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CHOW MOTIVES OF QUASI-ELLIPTIC SURFACES

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Abstract

We prove that the transcendental motive of any quasi-elliptic surface is trivial. To prove this, we focus on the uniruledness of quasi-elliptic surfaces.

1. Introduction

Let k be an algebraically closed field of characteristic $p \geq 0$. Let X be a smooth projective surface over k and $h(X)$ its Chow motive with $\mathbb{Q}$-coefficients. Kahn-Murre-Pedrini [4] proved that X admits a refined Chow-Künneth decompositon $h(X) \cong \oplus_{i=0}^{4} h_i(X)$ with

$$h_2(X) \cong h_2^{alg}(X) \oplus t_2(X).$$

The motive $t_2(X)$ is called the transcendental motive of X. It is a birational invariant and, for a prime number $l \neq p$,

$$H^*_{ét}(t_2(X)) = H^2_{ét}(X, \mathbb{Q}_l)_{tr} \quad \text{and} \quad \mathrm{CH}^*(t_2(X)) = T(X)_{\mathbb{Q}},$$

where $H^2_{ét}(X, \mathbb{Q}_l)_{tr}$ is the transcendental lattice and $T(X)_{\mathbb{Q}}$ is the Albanese kernel. The motives $h_i(X)$ (for $i \neq 2$) and $h_2^{alg}(X)$ are well understood, but the transcendental motive $t_2(X)$ is still mysterious. For example, there is the following conjecture:

Conjecture 1.1 (Conservativity). *If $H^*_{ét}(t_2(X)) = 0$, then $t_2(X) = 0$.*

When $k = \mathbb{C}$, Conjecture 1.1 is equivalent to the famous conjecture of Bloch [2]. It is known for surfaces over $\mathbb{C}$ of Kodaira dimension $\kappa < 2$, but is wide open for surfaces of $\kappa = 2$ (e.g. [8] for some examples of surfaces where Conjecture 1.1 is proved).

In this paper, we prove Conjecture 1.1 for quasi-elliptic surfaces, which can exist in characteristic 2 and 3, only. More precisely, the purpose of this paper is to prove the following:

Theorem 1.2. *Let $f : X \to C$ be a quasi-elliptic surface. Then*

$$t_2(X) = 0.$$

1.1. Organization. This paper is organized as follows. In Section 2, we recall the definitions and properties of uniruled surfaces, Shioda-supersingular surfaces, and quasi-elliptic surfaces. In this paper, we focus on the uniruledness of quasi-elliptic surfaces (Theorem 2.9). In Section 3, we prove two lemmas about homomorphisms between transcendental motives (Lemma 3.1 and Lemma 3.5). In Section 4, we prove Theorem 1.2. More precisely, we prove that the transcendental motive of any uniruled surface is trivial (Theorem 4.1).

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1.2. Notation. Throughout this paper, let k be an algebraically closed field of characteristic $p \geq 0$ and let $\mathcal{V}(k)$ be the category of smooth projective varieties over k.

2. The uniruledness of quasi-elliptic surfaces

2.1. Uniruled surfaces. In this subsection, we recall the notions of uniruledness and birationally ruledness.

Let $X \in \mathcal{V}(k)$ be a surface.

(i) We say X is *uniruled* if there exist a curve C and a dominant rational map

$$\phi\colon \mathbb{P}^1 \times C \dashrightarrow X.$$

We say X is *separably uniruled* (resp. *purely inseparable uniruled*) if there exists a such a rational map ϕ inducing a *separable* (resp. *purely inseparable*) extension of function fields.

(ii) We say X is *birationally ruled* if there exist a curve C and a *birational map*

$$\phi\colon \mathbb{P}^1 \times C \overset{\cong}{\dashrightarrow} X.$$

The following fact is well-known.

Proposition 2.1. *Let $X \in \mathcal{V}(k)$ be a surface. Then the following are equivalent:*

(i) *X is birationally ruled*;

(ii) *X is separably uniruled*;

(iii) *X has negative Kodaira dimension.*

Proof. (i) $\Rightarrow$ (ii): This is clear. (ii) $\Rightarrow$ (iii): If X is separably uniruled, then there exist a curve C and a dominant rational map $\phi : \mathbb{P}^1 \times C \dashrightarrow X$ such that the extension of function fields $k(\mathbb{P}^1 \times C)/k(X)$ is separable. Then $P_n(\mathbb{P}^1 \times C) = P_n(\mathbb{P}^1) \cdot P_n(C) = 0 \cdot P_n(C) = 0$ for every $n \geq 1$. Since $k(\mathbb{P}^1 \times C)/k(X)$ is separable, $P_n(\mathbb{P}^1 \times C) \geq P_n(X)$. Thus $P_n(X) = 0$ for every $n \geq 1$. Namely, X has negative Kodaira dimension.

(iii) $\Rightarrow$ (i): For example, see [1, Theorem 13.2, p.195] □

To derive the uniruledness of quasi-elliptic surfaces, we need the following:

Theorem 2.2 (Noether-Tsen)**.** *Let $\phi : Y \dashrightarrow B$ be a dominant rational map from a surface Y to a curve B satisfying the following conditions* :

(i) *$k(B)$ is algebraically closed in $k(Y)$*;

(ii) *The generic fiber of ϕ has arithmetic genus* 0.

Then Y is birationally-isomorphic to $\mathbb{P}^1 \times B$.

Proof. For example, see [1, Theorem 11.3, p.166]. □

2.2. Shioda-supersingular surfaces. In this subsection, we recall the notions of Lefschetz numbers and Shioda-supersingularity.

Let $X \in \mathcal{V}(k)$ be a surface. Let $\mathrm{Br}(X) := H^2_{\acute{e}t}(X, \mathbb{G}_m)$ denote the cohomological Brauer group of X. For a prime number $l \neq p$, we consider the l-adic Tate module

$$T_l(\mathrm{Br}(X)) := \varprojlim_n \mathrm{Ker}([l^n] : \mathrm{Br}(X) \to \mathrm{Br}(X)).$$

We call $\lambda(X) := \mathrm{rank}_{\mathbb{Z}_l}(T_l(\mathrm{Br}(X)))$ the *Lefschetz number* of X. It is a birational invariant. The Kummer sequence $0 \to \mu_{l^n} \to \mathbb{G}_m \overset{\times l^n}{\to} \mathbb{G}_m \to 0$ gives an exact sequence

$$0 \to \mathrm{NS}(X) \otimes \mathbb{Z}_l \to H^2_{\acute{e}t}(X, \mathbb{Z}_l(1)) \to T_l(\mathrm{Br}(X)) \to 0.$$

Thus, we have

$$\lambda(X) = b_2(X) - \rho(X),$$

where $b_2(X)$ and $\rho(X)$ denote the second Betti number and the Picard number of X, respectively. Since b_2 is the independent of l, so is λ.

DEFINITION 2.3. A surface X is *Shioda-supersingular* if $\lambda(X) = 0$ i.e., $b_2(X) = \rho(X)$.

In particular, Conjecture 1.1 becomes the following statement:

Conjecture 2.4 (= Conjecture 1.1)**.** *If X is a Shioda-supersingular surface, then*

$$t_2(X) = 0.$$

In this paper, we prove Conjecture 2.4 for quasi-elliptic surfaces (Theorem 1.2). Now, we recall the following property of Lefschetz numbers:

Lemma 2.5 ([9, Lemma, p.234])**.** *Let $\phi : Y \dashrightarrow X$ be a dominant rational map of surfaces over k. Then*

$$\lambda(Y) \geq \lambda(X).$$

For the reader's convenience, we include a proof of the following fact due to Shioda:

Corollary 2.6 ([9, Corollary 2, p.235])**.** *Any uniruled surface is Shioda-supersingular.*

Proof. Let X be a uniruled surface. By definition, there is a dominant rational map $\phi : \mathbb{P}^1 \times C \dashrightarrow X$ for some curve C. Now, one has

$$b_2(\mathbb{P}^1 \times C) = \rho(\mathbb{P}^1 \times C) = 2.$$

(Indeed, we have $H^2_{\acute{e}t}(\mathbb{P}^1 \times C, \mathbb{Q}_l) \cong \oplus_{i+j=2} H^i_{\acute{e}t}(\mathbb{P}^1, \mathbb{Q}_l) \otimes H^j_{\acute{e}t}(C, \mathbb{Q}_l)$ by the Künneth decomposition. Then, we have $H^2_{\acute{e}t}(\mathbb{P}^1, \mathbb{Q}_l) = H^2_{\acute{e}t}(C, \mathbb{Q}_l) = \mathbb{Q}_l$ by Poincare duality. Since both $\mathbb{P}^1$ and C are irreducible, we have $H^0_{\acute{e}t}(\mathbb{P}^1, \mathbb{Q}_l) = H^0_{\acute{e}t}(C, \mathbb{Q}_l) = \mathbb{Q}_l$. Thus, we get $b_2(\mathbb{P}^1 \times C) = 2$ by $H^1_{\acute{e}t}(\mathbb{P}^1, \mathbb{Q}_l) = 0$. On the other hand, we have $\mathrm{NS}(\mathbb{P}^1 \times C) = \mathrm{NS}(\mathbb{P}^1) \oplus \mathrm{NS}(C) \oplus \mathrm{Hom}(\mathrm{Jac}(\mathbb{P}^1), \mathrm{Jac}(C))$. Since both $\mathbb{P}^1$ and C have dimension 1, we have $\mathrm{NS}(\mathbb{P}^1) = \mathrm{NS}(C) = \mathbb{Q}_l$. Thus, we get $\rho(\mathbb{P}^1 \times C) = 2$ by $\mathrm{Jac}(\mathbb{P}^1) = 0$. Therefore, we get $b_2(\mathbb{P}^1 \times C) = \rho(\mathbb{P}^1 \times C) = 2$.)

Namely, $\lambda(\mathbb{P}^1 \times C) = 0$. Since ϕ is dominant, $\lambda(X) = 0$ by Lemma 2.5. Thus, X is Shioda-supersingular. □

2.3. Quasi-elliptic surfaces. In this subsection, we recall the uniruledness of quasi-elliptic surfaces (Theorem 2.9). Let us begin with the following definition:

DEFINITION 2.7. A *genus* 1 *fibration* from a surface is a proper morphism

$$f : X \to C$$

from a smooth, relatively-minimal surface X onto a normal curve C such that the generic fiber X_η is a normal, geometrically-integral, curve with arithmetic genus 1.

The fibration f is called *quasi-elliptic* (resp. *elliptic*) if the geometric generic fiber $X_{\bar\eta}$ is *not normal* (resp. *normal*).

Remark 2.8. In fact, if f is quasi-elliptic, then $X_{\bar\eta}$ is a singular rational curve with one cusp. Quasi-elliptic surfaces can occur only in characteristic 2 and 3 (e.g. [3]).

The following result plays a key role in the proof of Theorem 1.2.

Theorem 2.9. *Let $f : X \to C$ be a quasi-elliptic surface over an algebraically closed field k of characteristic $p > 0$. Then, there are a birationally ruled surface Y and a proper map $\pi : Y \to X$ of degree p. More precisely, any quasi-elliptic surface is (purely inseparable) uniruled.*

Proof. The ideas of the proof are based on [3, Section 1] or [5, Theorem 9.4, p.266]. Let $F : C^{(1/p)} \to C$ be the Frobenius morphism of degree p. Let K and L be the functions fields of C and $C^{(1/p)}$, respectively. Let X_η be the generic fiber of f. Since f is quasi-elliptic, $X_\eta \otimes_K L$ is not normal. Let Y_ξ be the normalization of $X_\eta \otimes_K L$. Then Y_ξ has arithmetic genus 0. Let $\phi : Y \to C^{(1/p)}$ be a regular, relatively minimal model of Y_ξ. Then, there are the following commutative diagrams

$$\begin{array}{ccccc} Y & \longrightarrow & X \times_C C^{(1/p)} & \xrightarrow{\ \pi\ } & X \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle f} \\ C^{(1/p)} & = & C^{(1/p)} & \xrightarrow[F]{} & C \end{array}$$

Now, the generic fiber of ϕ is Y_ξ, and $L = k(C^{(1/p)})$ is algebraically closed in $k(Y)$ (since $\phi_*\mathcal{O}_Y \cong \mathcal{O}_{C^{(1/p)}}$). By Noether-Tsen's theorem (Theorem 2.2), Y is birationally isomorphic to $\mathbb{P}^1 \times C^{(1/p)}$. Hence, we get a dominant rational map

$$\mathbb{P}^1 \times C^{(1/p)} \dashrightarrow X.$$

Since L/K is purely inseparable, X is purely inseparable uniruled. □

Remark 2.10. Any genus 1 fibration has Kodaira dimension $-\infty$, 0, or 1 (e.g. [1]). By Proposition 2.1 and Theorem 2.9, any quasi-elliptic surface has Kodaira dimension 0 or 1.

Remark 2.11. By Theorem 2.9 and Corollary 2.6, we have

$$\text{quasi-elliptic} \implies \text{uniruled} \implies \text{Shioda-supersingular}$$

3. Transcendental motives

3.1. Chow motives. In this subsection, we recall the notions of Chow motives and transcendental motives. Let k be an algebraically closed field of characteristic $p \geq 0$. Let $\mathcal{V}(k)$ be the category of smooth projective varieties over k. For every $V \in \mathcal{V}(k)$, we denote by $\mathrm{CH}^i(V)$ the Chow group of codimensional i-cycles with $\mathbb{Q}$-coefficients, and $\mathrm{CH}(V) = \oplus_i \mathrm{CH}^i(V)$. Let $U, V, W \in \mathcal{V}(k)$. For $\alpha \in \mathrm{CH}(U \times V)$, $\beta \in \mathrm{CH}(V \times W)$, define $\beta \circ \alpha := p_{UW*}(p^*_{UV}(\alpha) \cdot p^*_{VW}(\beta)) \in \mathrm{CH}(U \times W)$ where p_{UV}, p_{VW}, and p_{UW} are the appropriate projections. We denote by $\mathcal{M}_{rat}(k)$

the contravariant category of Chow motives with $\mathbb{Q}$-coefficients over k, which is $\mathbb{Q}$-linear, pseudoabelian, tensor category. An object M of $\mathcal{M}_{rat}(k)$ is the triple $M = (V, p, m)$, where $V \in \mathcal{V}(k)$, $p \in \mathrm{CH}^{\dim(V)}(V \times V)$ a projector (i.e., $p \circ p = p$), and $m \in \mathbb{Z}$. If $V, W \in \mathcal{V}(k)$ are irreducible, then

$$\mathrm{Hom}_{\mathcal{M}_{rat}(k)}((V, p, m), (W, q, n)) = q \circ \mathrm{CH}^{\dim(V)+n-m}(V \times W) \circ p.$$

For $M = (V, p, m), N = (W, q, n) \in \mathcal{M}_{rat}(k)$, we denote by $M \oplus N$ the sum and by the tensor product $M \otimes N$. In particular, if $m = n$, then $M \oplus N = (V \sqcup W, p \oplus q, m)$. For a non-negative integer n, let $\mathbb{L}^{\oplus n} := \mathbb{L} \oplus \cdots \oplus \mathbb{L}$ and $\mathbb{L}^{\otimes n} = \mathbb{L} \otimes \cdots \otimes \mathbb{L}$ (n-times). For a prime number $l \neq p$, we consider the l-adic étale cohomology theory $H^*_{ét}$ which induces a functor $H^*_{ét} : \mathcal{M}_{rat}(k) \to Vect^{gr}_{\mathbb{Q}_l}$ such that $H^i_{ét}((V, p, m)) = p^* H^{i-2m}_{ét}(V, \mathbb{Q}_l)$.

Let $V \in \mathcal{V}(k)$ be a variety of dimension d. We denote by $h(V) = (V, \Delta_V, 0)$ the Chow motive of V. Here Δ_V is the diagonal of V in $\mathrm{CH}^d(V \times V)$. If $d \leq 2$, then V admits a Chow-Künneth decomposition, that is, there is a decomposition

$$h(V) \cong \oplus_{i=0}^{2d} h_i(V)$$

such that $h_i(V) = (V, \pi_i(V), 0)$, π are pairwise orthogonal projectors, and $cl(\pi_i)$ coincides with $(2d - i, i)$-component of Δ_V in the Künneth component of $H^{2d}_{ét}(V \times V, \mathbb{Q}_l)$. Here $cl : \mathrm{CH}^d(V \times V)_{hom} \to H^{2d}_{ét}(V \times V, \mathbb{Q}_l)$ is the cycle map. Then $h_0(V) \cong 1$ and $h_{2d}(V) \cong \mathbb{L}^{\otimes 2d}$. In particular, $h(\mathbb{P}) = 1 \oplus \mathbb{L}$. Moreover, for two curves $C, D \in \mathcal{V}(k)$, by [6, 6.1.5, p.69],

$$h(C \times D) \cong \oplus_{i=0}^{2} \oplus_{j+k=i} h_j(C) \otimes h_k(D). \tag{1}$$

From now on, let $X \in \mathcal{V}(k)$ be a surface. The motive $h_1(X)$ (resp. $h_3(X)$) is controlled by the Picard (resp. Albanese) variety of X. Thus, h_i is well understood for $i \neq 2$. Let

$$h_2(X) = h_2^{alg}(X) \oplus t_2(X) = (X, \pi_2^{alg}(X), 0) \oplus (X, \pi_2^{tr}(X), 0)$$

be the decomposition of $h_2(X)$ as in [4]. The motive $t_2(X)$ is called the transcendental motive of X. It is a birational invariant and $H^2_{ét}(h_2^{alg}(X)) = \mathrm{NS}(X)_{\mathbb{Q}_l}$ and $H^2_{ét}(t_2(X)) = H^2_{ét}(X, \mathbb{Q}_l)_{tr}$. By construction, $h_2^{alg}(X) \cong \mathbb{L}^{\oplus \rho(X)}$, so h_2^{alg} is also well understood. However, t_2 is still mysterious. For example, see Conjecture 1.1 (= Conjecture 2.4).

Now, we prove a necessary and sufficient condition for $t_2 = 0$.

Lemma 3.1. *Let $X \in \mathcal{V}(k)$ be a surface. Then*

$$h_2(X) \cong \mathbb{L}^{\oplus b_2(X)} \quad \text{if and only if} \quad t_2(X) = 0.$$

In particular, if $b_2(X) \neq \rho(X)$, then $t_2(X) \neq 0$.

Proof. Assume $h_2(X) \cong \mathbb{L}^{\oplus b_2(X)}$. Since $h_2^{alg}(X) \cong \mathbb{L}^{\oplus \rho(X)}$, we have $t_2(X) \cong \mathbb{L}^{b_2 - \rho}$. Since $\mathrm{Hom}(\mathbb{L}, t_2(X)) = 0$ (e.g. [4]), we have $\mathrm{Hom}(\mathbb{L}^{b_2(X)-\rho(X)}, t_2(X)) = 0$, so $t_2(X) = 0$.

Conversely, assume $t_2(X) = 0$. Then $h_2(X) = h_2^{alg}(X) \cong \mathbb{L}^{\oplus \rho(X)}$. Take the cohomology: $H^2_{ét}(t_2(X)) = H^2_{ét}(X, \mathbb{Q}_l)_{tr} \cong \mathbb{Q}_l^{b_2 - \rho}$. Since $t_2(X) = 0$, we have $\mathbb{Q}_l^{b_2-\rho} = 0$, so $b_2 = \rho$. Thus, $h_2(X) \cong \mathbb{L}^{\oplus b_2(X)}$. On the contrary, if $b_2 \neq \rho$, then $t_2(X) \neq 0$. □

3.2. Homomorphisms between transcendental motives. In this subsection, we prove some results on homomorphisms between transcendental motives. Let k be an algebraically closed field. Let $X, Y \in \mathcal{V}(k)$ be surfaces.

$CH_2(X \times Y)_{\equiv}$: the subgroup of $CH_2(X \times Y)$ generated by the classes supported on subvarieties of the form $X \times N$ or $M \times Y$, with M a closed subvariety of X of dimension < 2 and N a closed subvariety of Y of dimension < 2.

We define a homomorphism

$$\Phi_{X,Y} : CH_2(X \times Y) \to \mathrm{Hom}_{\mathcal{M}_{rat}(k)}(t_2(X), t_2(Y))$$
$$\alpha \mapsto \pi_2^{tr}(Y) \circ \alpha \circ \pi_2^{tr}(X).$$

Theorem 3.2 ([4, Theorem 7.4.3, p.165])**.** *There is an isomorphism of groups*

$$CH_2(X \times Y)/CH_2(X \times Y)_{\equiv} \cong \mathrm{Hom}_{\mathcal{M}_{rat}(k)}(t_2(X), t_2(Y)).$$

To prove the functorial relation for $\Phi_{X,Y}$, we need the following lemma:

Lemma 3.3. *Let* $\alpha \in CH_2(X \times Y)$ *and* $\gamma \in CH_2(Y \times X)_{\equiv}$. *Then*

$$\text{(i) } \gamma \circ \alpha \in CH_2(X \times X)_{\equiv} \quad \text{and} \quad \text{(ii) } \alpha \circ \gamma \in CH_2(Y \times Y)_{\equiv}.$$

Proof. The proof of (ii) is similar to (i). Thus, it suffices to prove (i). Without loss of generality, we may assume that γ is irreducible and supported on $Y \times C$ with $\dim(C) \leq 1$.

First, assume $\dim(C) = 0$. Let $p \in X$ be a closed point. For $\gamma = [Y \times p]$, then

$$\gamma \circ \alpha = [Y \times p] \circ \alpha = p^{YXY}_{YY*}(\alpha \times Y \cdot Y \times X \times p) = p^{YXY}_{YY*}(\alpha \times p) = [p^{YX}_{Y*}(\alpha) \times p].$$

Thus $\gamma \circ \alpha \in CH_2(X \times X)_{\equiv}$. Next, assume $\dim(C) = 1$. Since γ is supported on $Y \times C$, there are a smooth irreducible curve C and a closed embedding $\iota : C \hookrightarrow X$ such that $\gamma = \Gamma_\iota \circ D$ in $CH_2(Y \times X)$, where $\Gamma_\iota \in CH_1(C \times X)$ is the graph of ι and $D \in CH_2(Y \times C)$. Since the support of the second projection of Γ_ι has dimension ≤ 1, the support of the second projection of $\gamma \circ \alpha$ has dimension ≤ 1, and hence $\gamma \circ \alpha \in CH_2(X \times X)_{\equiv}$. □

The following result is the functorial relation for $\Phi_{X,Y}$:

Proposition 3.4 ([7, p.62])**.** *For surfaces X, Y, Z* $\in \mathcal{V}(k)$*,*

$$\Psi_{Y,Z}(\beta) \circ \Psi_{X,Y}(\alpha) = \Psi_{X,Z}(\beta \circ \alpha) \ \text{ in } \ \mathrm{Hom}_{\mathcal{M}_{rat}(k)}(t_2(X), t_2(Z)).$$

Proof. Let $\Delta_Y = \pi_0 + \pi_1 + \pi_2^{alg} + \pi_2^{tr} + \pi_3 + \pi_4$ be the CK-decomposition in $CH_2(Y \times Y)$. Since $\pi_2^{tr}(Y) \circ \pi_2^{tr}(Y) = \pi_2^{tr}(Y)$, it suffices to prove in $\mathrm{Hom}(t_2(X), t_2(Z))$

$$\pi_2^{tr}(Z) \circ \beta \circ \pi_2^{tr}(Y) \circ \alpha \circ \pi_2^{tr}(X) = \pi_2^{tr}(Z) \circ \beta \circ \alpha \circ \pi_2^{tr}(X).$$

By Theorem 3.2, it suffices to prove

$$\beta \circ \pi_2^{tr}(Y) \circ \alpha - \beta \circ \alpha \in CH_2(X \times Z)_{\equiv}.$$

By the constructions of π_i for $i \neq 2$ and π_2^{alg} (e.g. [4]),

$$\pi_i(Y) \in CH_2(Y \times Y)_{\equiv} \quad \text{and} \quad \pi_2^{alg}(Y) \in CH_2(Y \times Y)_{\equiv}.$$

By Lemma 3.3,

$$\beta \circ \pi_i(Y) \circ \alpha \in CH_2(X \times Z)_{\equiv} \quad \text{and} \quad \beta \circ \pi_2^{alg}(Y) \circ \alpha \in CH_2(X \times Z)_{\equiv} \tag{2}$$

Therefore, we get

$$\beta\circ\pi_2^{tr}(Y)\circ\alpha-\beta\circ\alpha=\beta\circ(\Delta_Y-\pi_0(Y)-\pi_4(Y)-\pi_2^{alg}(Y)-\pi_1(Y)-\pi_3(Y))\circ\alpha-\beta\circ\alpha$$
$$\overset{(2)}{=}\alpha\circ(-\pi_0(Y)-\pi_4(Y)-\pi_2^{alg}(Y)-\pi_1(Y)-\pi_3(Y))\circ\beta \text{ in } \mathrm{CH}_2(X\times Z)_{\equiv}. \quad \square$$

Using Proposition 3.4, we prove the following:

Lemma 3.5. *Let $\pi : Y \to X$ be a finite morphism of surfaces. Let $\Gamma_\pi \in \mathrm{CH}^2(Y \times X)$ be the graph of π and ${}^t\Gamma_\pi$ its transpose. Then there is an isomorphism of Chow motives*

$$t_2(Y) \cong t_2(X) \oplus (Y, \pi_2^{tr}(Y) - \Psi_{X,Y}(\Gamma) \circ \Psi_{Y,X}({}^t\Gamma), 0).$$

Proof. Let d be the degree of π. We let $p := 1/d \cdot \Psi_{X,Y}({}^t\Gamma_\pi) \circ \Psi_{Y,X}(\Gamma_\pi)$.

(i) We prove that p and $\pi_2^{tr}(Y)-p$ are pairwise orthogonal projectors. In $\mathrm{Hom}(t_2(Y), t_2(Y))$,

$$\begin{aligned} p \circ p &= 1/d^2 \cdot \Psi_{X,Y}({}^t\Gamma_\pi) \circ \Psi_{Y,X}(\Gamma_\pi) \circ \Psi_{X,Y}({}^t\Gamma) \circ \Psi_{Y,X}(\Gamma) \\ &= 1/d^2 \cdot \Psi_{X,Y}({}^t\Gamma_\pi \circ \Gamma_\pi \circ^t \Gamma_\pi \circ \Gamma_\pi) && \text{by Proposition 3.4} \\ &= 1/d \cdot \Psi_{X,Y}({}^t\Gamma_\pi \circ \Gamma_\pi) && \text{by } \Gamma_\pi \circ^t\Gamma_\pi = d \cdot \Delta_X \\ &= 1/d \cdot \Psi_{X,Y}({}^t\Gamma_\pi) \circ \Psi_{Y,X}(\Gamma_\pi) && \text{by Proposition 3.4} \\ &= p. \end{aligned}$$

Thus p is a projector. Similarly, one has $p \circ \pi_2^{tr}(Y) = \pi_2^{tr}(Y) \circ p = p$. Thus, $\pi_2^{tr}(Y) - p$ is also a projector, and p and $\pi_2^{tr}(Y) - p$ are orthogonal.

(ii) We prove $t_2(X) \cong (Y, p, 0)$. We let

$$\alpha := 1/d \cdot p \circ \Phi_{X,Y}({}^t\Gamma_\pi) \circ \pi_2^{tr}(X) \in \mathrm{Hom}(t_2(X), (Y, p, 0))$$
$$\beta := 1/d \cdot \pi_2^{tr}(X) \circ \Psi_{Y,X}(\Gamma_\pi) \circ p \in \mathrm{Hom}((Y, p, 0), t_2(X)).$$

By the same way as in (i), we have $\alpha \circ \beta = p$ and $\beta \circ \alpha = \pi_2^{tr}(X)$, so we get $t_2(X) \cong (Y, p, 0)$.

(iii) We prove $t_2(Y) \cong t_2(X) \oplus (Y, \pi_2^{tr}(Y) - p, 0)$. By (i) and (ii), we get isomorphisms

$$t_2(Y) \overset{\text{(i)}}{\cong} (Y, p, 0) \oplus (Y, \pi_2^{tr}(Y) - p, 0) \overset{\text{(ii)}}{\cong} t_2(X) \oplus (Y, \pi_2^{tr}(Y) - p, 0).$$

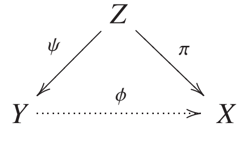
Thus, we complete the proof of Lemma 3.5. $\square$

To prove $t_2 = 0$ for uniruled surfaces (Theorem 4.1), we need the following:

Lemma 3.6 ([7, p.66])**.** *Let $\phi : Y \dashrightarrow X$ be a dominant rational map of surfaces. Then $t_2(X)$ is the direct summand of $t_2(Y)$, that is, there are a motive M and a decompostion*

$$t_2(Y) \cong t_2(X) \oplus M.$$

Proof. By the elimination of indeterminacy of ϕ (since $\dim(X) = 2$), there are a surface Z, a birational morphism $\psi : Z \to Y$, and a finite surjective morphism $\pi : Z \to X$ such that the diagram



is commutative. By Lemma 3.5, there is a decomposition $t_2(Z) \cong t_2(X) \oplus M$ for some motive

M. Since t_2 is a birational invariant, $t_2(Y) \cong t_2(Z)$. Therefore, we get $t_2(Y) \cong t_2(X) \oplus M$ for some motive M. □

4. Proof of Main theorem

To prove our main theorem, we prove the following:

Theorem 4.1. *Let X be a uniruled surface. Then $t_2(X) = 0$.*

Proof. Since X is uniruled, there are a curve C and a dominant rational map

$$\phi : \mathbb{P}^1 \times C \dashrightarrow X.$$

By Lemma 3.6, there are a motive M and a decomposition

$$t_2(\mathbb{P}^1 \times C) \cong t_2(X) \oplus M.$$

Thus, it suffices to prove $t_2(\mathbb{P}^1 \times C) = 0$. Indeed, there is a CK-decomposition

$$h_2(\mathbb{P}^1 \times C) \cong \oplus_{j+k=2} h_j(\mathbb{P}^1) \otimes h_k(C)$$

by (1). Since both $\mathbb{P}^1$ and C have dimension 1, we have $h_0(-) = 1$ and $h_2(-) = \mathbb{L}$, so we get $h_2(\mathbb{P}^1 \times C) \cong \mathbb{L}^{\oplus 2}$ because $h_1(\mathbb{P}) = 0$. By the argument as in the proof of Corollary 2.6, we have $b_2(\mathbb{P}^1 \times C) = \rho(\mathbb{P}^1 \times C) = 2$. Thus, we have $h_2(\mathbb{P}^1 \times C) \cong \mathbb{L}^{\oplus b_2(\mathbb{P}^1 \times C)}$. By Lemma 3.1, we get $t_2(\mathbb{P}^1 \times C) = 0$. This completes the proof of Theorem 4.1. □

REMARK 4.2. Let C and D be smooth projective curves over $\mathbb{C}$ with positive genus. Let $X = C \times D$. Then $p_g(X) = p_g(C) \cdot p_g(D) > 0$. By [10, pp.155-156], $b_2(X) \neq \rho(X)$. By Lemma 3.1, we get $t_2(X) \neq 0$, that is, $h_2(X) \neq \mathbb{L}^{\oplus b_2(X)}$.

Our main theorem is the following:

Theorem 4.3 (= Theorem 1.2)**.** *Let $f : X \to C$ be a quasi-elliptic surface. Then*

$$t_2(X) = 0.$$

Proof. By Theorem 2.9, X is uniruled. By Theorem 4.1, $t_2(X) = 0$. □

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