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ARC SCHEME AND HIGHER DIFFERENTIAL FORMS

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Abstract

Let k be a field. In this article, we identify the component of weight 2 of the natural $\mathbf{G}_{m,k}$-graduation on the k-algebra of the arc scheme attached to an affine algebraic variety X with the module of the 2-nd order derivations on X. We in particular deduce, from this property, characterizations of the geometry of hypersurfaces (in affine spaces) in terms of the nilpotency on arc scheme.

1. Introduction

1.1. Let k be a field. For every integer $m \in \mathbf{N}$, every $n \in \mathbf{N} \cup \{\infty\}$ let us note $A_n := k[x_1, \ldots, x_m]_n := k[(x_{i,j}); i \in \{1, \ldots, m\}, j \in \{0, \ldots, n\}]$ which has a structure of $A := k[x_1, \ldots, x_m]$-module *via* the identification of $A_0 = k[x_1, \ldots, x_m]_0$ and A. For every polynomial $f \in k[x_1, \ldots, x_m]$, there exists a unique family $(\Delta_s(f))_{s \in \mathbf{N}}$ of polynomials in $k[x_1, \ldots, x_m]_\infty$, only depending on the polynomial f, such that the following equality holds in the ring $k[x_1, \ldots, x_m]_n[t]$:

$$f\left(\left(\sum_{j=0}^{n} x_{i,j}t^j\right)_{i\in\{1,\ldots,m\}}\right) = \sum_{s=0}^{n} \Delta_s(f)\left((x_{i,j})_{\substack{i\in\{1,\ldots,m\}\\ j\in\{0,\ldots,s\}}}\right) t^s \pmod{t^{n+1}}. \tag{1.1}$$

For every affine k-variety $X = \mathrm{Spec}(k[x_1, \ldots, x_m]/I)$ and every $n \in \mathbf{N} \cup \{\infty\}$ the k-scheme $\mathscr{L}_n(X)$ defined by $\mathrm{Spec}(k[x_1, \ldots, x_m]_n/\langle \Delta_s(f),\ s \in \{0, \ldots, n\}, f \in I\rangle)$ is the associated *jet scheme of level* n when $n \in \mathbf{N}$ and the associated *arc scheme* when $n = \infty$. The natural $\mathbf{G}_{m,k}$-action on A_n, with $n \in \mathbf{N} \cup \{\infty\}$, defined to be with weight j on every variable $x_{i,j}$ for every integer $i \in \{1, \ldots, m\}$ and every integer $j \in \{0, \ldots, n\}$, induces a graduation on A_n for which the polynomial $\Delta_s(f)$ is a homogeneous element with weight s for every integer $s \in \mathbf{N}$ and every polynomial $f \in A$. We say that $\Delta_s(f)$ is *isobaric* with weight s. This usual observation gives rise to a $\mathbf{G}_{m,k}$-action on the k-scheme $\mathscr{L}_n(X)$, for every $n \in \mathbf{N} \cup \{\infty\}$ (which also is an action of the multiplicative monoid $\mathbf{A}^1_k$).

1.2. Let X be an affine k-variety. Attached to the former $\mathbf{G}_{m,k}$-action, we consider the *weight grading* on the k-algebra $\mathcal{O}(\mathscr{L}_\infty(X))$; we denote it by

$$\mathcal{O}(\mathscr{L}_\infty(X)) = \bigoplus_{n \geq 0} W^n_{\mathcal{O}(X)}.$$

In this decomposition, one can easily observe that the $\mathcal{O}(X)$-module $W^1_{\mathcal{O}(X)}$ can be naturally identified with the module of *Kähler differential forms* $\Omega^1_{\mathcal{O}(X)}$ on X.

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1.3. In this article, we extend this observation by constructing a natural isomorphism of $\mathcal{O}(X)$-modules between $W^2_{\mathcal{O}(X)}$ and the module $\Omega^{(2)}_{\mathcal{O}(X)/k}$ formed by the 2-nd order differential forms on X. Precisely, for every integer $n \geq 1$, we show how to use the universal property defining $\Omega^{(n)}_{\mathcal{O}(X)/k}$ in order to exhibit a morphism of $\mathcal{O}(X)$-modules

$$\varphi^n_{\mathcal{O}(X)} : \Omega^{(n)}_{\mathcal{O}(X)/k} \to W^n_{\mathcal{O}(X)} \tag{1.2}$$

and show the following statement:

Theorem 1.4. *Let k be a field. Let $I \subset A = k[x_1, \dots, x_m]$ be an ideal and $B = A/I$. The morphism of B-modules φ^2_B induces an isomorphism of B-modules from $\Omega^{(2)}_{B/k}$ to W^2_B.*

Let us stress that, for $n = 1$, the morphism $\varphi^n_{\mathcal{O}(X)}$ provides the identification mentionned above and that, for $n \geq 3$, the picture is much more complicated since $\varphi^n_{\mathcal{O}(X)}$ stops to be bijective in general. For example, when the k-variety is assumed to be smooth, the modules $\Omega^{(n)}_{\mathcal{O}(X)/k}$, $W^n_{\mathcal{O}(X)}$ are free $\mathcal{O}(X)$-modules but, in general, with nonequal ranks.

1.5. Theorem 1.4 has various geometric applications in the study of arc scheme. A by-product of our main result can be formulated as follows:

Corollary 1.6. *Let k be a perfect field. Let $m \geq 1$ be a positive integer. Let X be an integral hypersurface of $\mathbf{A}^m_k$.*

(1) *The following assertions are equivalent*:
   (a) *The hypersurface X is normal.*
   (b) *The $\mathcal{O}(X)$-module $W^2_{\mathcal{O}(X)}$ is torsionfree.*
   (c) *The $\mathcal{O}(X)$-module* $\mathrm{Nilrad}(\mathcal{O}(\mathscr{L}_\infty(X))) \cap W^2_{\mathcal{O}(X)} = (0)$.

(2) *The following assertions are equivalent*:
   (a) *The hypersurface X is regular.*
   (b) *The $\mathcal{O}(X)$-module $W^2_{\mathcal{O}(X)}$ is projective.*

In particular, if X is an integral affine plane curve, then $\mathcal{O}(X)$-module $W^2_{\mathcal{O}(X)}$ is torsionfree if and only if it is projective.

2. Notations, conventions

2.1. In this article, k is a field with an arbitrary characteristic. A *k-variety* is a k-scheme of finite type. If the field k is assumed to be perfect, every reduced k-variety X is geometrically reduced, then $\mathrm{Reg}(X)$ (which can be understood equivalently as the locus formed by the regular points or the smooth points) is not empty or, equivalently, $\mathrm{Sing}(X) \neq X$.

2.2. Let R be a k-algebra and M be a R-module. Let $n \geq 1$ be a positive integer. According to [11, Chapter I,§1], a *n-th order k-derivation* from R to M is a differential operator with a zero constant term, that is to say a morphism of k-vector spaces $D : R \longrightarrow M$ which satisfies the *Leibniz rule* with order n:

$$D(a_0 \cdots a_n) = \sum_{s=1}^{n} (-1)^{s-1} \sum_{0 \leq i_1 < \cdots < i_s \leq n} a_{i_1} \cdots a_{i_s} D(a_0 \cdots \check{a}_{i_1} \cdots \check{a}_{i_s} \cdots a_n) \tag{2.1}$$

for every element $a_0, \cdots, a_n \in R$. In this identity, one denotes by $a_0 \cdots \breve{a}_{i_1} \cdots \breve{a}_{i_s} \cdots a_n$ the element $\prod_{\substack{0 \le j \le n \\ j \ne i_1, \cdots, i_s}} a_j$. We denote by $\mathrm{Der}_k^{(n)}(R, M)$ the R-module formed by n-th order k-derivations from R to M, and simply $\mathrm{Der}_k^{(n)}(R, R)$ by $\mathrm{Der}_k^{(n)}(R)$. One has $\mathrm{Der}_k^{(1)}(R) = \mathrm{Der}_k(R)$.

Example 2.3. The datum of $f \mapsto (\Delta_s(f))_{s \in \mathbf{N}}$ induces a Hasse-Schmidt derivation (e.g., see [7, §27] or [2, Proposition 7.5.1]). In this way, one knows that the k-linear map $\Delta_n \colon f \mapsto \Delta_n(f)$, defines, for every integer $n \ge 1$, a n-th order derivation from A to W_A^n, by [11, Chapter I, Proposition 5].

2.4. By [12, Proposition 1.6], one knows that the functor attached to $R \mapsto \mathrm{Der}_k^{(n)}(R)$ is representable by a R-module $\Omega_{R/k}^{(n)}$ called the *module of Kähler differentials of order n*. (When $n = 1$, this construction corresponds to the usual notion of module of Kähler differentials.) We give a concrete description of the R-module $\Omega_{R/k}^{(n)}$ (simply denoted by $\Omega_R^{(n)}$) which is due to [11, Chapter II,§1] and [12, §1]. The k-algebra $R \otimes_k R$, endowed with the morphism of k-algebra $R \longrightarrow R \otimes_k R$ which maps $x \in R$ to $x \otimes 1$, can be considered as a R-algebra. Let J be the kernel of the product map $R \otimes_k R \longrightarrow R$. For every element $x \in R$, let us stress that the element $1 \otimes x - x \otimes 1$ belongs to the ideal J; the subset of J defined by the datum of the elements of the form $1 \otimes x - x \otimes 1$ forms a generating system of the ideal J. The module of Kähler differentials of order n then is constructed as the quotient J/J^{n+1}. It is equipped with the following derivation of order n

$$\begin{array}{rcl} d_R : R & \longrightarrow & \Omega_{R/k}^{(n)} = J/J^{n+1} \\ x & \longmapsto & [1 \otimes x - x \otimes 1]. \end{array}$$

For every element $x \in R$, we denote by $[1 \otimes x - x \otimes 1]$ the class of the element $1 \otimes x - x \otimes 1$ modulo J^{n+1}. Let us observe that, by construction the R-module $\Omega_{R/k}^{(n)}$ is generated by the family $(d_R(x))_{x \in R}$.

Example 2.5. Let $A = k[x_1, \ldots, x_m]$. The A-module $\Omega_{A/k}^{(n)}$ is free. A basis consists of the differential forms $(d_A(x))^\alpha := \prod_{i \in \{1,\ldots,m\}} d_A(x_i)^{\alpha_i}$ with $\alpha \in \mathbf{N}^m$. The universal derivation d_A is given by the formula :

$$d_A(f) = \sum_{1 \le |\alpha| \le n} \delta_\alpha(f) d(x)^\alpha \tag{2.2}$$

for every polynomial $f \in A$ (see [11, Chapter II,§2]). In this formula, the polynomial $\delta_\alpha(f)$ is obtained as the coefficient of $t_1^{\alpha_1} \cdots t_m^{\alpha_m}$ in the expression $f((x_i + t_i)) - f((x_i)_i)$.

3. Proof of theorem 1.4

3.1. Let $n \ge 1$ be an integer. Let $I \subset A$ be an ideal and $B = A/I$. Let $\pi : A \to B$ be the quotient morphism and $\pi_n \colon A_n \to B_n := A_n/\langle \Delta_s(f) \ : \ s \in \{0, \ldots, n\}, f \in I \rangle$ the induced morphism. The morphism of k-modules $\pi_n \circ \Delta_n : A \to W_B^n$ induces, by the universal property of quotient, a n-th order derivation from B to W_B^n. Hence, by [12, Proposition 1.6], we deduce, by adjunction, the existence of a canonical morphism of B-modules

$$\varphi_B^n : \Omega_B^{(n)} \longrightarrow W_B^n \tag{3.1}$$

which satisfies the formula $\varphi_B^n(d_B(\overline{f})) = \pi_n \circ \Delta_n(f)$ for every element $f \in A$.

3.2. Let us begin by recalling the proof of the corresponding statement when $n = 1$. We observe that the morphism φ_A^1, defined by $dx_i \mapsto x_{i,1}$ for every integer $i \in \{1, \dots, m\}$, induces an isomorphism from $\Omega_B^1 \cong \Omega_A^1/\langle df,\ f \in I\rangle + I\Omega_A^1$ to $W_B^1 \cong W_A^1/\langle x_{i,1}f,\ \Delta_1(f),\ i \in \{1, \dots, m\}, f \in I\rangle$ since $d_A(f) = \sum_{i=1}^m \partial_{x_i}(f) d_A(x_i)$ and $\Delta_1(f) = \sum_{i=1}^m \partial_{x_i}(f) x_{i,1}$.

3.3. Let us prove theorem 1.4. Let us begin by a preliminary observation. For every integer $i \in \{1, \dots, m\}$, we set $T_i = x_{i,1}t + x_{i,2}t^2$. Let us set, for every integer $i \in \{1, \dots, m\}$, $T^\alpha = \prod_{i=1}^m T_i^{\alpha_i}$ and $e_i = (0, \cdots, 1, \cdots, 0)$ for the i-th canonical basis vector in $\mathbf{N}^m$. We have

$$\begin{aligned} f((x_{i,0}+T_i)_i) &= f((x_{i,0})_i) + \left(\sum_{|\alpha|=1} \delta_\alpha(f) T^\alpha\right) + \left(\sum_{|\alpha|=2} \delta_\alpha(f) T^\alpha\right) + (\cdots) \\ &= f((x_{i,0})_i) + \left(\sum_{i=1}^m \delta_{e_i}(f) x_{i,1}\right) t + \left(\sum_{i=1}^m \delta_{e_i}(f) x_{i,2}\right) t^2 + \left(\sum_{i \le j} \delta_{e_i+e_j}(f) x_{i,1} x_{j,1}\right) t^2 + (\cdots). \end{aligned}$$

Because of the uniqueness of the $\Delta_i(f)$, we conclude that

$$\Delta_2(f) = \left(\sum_{i=1}^m \delta_{e_i}(f) x_{i,2}\right) + \left(\sum_{1 \le i \le j \le m} \delta_{e_i+e_j}(f) x_{i,1} x_{j,1}\right). \tag{3.2}$$

$\circ$ *Let us describe our main ingredients.* By subsection 3.1, we know that $B_2 = A_2/\langle \{f, \Delta_1(f), \Delta_2(f),\ f \in I\}\rangle$. We set $I_2 := \langle \{f, \Delta_1(f), \Delta_2(f),\ f \in I\}\rangle \subset A_2$. In this way, we deduce that

$$W_B^2 = \frac{W_A^2 + I_2}{I_2} = \frac{W_A^2}{I_2 \cap W_A^2} = \frac{\left(\oplus_{1 \le i \le j \le m} A \cdot x_{i,1} x_{j,1}\right) \bigoplus \left(\oplus_{i \in \{1,\dots,m\}} A \cdot x_{i,2}\right)}{IW_A^2 + \langle \{x_{i,1}\Delta_1(f), \Delta_2(f),\ f \in I,\ i \in \{1, \dots, m\}\}\rangle}.$$

On the other hand, by [1, Proposition 2.5] or [11, Chapter II, Corollary 14.1], we know that

$$\Omega_B^{(2)} \cong \frac{\Omega_A^{(2)} \otimes_A B}{\langle d_A(f) \otimes 1, d_A(x_i) d_A(f) \otimes 1,\ i \in \{1, \dots, m\},\ f \in I\rangle}.$$

In this end, by subsection 3.1, the morphism of A-modules φ_A^2 (resp. φ_B^2) is defined by $d_A(f) \mapsto \Delta_2(f)$ (resp. $\varphi_B^2(d_B(\overline{f})) = \pi_2 \circ \Delta_2(f)$) for every polynomial $f \in A$.

$\circ$ *Let us introduce the morphism of A-modules $\psi_A^2 \colon W_A^2 \to \Omega_A^{(2)}$.* Because of formula (3.2), we introduce the morphism of A-modules ψ_A^2 defined by $\psi_A^2(x_{i,2}) = d_A(x_i)$ and $\psi_A^2(x_{i,1}x_{j,1}) = d_A(x_i) d_A(x_j)$ for every pair of integers $(i, j) \in \{1, \dots, m\}^2$. Let us stress that, by the construction of the morphism ψ_A^2 and formula (3.2) , we have

$$\psi_A^2(\varphi_A^2(d_A(f))) = \psi_A^2(\Delta_2(f)) = d_A(f). \tag{3.3}$$

In other words, the morphism ψ_A^2 is a retraction of φ_A^2.

$\circ$ *Let us prove that ψ_A^2 induces a morphism of B-modules from W_B^2 to $\Omega_B^{(2)}$.* For every integer $j \in \{1, \dots, m\}$, we have

$$\psi_A^2(\Delta_1(f) x_{j,1}) = \psi_A^2\Big(\sum_{i=1}^m \partial_{x_i}(f) x_{i,1} x_{j,1}\Big) = \sum_{i=1}^m \partial_{x_i}(f) \psi_A^2(x_{i,1} x_{j,1}) = \sum_{i=1}^m \partial_{x_i}(f) d_A(x_i) d_A(x_j).$$

On the other hand, since the product of three terms of the form $d_A(x_s)$ is zero in $\Omega_A^{(2)}$, we have:

$$d_A(f)d_A(x_j)=d_A(x_j)\left(\sum_{1\leq|\alpha|\leq 2}\delta_\alpha(f)d_A(x)^\alpha\right)=d_A(x_j)\left(\sum_{|\alpha|=1}\delta_\alpha(f)d_A(x)^\alpha\right)=\sum_{i=1}^m \partial_{x_i}(f)d_A(x_i)d_A(x_j).$$

In other words, the formula $\psi_A^2(\Delta_1(f)x_{j,1}) = d_A(f)d_A(x_j)$ holds true for every integer $j \in \{1,\ldots,m\}$. In the end, for every integer $j \in \{1,\ldots,m\}$, we also have $\psi_A^2(fx_{j,2}) = fd_A(x_j)$. Hence, the morphism ψ_A^2 induces a morphism of B-modules $\psi_B^2\colon W_B^2 \to \Omega_B^{(2)}$.

◦ *Let us prove that the morphisms of B-modules φ_B^2, ψ_B^2 are mutually inverse.* By equaliy (3.3), we know that ψ_B^2 also is a retraction of φ_B^2. Let $\bar{P} \in W_B^2$. By the very definitions, for every lifting $P \in W_A^2$, there exist polynomials $a_i, b_i \in A$, with $i \in \{1,\ldots,m\}$, such that:

$$P = \sum_{i=1}^m a_i x_{i,2} + \sum_{1\leq i\leq j\leq m} b_{i,j}x_{i,1}x_{j,1}.$$

Let us observe that, since the family $(\Delta_s)_s$ is a high-order derivation, we have, for every $i, j \in \{1,\ldots,m\}$,

$$\Delta_2(x_ix_j) = \sum_{s=0}^2 \Delta_s(x_i)\Delta_{2-s}(x_j) = x_{i,0}x_{j,2} + x_{i,2}x_{j,0} + x_{i,1}x_{j,1}. \tag{3.4}$$

On the other hand, by the very definition of d_A, we have

$$d_A(x_ix_j) = x_id_A(x_j) + x_jd_A(x_i) + d_A(x_i)d_A(x_j). \tag{3.5}$$

By the definitions of the morphisms φ_A^2, ψ_A^2 and formulas (3.4) and (3.5), we obtain that

$$\begin{aligned}
(\varphi_B^2\circ\psi_B^2)(\bar{P}) &= (\pi_2\circ\varphi_A^2)\left(\sum_{i=1}^m a_id_A(x_i) + \sum_{1\leq i\leq j\leq m} b_{i,j}d_A(x_i)d_A(x_j)\right)\\
&= \pi_2\left(\sum_{i=1}^m a_ix_{i,2} + \sum_{1\leq i\leq j\leq m} b_{i,j}\varphi_A^2(d_A(x_i)d_A(x_j))\right)\\
&= \pi_2\left(\sum_{i=1}^m a_ix_{i,2} + \sum_{1\leq i\leq j\leq m} b_{i,j}\varphi_A^2(d_A(x_ix_j) - x_id_A(x_j) - x_jd_A(x_i))\right)\\
&= \pi_2\left(\sum_{i=1}^m a_ix_{i,2} + \sum_{1\leq i\leq j\leq m} b_{i,j}(\Delta_2(x_ix_j) - x_i\Delta_2(x_j) - x_j\Delta_2(x_i))\right)\\
&= \pi_2\left(\sum_{i=1}^m a_ix_{i,2} + \sum_{1\leq i\leq j\leq m} b_{i,j}x_{i,1}x_{j,1}\right)\\
&= \bar{P}.
\end{aligned}$$

REMARK 3.4. In general, there is no hope for W_B^n to be isomorphic to $\Omega_B^{(n)}$. We illustrate here this remark by several properties. By [1, Theorem 4.3], one knows that, for every integer $n \geq 1$, the k-variety $H = V(f)$, attached to $f \in A$, is normal if and only if $\Omega_{\mathcal{O}(H)}^{(n)}$ is torsion-free. (Let us stress that for $n = 1$ the former property is classical; e.g., see [6, Corollary 9.8].) In other hand, it is quite simple to find examples of such a normal hypersurface H with nonzero $\mathrm{Tors}(W_{\mathcal{O}(H)}^n)$. As an illustration, one can consider example 4.9, and, more generally, [5, Conjecture 9.1] suggests that any normal hypersurface H without rational singularity share this property. Another observation leads us to conclude that, in general, $W^n_B, \Omega^{(n)}_B$ are not isomorphic. If the k-algebra B is assumed to be smooth, then both B-modules $W^n_B, \Omega^{(n)}_B$ are free; but their ranks in general differ.

4. Applications

In this section, we show that theorem 1.4 and properties of the 2-nd order derivation module can be used to prove corollary 1.6. We also explain how to use theorem 1.4 to study the torsion submodule of the 2-nd order derivation module. Other general results on the interpretation of geometric properties on algebraic varieties in terms of nilpotency on arc scheme can be found, e.g., in [10, 13, 14, 15].

Lemma 4.1. *Let k be a field of characteristic zero. Let $n \geq 1$ be a positive integer. Let X be an integral affine k-variety. Then the $\mathcal{O}(X)$-module* $\mathrm{Tors}(W^n_{\mathcal{O}(X)})$ *is formed by the nilpotent isobaric functions on $\mathscr{L}_\infty(X)$ with weight n.*

Proof. Let us fix an embedding $X \hookrightarrow \mathbf{A}^m_k = \mathrm{Spec}(k[x_1, \ldots, x_m])$ defined by the datum of a prime ideal I of A. We denote by $[I]$ the ideal of A_∞ generated by the $\Delta_n(g)$ for every integer $n \in \mathbf{N}$ and every polynomial $g \in I$. By definition, one have $\mathscr{L}_\infty(X) = \mathrm{Spec}(A_\infty/[I])$. Let $\bar{f} \in \mathcal{O}(\mathscr{L}_\infty(X))$ be a function that we assumed to be isobaric with weight n. Then, the function $\bar{f}$ is torsion if and only if there a nonzero $\bar{a} \in \mathcal{O}(X)$ such that $\bar{a}\bar{f} = 0$; hence, the function $\bar{a}\bar{f}$ belongs to the nilradical of $\mathcal{O}(\mathscr{L}_\infty(X))$, which is prime ideal of $\mathcal{O}(\mathscr{L}_\infty(X))$ by the Kolchin irreducibility. We conclude that the function $\bar{f}$ belongs to the nilradical of $\mathcal{O}(\mathscr{L}_\infty(X))$. Indeed, if any polynomial lifting $a \in k[x_1, \ldots, x_m]$ belongs to the radical of $[I]$ in A_∞, then, because of a direct argument of weight, we shall have $a \in I$ which is impossible by the assumption on $\bar{a}$. Conversely, if $\bar{f}$ is nilpotent, e.g., by [8, Lemma 3.7], there exists a polynomial $h \notin I$ and an integer $s \in \mathbf{N}$ such that $h^s f \in [I]$, which implies that $\bar{f} \in \mathrm{Tors}(\mathcal{O}(\mathscr{L}_\infty(X)))$ by definition. That concludes the proof. □

4.2. For every R-module M, we denote by $M^\vee$ its *dual*, i.e., $M^\vee := \mathrm{Hom}_R(M, R)$. We assume from now on that R is a noetherian domain, $M \neq (0)$ is finitely generated. Let K be the fraction field of R. Let $\ell_K(M) : M \to M_K := M \otimes_R K$ be the localization morphism. One observes, because of the very definitions, that:

$$\mathrm{Tors}(M) := \mathrm{Tors}_R(M) = \mathrm{Ker}(\ell_K(M)). \tag{4.1}$$

Moreover, if $c_M : M \to M^{\vee\vee}$ is the canonical morphism of R-modules, one also has:

$$\mathrm{Tors}(M) = \mathrm{Ker}(c_M). \tag{4.2}$$

This formula needs a quick justification. The following diagram is commutative Since the

$$\begin{array}{ccc} M & \xrightarrow{\quad c_M \quad} & M^{\vee\vee} \\ \downarrow{\scriptstyle \ell_K(M)} & & \downarrow{\scriptstyle \ell_K(M^{\vee\vee})} \\ M_K := M \otimes_R K & \xrightarrow[\cong]{} & M^{\vee\vee} \otimes_R K \cong M_K^{\vee\vee}. \end{array}$$

bottom horizontal morphism is an isomorphism, then, by (4.1), it follows from the commutativity of the former diagram that $\mathrm{Tors}(M) = c_M^{-1}(\ell_K(M^{\vee\vee})^{-1}(0))$. But, since R is a domain and $M^{\vee\vee}$ a dual, we know $\ell_K^{-1}(M^{\vee\vee})(0) = \mathrm{Tors}(M^{\vee\vee}) = (0)$. In the end, let us observe that the morphism $\ell_K(M)$ factorizes into

$$M \xrightarrow{\ell_x(M)} M_x := M \otimes_R R_x \xrightarrow{\ell_K(M_x)} M_K$$

for every point $x \in \mathrm{Spec}(R)$. Thus, one has

$$\mathrm{Tors}(M) = \bigcap_{x \in \mathrm{Spec}(R)} (M \cap \mathrm{Tors}_{R_x}(M_x)). \tag{4.3}$$

Thus, the R_x-module $\mathrm{Tors}(M_x)$ is torsionfree for every point $x \in \mathrm{Spec}(R)$ if and only if $\mathrm{Tors}(M) = (0)$,

Proposition 4.3. *Let k be a field of characteristic zero. Let $n \geq 1$ be a positive integer. Let X be an integral affine k-variety. Then submodule of the nilradical of $\mathcal{O}(\mathscr{L}_\infty(X)))$ formed by the isobaric functions with weight n equals the submodule*

$$\bigcap_{\theta \in (W^n_{\mathcal{O}(X)})^\vee} \mathrm{Ker}(\theta).$$

Proof. By lemma 4.1, we need to prove that $\mathrm{Tors}(W^n_{\mathcal{O}(X)}) = \bigcap_{\theta \in (W^n_{\mathcal{O}(X)})^\vee} \mathrm{Ker}(\theta)$. Now, let us observe that $\bigcap_{\theta \in (W^n_{\mathcal{O}(X)})^\vee} \mathrm{Ker}(\theta)$ coincides with the kernel N of the canonical morphism $W^n_{\mathcal{O}(X)} \to (W^n_{\mathcal{O}(X)})^{\vee\vee}$. The proof concludes from the fact that $\mathrm{Tors}(W^n_{\mathcal{O}(X)}) = N$; see formula (4.2). □

Recall that the morphism of B-modules $\ell \mapsto \ell \circ d_B$ defined from $\mathrm{Hom}_B(\Omega_B^{(2)}, B)$ to $\mathrm{Der}_k^{(2)}(B)$ is an isomorphism; hence, by theorem 1.4, we deduce that $\mathrm{Hom}_B(W_B^2, B) \cong \mathrm{Der}_k^{(2)}(B)$. Let $\theta \in \mathrm{Der}_k^{(2)}(B)$ be a 2-nd order derivation such that $\theta = \ell \circ d_B$ with $\ell \in \mathrm{Hom}_B(\Omega_B^{(2)}, B)$. Thanks to the former remark, one can define the *image* of any element $\bar{P} \in W_B^2$ by θ by setting

$$\theta \cdot \bar{P} = \ell((\varphi_B^2)^{-1}(\bar{P})) \in B.$$

Proposition 4.3 asserts that $\bar{P} \in W_B^2$ is torsion if and only if its image by every 2-nd order derivation is zero. This property can be linked to [15, Corollary 1.4] or [4, Corollary 4.8].

EXAMPLE 4.4. To illustrate this point of view, let us consider the polynomial $f = x^3 + y^2 \in k[x,y]$, with $B = A/\langle f \rangle$. Let us set $g := 4x_0y_2 - x_1y_1 - 6x_2y_0$, $h := 8y_0y_2 + 12x_0^2x_2 + 3x_0x_1^2 \in A_2$ whose images in the ring B are respectively denoted by $\bar{g}, \bar{h}$. The relations in the ring A_2

$$\begin{aligned}
2y_0^3 g &= y_0^2 \cdot \left(4x_0(2y_0y_2) - x_1(2y_0y_1) - 12y_0^2x_2\right) \\
&\equiv y_0^2 \cdot \left(4x_0(-3x_0^2x_2 - 3x_0x_1^2 - y_1^2) - x_1(2y_0y_1) - 12y_0^2x_2\right) \pmod{\Delta_2(f)} \\
&\equiv y_0^2 \cdot \left(-9x_0^2x_1^2 - 4x_0y_1^2 - x_1(3x_0^2x_1 + 2y_0y_1) - 12x_2(x_0^3 + y_0^2)\right) \pmod{\Delta_2(f)} \\
&\equiv -x_0 \cdot (9x_0y_0^2x_1^2 + (2y_0y_1)^2) \pmod{f, \Delta_1(f), \Delta_2(f)} \\
&\equiv -x_0 \cdot (9x_0y_0^2x_1^2 + 9x_0^4x_1^2) \pmod{f, \Delta_1(f), \Delta_2(f)} \\
&\equiv -9x_0^2x_1^2 \cdot (y_0^2 + x_0^3) \pmod{f, \Delta_1(f), \Delta_2(f)} \\
&\equiv 0 \pmod{f, \Delta_1(f), \Delta_2(f)}
\end{aligned}$$

imply that g is a torsion element in the ring B_2 (which is nonzero). In the same spirit, we observe that

$$\begin{aligned} h &\equiv -4(3x_0^2x_2 + 3x_0x_1^2 + y_1^2) + 12x_0^2x_2 + 3x_0x_1^2 \pmod{\Delta_2(f)} \\ &\equiv -(9x_0x_1^2 + 4y_1^2) \pmod{\Delta_2(f)}. \end{aligned}$$

Then, we conclude, in the same way, that $y_0^2 h \in I_2$; hence, $\bar{h}$ is a (nonzero) torsion element in B_2. Let us consider the 2-nd order derivation $(3x^2\partial_y - 2y\partial_x)^2 \in \mathrm{Der}_k^{(2)}(A)$. It clearly induces a 2-nd order derivation $\theta \in \mathrm{Der}_k^{(2)}(B)$ such that $\theta = \ell \circ d_B$ with $\ell\colon \Omega_B^{(2)} \to B$ defined by $d_B(\bar{x}) \mapsto -6\bar{x}^2$, $d_B(\bar{y}) \mapsto -12\bar{x}\bar{y}$, $d_B(\bar{x})^2 \mapsto 8\bar{y}^2$, $d_B(\bar{y})^2 \mapsto 18\bar{x}^4$, $d_B(\bar{x})d_B(\bar{y}) \mapsto -12\bar{x}^2\bar{y}$. Then, we obtain, by the very definition, that

$$\begin{cases} \theta \cdot \bar{g} &= 4x(-12\bar{x}\bar{y}) - (-12\bar{x}^2\bar{y}) - 6y(-6\bar{x}^2) \\ &= 0, \\ \theta \cdot \bar{h} &= 8y(-12\bar{x}\bar{y}) + 12x^2(-6\bar{x}^2) + 3x(8\bar{y}^2) \\ &= -72\bar{x}(\bar{y}^2 + \bar{x}^3) \\ &= 0. \end{cases}$$

REMARK 4.5. Let us note that one can attach, to every $\ell \in (W^n_{\mathcal{O}(X)})^\vee$, a n-th order derivation $\theta_\ell \in \mathrm{Der}_k^{(n)}(\mathcal{O}(X))$ defined by $\ell \circ \varphi^n_{\mathcal{O}(X)} \circ d^n_{\mathcal{O}(X)}$. This observation suggests the following question: *does every n-th order derivation $\theta \in \mathrm{Der}_k^{(n)}(\mathcal{O}(X))$ factorize through $W^n_{\mathcal{O}(X)}$ (in a non-unique way)*? Since every differential operator on smooth varieties are generated by derivations, we can deduce that this question admits a positive answer for smooth varieties X. This question is also related to the following one, which is stronger[1]: *does the morphism $\varphi^n_{\mathcal{O}(X)}$ admit a retraction $\psi^n_{\mathcal{O}(X)} : W^n_{\mathcal{O}(X)} \to \Omega^{(n)}_{\mathcal{O}(X)}$*? Once again, we can prove that, if the k-variety X is assumed to be smooth, this second question also admits a positive answer. It seems to us plausible that such questions are related to the singularities of X.

4.6. The existence of an isomorphism $W_B^2 \to \Omega_B^{(2)}$ for every k-algebra $B = A/I$ of finite type provides new algorithms to compute $\mathrm{Tors}(\Omega_B^{(2)})$. Indeed, after identifying $\mathrm{Tors}(\Omega_B^{(2)})$ with $\mathrm{Tors}(W_B^2)$, one can apply the algorithms introduced in [9, §5] whose output will provide a presentation for $\mathrm{Tors}(W_B^2)$. We denote by $[I]$ the ideal generated by the $\Delta_s(f)$, with $f \in I$ and $s \in \mathbf{N}$, in the ring A_∞. Precisely, these algorithms will compute, in this particular case, a Groebner basis for the ideal $\mathcal{N}_2 = \sqrt{[I]} \cap A_2$ in the ring A_2. This Groebner basis obviously gives rise to a generating system for $\mathrm{Tors}(W_B^2)$ by lemma 4.1. See example 4.7. (See also [5, 8] for related considerations).

EXAMPLE 4.7. To illustrate this remark, let us consider the polynomial $f = x^3 + y^2 \in k[x, y]$, with $B = A/\langle f\rangle$. We set $E(f) = 3y_0x_1 - 2x_0y_1$. Here, [9, §5] applied with the lexicographic order and ordering $y_2 > y_1 > y_0 > x_2 > x_1 > x_0$, provides a Groebner basis for the nilpotent functions in $\mathcal{O}(B_\infty)$ induced by polynomials in A_2. From this computation we deduce in particular a presentation of $\mathrm{Tors}(W_B^2)$ by "picking out" the elements with weight $w \leq 2$ (see lemma 4.1). We obtain that $\mathrm{Tors}(W_B^2)$ coincides with

$$\pi_2(\langle fW_A^2, x_1E(f), y_1E(f), 9x_0x_1^2 + 4y_1^2, 4x_0y_2 - x_1y_1 - 6x_2y_0, 8y_0y_2 + 12x_0^2x_2 + 3x_0x_1^2\rangle)$$

Then we deduce that $\mathrm{Tors}(\Omega_B^{(2)})$ is isomorphic to the quotient of $\Omega_A^{(2)} \otimes_A B$ by the submodule

[1]Actually, this second question is equivalent to the problem to determine whether, for every $\mathcal{O}(X)$-module M, for every n-th order derivation $\theta \in \mathrm{Der}_k^{(n)}(\mathcal{O}(X), M)$, there exists a morphism $\ell \in \mathrm{Hom}_{\mathcal{O}(X)}(W^n_{\mathcal{O}(X)}, M)$ such that $\theta = \ell \circ \varphi^n_{\mathcal{O}(X)} \circ d^n_{\mathcal{O}(X)}$.

generated by the images of the following elements:

$$\begin{cases} 3yd_A(x)^2 - 2xd_A(x)d_A(y), \\ 3yd_A(x)d_A(y) - 2xd_A(y)^2, \\ 9xd_A(x)^2 + 4d_A(y)^2, \\ 4xd_A(y) - d_A(x)d_A(y) - 6yd_A(x), \text{ and} \\ 8yd_A(y) + 12x^2d_A(x) + 3xd_A(x)^2. \end{cases}$$

4.8. Let us prove corollary 1.6. We set $B = \mathcal{O}(X)$. By theorem 1.4, we need to prove the corresponding properties for the $\mathcal{O}(X)$-module $\Omega^{(2)}_B$. By [11, Theorem 9], one knows that $\Omega^{(2)}_{B_x} \cong \Omega^{(2)}_B \otimes_B B_x$ for every point $x \in X$

∘ Since the noetherian ring B is regular if and only if B_x is regular for every point $x \in X$, [3, Proposition 4.1] proves assertion (2).

∘ From [1, Theorem 4.3], following the same argument, we also deduce that X is normal if and only if $\Omega^{(2)}_{B_x}$ is torsionfree for every point $x \in X$. We conclude the proof of the first equivalence in assertion (1) by applying (4.3) to $M = \Omega^{(2)}_B$. The last equivalence in assertion (1) directly follows from lemma 4.1.

Example 4.9. Let k be a field of characteristic zero. Let us consider the polynomial $f = x_1^3 + x_2^3 + x_3^3$ in the ring $k[x_1, x_2, x_3]$ with associated surface $H \subset \mathbf{A}^3_k$. It is well-known that this k-variety is a normal variety with a singular point at the origin which is not a rational singularity. Let us also note that its tangent space is reduced, as every normal hypersurface of an affine space. In particular, $W^1_{\mathcal{O}(H)}$ is torsionfree, i.e., there is no nontrivial isobaric function on $\mathscr{L}_\infty(X)$ with weight 1 which are nilpotent. Indeed, by subsection 3.2, we know that it means that $\Omega^1_{\mathcal{O}(H)}$ is torsionfree; this property is implied by the normality of H (see [6, Corollary 9.8]). There also is no nontrivial nilpotent isobaric function on $\mathscr{L}_\infty(X)$ with weight 2 by corollary 1.6. This observation can also be checked by a direct computation. Indeed, the algorithms introduced in [9] confirms this result. Moreover, with this tool, we observe for example that the regular function induced by the polynomial $g := x_{10}^2x_{20}x_{21}x_{30}x_{32} - x_{10}x_{11}x_{20}^2x_{30}x_{32} + x_{10}^2x_{20}x_{21}x_{31}^2 - x_{10}x_{11}x_{20}^2x_{31}^2 - x_{10}^2x_{20}x_{22}x_{30}x_{31} - x_{10}^2x_{21}^2x_{30}x_{31} + x_{10}x_{12}x_{20}^2x_{30}x_{31} + x_{11}^2x_{20}^2x_{30}x_{31} + x_{10}x_{11}x_{20}x_{22}x_{30}^2 + x_{10}x_{11}x_{21}^2x_{30}^2 - x_{10}x_{12}x_{20}x_{21}x_{30}^2 - x_{11}^2x_{20}x_{21}x_{30}^2$ induces a nilpotent function on $\mathscr{L}_\infty(X)$ (see lemma 4.1); but it is isobaric with weight 3.

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