r = \|A^r z\| = \left\{ \sum_{j=1}^{\infty} \lambda_j^{2r} \|E_j z\|^2 \right\}^{1/2}, \quad z \in Z^r,$$

and

$$A^r z = \sum_{j=1}^{\infty} \lambda_j^r E_j z. \tag{2.6}$$

Also, for $r \geq 0$ we define $Z_r = Z^r \times Z$, which is a Hilbert Space with norm given by

$$\left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\|_{Z_r}^2 = \|u\|_r^2 + \|v\|^2.$$

Hence, (1.1) can be written as an abstract system of ordinary differential equations in the Hilbert space $Z_1 = Z^1 \times Z$ as follows:

$$\begin{cases} u' = v \\ v' = -A^2 u - 2\beta A v + 1_\omega u \end{cases} \tag{2.7}$$

Finally, system (1.1) can be rewritten as a first order system of ordinary differential equations in the Hilbert space $Z_1 = Z^1 \times Z$ as follows:

$$z' = \mathcal{A} z + B_\omega u, \quad z \in Z_1 \quad t \geq 0, \tag{2.8}$$

where $u \in L^2([0,\tau]; U)$, $U = L^2(\Omega)$,

$$\mathcal{A} = \begin{bmatrix} 0 & I_Z \\ -A^2 & -2\beta A \end{bmatrix}, \tag{2.9}$$

is an unbounded linear operator with domain

$$D(\mathcal{A}) = \{u \in H^4(\Omega) : u = \Delta w = 0\} \times D(A),$$

and $B : U \longrightarrow Z_1$, $B_\omega = \begin{bmatrix} 0 \\ 1_\omega \end{bmatrix}$ is a bounded linear operator.

Proposition 2.1. *The adjoint of operators B_Ω and B_ω are given by*

$$B_\Omega^* = \begin{bmatrix} 0 & I_Z \end{bmatrix}, \quad B_\omega^* = \begin{bmatrix} 0 & 1_\omega \end{bmatrix}$$

Now, we shall prove that the linear unbounded operator $\mathcal{A}$ given by the linear beam equation (2.9) generates a strongly continuous semigroup which decays exponentially to zero. In fact, using Lemma 2.1 from [6] we can prove the following theorem.

Theorem 2.2. *The operator $\mathcal{A}$, given by (2.9), is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ represented by*

$$T(t)z = \sum_{j=1}^{\infty} e^{A_j t} P_j z, \quad z \in Z_1, \quad t \geq 0 \tag{2.10}$$

where $\left\{P_j\right\}_{j\geq 0}$ is a complete family of orthogonal projections in the Hilbert space Z_1 given by

$$P_j = \begin{bmatrix} E_j & 0 \\ 0 & E_j \end{bmatrix}, \quad j = 1, 2, \ldots, \infty, \tag{2.11}$$

and

$$A_j = B_j P_j, \quad B_j = \begin{bmatrix} 0 & 1 \\ -\lambda_j^2 & 2\beta\lambda_j. \end{bmatrix}, \ j \geq 1. \tag{2.12}$$

Moreover, the eigenvalues $\sigma_1(j)$, $\sigma_2(j)$, of the matrix B_j are simple and given by:

$$\sigma_1(j) = -\lambda_j \rho_1, \quad \sigma_2(j) = -\lambda_j \rho_2,$$

where $0 < \rho_1 < \rho_2$ are given by

$$\rho_1 = \beta - \sqrt{\beta^2 - 1} \quad \textit{and} \quad \rho_2 = \beta + \sqrt{\beta^2 - 1}$$

and this semigroup decays exponentially to zero

$$||T(t)|| \leq M e^{-\mu t}, \quad t \geq 0, \tag{2.13}$$

where

$$\mu = \lambda_1 \rho_1$$

The following gap condition plays an important role in this paper

$$\frac{\lambda_{j+1}}{\lambda_j} > \frac{\rho_2}{\rho_1}. \tag{2.14}$$

Proposition 2.3. *The operator $P_j : Z_r \to Z_r, \quad j \geq 0$, defined by*

$$P_j = \begin{bmatrix} E_j & 0 \\ 0 & E_j \end{bmatrix}, \ j \geq 1\,, \tag{2.15}$$

is a continuous(bounded) orthogonal projections in the Hilbert space Z_r.

Proof First we shall show that $P_j(Z_r) \subset Z_r$, which is equivalent to show that $E_j(Z^r) \subset Z^r$. In fact, let z be in Z^r and consider $E_j z$. Then

$$\sum_{n=1}^{\infty} \lambda_n^{2r} ||E_n E_j z||^2 = \lambda_j^{2r} ||E_j z||^2 < \infty$$

Therefore, $E_j z \in Z^r, \forall z \in Z^r$.
Now, we shall prove that this projection is bounded. In fact, from the continuous inclusion $Z^r \subset Z$, there exists a constant $k > 0$ such that

$$||z|| \le k||z||_r, \ \ \forall z \in Z^r.$$

Then, for all $z \in Z^r$ we have the following estimate

$$\begin{aligned} ||E_j z||_r^2 &= \sum_{n=1}^{\infty} \lambda_n^{2r} ||E_n E_j z||^2 = \lambda_j^{2r} ||E_j z||^2 \\ &\le \lambda_j^{2r} ||z||^2 \le \lambda_j^{2r} k^2 ||z||_r^2 \end{aligned}$$

Hence $||E_j z|| \le \lambda_j^r k ||z||_r$, which implies the continuity of $E_j : Z^r \to Z^r$. So, P_j is a continuous projection on Z_r.

□

3 Proof of the Main Theorem

In this section we shall prove the main result of this paper on the controllability of the linear system (2.8). But, before we shall give the definition of approximate controllability for this system. To this end, for all $z_0 \in Z_1$ and $u \in L^2(0,\tau;U)$ the the initial value problem

$$\begin{cases} z' = \mathcal{A}z + B_\omega u(t), z \in Z_1, \\ z(0) = z_0, \end{cases} \tag{3.1}$$

where the control function u belong to $L^2(0,\tau;U)$, admits only one mild solution given by

$$z(t) = T(t)z_0 + \int_0^t T(t-s)B_\omega u(s)ds, \ \ t \in [0,\tau]. \tag{3.2}$$

Definition 3.1. (**Approximate Controllability**) The system (2.8) is said to be approximately controllable on $[0,\tau]$ if for every z_0, $z_1 \in Z_1$, $\varepsilon > 0$ there exists $u \in L^2(0,\tau;U)$ such that the solution $z(t)$ of (3.2) corresponding to u verifies:

$$z(0) = z_0 \ \text{ and } \ ||z(\tau) - z_1|| < \varepsilon.$$

Consider the following bounded linear operator:

$$G : L^2(0,\tau;Z) \to Z_1, \ \ Gu = \int_0^\tau T(\tau - s)B_\omega u(s)ds, \tag{3.3}$$

whose adjoint operator $G^* : Z_1 \longrightarrow L^2(0,\tau;Z)$ is given by

$$(G^*z)(s) = B_\omega^* T^*(\tau - s)z, \quad \forall s \in [0,\tau], \quad \forall z \in Z_1. \tag{3.4}$$

Lemma 3.2. *(see [4] and [5]) The equation (2.8) is approximately controllable on* $[0,\tau]$ *if, and only if, one of the following statements holds:*

a) $\overline{\text{Rang}(G)} = Z_1$.

b) $Ker(G^*) = \{0\}$.

c) $\langle GG^*z, z\rangle > 0$*,* $z \neq 0$ *in* Z_1.

d) $\lim_{\alpha\to 0^+} \alpha(\alpha I + GG^*)^{-1}z = 0$.

e) $\sup_{\alpha>0} \|\alpha(\alpha I + GG^*)^{-1}\| \leq 1$.

f) $B_\omega^* T^*(t)z = 0, \quad \forall t \in [0,\tau], \quad \Rightarrow z = 0$.

g) *For all* $z \in Z_1$ *we have* $Gu_\alpha = z - \alpha(\alpha I + GG^*)^{-1}z$*, where*

$$u_\alpha = G^*(\alpha I + GG^*)^{-1}z, \quad \alpha \in (0,1].$$

So, $\lim_{\alpha\to 0} Gu_\alpha = z$ *and the error* $E_\alpha z$ *of this approximation is given by*

$$E_\alpha z = \alpha(\alpha I + GG^*)^{-1}z, \quad \alpha \in (0,1].$$

For the proof of the main theorem of this paper we shall use the following version of Lemma 3.14 from [2] and Lemma 4.4 from [1].

Lemma 3.3. *Let* $\{\alpha_1(j)\}_{j\geq 1}$, $\{\beta_{1j}\}_{j\geq 1}$ *and* $\{\alpha_2(j)\}_{j\geq 1}$,$\{\beta_{2j}\}_{j\geq 1}$ *be sequences of real numbers such that* $\alpha_2(j) < \alpha_1(j)$ *and*

$$\alpha_s(j+1) < \alpha_s(j), \quad \alpha_1(j+1) < \alpha_2(j). \tag{3.5}$$

for $s = 1,2; j = 1,2,3,\ldots$*. Then, for any* $\tau > 0$ *we have that*

$$\sum_{j=1}^{\infty} (e^{\alpha_1(j)t}\beta_{1j} + e^{\alpha_2(j)t}\beta_{2j}) = 0, \quad \forall t \in [0,\tau] \tag{3.6}$$

if, and only if,

$$\beta_{1j} = \beta_{2j} = 0, \forall j \geq 1. \tag{3.7}$$

Now, we are ready to formulate and prove the main theorem of this work.

Theorem 3.4. *(Main Result) Under condition (2.14), for all nonempty open subset* ω *of* Ω *and* $\tau > 0$ *the system (2.8) is approximately controllable on* $[0,\tau]$*. Moreover, a sequence of controls steering the system (2.8) from initial state* z_0 *to an* ϵ *neighborhood of the final state* z_1 *at time* $\tau > 0$ *is given by*

$$u_\alpha(t) = B_\omega^* T(\tau - t)(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0), \quad \alpha \in (0,1],$$

and the error of this approximation E_α *is given by*

$$E_\alpha = \alpha(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0), \quad \alpha \in (0,1].$$

Proof . We shall apply part f) of lemma 3.2 to prove the controllability of system (2.8). To this end, we observe that

$$T^*(t)z = \sum_{j=1}^{\infty} e^{A_j^* t} P_j^* z, \quad z \in Z, \quad t \geq 0,$$

and, since the eigenvalues of the matrix A_j are simple, there exists a family of complete complementary projections $\{q_1(j), q_2(j)\}$ on $\mathbb{R}^2$ such that

$$e^{A_j^* t} = e^{\sigma_1(j)t} q_1^*(j) P_j^* + e^{\sigma_2(j)t} q_2^*(j) P_j^*.$$

Therefore,

$$B_\omega^* T^*(t)z = \sum_{j=1}^{\infty} B_\omega^* e^{A_j^* t} P_j^* z = \sum_{j=1}^{\infty} \sum_{s=1}^{2} e^{\sigma_s(j)t} B_\omega^* P_{s,j}^* z,$$

where $P_{s,j} = q_s(j) P_j = P_j q_s(j)$.

Now, suppose that $B_\omega^* T^*(t)z = 0, \ \forall t \in [0, \tau]$. Then,

$$\begin{aligned} B_\omega^* T^*(t)z \quad &= \quad \sum_{j=1}^{\infty} B_\omega^* e^{A_j^* t} P_j^* z = \sum_{j=1}^{\infty} \sum_{s=1}^{2} e^{\sigma_s(j)t} B_\omega^* P_{s,j}^* z = 0. \\ &\Longleftrightarrow \quad \sum_{j=1}^{\infty} \sum_{s=1}^{2} e^{\sigma_s(j)t} (B_\omega^* P_{s,j}^* z)(x) = 0, \quad \forall x \in \Omega. \end{aligned}$$

The assumption (2.14) implies that the sequence $\{\alpha_s(j) = -\lambda_j \rho_s : s = 1,2; j = 1,2,\dots\}$ satisfies the conditions on Lemma 3.3. In fact, we have trivially that
$\alpha_2(j) < \alpha_1(j)$ and from (2.14) we obtain $-\lambda_{j+1}\rho_1 < -\lambda_j \rho_2$ Therefore,

$$\alpha_s(j+1) < \alpha_s(j), \quad \alpha_1(j+1) < \alpha_2(j).$$

Then, from Lemma 3.3 we obtain for all x $\in \Omega$ that

$$(B_\omega^* P_{s,j}^* z)(x) = 0, \quad \forall x \in \Omega, \quad s = 1,2; \quad j = 1,2,3,\dots.$$

Since

$$q_i^*(j) = \begin{bmatrix} a_{11}^{ij} & a_{12}^{ij} \\ a_{21}^{ij} & a_{22}^{ij} \end{bmatrix}, i = 1,2; \quad j = 1,2,3,4,\dots,$$

we get $\forall x \in \Omega, \quad i = 1,2; \quad j = 1,2,3,4,\dots$ that

$$(B_\omega^* P_{s,j}^* z)(x) = \begin{bmatrix} 1_\omega [a_{21}^{ij} E_j z_1(x) + a_{22}^{ij} E_j z_2(x)] \end{bmatrix} = 0$$

That is to say,

$$(B_\omega^* P_{s,j}^* z)(x) = \begin{bmatrix} a_{21}^{ij} E_j z_1(x) + a_{22}^{ij} E_j z_2(x) \end{bmatrix} = 0, \quad \forall x \in \omega.$$

Now, putting $f(x) = a_{21}^{ij} E_j z_1(x) + a_{22}^{ij} E_j z_2(x), \quad \forall x \in \Omega$, we obtain that

$$\begin{cases} (\Delta + \lambda_j I) f \equiv 0 \quad \text{in} \quad \Omega, \\ f(x) = 0 \quad \forall x \in \omega. \end{cases}$$

Then, from the classical Unique Continuation Principle for Elliptic Equations (see [10]), it follows that $f(x) = 0, \quad \forall x \in \Omega$. So, we get for $i = 1,2;\; j = 1,2,3,4,\dots$ that

$$(B_\omega^* P_{s,j}^* z)(x) = \left[\; a_{21}^{ij} E_j z_1(x) + a_{22}^{ij} E_j z_2(x) \;\right] = 0, \quad \forall x \in \Omega.$$

Hence

$$B_\Omega^* T^*(t) z = \sum_{j=1}^{\infty} B_\Omega^* e^{A_j^* t} P_j^* z = \sum_{j=1}^{\infty} \sum_{s=1}^{2} e^{\sigma_s(j)t} B_\Omega^* P_{s,j}^* z = 0, \;\; \forall t \in [0,\tau].$$

Since system (1.2)(see [3]) is approximately controllable, then from part f) of lemma 3.2 we get that $z = 0$.
So, putting $z = z_1 - T(\tau)z_0$, using (3.2) and part g) of Lemma 3.2, we obtain the nice result:

$$z_1 = \lim_{\alpha \to 0^+} \{T(\tau)z_0 + \int_0^\tau T(\tau - s) B_\omega u_\alpha(s) ds\}.$$

□

Proof of Lemma 3.3. By analytic extension we obtain

$$\sum_{j=1}^{\infty} (e^{\alpha_1(j)t} \beta_{1j} + e^{\alpha_2(j)t} \beta_{2j}) = 0, \quad \forall t \in [0,\infty).$$

Now, dividing this expression by $e^{\alpha_1(1)t}$ we get

$$\beta_{11} + \sum_{j=2}^{\infty} e^{(\alpha_1(j) - \alpha_1(1))t} \beta_{1j} + \sum_{j=1}^{\infty} e^{(\alpha_2(j) - \alpha_1(1))t} \beta_{2j} = 0, \quad \forall t \in [0,\infty).$$

Since $\alpha_1(j) - \alpha_1(1)) < 0$ for $j > 1$ and $\alpha_2(j) - \alpha_1(1) < 0$ for $j \geq 1$, then passing to the limit when $t \to \infty$ we obtain that $\beta_{11} = 0$
Then, we have that

$$\sum_{j=2}^{\infty} e^{\alpha_1(j)t} \beta_{1j} + \sum_{j=1}^{\infty} e^{\alpha_2(j)t} \beta_{2j}, \quad \forall t \in [0,\infty).$$

Now, dividing this expression by $e^{\alpha_2(1)t}$ we get

$$\beta_{21} + \sum_{j=2}^{\infty} e^{(\alpha_1(j) - \alpha_2(1))t} \beta_{1j} + \sum_{j=2}^{\infty} e^{(\alpha_2(j) - \alpha_2(1))t} \beta_{2j} = 0, \quad \forall t \in [0,\infty).$$

From (3.5) we have that $\alpha_1(j) - \alpha_2(1)) < 0$ and $\alpha_2(j) - \alpha_2(1) < 0$ for $j \geq 2$. Then passing to the limit when $t \to \infty$ we obtain that $\beta_{21} = 0$
Then, we have that

$$\sum_{j=2}^{\infty} e^{\alpha_1(j)t} \beta_{1j} + \sum_{j=2}^{\infty} e^{\alpha_2(j)t} \beta_{2j} = 0, \quad \forall t \in [0,\infty).$$

Repeating this procedure from here, we would obtain that $\beta_{12} = \beta_{22} = 0$, and continuing this way we get $\beta_{1j} = \beta_{2j} = 0, \forall j \geq 1$.

□

References

[1] S. BADRAOUI,Approximate Controllability of a Reaction-Diffusion System with a Cross Diffusion Matrix and Fractional Derivatives on Bounded Domains. Journal of Boundary Value Problems, Vol. 2010(2010), Art. ID 281238, 14pgs.

[2] R.F. CURTAIN, A.J. PRITCHARD, *Infinite Dimensional Linear Systems.* Lecture Notes in Control and Information Sciences, **8**. Springer Verlag, Berlin (1978).

[3] H. LÁREZ, H. LEIVA AND J. UZCÁTEGUI, Controllability of Block Diagonal Systems and Applications. Int. J. Systems, Control and Communications, Vol. 3, N. 1, pp. 64-81.

[4] H. LAREZ AND H. LEIVA, Interior controllability of a 2×2 reaction-diffusion system with cross-diffusion matrix. Boundary Value Problems, Vol. 2009, Article ID 560407, 9 pages, doi:10.1155/2009/560407.

[5] H. LEIVA AND Y. QUINTANA, Interior controllability of a broad class of reaction diffusion equation. Mathematical Problems in Engineering, Vol. 2009, Article ID 708516, 8 pages, doi:10.1155/2009/708516.

[6] H. LEIVA, A Lemma on C_0-Semigroups and Applications PDEs Systems. Quaestions Mathematicae, Vol. 26, pp. 247-265 (2003).

[7] H. LEIVA AND N. MERENTES, Controllability of Second Order Equation in L2. Mathematical Problems in Engineering, Vol. 2010, Art. ID 147195, 11pages, doi:10.1155/2010/147195.

[8] H. LEIVA AND N. MERENTES, Interior Controllability of the Thermoelastic Plate Equation. African Diaspora Journal of Mathematics, Vol. 12, N 1, pp. 1-14(2011).

[9] H. LEIVA, N. MERENTES AND J.L. SANCHEZ, Interior Controllability of the nD Semilinear Heat Equation. African Diaspora Journal of Mathematics, Special Volume in Honor fo Profs. C. Corduneanu, A. Fink, and S. Zaidman, Vol. 12, Number 2, pp. 1-12 (2011).

[10] M.H. PROTTER, *Unique continuation for elliptic equations.* Transaction of the American Mathematical Society, Vol. 95, N 1, Apr., 1960.

[11] D. SEVICOVIC, Existence and limiting behaviour for damped nonlinear evolution equations with nonlocal terms. *Comment. Math. Univ. Carolinae* 31, 2, 283-293, 1990.