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Finite and Infinite Time Interval of BDSDEs Driven by Lévy Processes

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Abstract

In this work we deal with a backward doubly stochastic differential equation (BDSDE) associated to a Poisson random measure. We establish existence and uniqueness of solution in the case of non-Lipschitz coefficients. The novelty of our result lies in the fact that we allow the time interval to be infinite.

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1 Introduction

It is well known that backward stochastic differential equations (BSDEs in short) provide a stochastic representation of solutions of semilinear partial differential equations (PDEs). As far as we know, in these works, Lipschitz or at least monotonicity condition is required on the drift of the BSDEs and the horizon time is fixed. Recently several authors investigate successfully in weakening these conditions (see among others [3], [10]). The assumption usually satisfied by the drift is replaced by a rather smooth one which ensures existence and uniqueness result. Inspired by the method developed by Wang and Huang [10], Sow [8], extended their result to BSDE with jumps and proved a large deviation principle of such family of equations. Recently, Fan and Jiang [2] under similar conditions required in [10], prove existence and uniqueness of solution of a class of BSDEs with non-Lipschitz coefficients. They allow the time interval of the equation to be infinite.

Backward doubly stochastic differential equations (BDSDE in short) appear as a natural extension of backward stochastic differential equations (BSDE). Their link with stochastic

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partial differential equations (SPDEs) in the case of Lipchitzian drift was established by Pardoux and Peng [6]. The key point of solvency of such equations is the martingale representation theorem. Using the generalized martingale representation theorem associated to Lévy process established by Nualart and Schoutens [5], several authors investigate in extending this result to BDSDEs driven by Lévy processes. In this spirit Ren *et al* [7] established an existence and uniqueness of solutions and provided a stochastic representation of solutions of stochastic partial differential integral equations (SPDIEs) under Lipschitz condition on the drift. It has been known that there is an intrinsic connection between infinite time interval BDSDEs and stationary solutions of SPDEs (see for example [12]).

In this paper, inspired by the method introduced by Fan and Jiang [2], we solve an infinite time interval BDSDEs driven by Lévy processes and non-Lipschitz coefficients. We prove an existence and uniqueness result which extend the result of Nualart and Schoutens [5] in the case of coefficients satisfying rather weaker conditions. The paper is organized as follows. In section 2, we prove some useful results on BDSDEs driven by Poisson random measure. In section 3, we establish our main result.

2 Backward doubly SDE and Poisson random measure

2.1 Definitions and notations

Let Ω be a non-empty set, $\mathcal{F}$ a $\sigma-$algebra of sets of Ω and $\mathbb{P}$ a probability measure defined on $\mathcal{F}$. The triplet $(\Omega,\mathcal{F},\mathbb{P})$ defines a probability space, which is assumed to be complete. For a fix real $0 < T \leq \infty$, we assume given three mutually independent processes :

- a $\ell-$dimensional Brownian motion $(B_t)_{0\leq t\leq T}$,
- a $d-$dimensional Brownian motion $(W_t)_{0\leq t\leq T}$,
- a Poisson random measure μ on $E\times\mathbb{R}_+$.

The space $E = \mathbb{R}^\ell - \{0\}$ is equipped with its Borel field $\mathcal{E}$ with compensator $\nu(dt,de) = \lambda(de)dt$ such that $\{\widetilde{\mu}([0,t]\times A) = (\mu-\nu)[0,t]\times A\}$ is a martingale for any $A \in \mathcal{E}$ satisfying $\lambda(A) < \infty$. λ is a $\sigma-$finite measure on $\mathcal{E}$ and satisfies

$$\int_E (1\wedge|e|^2)\lambda(de) < \infty.$$

We consider the family $(\mathcal{F}_t)_{0\leq t\leq T}$ given by

$$\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B \vee \mathcal{F}_t^\mu, \quad 0\leq t\leq T,$$

where for any process $\{\eta_t\}_{t\geq 0}$, $\mathcal{F}_{s,t}^\eta = \sigma\{\eta_r-\eta_s,\ s\leq r\leq t\}\vee\mathcal{N}$, $\mathcal{F}_t^\eta = \mathcal{F}_{0,t}^\eta$. $\mathcal{N}$ denotes the class of $\mathbb{P}-$null sets of $\mathcal{F}$. Note that $(\mathcal{F}_t)_{0\leq t\leq T}$ does not constitute a classical filtration.

Let $g:\Omega\times[0,T]\times\mathbb{R}^k\times\mathbb{R}^{k\times d}\times\mathbb{R}^k\to\mathbb{R}^{k\times\ell}$ and $f:\Omega\times[0,T]\times\mathbb{R}^k\times\mathbb{R}^{k\times d}\times\mathbb{R}^k\to\mathbb{R}^k$ be jointly measurable. Given ξ a $\mathcal{F}_T-$measurable $\mathbb{R}^k$ valued random variable, we are interested in the backward doubly stochastic differential equation with Poisson random measure (BDSDEP in short)

$$Y_t = \xi + \int_t^T f(r,\Theta_r)\,dr + \int_t^T g(r,\Theta_r)\,dB_r - \int_t^T Z_r dW_r - \int_t^T\int_E U_r(e)\widetilde{\mu}(dr,de), \quad 0\leq t\leq T, \tag{2.1}$$

where $\Theta_r = (Y_r, Z_r, U_r)$.
Our motivation in studying such equations comes from their strong link with SPDIEs. Indeed if we consider the forward SDE with Poisson jumps given by

$$X_t = x + \int_s^t b(r, X_r)dr + \int_s^t \sigma(r, X_r)dWr + \int_s^t \int_E h(r_-, X_{r_-}, e)\widetilde{\mu}(dr, de), \quad 0 \leq s \leq t \leq T, \tag{2.2}$$

where $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$, $\quad h : [0,T] \times \mathbb{R}^d \times E \to \mathbb{R}^d$, one can associated this equation to the following BDSDEP (where $\Theta_r^X = (X_r, Y_r, Z_r, U_r)$ and $\Phi : \mathbb{R}^d \to \mathbb{R}^k$)

$$Y_t = \Phi(X_T) + \int_t^T f\left(r, \Theta_r^X\right)dr + \int_t^T g\left(r, \Theta_r^X\right)dB_r - \int_t^T Z_r dW_r - \int_t^T \int_E U_r(e)\widetilde{\mu}(dr, de) \tag{2.3}$$

It is well known that the BDSDEP (2.3) is related to the system of parabolic SPDIEs

$$\begin{cases} \partial_t u_i(t,x) = \mathcal{L}u_i(t,x) + f_i(t,x,u(t,x), \partial_x u(t,x)\sigma(t,x), u(t,x+h(t,x,\cdot)) - u(t,x)) \\ \qquad\qquad + g_i(t,x,u(t,x), \partial_x u(t,x)\sigma(t,x), u(t,x+h(t,x,\cdot)) - u(t,x))dB_t, \quad i = 1,\dots,k. \\ u(T,x) = \Phi(x) \end{cases}$$

where

$$\begin{aligned} \mathcal{L}u_i(t,x) &= \frac{1}{2} Tr\left(a(t,x)\partial_{xx}^2 u_i(t,x)\right) + \langle b(t,x), \partial_x u_i(t,x)\rangle \\ &\quad + \int_E [u_i(t,x+h(t,x,e)) - u_i(t,x) - \langle h(t,x,e), \partial_x, u_i(t,x)\rangle]\,\lambda(de), \\ a_{ij}(t,x) &= (\sigma(t,x)\sigma(t,x)^*)_{i,j}. \end{aligned}$$

Using essentially Itô formula and assumptions on the coefficients, one can prove that the solutions of the SPDIEs have the stochastic representation : $u(t,x) = Y_t^{t,x}$.

For $Q \in \mathbb{N}^*$, $|\,.\,|$ and $\langle\cdot\rangle$ stand for the euclidian norm and the inner product in $\mathbb{R}^Q$.
We consider the following sets (where $\mathbb{E}$ denotes the mathematical expectation with respect to the probability measure $\mathbb{P}$):

- $S^2_{[0,T]}(\mathbb{R}^Q)$ the space of $\mathcal{F}_t-$adapted càdlàg processes

$$\Psi : [0,T] \times \Omega \longrightarrow \mathbb{R}^Q, \; \|\Psi\|^2_{S^2} = \mathbb{E}\left(\sup_{0\leq t\leq T} |\Psi_t|^2\right) < \infty.$$

- $H^2_{[0,T]}(\mathbb{R}^Q)$ the space of $\mathcal{F}_t-$progressively measurable processes

$$\Psi : [0,T] \times \Omega \longrightarrow \mathbb{R}^Q, \; \|\Psi\|^2_{H^2} = \mathbb{E}\int_0^T |\Psi_t|^2\,dt < \infty.$$

- $L^2_{[0,T]}(\widetilde{\mu}, \mathbb{R}^Q)$ the space of mappings $U : \Omega \times [0,T] \times E \longrightarrow \mathbb{R}^Q$ which are $\mathcal{P}\otimes\mathcal{E}$-measurable s.t.

$$\|U\|^2_{L^2} = \mathbb{E}\int_0^T \int_E |U_t(e)|^2\,\lambda(de)dt < \infty,$$

where $\mathcal{P}\otimes\mathcal{E}$ denotes the $\sigma-$algebra of predictable sets of $\Omega\times[0,T]$.

Notice that the space $\mathcal{B}^2 = \mathcal{B}^2_{[0,T]}(\mathbb{R}^Q) = S^2_{[0,T]}(\mathbb{R}^Q) \times H^2_{[0,T]}(\mathbb{R}^Q) \times L^2_{[0,T]}(\widetilde{\mu}, \mathbb{R}^Q)$ is a Banach space. For notational simplicity we note for a function $\psi : [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \times \mathbb{R}^k \to \mathbb{R}^k$, $\psi(r,0) = \psi(r,0,0,0)$.

Definition 2.1. A triplet of processes $(Y_t, Z_t, U_t)_{0\le t\le T}$ is called a solution to eq. (2.1), if $(Y_t, Z_t, U_t) \in \mathcal{B}^2$ and satisfies (2.1).

2.2 Preliminary results

In the following, we assume that f and g satisfy assumptions **(H1)** :

(H1.1) For any $(y,z,u) \in \mathbb{R}^k \times \mathbb{R}^{k\times d} \times \mathbb{R}^k$, $f(.,y,z,u)$ is a progressively measurable process such that $\mathbb{E}(\int_0^\infty |f(s,y,z,u)|ds)^2 < \infty$ and $g(.,y,z,u) \in H^2_{[0,\infty]}(\mathbb{R}^{k\times\ell})$.

(H1.2) There exists a constant $0 < \alpha < 1$ and positive non random continuous functions $\{\nu(t)\}$, $\{\varphi(t)\}$ and $\{\varrho(t)\}$ such that for $t \ge 0$, $(y,y') \in (\mathbb{R}^k)^2$, $(z,z') \in (\mathbb{R}^{k\times d})^2$, $(u,u') \in (\mathbb{R}^k)^2$
$|f(t,y,z,u) - f(t,y',z',u')| \le \nu(t)|y-y'| + \varphi(t)(|z-z'| + |u-u'|)$
$|g(t,y,z,u) - g(t,y',z',u')|^2 \le \varrho(t)|y-y'|^2 + \alpha(|z-z'|^2 + |u-u'|^2)$

(H1.3) $\int_0^\infty \nu(s)ds < \infty, \quad \int_0^\infty [\varphi^2(s) + \varrho(s)]ds < \infty.$

First let us recall the following Gronwall's lemma which will be useful in the sequel.

Lemma 2.2. *Assume given $T \ge 0$, $K \ge 0$ and $\Phi, \Psi : [0,T] \to \mathbb{R}^+$ such that $\int_0^T \Psi(r)dr < \infty$. If*

$$\forall 0 \le t \le T, \qquad \Phi(t) \le K + \int_0^t \Psi(r)\Phi(r)dr < \infty,$$

then we have

$$\forall 0 \le t \le T, \qquad \Phi(t) \le K \exp\left(\int_0^t \Psi(r)dr\right).$$

We have also the following version of Itô's formula. The proof is omitted since it is an adaptation of [6, Lemma 1.3].

Lemma 2.3. *Let $X \in S^2_{[0,T]}(\mathbb{R}^k)$, $\vartheta \in H^2_{[0,T]}(\mathbb{R}^k)$, $\zeta \in H^2_{[0,T]}(\mathbb{R}^{k\times\ell})$, $\pi \in H^2_{[0,T]}(\mathbb{R}^{k\times d})$ and $\phi \in L^2_{[0,T]}(\widetilde{\mu}, \mathbb{R}^k)$ be such that*

$$X_t = X_0 + \int_0^t \vartheta_r dr + \int_0^t \zeta_r dB_r + \int_0^t \pi_r dW_r + \int_0^t \int_E \phi_r(e)\widetilde{\mu}(dr,de), \quad 0 \le t \le T.$$

Then we have for any $0 \le t \le T$ *and* $\beta \in \mathbb{R}$,

$$\textbf{(i)}\quad |X_t|^2 = |X_0|^2 + 2\int_0^t \langle X_r, \vartheta_r\rangle dr + 2\int_0^t \langle X_r, \zeta_r dB_r\rangle + 2\int_0^t \langle X_r, \pi_r dW_r\rangle + 2\int_0^t\int_E \langle X_{r-}, \phi_r(e)\widetilde{\mu}(dr,de)\rangle - \int_0^t |\zeta_r|^2 dr + \int_0^t |\pi_r|^2 dr + \int_0^t\int_E |\phi_r(e)|^2\lambda(de)dr + \sum_{0<s\le t}(\Delta X_s)^2.$$

$$\textbf{(ii)}\quad \mathbb{E}|X_t|^2 + \mathbb{E}\int_t^T |\pi_r|^2 dr + \mathbb{E}\int_t^T\int_E |\phi_r(e)|^2\lambda(de)dr \le \mathbb{E}|X_T|^2 + 2\mathbb{E}\int_t^T \langle X_r, \vartheta_r\rangle dr + \mathbb{E}\int_t^T |\zeta_r|^2 dr,$$

$$\textbf{(iii)}\quad \mathbb{E}(e^{\beta t}|X_t|^2) + \mathbb{E}\int_t^T \beta e^{\beta r}|X_r|^2 dr + \mathbb{E}\int_t^T e^{\beta r}|\pi_r|^2 dr + \mathbb{E}\int_t^T\int_E e^{\beta r}|\phi_r(e)|^2\lambda(de)dr \le \mathbb{E}(e^{\beta T}|X_T|^2) + 2\mathbb{E}\int_t^T e^{\beta r}\langle X_r, \vartheta_r\rangle dr + \mathbb{E}\int_t^T e^{\beta r}|\zeta_r|^2 dr.$$

In what follows, we study our equation in the finite case under our standing conditions.

Proposition 2.4. *Assume that* $T < \infty$ *and (H1) is in force. Then BDSDEP* (2.1) *admits a unique solution.*

Proof. To prove existence of solution, let us consider the sequence of processes $\Theta_t^n = (Y_t^n, Z_t^n, U_t^n)_{n\ge 0}$ given by

$$\begin{cases} Y_t^0 = 0; \\ Y_t^{n+1} = \xi + \int_t^T f(r,\Theta_r^n)dr + \int_t^T g(r,\Theta_r^n)dB_r - \int_t^T Z_r^{n+1}dW_r - \int_t^T\int_E U_r^{n+1}(e)\widetilde{\mu}(dr,de). \end{cases} \tag{2.4}$$

Since for a fix $n \in \mathbb{N}$, the coefficients f and g of the BDSDEP (2.4) do not depend on the solution $(Y_t^{n+1}, Z_t^{n+1}, U_t^{n+1})$, it follows from [9, Proposition 2.1] that the sequence $(\Theta^n)_{n\ge 0}$ is well defined.

We want to prove that $(\Theta^n)_{n\ge 0}$ is a Cauchy sequence in $\mathcal{B}^2$. For this end define

$$\overline{Y}_t^{n+1} = Y_t^{n+1} - Y_t^n, \quad \overline{Z}_t^{n+1} = Z_t^{n+1} - Z_t^n \quad \text{and} \quad \overline{U}_t^{n+1} = U_t^{n+1} - U_t^n, \quad 0 \le t \le T.$$

Fix $\beta \in \mathbb{R}$. Applying **(iii)** in Lemma 2.3, we obtain

$$\mathbb{E}\Big(e^{\beta t}|\overline{Y}_t^{n+1}|^2\Big) + \mathbb{E}\int_t^T \beta e^{\beta r}|\overline{Y}_r^{n+1}|^2 dr + \mathbb{E}\int_t^T e^{\beta r}|\overline{Z}_r^{n+1}|^2 dr + \mathbb{E}\int_t^T\int_E e^{\beta r}|\overline{U}_r^{n+1}(e)|^2\lambda(de)dr \le 2\mathbb{E}\int_t^T e^{\beta r}\langle \overline{Y}_r^{n+1}, \Delta f^n(r)\rangle dr + \mathbb{E}\int_t^T e^{\beta r}|\Delta g^n(r)|^2 dr, \tag{2.5}$$

where for a function $h \in \{f, g\}$, $\Delta h^n(r) = h(r, Y_r^n, Z_r^n, U_r^n) - h(r, Y_r^{n-1}, Z_r^{n-1}, U_r^{n-1})$, $0 \le r \le T$. Using standard estimates we have (where $\varepsilon > 0$ will be chosen later)

$$2|\overline{Y}_r^{n+1}||\Delta f^n(r)| \le (\nu(r) + \frac{2}{\varepsilon}\varphi^2(r))|\overline{Y}_r^{n+1}|^2 + \nu(r)|\overline{Y}_r^n|^2 + \varepsilon(|\overline{Z}_r^n|^2 + |\overline{U}_r^n|^2),$$

and

$$|\Delta g^n(r)|^2 \leq \varrho(r)|\overline{Y}_r^n|^2 + \alpha(|\overline{Z}_r^n|^2 + |\overline{U}_r^n|^2).$$

Hence putting pieces together, we deduce that the right hand side in (2.5) is less than

$$\gamma\mathbb{E}\int_t^T |\overline{Y}_r^{n+1}|^2 e^{\beta r}dr + \mathbb{E}\int_t^T [c|\overline{Y}_r^n|^2 + (\varepsilon+\alpha)(|\overline{Z}_r^n|^2 + |\overline{U}_r^n|^2)]e^{\beta r}dr.$$

where $\gamma = m_\nu + (2/\varepsilon)m_\varphi^2$, $c = m_\nu + m_\varrho$ and for a function $\delta \in \{\nu, \varphi, \varrho\}$, m_δ is s.t $|\delta(r)| \leq m_\delta$. Now, let $\varepsilon = \frac{1-\alpha}{2}$ and $\beta = \gamma + \overline{c}$ (where $\overline{c} = \frac{2c}{1+\alpha}$), we deduce that

$$\mathbb{E}\int_t^T [\overline{c}|\overline{Y}_r^{n+1}|^2 + |\overline{Z}_r^{n+1}|^2 + |\overline{U}_r^{n+1}|^2]e^{\beta r}dr \leq (\frac{1+\alpha}{2})^n\, \mathbb{E}\int_t^T [\overline{c}|\overline{Y}_r^1|^2 + |\overline{Z}_r^1|^2 + |\overline{U}_r^1|^2]e^{\beta r}dr.$$

Since $\frac{1+\alpha}{2} < 1$, we deduce that $(\Theta_t^n)_{n\geq 0}$ is a Cauchy sequence in $\mathcal{B}^2$. Hence $\Theta_t = \lim_{n\longrightarrow+\infty}\Theta_t^n$, satisfies (2.1) in $[0,T]$.

Let us prove uniqueness. Let (Y,Z,U) and $(\widetilde{Y},\widetilde{Z},\widetilde{U})$ be two solutions of eq. (2.1) and define for $0 \leq r \leq T$, $h \in \{f,g\}$, $\delta \in \{Y,Z,U\}$, $\Delta h(r) = h(r,Y_r,Z_r,U_r) - h(r,\widetilde{Y}_r,\widetilde{Z}_r,\widetilde{U}_r)$, $\overline{\delta} = \delta - \tilde{\delta}$.

It is readily seen that the triplet $(\overline{Y},\overline{Z},\overline{U})$ solves the BDSDEP

$$\overline{Y}_t = \int_t^T \Delta f(r)dr + \int_t^T \Delta g(r)dB_r - \int_t^T \overline{Z}_r dW_r - \int_t^T\int_E \overline{U}_r(e)\widetilde{\mu}(dr,de),\ 0 \leq t \leq T.$$

Applying **(ii)** in Lemma 2.3, we deduce that

$$\mathbb{E}|\overline{Y}_t|^2 + \mathbb{E}\int_t^T |\overline{Z}_r|^2 dr + \mathbb{E}\int_t^T\int_E |\overline{U}_r(e)|^2\lambda(de)dr \leq 2\mathbb{E}\int_t^T \langle\overline{Y}_r, \Delta f(r)\rangle dr + \mathbb{E}\int_t^T |\Delta g(r)|^2 dr.$$

Using assumptions (H1), we have

$$|\Delta g(r)|^2 \leq \varrho(r)|\overline{Y}_r|^2 + \alpha[|\overline{Z}_r|^2 + |\overline{U}_r|^2],$$
$$2\langle\overline{Y}_r, \Delta f(r)\rangle \leq \nu(r)|\overline{Y}_r|^2 + \frac{\varphi^2(r)}{1-\alpha}|\overline{Y}_r|^2 + \frac{1-\alpha}{2}[|\overline{Z}_r|^2 + |\overline{U}_r|^2].$$

Hence we deduce that

$$\mathbb{E}|\overline{Y}_t|^2 + \frac{1-\alpha}{2}\Big(\mathbb{E}\int_t^T |\overline{Z}_r|^2 dr + \mathbb{E}\int_t^T\int_E |\overline{U}_r(e)|^2\lambda(de)dr\Big) \leq \int_t^T \Big(\nu(r) + \varrho(r) + \frac{1}{1-\alpha}\varphi^2(r)\Big)\mathbb{E}|\overline{Y}_r|^2 dr$$

which implies in particular $\mathbb{E}|\overline{Y}_t|^2 \leq \int_t^T \Psi(r)\mathbb{E}|\overline{Y}_r|^2 dr$ with $\Psi(r) = \nu(r) + \varrho(r) + \frac{1}{1-\alpha}\varphi^2(r)$.

Applying Lemma 2.2, we deduce that $\overline{Y} = 0$. As a consequence we derive that $\overline{Z} = 0$ and $\overline{U} = 0$. Uniqueness follows.

Before proving solvency in the infinite time horizon, we need some technical results. If (Y^i,Z^i,U^i) and $(y_i,z_i,u_i) \in \mathcal{B}^2, i = 1,2$, we note for $v \in \{Y,Z,U,y,u,z\}$, $\widehat{v} = v^1 - v^2$. We have

Lemma 2.5. *Assume that (H1) is in force. For any $T \in [0,\infty]$, let $Y_T^i \in L^2(\Omega,\mathcal{F}_T,\mathbb{P})$, (Y^i,Z^i,U^i) and $(y_i,z_i,u_i) \in \mathcal{B}^2, i=1,2$, satisfy the following equations*

$$Y_t^i = Y_T^i + \int_t^T f(s,y_s^i,z_s^i,u_s^i)ds + \int_t^T g(s,y_s^i,z_s^i,u_s^i)dB_s - \int_t^T Z_s^i dW_s - \int_t^T \int_E U_s(e)\widetilde{\mu}(ds,de), \quad 0 \le t \le T \le \infty, \quad i=1,2. \tag{2.6}$$

Then there exists a constant $K(\alpha) > 0$ such that for any $\tau \in [0,T]$,

$$\left\|(\widehat{Y}\mathbb{1}_{[\tau,T]},\widehat{Z}\mathbb{1}_{[\tau,T]},\widehat{U}\mathbb{1}_{[\tau,T]})\right\|_{\mathcal{B}^2}^2 \le K(\alpha)\left[\mathbb{E}|\widehat{Y}_T|^2 + l^2(\tau,T)\left\|(\widehat{y}\mathbb{1}_{[\tau,T]},\widehat{z}\mathbb{1}_{[\tau,T]},\widehat{u}\mathbb{1}_{[\tau,T]})\right\|_{\mathcal{B}^2}^2\right] \tag{2.7}$$

where $l^2(\tau,T) = (\int_\tau^T \nu(s)ds)^2 + \int_\tau^T [\varphi^2(s)+\varrho(s)]ds, \quad 0 \le \tau < T \le +\infty.$

Proof. Let us consider the filtration $(\mathcal{G}_t)_{t\ge 0}$ given by $\mathcal{G}_t = \mathcal{F}_t^W \vee \mathcal{F}_T^B \vee \mathcal{F}_t^\mu,\ 0 \le t \le T \le \infty$. Without loss of generality, we assume that $\tau = 0,\ T = \infty$, otherwise we can replace g by $g\mathbb{1}_{[\tau,T]}$.
Since $(\widehat{Y},\widehat{Z},\widehat{U}) \in \mathcal{B}^2$, the process $\{\int_0^t \widehat{Z}_s dW_s + \int_0^t \int_E \widehat{U}_s(e)\widetilde{\mu}(ds,de)\}_{t\ge 0}$ is a martingale and from (2.6) we have

$$\widehat{Y}_t = \mathbb{E}^{\mathcal{G}_t}\left[\widehat{Y}_T + \int_t^\infty \mathbb{1}_{[t,T]}\widehat{f}_s ds + \int_t^\infty \mathbb{1}_{[t,T]}\widehat{g}_s dB_s\right], \quad 0 \le t \le T.$$

where for $h \in \{f,g\}$, $\widehat{h}_s = h(s,y_s^1,z_s^1,u_s^1) - h(s,y_s^2,z_s^2,u_s^2)$. It's readily seen that

$$\begin{aligned}\left\|\widehat{Y}\right\|_{S^2}^2 &= \mathbb{E}\left[\sup_{t\ge 0}\left|\mathbb{E}^{\mathcal{G}_t}\left(\widehat{Y}_T + \int_t^\infty \mathbb{1}_{[t,T]}\widehat{f}_s ds + \int_t^\infty \mathbb{1}_{[t,T]}\widehat{g}_s dB_s\right)\right|\right]^2 \\ &\le 2\mathbb{E}\left[\sup_{t\ge 0}\mathbb{E}^{\mathcal{G}_t}\left(|\widehat{Y}_T| + \int_t^\infty |\widehat{f}_s|ds\right)\right]^2 + 2\mathbb{E}\left[\sup_{t\ge 0}\mathbb{E}^{\mathcal{G}_t}\left(\left|\int_t^\infty \widehat{g}_s dB_s\right|\right)\right]^2. \end{aligned}\tag{2.8}$$

Furthermore by Doob and Burkhölder-Davis-Gundy inequalities there exists $c > 0$ s. t.

$$\begin{aligned}2\mathbb{E}\left[\sup_{t\ge 0}\mathbb{E}^{\mathcal{G}_t}\left(|\widehat{Y}_T| + \int_t^\infty |\widehat{f}_s|ds\right)\right]^2 &\le 8\mathbb{E}\left(|\widehat{Y}_T| + \int_0^\infty |\widehat{f}_s|ds\right)^2 \\ &\le 16\mathbb{E}\left[|\widehat{Y}_T|^2 + \left(\int_0^\infty |\widehat{f}_s|ds\right)^2\right]\end{aligned}$$

and

$$2\mathbb{E}\left[\sup_{t\ge 0}\mathbb{E}^{\mathcal{G}_t}\left(\left|\int_t^\infty \widehat{g}_s dB_s\right|\right)\right]^2 \le 2c\mathbb{E}\int_0^\infty |\widehat{g}_s|^2 ds.$$

Plugging these inequalities in (2.8), we derive that

$$\left\|\widehat{Y}\right\|_{S^2}^2 \le 16\mathbb{E}\left[|\widehat{Y}_T|^2 + \left(\int_0^\infty |\widehat{f}_s|ds\right)^2\right] + 2c\mathbb{E}\int_0^\infty |\widehat{g}_s|^2 ds. \tag{2.9}$$

On the other hand, from (2.6) it follows that

$$\left\|\widehat{Z}\right\|_{H^2}^2+\left\|\widehat{U}\right\|_{L^2}^2=\mathbb{E}\left\langle\int_0^{\cdot} Z_s dW_s+\int_0^{\cdot}\int_E U_s(e)\widetilde{\mu}(ds,de)\right\rangle_{\infty}$$
$$=\mathbb{E}\left(|\widehat{Y}_T|+\int_0^{\infty}|\widehat{f}_s|ds+\int_0^{\infty}\widehat{g}_s dB_s\right)^2-\left[\mathbb{E}\left(|\widehat{Y}_T|+\int_0^{\infty}|\widehat{f}_s|ds+\int_0^{\infty}\widehat{g}_s dB_s\right)\right]^2.$$

By standard estimates, the right hand side of the previous equality is less than

$$4\mathbb{E}\left[|\widehat{Y}_T|^2+\left(\int_0^{\infty}|\widehat{f}_s|ds\right)^2\right]+2\mathbb{E}\left(\sup_{t\geq 0}\left|\int_0^t\widehat{g}_s dB_s\right|\right)^2.$$

Using assumption (H1.1), we have

$$\mathbb{E}\left(\int_0^{\infty}|\widehat{f}_s|ds\right)^2\leq\mathbb{E}\left(\int_0^{\infty}(\nu(s)|\widehat{y}_s|+\varphi(s)(|\widehat{z}_s|+|\widehat{u}_s|))ds\right)^2$$
$$\leq 2\mathbb{E}\left(\int_0^{\infty}\nu(s)ds.\sup_{t\geq 0}|\widehat{y}_t|\right)^2+2\mathbb{E}\left(\int_0^{\infty}\varphi^2(s)ds.\int_0^{\infty}(|\widehat{z}_s|+|\widehat{u}_s|)^2ds\right),$$

which implies

$$\mathbb{E}\left(\int_0^{\infty}|\widehat{f}_s|ds\right)^2\leq 4\left[\left(\int_0^{\infty}\nu(s)ds\right)^2+\int_0^{\infty}\varphi^2(s)ds\right]\left\|(\widehat{y},\widehat{z},\widehat{u})\right\|_{\mathcal{B}^2}^2. \tag{2.10}$$

Moreover by Doob inequality and standard estimates we have (where $c'(\alpha)$ is a positive constant depending on α)

$$\mathbb{E}\left(\sup_{t\geq 0}\left|\int_0^t\widehat{g}_s dB_s\right|\right)^2\leq 4\mathbb{E}\int_0^{\infty}\left|\widehat{g}_s\right|^2ds\leq 8\mathbb{E}\int_0^{\infty}\varrho(s)|\widehat{y}_s|^2+8\mathbb{E}\int_0^{\infty}\alpha\left(|\widehat{z}_s|^2+|\widehat{u}_s|^2\right)ds$$
$$\leq c'(\alpha)\left\|(\widehat{y},\widehat{z},\widehat{u})\right\|_{\mathcal{B}^2}^2[\int_0^{\infty}\varrho(s)ds]. \tag{2.11}$$

Consequently, from inequalities (2.9)–(2.11), there exists $K(\alpha)>0$ such that

$$\left\|(\widehat{Y},\widehat{Z},\widehat{U}\right\|_{\mathcal{B}^2}^2\leq 20\mathbb{E}\left(|\widehat{Y}_T|^2+\left(\int_0^{\infty}|\widehat{f}_s|ds\right)^2\right)+2(2+c)\mathbb{E}\int_0^{\infty}|\widehat{g}_s|^2ds$$
$$\leq K(\alpha)\left[\mathbb{E}|\widehat{Y}_T|^2+l^2(0,\infty)\left\|(\widehat{y},\widehat{z},\widehat{u})\right\|_{\mathcal{B}^2}^2\right]. \tag{2.12}$$

This completes the proof.

We have

Theorem 2.6. *Assume that (H1) is in force and $\xi\in L^2(\Omega,\mathcal{F}_T,\mathbb{P})$. Then the BDSDEP* (2.1) *admits a unique solution* $(Y,Z,U)\in\mathcal{B}^2$.

Proof. We prove the theorem in two steps.

Step 1. We assume that $\left[(\int_0^{\infty}\nu(s)ds)^2+\int_0^{\infty}(\varphi^2(s)+\varrho(s))ds\right]^{1/2}<[K(\alpha)]^{-1/2}$.

For any triplet $\Theta_t=(Y_t,Z_t,U_t)_{0\le t\le T}\in\mathcal{B}^2$, we have

$$\mathbb{E}\Big(\xi+\int_0^\infty f(s,\Theta_s)ds+\int_0^\infty g(s,\Theta_s)dB_s\Big)^2\le\mathbb{E}\Big(|\xi|+\int_0^\infty |f(s,\Theta_s)|ds+|\int_0^\infty g(s,\Theta_s)dB_s|\Big)^2$$
$$\le 2\mathbb{E}\Big(|\xi|+\int_0^\infty |f(s,\Theta_s)|ds\Big)^2+2\mathbb{E}\Big(\sup_{t\ge0}|\int_0^t g(s,\Theta_s)dB_s|^2\Big).$$

By assumption (H1) and standard estimates, we deduce that

$$2\mathbb{E}\Big(|\xi|+\int_0^\infty |f(s,\Theta_s)|ds\Big)^2+2\mathbb{E}\Big(\sup_{t\ge0}|\int_0^t g(s,\Theta_s)dB_s|^2\Big)\le 8\mathbb{E}|\xi|^2+8\mathbb{E}\Big(\int_0^\infty |f(s,0)|ds\Big)^2$$
$$+8\mathbb{E}\Big(\int_0^\infty \nu(s)|Y_s|ds\Big)^2+8\mathbb{E}\Big(\int_0^\infty \varphi(s)(|Z_s|+|U_s|)ds\Big)^2$$
$$+8\mathbb{E}\int_0^\infty |g(s,0)|^2ds+8\mathbb{E}\Big(\int_0^\infty \varrho(s)|Y_s|^2ds\Big)$$
$$+8\mathbb{E}\int_0^\infty \alpha(|Z_s|^2+|U_s|^2)ds.$$

Using standard estimates once again, we have

$$\mathbb{E}\Big(\int_0^\infty \nu(s)|Y_s|ds\Big)^2+\mathbb{E}\int_0^\infty \varrho(s)|Y_s|^2ds\le\mathbb{E}\Big(\int_0^\infty \nu(s).(\sup_{t\ge0}|Y_t|)ds\Big)^2+\mathbb{E}\int_0^\infty \varrho(s).(\sup_{t\ge0}|Y_t|^2)ds$$
$$\le\Big[\Big(\int_0^\infty \nu(s)ds\Big)^2+\int_0^\infty \varrho(s)ds\Big]\|Y\|^2_{S^2}<\infty,$$

and Hölder inequality implies

$$\mathbb{E}\Big(\int_0^\infty \varphi(s)(|Z_s|+|U_s|)ds\Big)^2+\mathbb{E}\int_0^\infty \alpha(|Z_s|^2+|U_s|^2)ds\le$$
$$+\mathbb{E}\Big(\int_0^\infty \varphi^2(s)ds.\int_0^\infty (|Z_s|+|U_s|)^2ds\Big)+\alpha(\|Z\|^2_{H^2}+\|U\|^2_{L^2})$$
$$\le 2\int_0^\infty \varphi^2(s)ds.(\|Z\|^2_{H^2}+\|U\|^2_{L^2})+\alpha(\|Z\|^2_{H^2}+\|U\|^2_{L^2})<\infty.$$

Hence we deduce that

$$\mathbb{E}\Big(\xi+\int_0^\infty f(s,\Theta_s)ds+\int_0^\infty g(s,\Theta_s)dB_s\Big)^2<\infty.$$

It follows that the process $(M_t)_{0\le t\le\infty}$ given by $\{M_t=\mathbb{E}^{\mathcal{G}_t}\big(\xi+\int_0^\infty f(s,\Theta_s)ds+\int_0^\infty g(s,\Theta_s)dB_s\big)\}_{t\ge0}$ is a square integrable martingale. According to an extension martingale representation theorem, there exists $(z,u)\in H^2_{[0,\infty]}(\mathbb{R}^{k\times d})\times L^2_{[0,\infty]}(\widetilde{\mu},\mathbb{R}^k)$ such that

$$M_t=M_0+\int_0^t z_sdW_s+\int_0^t\int_E u_s(e)\widetilde{\mu}(ds,de)\quad 0\le t\le\infty. \tag{2.13}$$

Let

$$y_t = \mathbb{E}^{\mathcal{G}_t}\left(\xi + \int_t^\infty f(s,\Theta_s)ds + \int_t^\infty g(s,\Theta_s)dB_s\right), \quad 0 \le t \le \infty. \tag{2.14}$$

Obviously, $(y,z,u) \in \mathcal{B}^2$. So equations (2.13) and (2.14) have constructed a mapping ϕ from $\mathcal{B}^2$ to $\mathcal{B}^2$, given by

$$\phi : (Y,Z,U) \to (y,z,u).$$

Then if ϕ is a contractive mapping with respect to the norm $\|.\|^2_{\mathcal{B}}$, by the fixed point theorem, there exists a unique triple $(Y,Z,U) \in \mathcal{B}^2$ satisfying (2.13) and (2.14), that is,

$$\begin{cases} M_t = M_0 + \int_0^t Z_s dW_s + \int_0^t \int_E U_s(e)\widetilde{\mu}(ds,de) \\ Y_t = \mathbb{E}^{\mathcal{G}_t}\left(\xi + \int_t^\infty f(s,\Theta_s)ds + \int_t^\infty g(s,\Theta_s)dB_s\right), \quad 0 \le t \le \infty. \end{cases}$$

which is equivalent to BDSDEP (2.1).

We now prove that ϕ is a contractive mapping. Suppose $(Y^i,Z^i,U^i) \in \mathcal{B}^2$, let (y^i,z^i,u^i) such that

$$\phi(Y^i,Z^i,U^i) = (y^i,z^i,u^i), \quad i = 1,2.$$

By lemma 2.5 we have

$$\left\|(\widehat{y},\widehat{z},\widehat{u})\right\|_{\mathcal{B}^2} = \left\|\phi(Y^1,Z^1,U^1) - \phi(Y^2,Z^2,U^2)\right\|_{\mathcal{B}^2} \le (K(\alpha))^{1/2}\, l(0,\infty)\left\|(\widehat{Y},\widehat{Z},\widehat{U})\right\|_{\mathcal{B}^2}.$$

Note that $l(0,\infty) = \left[(\int_0^\infty \nu(s)ds)^2 + \int_0^\infty(\varphi^2(s)+\varrho(s))ds\right]^{1/2} < (K(\alpha))^{-1/2}$. Thus ϕ is a contractive mapping from $\mathcal{B}^2$ to $\mathcal{B}^2$.

Step 2. Since $\int_0^\infty \nu(s)ds < \infty$ and $\int_0^\infty[\varphi^2(s)+\varrho(s)]ds < \infty$, then there exists a sufficiently large constant $\Gamma > 0$ such that

$$\left[\left(\int_\Gamma^\infty \nu(s)ds\right)^2 + \int_\Gamma^\infty[\varphi^2(s)+\varrho(s)]ds\right]^{1/2} < (K(\alpha))^{-1/2}.$$

Let $\widetilde{f}(t,y,z,u) = \mathbb{1}_{[\Gamma,\infty]}(t)f(t,y,z,u)$ and $\widetilde{g}(t,y,z,u) = \mathbb{1}_{[\Gamma,\infty]}(t)g(t,y,z,u)$, then (H1) hold on $\widetilde{f}$ and $\widetilde{g}$ whose Lipschitzian functions coefficients are $\overline{\nu}(t) = \mathbb{1}_{[\Gamma,\infty]}\nu(t)$, $\overline{\varphi}(t) = \mathbb{1}_{[\Gamma,\infty]}\varphi(t)$ (for $\widetilde{f}$) and $\overline{\varrho}(t) = \mathbb{1}_{[\Gamma,\infty]}\varrho(t)$ (for $\widetilde{g}$). It is straightforward that

$$\left[\left(\int_\Gamma^\infty \overline{\nu}(s)ds\right)^2 + \int_\Gamma^\infty[\overline{\varphi}^2(s)+\overline{\varrho}(s)]ds\right]^{1/2} < (K(\alpha))^{-1/2}$$

and by Step 1, there exists a unique triple of processes $(\widehat{Y},\widehat{Z},\widehat{U}) \in \mathcal{B}^2$ such that

$$\widehat{Y}_t = \xi + \int_t^\infty \widetilde{f}(s,\widehat{Y}_s,\widehat{Z}_s,\widehat{U}_s)ds + \int_t^\infty \widetilde{g}(s,\widehat{Y}_s,\widehat{Z}_s,\widehat{U}_s)dB_s - \int_t^\infty \widehat{Z}_s dW_s - \int_t^\infty\int_E \widehat{U}_s(e)\widetilde{\mu}(ds,de).$$

For $(\widehat{Y}_t,\widehat{Z}_t,\widehat{U}_t)$ given as above, let us consider the following finite BDSDEP:

$$\begin{cases} \overline{Y}_t = \xi + \int_t^\Gamma f(s,\widehat{\Theta}_s)ds + \int_t^\Gamma g(s,\widehat{\Theta}_s)dB_s - \int_t^\Gamma \overline{Z}_s dW_s - \int_t^\Gamma\int_E \overline{U}_s(e)\widetilde{\mu}(ds,de) \quad 0 \le t \le \Gamma, \\ \overline{Y}_t \equiv 0, \quad \overline{Z}_t \equiv 0, \quad \overline{U}_t \equiv 0, \quad t > \Gamma \end{cases}$$

where we define $\widehat{\Theta}_s = (\overline{Y}_s + \widehat{Y}_s, \overline{Z}_s + \widehat{Z}_s, \overline{U}_s + \widehat{U}_s)$. Thus by virtue of Proposition 2.4, the above BDSDEP has a unique solution $(\overline{Y}_t, \overline{Z}_t, \overline{U}_t)$ in $[0,\Gamma]$, and satisfies $(\overline{Y}_t, \overline{Z}_t, \overline{U}_t) = (0,0,0)$ for every $t \geq \Gamma$.
Putting $Y = \overline{Y} + \widehat{Y}, \quad Z = \overline{Z} + \widehat{Z}, \quad U = \overline{U} + \widehat{U}$, it is easy to check that (Y_t, Z_t, U_t) is the unique solution of the BDSDEP (2.1).

We are now in position to study our main subject.

3 Main Result

In the following we assume that f and g satisfy assumptions **(H2)** where $0 \leq T \leq +\infty$:
(H2.1) : For all $(y,z,u) \in \mathbb{R}^k \times \mathbb{R}^{k\times d} \times \mathbb{R}^k$, $\quad g(\cdot,y,z,u) \in H^2_{[0,T]}(\mathbb{R}^{k\times \ell})$ and $f(\cdot,y,z,u)$ is progressively measurable process such that $\mathbb{E}(\int_0^\infty |f(s,y,z,u)|ds)^2 < \infty$.
(H2.2) : There exists $0 < \alpha < 1$ and positive non random continuous functions $\{\nu(t)\}$, $\{\beta(t)\}$ such that for $t \geq 0$, $(y,y') \in (\mathbb{R}^k)^2$, $(z,z') \in (\mathbb{R}^{k\times d})^2$, $(u,u') \in (\mathbb{R}^k)^2$,

$$|f(t,y,z,u) - f(t,y',z',u')|^2 \leq \nu(t)\rho(t,|y-y'|^2) + \beta(t)(|z-z'|^2 + |u-u'|^2)$$
$$|g(t,y,z,u) - g(t,y',z',u')|^2 \leq \rho(t,|y-y'|^2) + \alpha(|z-z'|^2 + |u-u'|^2)$$

where $\rho(\cdot,\cdot) : [0,T] \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies

- For fixed $t \in [0,\infty]$, $\rho(t,\cdot)$ is a continuous, concave and nondecreasing function s.t. $\rho(t,0) = 0$.
- For any $\widetilde{\Gamma} > 0$, the ordinary differential equation
$$v' = -\widetilde{\Gamma}\rho(t,v), \quad v(T) = 0, \tag{3.1}$$
has a unique solution $v(t) = 0, \quad 0 \leq t \leq T$.
- There exists $a(\cdot), b(\cdot) : [0,T] \to \mathbb{R}^+$ s.t. $\rho(t,v) \leq a(t) + b(t)v$ and $\displaystyle\int_0^T [a(t) + b(t)]dt < \infty$.

(H2.3): $\displaystyle\int_0^\infty \nu(s)ds < \infty$ and $\displaystyle\int_0^\infty \beta(s)ds < \infty$.

We have the following result

Proposition 3.1. *Let $0 < T \leq +\infty$, f, g satisfy* **(H2)** *and $(Y_t, Z_t, U_t)_{t\in[0,T]}$ be a solution to the BDSDEP* (2.1) *with parameters (ξ, T, f, g). Then there exists three positive constants C_1, C_2 and C_3 only depending on α such that*

$$\mathbb{E}\left(\sup_{t\leq s\leq T} |Y_s|^2\right) + \mathbb{E}\left(\int_t^T |Z_s|^2 ds\right) + \mathbb{E}\left(\int_t^T \int_E |U_s(e)|^2 \lambda(de)ds\right)$$
$$\leq C_1\left(\mathbb{E}|\xi|^2 + \mathbb{E}\left(\int_t^T |f(s,0)|ds\right)^2 + \mathbb{E}\int_t^T |g(s,0)|^2 ds\right)$$
$$+ C_2 \int_t^T \rho(s, \mathbb{E}|Y_s|^2)ds$$

hold true for each $t \in [0,T]$ satisfying $\int_t^T (\nu(s) + \beta(s))ds \leq C_3$.

Proof. Using Lemma 2.3, we deduce from eq. (2.1)

$$|Y_t|^2 + \int_t^T |Z_r|^2 dr + \int_t^T \int_E |U_r(e)|^2 \lambda(de)dr + \sum_{t<s\le T} (\Delta Y_s)^2 = |\xi|^2 + 2\int_t^T \langle Y_r, f(r,\Theta_r)\rangle dr$$
$$+ 2\int_t^T \langle Y_r, g(r,\Theta_r)dB_r\rangle - 2\int_t^T \langle Y_r, Z_r dW_r\rangle$$
$$- 2\int_t^T \int_E \langle Y_{r-}, U_r(e)\widetilde{\mu}(de,dr)\rangle + \int_t^T |g(r,\Theta_r)|^2 dr. \tag{3.2}$$

Using the fact that $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ and $2ab \le \varepsilon a^2 + b^2/\varepsilon$ for $\varepsilon > 0$, we deduce from assumption (H2)

$$2\langle Y_r, f(r,\Theta_r)\rangle \le 2|Y_r| \left[\sqrt{\nu(r)\rho(r,|Y_r|^2) + \beta(r)(|Z_r|^2 + |U_r|^2)} + |f(r,0)| \right]$$
$$\le \frac{1}{\varepsilon}[\nu(r)+\beta(r)]|Y_r|^2 + \varepsilon\rho(r,|Y_r|^2) + 2|Y_r||f(r,0)|$$
$$+ \varepsilon(|Z_r|^2 + |U_r|^2), \tag{3.3}$$

where ε will be chosen later. Furthermore thanks to standard estimates, we have

$$|g(r,\Theta_r)|^2 \le (2-\alpha)|g(r,\Theta_r) - g(r,0)|^2 + \frac{2-\alpha}{1-\alpha}|g(r,0)|^2$$
$$\le (2-\alpha)\rho(r,|Y_r|^2) + \alpha(2-\alpha)(|Z_r|^2 + |U_r|^2) + \frac{2-\alpha}{1-\alpha}|g(r,0)|^2. \tag{3.4}$$

Putting pieces together, we deduce from eq. (3.2)

$$\mathbb{E}\int_t^T |Z_r|^2 dr + \mathbb{E}\int_t^T \int_E |U_r(e)|^2 \lambda(de)dr \le \mathbb{E}|\xi|^2 + (\varepsilon + 2 - \alpha)]\mathbb{E}\int_t^T \rho(r,|Y_r|^2)dr$$
$$+ \frac{1}{\varepsilon}\int_t^T [\nu(r)+\beta(r)]\cdot \mathbb{E}(\sup_{t\le u\le T} |Y_u|^2)dr$$
$$+ (\varepsilon + \alpha(2-\alpha))\mathbb{E}\int_t^T |Z_r|^2 dr + (\varepsilon + \alpha(2-\alpha))\mathbb{E}\int_t^T |U_r|^2 dr$$
$$+ \frac{2-\alpha}{1-\alpha}\mathbb{E}\int_t^T |g(r,0)|^2 dr + 2\mathbb{E}\int_t^T |Y_r||f(r,0)|dr.$$

Choosing $\varepsilon = (\alpha-1)^2/2$, we obtain

$$\frac{(\alpha-1)^2}{2}\left[\mathbb{E}\int_t^T |Z_r|^2 dr + \mathbb{E}\int_t^T \int_E |U_r(e)|^2 \lambda(de)dr\right] \le X_t + 2\mathbb{E}\int_t^T |Y_r||f(r,0)|dr$$
$$+ \frac{2-\alpha}{1-\alpha}\mathbb{E}\int_t^T |g(r,0)|^2 dr, \tag{3.5}$$

where

$$X_t = \mathbb{E}|\xi|^2 + (\alpha-1)^2/2\int_t^T [\nu(r)+\beta(r)]\cdot\mathbb{E}(\sup_{t\le u\le T}|Y_u|^2)dr + [((\alpha-2)^2+1)/2]\mathbb{E}\int_t^T \rho(r,|Y_r|^2)dr. \tag{3.6}$$

Moreover from eq.(3.2), we have

$$\begin{aligned}\mathbb{E}(\sup_{t\le r\le T}|Y_r|^2) &\le \mathbb{E}|\xi|^2 + 2\mathbb{E}\sup_{t\le s\le T}\left(\int_s^T \langle Y_r, f(r,\Theta_r)\rangle dr\right)\\
&+2\mathbb{E}\sup_{t\le s\le T}\left|\int_s^T \langle Y_r, g(r,\Theta_r)dB_r\rangle\right| + 2\mathbb{E}\sup_{t\le s\le T}\left|\int_s^T \langle Y_r, Z_r dW_r\rangle\right|\\
&+2\mathbb{E}\sup_{t\le s\le T}\left|\int_s^T\int_E \langle Y_{r-}, U_r(e)\widetilde{\mu}(de,dr)\rangle\right| + \mathbb{E}\int_t^T |g(r,\Theta_r)|^2 dr. \qquad (3.7)\end{aligned}$$

By Burkhölder-Davis-Gundy inequality, there exists $c > 0$ which may vary from line to line such that

$$2\mathbb{E}\sup_{t\le s\le T}\left|\int_s^T \langle Y_r, g(r,\Theta_r)dB_r\rangle\right| \le \frac{1}{8}\mathbb{E}\left[\sup_{t\le r\le T}|Y_r|^2\right] + c\mathbb{E}\int_t^T |g(r,\Theta_r)|^2 dr, \qquad (3.8)$$

$$2\mathbb{E}\sup_{t\le s\le T}\left|\int_s^T \langle Y_r, Z_r dW_r\rangle\right| \le \frac{1}{8}\mathbb{E}\left[\sup_{t\le r\le T}|Y_r|^2\right] + c\mathbb{E}\int_t^T |Z_r|^2 dr. \qquad (3.9)$$

Similarly, for the discontinuous martingale, we have

$$\begin{aligned}2\mathbb{E}\sup_{t\le s\le T}\left|\int_s^T\int_E \langle Y_{r-}, U_r(e)\widetilde{\mu}(de,dr)\rangle\right| &\le \frac{1}{8}\mathbb{E}\left[\sup_{t\le r\le T}|Y_r|^2\right]\\
&+c\mathbb{E}\int_t^T\int_E |U_r(e)|^2\lambda(de)dr. \qquad (3.10)\end{aligned}$$

Using equations (3.5) and (3.8)-(3.10), we deduce from (3.7)

$$\begin{aligned}\frac{5}{8}\mathbb{E}\left[\sup_{t\le r\le T}|Y_r|^2\right] &\le \mathbb{E}|\xi|^2 + 2\mathbb{E}\sup_{t\le s\le T}\left(\int_s^T \langle Y_r, f(r,\Theta_r)\rangle dr\right) + c\mathbb{E}\int_t^T |g(r,\Theta_r)|^2 dr\\
&+\frac{2c}{(\alpha-1)^2}\left[X_t + 2\mathbb{E}\int_t^T |Y_r||f(r,0)|dr + \frac{2-\alpha}{1-\alpha}\mathbb{E}\int_t^T |g(r,0)|^2 dr\right]. \qquad (3.11)\end{aligned}$$

Furthermore exploiting eq. (3.3), (3.4) and gathering (3.5) and (3.11) we obtain

$$\begin{aligned}\frac{5}{8}\mathbb{E}\left[\sup_{t\le r\le T}|Y_r|^2\right] + \mathbb{E}\int_t^T |Z_r|^2 dr + \mathbb{E}\int_t^T\int_E |U_r(e)|^2\lambda(de)dr &\le K_1(\alpha)X_t\\
+K_2(\alpha)\mathbb{E}\int_t^T |Y_r||f(r,0)|dr + K_2(\alpha)\mathbb{E}\int_t^T |g(r,0)|^2 dr, \qquad (3.12)\end{aligned}$$

where $K_1(\alpha)$ and $K_2(\alpha)$ are two positive constants (which may change from line to line) depending only on α.

From the following inequality

$$K_2(\alpha)\mathbb{E}\int_t^T |Y_r||f(r,0)|dr \le \frac{1}{8}\mathbb{E}\left[\sup_{t\le r\le T}|Y_r|^2\right] + K_2(\alpha)\mathbb{E}\left(\int_t^T |f(r,0)|dr\right)^2, \qquad (3.13)$$

we deduce that

$$\frac{1}{2}\mathbb{E}(\sup_{t\le r\le T}|Y_r|^2)+\mathbb{E}\int_t^T|Z_r|^2dr+\mathbb{E}\int_t^T\int_E|U_r(e)|^2\lambda(de)dr\le K_1(\alpha)X_t$$
$$+K_2(\alpha)\left(\mathbb{E}\int_t^T|f(r,0)|dr\right)^2+K_2(\alpha)\mathbb{E}\int_t^T|g(r,0)|^2dr.$$

Let us define

$$C_1=4(K_1(\alpha)+K_2(\alpha)),\quad C_2=4K_1(\alpha)\quad\text{and}\quad C_3=\frac{(\alpha-1)^{-2}}{2K_1(\alpha)}.$$

Then if $t\in[0,T]$, $\int_t^T(\nu(s)+\beta(s))ds\le C_3$, we deduce that

$$\mathbb{E}(\sup_{t\le r\le T}|Y_r|^2)+\mathbb{E}\int_t^T|Z_r|^2dr+\mathbb{E}\int_t^T\int_E|U_r(e)|^2\lambda(de)dr$$
$$\le C_1\left[\mathbb{E}|\xi|^2+\mathbb{E}\left(\int_t^T|f(r,0)|dr\right)^2+\mathbb{E}\int_t^T|g(r,0)|^2dr\right]$$
$$+C_2\mathbb{E}\int_t^T\rho(r,|Y_r|^2)dr.$$

The proof is completed by Fubini's theorem and Jensen's inequality in the last integral of the right hand side.

Our strategy in the proof of existence of solutions of eq. (2.1) is to use the Picard approximate sequence.
Let us consider now the sequence $(Y_t^n,Z_t^n,U_t^n)_{n\ge0}$ given by

$$\begin{cases}Y_t^0=0;\\ Y_t^n=\xi+\int_t^Tf(s,Y_s^{n-1},Z_s^n,U_s^n)ds+\int_t^Tg(s,Y_s^{n-1},Z_s^n,U_s^n)dB_s-\int_t^TZ_s^ndW_s\\ \quad-\int_t^T\int_EU_s^n(e)\widetilde{\mu}(ds,de),\ n\ge1.\end{cases}\tag{3.14}$$

By assumptions (H2), the generators $f(s,Y_s^{n-1},z,u)$ and $g(s,Y_s^{n-1},z,u)$ of the BDSDEP (3.14) satisfy also (H1) with

$$\nu(t)=0,\quad\varphi(t)=\sqrt{\beta(t)},\quad\text{and}\quad\varrho(t)=0.$$

Hence it follows from Theorem 2.6 that this sequence is well defined whether $T<\infty$ or $T=+\infty$. Moreover since $\int_0^T(\nu(s)+\beta(s)+b(s))ds<\infty$, putting (where C_2 and C_3 are taken from Proposition 3.1)

$$\delta=\min(\frac{1}{2\|b\|_\infty C_2},\frac{C_3}{2\|\nu+\beta\|_\infty}),\quad N=[T/\delta]+1,$$

the uniform subdivision of $[0,T]$ $(T_j)_{0\le j\le N}$ given by $T_0 = 0,\quad T_j = T-(N-j)\delta,\quad j = 1,\dots,N$, satisfies

$$\int_{T_j}^{T_{j+1}} (\nu(s)+\beta(s))ds \le \frac{C_3}{2} \quad \text{and} \quad \int_{T_j}^{T_{j+1}} b(s)ds \le \frac{1}{2C_2}. \tag{3.15}$$

We intend to prove that $(Y_t^n, Z_t^n, U_t^n)_{n\ge 0}$ is a Cauchy sequence. To this end we need two lemmas.

Lemma 3.2. *Assume that $\xi \in L^2(\Omega,\mathcal{F}_T,\mathbb{P})$ and (H2) is in force. Let $T_{N-1}\le t\le T$ satisfying $\int_t^T (\nu(s)+\beta(s))ds \le C_{(\alpha)}^{(2)}$. Then there exists two positive constants $C_{(\alpha)}^{(1)}$ and $C_{(\alpha)}^{(2)}$ depending only on α such that for any n, $m\ge 1$,*

$$\mathbb{E}(\sup_{t\le r\le T} |Y_r^{n+m}-Y_r^n|^2) \le C_{(\alpha)}^{(1)} \int_t^T \rho\Big(s, \mathbb{E}|Y_s^{n+m-1}-Y_s^{n-1}|^2\Big)ds.$$

Proof. Let us define

$$\overline{Y}_t^{n,m} = Y_t^{n+m}-Y_t^n, \quad \overline{Z}_t^{n,m} = Z_t^{n+m}-Z_t^n \quad \text{and} \quad \overline{U}_t^{n,m} = U_t^{n+m}-U_t^n, \quad 0\le t\le T.$$

By Lemma 2.3, we have for $n,m\ge 1$ and $0\le t\le T$,

$$\begin{aligned}|\overline{Y}_t^{n,m}|^2 + \int_t^T |\overline{Z}_s^{n,m}|^2 ds + \int_t^T\int_E |\overline{U}_s^{n,m}(e)|^2\lambda(de)ds + \sum_{t<s\le T}(\Delta\overline{Y}_s^{n,m})^2 &= 2\int_t^T \langle \overline{Y}_s^{n,m}, \Delta f^{(n,m)}(s)\rangle ds \\ &\quad - 2\int_t^T \langle \overline{Y}_s^{n,m}, \overline{Z}_s^{n,m} dW_s\rangle - 2\int_t^T\int_E \langle \overline{Y}_{s-}^{n,m}, \overline{U}_s^{n,m}\widetilde{\mu}(ds,de)\rangle \\ &\quad + 2\int_t^T \langle \overline{Y}_s^{n,m}, \Delta g^{(n,m)}(s)dB_s\rangle + \int_t^T |\Delta g^{(n,m)}(s)|^2 ds,\end{aligned} \tag{3.16}$$

where for $h\in\{f,g\}$, $\Delta h^{(n,m)}(s) = h(s,Y_s^{n+m-1},Z_s^{n+m},U_s^{n+m}) - h(s,Y_s^{n-1},Z_s^n,U_s^n)$, $0\le s\le T$. Using standard estimates and assumption (H2.2), we have for $\varepsilon>0$,

$$2\langle \overline{Y}_s^{n,m}, \Delta f^{(n,m)}(s)\rangle \le \frac{1}{\varepsilon}(\nu(s)+\beta(s))|\overline{Y}_s^{n,m}|^2 + \varepsilon\rho(s,|\overline{Y}_s^{n-1,m}|^2) + \varepsilon(|\overline{Z}_s^{n,m}|^2 + |\overline{U}_s^{n,m}|^2)$$

and

$$|\Delta g^{(n,m)}(s)|^2 \le \frac{2-\alpha}{1-\alpha}\rho(s,|\overline{Y}_s^{n-1,m}|^2) + \alpha(2-\alpha)(|\overline{Z}_s^{n,m}|^2 + |\overline{U}_s^{n,m}|^2)$$

Plugging these two inequalities in (3.16), there exists a constant $K_{(\alpha)}>0$ depending only on α (where we choose $\varepsilon = (\alpha-1)^2/2$) such that

$$\begin{aligned}\frac{(\alpha-1)^2}{2}\mathbb{E}\left[\int_t^T |\overline{Z}_s^{n,m}|^2 dr + \int_t^T\int_E |\overline{U}_s^{n,m}(e)|^2\lambda(de)ds\right] &\le K_{(\alpha)}\mathbb{E}\int_t^T (\nu(s)+\beta(s))|\overline{Y}_s^{n,m}|^2 ds \\ &\quad + K_{(\alpha)}\mathbb{E}\int_t^T \rho(s,|\overline{Y}_s^{n-1,m}|^2)ds.\end{aligned} \tag{3.17}$$

Moreover using again eq. (3.16), we deduce that

$$\begin{aligned}\mathbb{E}(\sup_{t\le s\le T}|\overline{Y}_s^{n,m}|^2) &\le 2\mathbb{E}\sup_{t\le r\le T}\left(\int_r^T\langle\overline{Y}_s^{n,m},\Delta f^{(n,m)}(s)\rangle ds\right)\\ &+2\mathbb{E}\sup_{t\le r\le T}\left|\int_r^T\langle\overline{Y}_s^{n,m},\overline{Z}_s^{n,m}dW_s\rangle\right|+2\mathbb{E}\sup_{t\le r\le T}\left|\int_r^T\int_E\langle\overline{Y}_{s-}^{n,m},\overline{U}_s^{n,m}(e)\widetilde{\mu}(ds,de)\rangle\right|\\ &+2\mathbb{E}\sup_{t\le r\le T}\left|\int_r^T\langle\overline{Y}_s^{n,m},\Delta g^{(n,m)}(s)dB_s\rangle\right|+\mathbb{E}\int_t^T|\Delta g^{(n,m)}(s)|^2ds. \qquad (3.18)\end{aligned}$$

So using the same computations as in the proof of Proposition 3.1 and Burkhölder-Davis-Gundy inequality, we deduce that

$$\frac{1}{2}\mathbb{E}(\sup_{t\le s\le T}|\overline{Y}_s^{n,m}|^2\le K'_{(\alpha)}\int_t^T(\nu(s)+\beta(s))ds.\mathbb{E}(\sup_{t\le r\le T}|\overline{Y}_r^{n,m}|^2)+K'_{(\alpha)}\mathbb{E}\int_t^T\rho(s,|\overline{Y}_s^{n-1,m}|^2)ds, \qquad (3.19)$$

where $K'_{(\alpha)}>0$ depend only on α. Hence putting $C^{(1)}_{(\alpha)}=4K'_{(\alpha)}$ and $C^{(2)}_{(\alpha)}=(4K'_{(\alpha)})^{-1}$, we get the desired result thanks to Fubini's theorem, Jensen's inequality and assumptions on the function $\rho(t,.)$.

Lemma 3.3. *Assume that $\xi\in L^2(\Omega,\mathcal{F}_T,\mathbb{P})$ and (H2) is in force. Then there exists a constant $M\ge 0$ such that for each $n\ge 1$ and $T_{N-1}\le t\le T$,*

$$\mathbb{E}\left(\sup_{t\le r\le T}|Y_r^n|^2\right)\le M.$$

Proof. Applying Proposition 3.1, we deduce for any $T_{N-1}\le t\le T$,

$$\begin{aligned}\mathbb{E}\left(\sup_{t\le s\le T}|Y_s^n|^2\right)+\mathbb{E}\left[\int_t^T|Z_s^n|^2ds\right]+\mathbb{E}\left[\int_t^T\int_E|U_s^n(e)|^2\lambda(de)ds\right]\\ \le C_1\left(\mathbb{E}|\xi|^2+\mathbb{E}\left(\int_t^T|f(s,0)|ds\right)^2+\mathbb{E}\int_t^T|g(s,0)|^2ds\right)\\ +C_2\int_t^T\rho(s,\mathbb{E}|Y_s^{n-1}|^2)ds.\end{aligned}$$

Hence we deduce that

$$\mathbb{E}\left(\sup_{t\le s\le T}|Y_s^n|^2\right)\le\mu_t+C_2\int_t^T\rho(s,\mathbb{E}|Y_s^{n-1}|^2)ds, \qquad (3.20)$$

where $\mu_t=C_1\left(\mathbb{E}|\xi|^2+\mathbb{E}\left(\int_t^T|f(s,0)|ds\right)^2+\mathbb{E}\int_t^T|g(s,0)|^2ds\right)$. Now let

$$M=2\mu_0+2C_2\int_0^Ta(s)ds. \qquad (3.21)$$

It follows from assumptions (H2) and (3.15) that for each $T_{N-1}\le t\le T$,

$$\mu_0+C_2\int_t^T\rho(s,M)ds\le\mu_0+C_2\int_t^Ta(s)ds+MC_2\int_t^Tb(s)ds\le M/2+M/2\le M. \quad (3.22)$$

So using (3.20), we deduce that for each $T_{N-1} \le t \le T$,

$$\mathbb{E}(\sup_{t\le s\le T}|Y_s^1|^2) \le \mu_0 \le M,$$
$$\mathbb{E}(\sup_{t\le s\le T}|Y_s^2|^2) \le \mu_0 + C_2\int_t^T \rho(s,\mathbb{E}|Y_s^1|^2)ds \le \mu_0 + C_2\int_t^T \rho(s,M)ds \le M,$$
$$\mathbb{E}(\sup_{t\le s\le T}|Y_s^3|^2) \le \mu_0 + C_2\int_t^T \rho(s,\mathbb{E}|Y_s^2|^2)ds \le \mu_0 + C_2\int_t^T \rho(s,M)ds \le M.$$

Thus the result follows by induction.

We claim :

Theorem 3.4. *Let $0 \le T \le +\infty$ and assume that (H2) is in force. Then for each $\xi \in L^2(\Omega,\mathcal{F}_T,\mathbb{P})$ the BDSDEP* (2.1) *has a unique solution.*

Proof. Using the constant M given by (3.21), we can consider the sequence $(\psi_n)_{n\ge1}$ given by

$$\psi_0(t) = C_2\int_t^T \rho(s,M)ds, \qquad \psi_{n+1}(t) = C_2\int_t^T \rho(s,\psi_n(s))ds, \quad n\ge 0, \quad T_{N-1}\le t\le T.$$

For each $T_{N-1} \le t \le T$, it follows from (3.22) that $\psi_0(t) \le M$. This implies that the sequence $\{\psi_n(t)\}_{n\ge1}$ satisfies

$$0 \le \psi_{n+1}(t) \le \psi_n(t) \le \cdots \le \psi_0(t) \le M.$$

Thus the limit $\psi(t)$ exists and applying Lebesgue's convergence dominated theorem, we deduce that $\psi(t) = C_2\int_t^T \rho(s,\psi(s))ds$, $\quad T_{N-1}\le t\le T$, whether $T<\infty$ or $T=+\infty$. Then by assumption (H2.2), one gets that $\psi(t)=0$, $\quad T_{N-1}\le t\le T$.

Using the same computations as in [2, Theorem 1], we prove that the sequence $(Y_t^n)_{n\ge1}$ is a Cauchy sequence in $S^2_{[T_{N-1},T]}(\mathbb{R}^k)$, $(Z_t^n)_{n\ge1}$ is a Cauchy sequence in $H^2_{[T_{N-1},T]}(\mathbb{R}^{k\times d})$ and $(U_t^n)_{n\ge1}$ is a Cauchy sequence in $L^2_{[T_{N-1},T]}(\widetilde{\mu},\mathbb{R}^k)$. Letting $n\to\infty$ in eq. (3.14), we obtain

$$Y_t = \xi + \int_t^T f(s,\Theta_s)ds + \int_t^T g(s,\Theta_s)dB_s - \int_t^T Z_s dW_s - \int_t^T\int_E U_s(e)\widetilde{\mu}(ds,de), \quad T_{N-1}\le t\le T.$$

Hence $\Theta_t = (Y_t,Z_t,U_t)$ satisfies (2.1) on $[T_{N-1},T]$. Moreover by virtue of (H2), we have $(Z_t,U_t) \in H^2_{[T_{N-1},T]}(\mathbb{R}^{k\times d})\times L^2_{[T_{N-1},T]}(\widetilde{\mu},\mathbb{R}^k)$. As a consequence, we deduce by Doob's inequality,

$$\mathbb{E}\left(\int_t^T f(s,\Theta_s)ds\right)^2 + \mathbb{E}\int_t^T |g(s,\Theta_s)|^2 ds + \mathbb{E}\left(\sup_{T_{N-1}\le t\le T}\left|\int_{T_{N-1}}^t Z_s dW_s\right|^2\right)$$
$$+\mathbb{E}\left(\sup_{T_{N-1}\le t\le T}\left|\int_{T_{N-1}}^t\int_E U_s(e)\widetilde{\mu}(de,ds)\right|^2\right) < \infty.$$

This implies essentially that $\mathbb{E}(\sup_{T_{N-1}\le t\le T}|Y_t|^2)<\infty$. Thus the triplet (Y_t,Z_t,U_t) solves eq. (2.1) on $[T_{N-1},T]$. By iteration we prove existence of solution on $[0,T]$.

Let us prove uniqueness. Let (Y_t, Z_t, U_t) and $(\widetilde{Y}_t, \widetilde{Z}_t, \widetilde{U}_t)$ two solutions of eq. (2.1). Applying Lemma 2.3 and using the same computations as in the proof of Lemma 3.2, we deduce that (where $\overline{D}_s = D_s - \widetilde{D}_s,\ \ D \in \{Y, Z, U\}$)

$$\frac{(\alpha-1)^2}{2}\mathbb{E}\left[\int_t^T |\overline{Z}_s|^2 ds + \int_t^T \int_E |\overline{U}_s(e)|^2 \lambda(de) ds\right] \leq K_{(\alpha)}\mathbb{E}\int_t^T (\nu(s)+\beta(s))|\overline{Y}_s|^2 ds + K_{(\alpha)}\mathbb{E}\int_t^T \rho(s, |\overline{Y}_s|^2) ds \tag{3.23}$$

and

$$\mathbb{E}|\overline{Y}_t|^2 \leq \mathbb{E}(\sup_{t\leq r\leq T} |\overline{Y}_r|^2) \leq C^{(1)}_{(\alpha)} \int_t^T \rho\Big(s, \mathbb{E}|\overline{Y}_s|^2\Big) ds, \quad T_{N-1} \leq t \leq T.$$

This implies from the comparison theorem of ordinary differential equation, $\mathbb{E}|\overline{Y}_t|^2 \leq r(t)$ where $r(t)$ is the maximum of solution of eq. (3.1) (with $\widetilde{\Gamma} = C^{(1)}_{(\alpha)}$). As a consequence, we have $\overline{Y}_t = 0$ i.e. $Y_t = \widetilde{Y}_t, \quad T_{N-1} \leq t \leq T$. From (3.23), we deduce $Z_t = \widetilde{Z}_t$ and $U_t = \widetilde{U}_t, \quad T_{N-1} \leq t \leq T$. Using the same scheme, we prove uniqueness on $[T_j, T_{j+1}],\ j = 0, \ldots, N-2$. This completes the proof.

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