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REMARKS ON BITRATIONAL RELATIONS OF TORIC MORI FIBER SPACES

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Abstract

We can run the MMP for any divisor on any $\mathbb{Q}$-factorial projective toric variety. We show that two Mori fiber spaces, which are outputs of the above MMP, are connected by finitely many elementary transforms.

Contents

1. Introduction

The minimal model program works for $\mathbb{Q}$-factorial projective toric varieties. Namely, for any $\mathbb{Q}$-factorial projective toric variety X and any $\mathbb{R}$-divisor D on X, one can run the D-MMP and it ends up with either a minimal model or a Mori fiber space. For details, see [5].

The purpose of this paper is to establish the following theorem.

Theorem 1.1. *Let Z be a $\mathbb{Q}$-factorial projective toric variety and let D_Z be an $\mathbb{R}$-divisor on Z. Let $\phi : X \to S$ and $\psi : Y \to T$ be two Mori fiber spaces, which are outputs of the D_Z-MMP.*

Then the induced birational map $\sigma : X \dashrightarrow Y$ is a composition of finitely many Sarkisov links (cf. Definition 1.6*).*

Theorem 1.1 is known as the log Sarkisov program (cf. [2]). By combining Theorem 1.1 and the following easy lemma, we obtain Theorem 1.3 called the Sarkisov program for toric varieties.

Lemma 1.2 (cf. [2])**.** *Let $\phi : X \to S$ and $\psi : X \to Y$ be two Mori fiber spaces with $\mathbb{Q}$-factorial terminal singularities. If X and Y are birational, then there is a smooth projective variety Z such that ϕ and ψ are outputs of the K_Z-MMP.*

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Theorem 1.3 (Sarkisov program for toric varieties, [10])**.** *Let $\phi : X \to S$ and $\psi : Y \to T$ be two toric Mori fiber spaces with $\mathbb{Q}$-factorial terminal singularities. If X and Y are birational, then the induced birational map $\sigma : X \dashrightarrow Y$ is a composition of finitely many Sarkisov links.*

Let X be a $\mathbb{Q}$-factorial projective toric variety and let D be a Weil divisor on X. Then there exists a positive integer r such that rD is linearly equivalent to a torus-invariant Cartier divisor on X. Hence, any $\mathbb{R}$-divisor on X is $\mathbb{R}$-linearly equivalent to a torus-invariant $\mathbb{R}$-Cartier divisor. Therefore, it is sufficient to prove the following theorem for Theorem 1.1.

Theorem 1.4. *Let Z be a $\mathbb{Q}$-factorial projective toric variety and let D_Z be a* torus-invariant *$\mathbb{R}$-divisor on Z. Let $\phi : X \to S$ and $\psi : Y \to T$ be two Mori fiber spaces, which are outputs of the D_Z-MMP.*

Then the induced birational map $\sigma : X \dashrightarrow Y$ is a composition of finitely many Sarkisov links.

The following corollary immediately follows from Proposition 2.7.

Corollary 1.5 (Log Sarkisov program for toric lc pairs)**.** *Let (Z, B) be a $\mathbb{Q}$-factorial projective toric lc pair. Let $\phi : X \to S$ and $\psi : Y \to T$ be two Mori fiber spaces, which are outputs of the $(K_Z + B)$-MMP.*

Then the induced birational map $\sigma : X \dashrightarrow Y$ is a composition of finitely many Sarkisov links.

We note that toric lc pairs defined in Definition 2.4. Corollary 1.5 was first established by Matsuki and Sharmov (cf. [10, Chapter 14] and [13]). This proof is based on the original idea by Sarkisov (cf. [3]). In their proof, we keep track of three invariants, called the Sarkisov degree, associated with the singularities and we need to check that the Sarkisov degree satisfies the ascending chain condition. Thus, this method heavily depends on a detailed study of the singularities. On the other hand, our approach is quite different from this. We use "the geography of models" instead of the Sarkisov degree. We remark that we can not use the traditional approach by Corti and Matsuki as we treat (not necessarily effective) divisors in this paper. We note that this idea is based on [6] and [11].

At first, we introduce the notation of Sarkisov links for toric varieties.

DEFINITION 1.6 (SARKISOV LINKS). Let Z be a $\mathbb{Q}$-factorial projective toric variety and let D_Z be an $\mathbb{R}$-divisor on Z. Let $\phi : X \to S$ and $\psi : Y \to T$ be two Mori fiber spaces, which are outputs of the D_Z-MMP.

The induced birational map $\sigma : X \dashrightarrow Y$ between ϕ and ψ is called a *Sarkisov link* if it is one of the following four types:

In the above commutative diagram, the vertical arrows p and q are divisorial contractions, and the horizontal dotted arrows are compositions of finitely many flops for the D'_Z-MMP, where D'_Z is an $\mathbb{R}$-divisor on the top left space, that is, X' or X. The spaces X', Y' and R are realized as the ample models of $\mathbb{R}$-divisors on Z (cf. Definition 2.8 for definition of ample models). Moreover, these spaces are the results of running the D_Z-MMP (see Definition 2.11 and Lemma 3.8). Links of Type (IV) are separated into two types: (IV$_m$) and (IV$_s$). In a link of Type (IV$_m$), s and t have both Mori fiber structures and R is $\mathbb{Q}$-factorial. In a link of Type (IV$_s$), s and t are small birational contractions and R is not $\mathbb{Q}$-factorial. We note that

Type (I)

$$\begin{tikzcd} X' \arrow[r, dashed] \arrow[d, "p"'] & Y \arrow[d, "\psi"] \\ X \arrow[d, "\phi"'] & T \arrow[dl] \\ S & \end{tikzcd}$$

Type (II)

$$\begin{tikzcd} X' \arrow[r, dashed] \arrow[d, "p"'] & Y' \arrow[d, "q"] \\ X \arrow[d, "\phi"'] & Y \arrow[d, "\psi"] \\ S \arrow[r, equal] & T \end{tikzcd}$$

Type (III)

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d, "\phi"'] & Y' \arrow[d, "q"] \\ S \arrow[dr] & Y \arrow[d, "\psi"] \\ & T \end{tikzcd}$$

Type (IV)

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[d, "\phi"'] & & Y \arrow[d, "\psi"] \\ S \arrow[dr, "s"'] & & T \arrow[dl, "t"] \\ & R & \end{tikzcd}$$

links of Type (IV_s) do not appear for $\dim Z \leq 3$.

Example 1.7. For toric 3-folds with terminal singularities, links of Type (I), (II), (III) and (IV_m) are completely classified and we can find various examples (e.g., [12]).

Next, we construct an easy example of links of Type (IV_s). Let $S \to R \leftarrow T$ be a flop for a divisor and we put $X = S \times \mathbb{P}^1$ and $Y = T \times \mathbb{P}^1$. Then here is a link of Type (IV_s).

The contents of this paper are as follows: In Section 2, we quickly recall some basic definitions and properties of the minimal model theory. In Section 3, we prove Theorem 1.4. In this paper, we will work over an arbitrary algebraically closed field of any characteristic.

2. Preliminaries

Notation and Conventions. A *contraction morphism* is a proper morphism $g : X \to Y$ between varieties with $g_*\mathcal{O}_X = \mathcal{O}_Y$. If X and Y are both normal, the condition above is equivalent to the one that g is a surjective morphism with connected fibers. A rational map $g' : X \dashrightarrow Y'$ to a variety is a *rational contraction* if there is a common resolution $p : W \to X$ and $q : W \to Y'$ which are contraction morphisms.

A birational map $f : X \dashrightarrow Z$ between normal varieties is a *birational contraction* if f is proper and f^{-1} does not contract any divisors. We say that f is *small* if f and f^{-1} are both birational contractions.

Let C be a polytope in a finite-dimensional real vector space. The *span* of C is the span of C as an affine subspace. The *relative interior* of C is the interior of C in the affine space spanned by C.

We say that a real vector space $\mathcal{V}_0$ *is defined over* $\mathbb{Q}$ if there is a rational vector space $\mathcal{V}'$ such that $\mathcal{V}_0 = \mathcal{V}' \otimes_{\mathbb{Q}} \mathbb{R}$. We say that an affine subspace $\mathcal{H}$ of a real vector space $\mathcal{V}_0$, which is defined over $\mathbb{Q}$, *is defined over* $\mathbb{Q}$ if $\mathcal{H}$ is spanned by rational vectors of $\mathcal{V}_0$.

DEFINITION 2.1 (DIVISORS). Let $\pi : X \to U$ be a projective morphism from a normal variety to a variety. Two $\mathbb{R}$-Divisors D_1 and D_2 on X are *$\mathbb{R}$-linearly equivalent over U* (denoted by $D_1 \sim_{\mathbb{R},U} D_2$) if there is an $\mathbb{R}$-Cartier divisor B on U such that $D_1 - D_2 \sim_{\mathbb{R}} \pi_* B$. Two $\mathbb{R}$-Cartier divisors D_1 and D_2 on X are *numerically equivalent over U* (denoted by $D_1 \equiv_U D_2$) if $D_1 - D_2 \cdot C = 0$ for any curve $C \subset X$ contained in a fiber of π. The *real linear system* of an $\mathbb{R}$-divisor D on X over U is defined as

$$|D/U|_{\mathbb{R}} = \{D' \geq 0 \mid D' \sim_{\mathbb{R},U} D\}.$$

Moreover, the *stable base locus* of D over U is defined as

$$\mathbf{B}(D/U) = \bigcap_{D' \in |D/U|_{\mathbb{R}}} D'.$$

We consider $\mathbf{B}(D/U)$ with the reduced scheme structure. When U is a point, we drop U from the notation, e.g., we simply write $\equiv$ and $\mathbf{B}(D)$ instead of $\equiv_U$ and $\mathbf{B}(D/U)$, respectively.

For an $\mathbb{R}$-divisor on X and its prime decomposition $D = \sum a_i D_i$, we define

$$\|D\| = \max\{|a_i|\}.$$

Moreover, for a subset $S \subset \mathbb{R}$, we denote $D \in S$ if $a_i \in S$.

DEFINITION 2.2. Let $N \simeq \mathbb{Z}^n$ be a lattice of rank n. A *toric variety* $X(\Delta)$ is associated to a fan Δ, a finite collection of convex cones $\sigma \subset N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ satisfying:

(i) Each convex cone σ is rational polyhedral in the sense that there are finitely many $v_1, \dots, v_k \in N \subset N_{\mathbb{R}}$ such that

$$\sigma = \{r_1 v_1 + \cdots + r_k v_k | r_i \in \mathbb{R}_{\geq 0} \text{ for all } i\}$$

and it is strongly convex in the sense

$$\sigma \cap -\sigma = \{0\}.$$

(ii) Each face τ of a convex cone $\sigma \in \Delta$ is again contained in Δ.

(iii) The intersection of two cones in Δ is a face of each.

DEFINITION 2.3 (RELATIVE PICARD NUMBERS). Let $f : X \to Y$ be a proper morphism between normal varieties. We define

$$N^1(X/Y) = \{\mathrm{Pic}(X)/\equiv_Y\} \otimes_{\mathbb{Z}} \mathbb{R}$$

and

$$N_1(X/Y) = \{Z_1(X/Y)/\equiv_Y\} \otimes_{\mathbb{Z}} \mathbb{R},$$

where $Z_1(X/Y)$ is the free abelian group of 1-cycles of X over Y. These are inducing the following non-degenerate bilinear pairing:

$$N^1(X/Y) \times N_1(X/Y) \to \mathbb{R}.$$

It is well-known that $N^1(X/Y)$ and $N_1(X/Y)$ are finite-dimensional real vector spaces. We write

$$\rho(X/Y) = \dim_{\mathbb{R}} N^1(X/Y) = \dim_{\mathbb{R}} N_1(X/Y)$$

and call it the *relative Picard number* of X over Y. We write $\rho(X) = \rho(X/Y)$ and $N^1(X) = N^1(X/Y)$ when Y is a point. We simply call $\rho(X)$ the *Picard number* of X.

If f is a surjective morphism of projective toric varieties with connected fibers, then

$$\rho(X/Y) = \rho(X) - \rho(Y)$$

by [4, Theorem 6.3.12]. For details, see [7, Lemma 3-2-5 (3)].

Definition 2.4 (Singularities of pairs). Let X be a normal variety and $D \geq 0$ be an $\mathbb{R}$-divisor on X. We say that (X, D) is a *pair* if $K_X + D$ is $\mathbb{R}$-Cartier. In addition, we say that a pair (X, D) is *toric* if X is toric and D is consisting of torus-invariant divisors.

Let (X, D) be a pair and let $f : Y \to X$ be a proper birational morphism from a normal variety Y. Then we can write

$$K_Y = f^*(K_X + D) + \sum a_i E_i.$$

We say that (X, D) is *klt* (resp. *lc*) if $a_i > -1$ (resp. $a_i \geq -1$) for any f and i. We say that X is $\mathbb{Q}$-*factorial* if every Weil divisor on X is $\mathbb{Q}$-Cartier. In addition, we say that a pair (X, D) is $\mathbb{Q}$-*factorial* if so is X.

Remark 2.5. For any normal toric variety X and any torus-invariant $\mathbb{R}$-divisor D on X, there exists a log resolution $f : Y \to X$ of D. More precisely, f is a projective birational morphism from a smooth toric variety Y such that $\mathrm{Exc}(f)$ is a divisor and $\mathrm{Exc}(f) \cup f^{-1}(\mathrm{Supp}\, D)$ is an SNC divisor. For details, see [4, Chapter 11].

The following lemma is the combinatorial characterization of $\mathbb{Q}$-factoriality in toric geometry.

Lemma 2.6 ([4, Proposition 4.2.7])**.** *Let $X = X(\Delta)$ be a toric variety. Then X is $\mathbb{Q}$-factorial if and only if each of $\sigma \in \Delta$ is simplicial.*

The following proposition is the well-known characterization of toric lc pairs.

Proposition 2.7 ([4, Proposition 11.4.24])**.** *Let (X, D) be a toric pair. If $D \in [0, 1]$, then (X, D) is lc. In addition, if $D \in [0, 1)$, then it is klt.*

Definition 2.8 (Ample models). Let X be a normal projective variety and let D be an $\mathbb{R}$-Cartier divisor on X. Then a rational contraction $g : X \dashrightarrow Y$ is the *ample model* of D if

- Y is normal and projective, and
- there is an ample $\mathbb{R}$-Cartier divisor H on Y such that if $p : W \to X$ and $q : W \to Y$ are a common resolution, and we write $p^*D \sim_{\mathbb{R}} q^*H + E$, where $E \geq 0$, then $B \geq E$ for any $B \in |p^*D|_{\mathbb{R}}$.

For the basic properties of ample models, see [1, Lemma 3.6.6].

Remark 2.9. If g is birational, then E is q-exceptional. If X is toric and D is pseudo-effective, then we can construct the ample model of D in toric geometry. By the uniqueness of ample models (cf. [1, Lemma 3.6.6 (1)]), Y is always toric. We remark that in Section 3, we always assume that D is pseudo-effective when we treat ample models.

Definition 2.10. Let $f : X \dashrightarrow Y$ be a proper birational contraction of normal varieties and let D be an $\mathbb{R}$-Cartier divisor on X such that f_*D is also $\mathbb{R}$-Cartier. Then we say that f is *D-non-positive* (resp. *D-negative*) if there is a common resolution $p : W \to X$ and $q : W \to Y$ such that

$$p^*D = q^* f_* D + E,$$

where $E \geq 0$ is q-exceptional (resp. $E \geq 0$ is q-exceptional and whose support contains the strict transform of the f-exceptional divisors).

We close this section with definition of minimal models and Mori fiber spaces.

Definition 2.11. Let $f : X \dashrightarrow Y$ be a birational contraction of normal projective varieties and let D be an $\mathbb{R}$-Cartier divisor on X such that f_*D is also $\mathbb{R}$-Cartier.

We say that f is a *weak log canonical model* of D if

- f is D-non-positive and
- f_*D is nef.

We say that f is a *minimal model* of D if

- Y is $\mathbb{Q}$-factorial,
- f is D-negative and
- f_*D is nef.

Let $\phi : X \to S$ be a contraction morphism to a normal projective variety. We say that ϕ is a *Mori fiber space* of D if

- X is $\mathbb{Q}$-factorial,
- $-D$ is ϕ-ample,
- $\rho(X/S) = \rho(X) - \rho(S) = 1$ and
- $\dim S < \dim X$.

We say that ϕ has a *Mori fiber structure* if ϕ is a Mori fiber space of some $\mathbb{R}$-Cartier divisor.

We say that f is the *output* of the D-MMP if f is a minimal model of D or a Mori fiber space of D. On the other hand, we say that f is the *result* of running the D-MMP if f is any sequence of divisorial contractions and flips for the D-MMP. We emphasize that the result of running the D-MMP is not necessarily a minimal model of D or a Mori fiber space of D.

3. Proof of Theorem 1.4

In this section, we will closely follow [6, Section 3, 4].

Symbols 3.1. Let Z be a $\mathbb{Q}$-factorial projective toric variety.

- $\mathcal{V}(Z)$ is the real vector space generated by all torus-invariant prime divisors on Z.

Let $\mathcal{B}$ be a convex polytope in $\mathcal{V}(Z)$. Let $f : Z \dashrightarrow X$ be a birational contraction to a normal projective variety X and let $g : Z \dashrightarrow Y$ be a rational contraction to a normal projective variety Y. Then we define

$$\mathcal{E}(\mathcal{B}) = \{D_Z \in \mathcal{B} \mid D_Z \text{ is pseudo-effective}\},$$
$$\mathcal{M}_f(\mathcal{B}) = \{D_Z \in \mathcal{E}(\mathcal{B}) \mid f \text{ is a minimal model of } D_Z\},$$

$$\mathcal{A}_g(\mathcal{B}) = \{D_Z \in \mathcal{E}(\mathcal{B}) \mid g \text{ is the ample model of } D_Z\},$$
$$\mathcal{N}(\mathcal{B}) = \{D_Z \in \mathcal{E}(\mathcal{B}) \mid D_Z \text{ is nef}\}$$

and we denote the closure of $\mathcal{A}_g(\mathcal{B})$ by $\mathcal{C}_g(\mathcal{B})$. We simply write $\mathcal{A}_g$ to denote $\mathcal{A}_g(\mathcal{B})$ if there is no risk of confusion.

In this section, we fix the following notation unless otherwise mentioned:

- Z is a $\mathbb{Q}$-factorial projective toric variety and
- $\mathcal{B}$ is a convex polytope of $\mathcal{V}(Z)$, which is defined over $\mathbb{Q}$.

Proposition 3.2. *There are only finitely many rational contractions $g_i : Z \dashrightarrow X_i$ $(1 \leq i \leq l)$ such that*

$$\mathcal{E}(\mathcal{B}) = \bigcup_{i=1}^{l} \mathcal{A}_{g_i},$$

where $\mathcal{A}_{g_i} \neq \mathcal{A}_{g_j}$ for $i \neq j$.

Proof. It follows from the finiteness of minimal models (see [4, Theorem 15.5.15]) and the property of ample models (cf. [1, Lemma 3.6.6]). □

The following two statements come from [6, Theorem 3.3] and these are easy consequences of the minimal model theory. Thus, we sketch the idea of the proofs. For the details, see [6, Theorem 3.3 (2), (3)].

Proposition 3.3. *With notation as in* Proposition 3.2. *If $\mathcal{A}_{g_j} \cap \mathcal{C}_{g_i} \neq \emptyset$ for $1 \leq i, j \leq l$, then there is a contraction morphism $g_{i,j} : X_i \to X_j$ such that $g_j = g_{i,j} \circ g_i$.*

Sketch of Proof. We take $D_Z \in \mathcal{A}_{g_i}$. Running the D_Z-MMP, we end up with a minimal model $f : Z \dashrightarrow X$ of D_Z. Then there is a contraction morphism $g : X \to X_i$ such that $g_i = g \circ f$. Using this morphism g, we can construct a semi-ample $\mathbb{R}$-divisor on X_i associated to the contraction morphism satisfying the desired property. □

Proposition 3.4. *With notation as in* Proposition 3.2. *Assume that $\mathcal{B}$ spans $N^1(Z)$. For any $1 \leq i \leq l$, the following are equivalent:*

- *there is a rational polytope C contained in $\mathcal{C}_{g_i}$ which intersects the interior of $\mathcal{B}$ and spans $\mathcal{B}$.*
- *g_i is birational and X_i is $\mathbb{Q}$-factorial.*

Sketch of Proof. Suppose that C spans $\mathcal{B}$. We take D_Z belonging to the relative interior of $C \cap \mathcal{A}_{g_i}$ and belonging to the interior of $\mathcal{B}$. Running the D_Z-MMP, we end up with a minimal model $f : Z \dashrightarrow X$ of D_Z. Then there is the index $1 \leq j \leq l$ such that $f = g_j$. Since D_Z belongs to the relative interior of $\mathcal{A}_{g_i}$, we see that $i = j$. Thus, $g_i = f$ is birational and $X_i = X$ is $\mathbb{Q}$-factorial. It is easy to see the converse. □

The following proposition is the key ingredient of this paper.

Proposition 3.5 (cf. [6, Theorem 3.3 (4)])**.** *With notation as in* Proposition 3.2. *Assume that $\mathcal{B}$ spans $N^1(Z)$. If $\mathcal{C}_{g_i}$ spans $\mathcal{B}$ and D_Z is a general point of $\mathcal{A}_{g_j} \cap \mathcal{C}_{g_i}$, which is also a point of the interior of $\mathcal{B}$ for $1 \leq i, j \leq l$, then $\rho(X_i/X_j) = \dim \mathcal{C}_{g_i} - \dim \mathcal{C}_{g_j} \cap \mathcal{C}_{g_i}$.*

Proof. Putting $X = X_i$ and $f = g_i$, by Proposition 3.4, X is $\mathbb{Q}$-factorial and f is birational. Let $E_1, \dots, E_k$ be all f-exeptional prime divisors. Since $\mathcal{B}$ spans $N^1(Z)$, we can take $B_i \in \mathcal{V}(Z)$, which are linear combinations of the elements of $\mathcal{B}$, such that $B_i \equiv E_i$, and we put $B_0 = \sum B_i$ and $E_0 = \sum E_i$. Since D_Z is contained in the interior of $\mathcal{B}$, there is a sufficiently small rational number $\delta > 0$ such that $D_Z + \delta B_0 \in \mathcal{B}$. Then f is $(D_Z + \delta E_0)$-negative and so it is a minimal model of $D_Z + \delta E_0$ and g_j is the ample model of $D_Z + \delta E_0$. Thus, $D_Z + \delta B_0 \in \mathcal{M}_f(\mathcal{B})$ and $D_Z + \delta B_0 \in \mathcal{A}_{g_j}$. In particular, $D_Z + \delta B_0 \in \mathcal{A}_{g_j} \cap \mathcal{C}_f$. Since we have $D_Z \in \mathcal{A}_{g_j} \cap \mathcal{C}_f$ in general, $D_Z \in \mathcal{M}_f(\mathcal{B})$ and so f is D_Z-negative.

We fix a sufficiently small rational number $\epsilon > 0$ such that if $D'_Z \in \mathcal{B}$ with $\|D'_Z - D_Z\| < \epsilon$, then $D'_Z \in \mathcal{B}$ and f is D_Z-negative. Then $D'_Z \in \mathcal{C}_f$ if and only if $D'_X = f_* D'_Z$ is nef.

For any $(a_1, \dots, a_k) \in \mathbb{R}^k$, we put $E = \sum a_i E_i$ and $B = \sum a_i B_i$. We put $\mathcal{B}_X = \{D'_X = f_* D'_Z \mid D'_Z \in \mathcal{B}\} \subset \mathcal{V}(X)$. Then $D'_X \in \mathcal{N}(\mathcal{B}_X)$ if and only if $D'_X + f_* B \in \mathcal{N}(\mathcal{B}_X)$ as $D'_Z + B$ is numerically equivalent to $D'_Z + E$. This means that

$$\mathcal{C}_f \simeq \mathcal{N}(\mathcal{B}_X) \times \mathbb{R}^k$$

in a neighbourhood of D_Z.

By the above argument and [1, Lemma 3.6.6], $D'_Z \in \mathcal{A}_{g_j} \cap \mathcal{C}_f$ if and only if it holds that $D'_X = f_* D'_Z \in \mathcal{N}(\mathcal{B}_X)$, and there is an ample $\mathbb{R}$-Cartier divisor H on X_j such that $f_* D'_Z = (g_{i,j})^* H$, where $g_{i,j} : X \to X_j$ is a contraction morphism. Since $D_Z \in \mathcal{A}_{g_j} \cap \mathcal{C}_f$, there is an ample $\mathbb{R}$-Cartier divisor H_0 on X_j such that $f_* D_Z = (g_{i,j})^* H_0$ and so there are ample $\mathbb{R}$-Cartier divisors $H_1, \dots, H_{\rho(X_j)}$, whose images on $N^1(X_j)$ are linearly independent, such that $f_* D'_Z = (g_{i,j})^*(H_0 + \sum b_i H_i)$ for any $(b_1, \dots, b_{\rho(X_j)}) \in \mathbb{R}^{\rho(X_j)}$ with $H_0 + \sum b_i H_i$ is ample. Thus

$$\dim \mathcal{C}_{g_j} \cap \mathcal{C}_f = k + \rho(X_j).$$

Therefore, we obtain

$$\begin{aligned} \rho(X_i/X_j) &= \rho(X_i) - \rho(X_j) \\ &= \dim \mathcal{N}(\mathcal{B}_X) - \rho(X_j) \\ &= \dim \mathcal{C}_f - \dim \mathcal{C}_{g_j} \cap \mathcal{C}_f. \end{aligned}$$

□

We recall the following Bertini-type statement for the reader's convenience.

Lemma 3.6 (cf. [6, Corollary 3.4]). *Let $\mathcal{P}$ be a convex polytope in $\mathcal{V}(Z)$ which spans $N^1(Z)$. Then for any general affine subspace $\mathcal{H} \subset \mathcal{V}(Z)$, the intersection $\mathcal{P} \cap \mathcal{H}$ of $\mathcal{P}$ and $\mathcal{H}$ satisfies the conclusions of* Proposition 3.4 *and* 3.5.

Lemma 3.7. *Assume that $\mathcal{B}$ satisfies the conclusion of* Propositions 3.4 *and* 3.5, *and that* $\dim \mathcal{B} = 2$. *Let $f : Z \dashrightarrow X$ and $g : Z \dashrightarrow Y$ be two rational contractions such that* $\dim \mathcal{C}_f = 2$ *and* $\dim \mathcal{O} = 1$, *where $\mathcal{O} = \mathcal{C}_f \cap \mathcal{C}_g$. Assume that $\rho(X) \geq \rho(Y)$ and that $\mathcal{O}$ is not contained in the boundary of $\mathcal{B}$. Let D_Z be a point in the relative interior of $\mathcal{O}$ and $D_X = f_* D_Z$.*

Then there is a rational map $\pi : X \dashrightarrow Y$ with $g = \pi \circ f$ such that

(I) *$\rho(X) = \rho(Y) + 1$, π is D_X-trivial and one of the following properties holds.*

(I. i) *π is birational and $\mathcal{O}$ is not contained in the boundary of $\mathcal{E}(\mathcal{B})$, and either*

(I. i. a) *π is divisorial and $\mathcal{O} \neq \mathcal{C}_g$, or*

(I. i. b) *π is small and $\mathcal{O} = \mathcal{C}_g$,*

(I. ii) *π has a Mori fiber structure and $\mathcal{O} = \mathcal{C}_g$ is contained in the boundary of $\mathcal{E}(\mathcal{B})$,*

(II) *$\rho(X) = \rho(Y)$, π is a D_X-flop and $\mathcal{O} \neq \mathcal{C}_g$ is not contained in the boundary of $\mathcal{E}(\mathcal{B})$.*

Proof. By Proposition 3.4, f is birational and X is $\mathbb{Q}$-factorial.

If $\mathcal{O}$ is contained in the boundary of $\mathcal{E}(\mathcal{B})$, then $\dim \mathcal{C}_g = 1$ and $\mathcal{O} = \mathcal{C}_g$. By Proposition 3.5, there is a contraction $\pi : X \to Y$ which has a Mori fiber structure. This is (I. ii).

In the rest of proof, we may assume that $\mathcal{O}$ is not contained in the boundary of $\mathcal{E}(\mathcal{B})$. If $\dim \mathcal{C}_g = 1$, then $\mathcal{O} = \mathcal{C}_g$. By Proposition 3.5, there is a contraction $\pi : X \to Y$ with $\rho(X/Y) = 1$. Since D_Z is not contained in the boundary of $\mathcal{E}(\mathcal{B})$, D_Z is big and so π is birational. Thus, by Proposition 3.4, Y is not $\mathbb{Q}$-factorial and so π is small. This is (I. i. b).

We assume that $\dim \mathcal{C}_g = 2$. Then g is birational and Y is $\mathbb{Q}$-factorial. Let $h : Z \dashrightarrow W$ be the ample model of D_Z. By Proposition 3.5, there are two contractions $p : X \to W$ and $q : Y \to W$ with $\rho(X/W), \rho(Y/W) \leq 1$. Then we can explicitly calculate the Picard numbers of X and Y and there are only two cases below:

(1) $\rho(X) = \rho(Y) + 1$, or

(2) $\rho(X) = \rho(Y)$.

In (1), $h = g$ and we put $\pi = p$. Then π is divisorial and this is (I. i. a).

In (2), $\rho(X/W) = \rho(Y/W) = 1$. Then $\dim \mathcal{C}_h = 1$ since $\dim \mathcal{O} = 1$. By Theorem 3.4, W is not $\mathbb{Q}$-factorial. Thus, p and q are small and so π is D_X-flop. This is (II). □

Lemma 3.8 (cf. [6, Lemma 3.6]). *Let $f : Z \dashrightarrow X$ be a birational contraction between $\mathbb{Q}$-factorial projective toric varieties. Let D_Z and D'_Z be two torus-invariant $\mathbb{R}$-divisors on Z. If f is the ample model of D'_Z and $D'_Z - D_Z$ is ample, then f is the result of running the D_Z-MMP.*

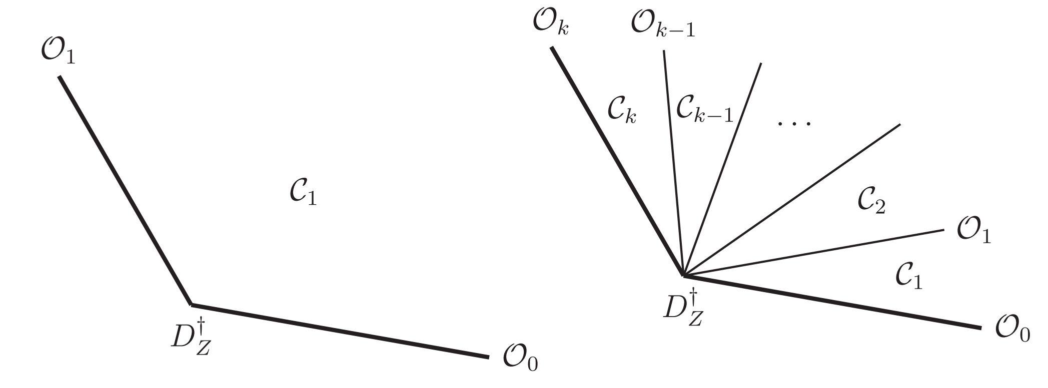
Before we will see that a certain point contained in the boundary of $\mathcal{E}(V)$ corresponds to a Sarkisov link, we introduce the following additional notation.

Notation 3.9. Assume that $\mathcal{B}$ satisfies the conclusion of Propositions 3.4 and 3.5, and that $\dim \mathcal{B} = 2$. Let $D_Z^{\dagger}$ be a point contained in the boundary of $\mathcal{E}(\mathcal{B})$ and the interior of $\mathcal{B}$. If $D_Z^{\dagger}$ is contained in only one polytope of the form $\mathcal{C}_{\bullet}$ of two-dimensional, then we assume that it is a vertex of $\mathcal{E}(\mathcal{B})$.

Let $\mathcal{C}_{f_1}, \ldots, \mathcal{C}_{f_k}$ be all two-dimensional rational polytopes containing $D_Z^{\dagger}$, where $f_i : Z \dashrightarrow X_i$ are rational contractions. Note that f_i is birational and X_i is $\mathbb{Q}$-factorial by Proposition 3.4. Renumbering $\mathcal{C}_{f_i}$ to $\mathcal{C}_i$, let $\mathcal{O}_0$ (resp. $\mathcal{O}_k$) be the intersection of $\mathcal{C}_1$ (resp. $\mathcal{C}_k$) with the boundary of $\mathcal{E}(\mathcal{B})$, and let $\mathcal{O}_i := \mathcal{C}_i \cap \mathcal{C}_{i+1}$ $(1 \leq i \leq k-1)$. Then we may assume that $\mathcal{O}_i$ is one-dimensional for any i. Let $g_i : Z \dashrightarrow S_i$ be the rational contractions associated to $\mathcal{O}_i$. We put $f = f_1 : Z \dashrightarrow X = X_1$, $g = f_k : Z \dashrightarrow Y = X_k$, $X' = X_2$ and $Y' = X_{k-1}$. Then, by Proposition 3.3, there are contraction morphisms $\phi : X \to S = S_0$ and $\psi : Y \to T = S_k$. Let $h : Z \dashrightarrow R$ be the ample model of $D_Z^{\dagger}$.

Theorem 3.10. *Let $\mathcal{B}$ and $D_Z^{\dagger}$ be notation as above. Let D_Z be an $\mathbb{R}$-divisor on Z with $D_Z^{\dagger} - D_Z$ is ample.*

Then ϕ and ψ are Mori fiber spaces, which are outputs of the D_Z-MMP, and f_i are the



$k = 1$ $\qquad$ $k \geq 2$

result of running the D_Z-MMP. Moreover, if $D_Z^\dagger$ is contained in more than two polytopes, then ϕ and ψ are connected by a Sarkisov link.

Proof. By Lemma 3.7, we have the following commutative diagram:

$$\begin{tikzcd}
X' = X_2 \arrow[rr, dashed] \arrow[d, dashed, "p"'] & & Y' = X_{k-1} \arrow[d, dashed, "q"] \\
X = X_1 \arrow[d, "\phi"'] & & Y = X_k \arrow[d, "\psi"] \\
S \arrow[dr, "s"'] & & T \arrow[dl, "t"] \\
& R &
\end{tikzcd}$$

where p, q and the horizontal arrow $X' \dashrightarrow Y'$ are birational, and ϕ and ψ have Mori fiber structures. Since $D_Z^\dagger - D_Z$ is ample, for any i we can take $D_i \in \mathcal{C}_i$ such that $D_i - D_Z$ is ample. By Lemma 3.8, f_i is the result of running the D_Z-MMP. By Proposition 3.5, there is a contraction $X_i \to R$ with $\rho(X_i/R) \leq 2$. If $\rho(X_i/R) = 0$, then $f_i = h$ and this case does not happen. If $\rho(X_i/R) = 1$, then $X_i \to R$ gives a Mori fiber structure. By Lemma 3.7, $\dim \mathcal{C}_h = 1$ and there is a facet of $\mathcal{C}_i$ contained in the boundary of $\mathcal{E}(V)$ and so $i = 1$ or k. Therefore, if $k \geq 3$, then $\rho(X_i/R) = 2$ for any $1 < i < k$. Thus, by Lemma 3.7 again, $X' \dashrightarrow Y'$ is connected by flops. Moreover, since $\rho(X'/R) = 2$, p is divisorial and s is the identity, or p is flop and s is not the identity. For q and t, similar conditions follow and there are only 7 possibilities below:

(1) $k = 1$.
(2) $k = 2$, $\rho(X/R) = 1$ and $\rho(Y/R) = 2$.
(3) $k = 2$, $\rho(X/R) = 2$ and $\rho(Y/R) = 1$.
(4) $k \geq 3$, p and q are divisorial, and s and t are the identities.
(5) $k \geq 3$, p divisorial, q is flop, s is the identity and t is not the identity.
(6) $k \geq 3$, p is flop, q is divisorial, s is not the identity and t is the identity.
(7) $k \geq 3$, p and q are flops, and s and t are not the identities.

In (1), $X = Y$ and so this is a link of Type (IV). In (2), s is the identity and $\rho(Y) \geq \rho(X)$.

By Lemma 3.7, there is a divisorial contraction $X' = Y \to X$. Thus, this is a special case of a link of Type (I). In (3), this is similar to (2) and we obtain a special case of a link of Type (III). In (4), this is a link of Type (II). In (5), this is a link of Type (I). In (6), this is a link of Type (III). In (7), this is a link of Type (IV).

The rest of the proof is that a link of Type (IV) is splitting into two types (IV_m) and (IV_s) in (1) and (7). We assume that s is a divisorial contraction. Then there is a prime divisor F on S which is contracted by s. Since $\rho(X/S) = 1$, there is a prime divisor E on X such that $mE = \phi^*F$ for some non-negative integer m. Since $D_X^\dagger = f_*D_Z^\dagger$ is numerically trivial over R, $\mathbf{B}(D_X^\dagger + E/R) = E$. Since $\rho(X/R) = 2$, by the 2-ray game (cf. [9, Chapter 6]), there are birational contractions $X \dashrightarrow V \xrightarrow{f'} W \xrightarrow{g} U$ such that f' is a divisorial contraction and g has a Mori fiber space. In (1), this is a contradiction since $X = Y$, ϕ and ψ are Mori fiber spaces, and $\rho(X/R) = 2$. In (7), we have $W = Y$ and $U = T$. Hence, we obtain a link of Type (III) and this is a contradiction. Similarly, t is not divisorial. Thus, s and t are not divisorial. If s has a Mori fiber structure, then R is $\mathbb{Q}$-factorial and so t has also a Mori fiber structure. Hence, this is a link of Type (IV_m). If s is a small contraction, then R is not $\mathbb{Q}$-factorial and so t is also small. Thus, this is a link of Type (IV_s). □

Lemma 3.11. *Let D_Z be a torus-invariant $\mathbb{R}$-divisor on Z. Let $f : Z \dashrightarrow X$ and $g : Z \dashrightarrow Y$ be the results of the D_Z-MMP. Let $\phi : X \to S$ and $\psi : Y \to T$ be two Mori fiber spaces, which are outputs of the D_Z-MMP.*

Then we can find a two-dimensional convex polytope $\mathcal{B} \subset \mathcal{V}(Z)$, which is defined over $\mathbb{Q}$, with the following properties:

(1) *$D'_Z - D_Z$ is ample for any $D'_Z \in \mathcal{E}(\mathcal{B})$,*
(2) *$\mathcal{A}_{\phi\circ f}$ and $\mathcal{A}_{\psi\circ g}$ are not contained in the boundary of $\mathcal{B}$,*
(3) *C_f and C_g are two-dimensional,*
(4) *$C_{\phi\circ f}$ and $C_{\psi\circ g}$ are one-dimensional, and*
(5) *$L := \{D'_Z \in \mathcal{E}(\mathcal{B}) \mid D'_Z$ is not big$\}$ is connected.*

Proof. We take ample torus-invariant divisors $H_1, \ldots, H_r \geq 0$, which generate $N^1(Z)$, and we put $H = H_1 + \cdots + H_r$. By assumption, there are ample divisors C on S and D on T, respectively, such that

$$-f_*D_Z + \phi^*C \text{ and } -g_*D_Z + \psi^*D$$

are both ample. Let $\epsilon > 0$ be a sufficiently small rational number. Then

$$-f_*D_Z + \epsilon f_*H + \phi^*C \text{ and } -g_*D_Z + \epsilon g_*H + \psi^*D$$

are both ample, and f and g are both $(D_Z + \epsilon H)$-negative. Replacing H by ϵH, we may assume that $\epsilon = 1$. We take a torus-invariant $\mathbb{Q}$-divisor $\widetilde{D}_Z$ on X sufficiently close to D_Z. Then

$$-f_*\widetilde{D}_Z + f_*H + \phi^*C \text{ and } -g_*\widetilde{D}_Z + g_*H + \psi^*D$$

are both ample and f and g are both $(\widetilde{D}_Z + H)$-negative. We take torus-invariant $\mathbb{Q}$-divisors H'_{r+1} on X and H'_{r+2} on Y, respectively, such that

$$H'_{r+1} \in |-f_*\widetilde{D}_Z + f_*H + \phi^*C|_{\mathbb{Q}} \text{ and } H'_{r+2} \in |-g_*\widetilde{D}_Z + g_*H + \phi^*D|_{\mathbb{Q}}.$$

There exist torus-invariant $\mathbb{Q}$-divisors H_{r+1} and H_{r+2} on Z such that

$$H_{r+1} \sim_{\mathbb{Q}} f^* H'_{r+1} \text{ and } H_{r+2} \sim_{\mathbb{Q}} g^* H'_{r+2}.$$

Let $a > 0$ be a sufficiently large rational number and we put a rational convex polytope

$$\mathcal{B}_0 = \left\{ \widetilde{D}_Z + a \sum_{i=1}^{r+2} t_i H_i \,\middle|\, \sum_{i=1}^{r+2} t_i \leq 1, t_i \geq 0 \right\}.$$

Possibly replacing H_i by suitable ones, we may assume that (2) holds for $\mathcal{B}_0$.

On the other hand, since f is $(\widetilde{D}_Z + H + H_{r+1})$-negative and $(\phi \circ f)_*(\widetilde{D}_Z + H + H_{r+1})$ is ample, $\widetilde{D}_Z + H + H_{r+1} \in \mathcal{A}_{\phi\circ f}(\mathcal{B}_0)$. Similarly, $\widetilde{D}_Z + H + H_{r+2} \in \mathcal{A}_{\psi\circ g}(\mathcal{B}_0)$. Since f is a weak log canonical model of $\widetilde{D}_Z + H + H_{r+1}$, $\widetilde{D}_Z + H + H_{r+1} \in \mathcal{C}_f(\mathcal{B}_0)$. Similarly, $\widetilde{D}_Z + H + H_{r+2} \in \mathcal{C}_g(\mathcal{B}_0)$.

Let $\mathcal{H}_0$ be the translation by $\widetilde{D}_Z$ of the affine subspace generated by $H+H_{r+1}$ and $H+H_{r+2}$ and let $\mathcal{H}$ be a small perturbation of $\mathcal{H}_0$, which is defined over $\mathbb{Q}$. Putting $\mathcal{B} = \mathcal{B}_0 \cap \mathcal{H}$, $\mathcal{B}$ satisfies (1) and (2). Since $\mathcal{B}_0$ spans $N^1(Z)$, $\mathcal{C}_{\phi\circ f}(\mathcal{B}_0)$ spans $\mathcal{B}_0$. Thus, by Lemma 3.6, $\mathcal{B}$ satisfies (3). By Proposition 3.5, $\dim \mathcal{C}_{\phi\circ f}(\mathcal{B}) = \dim \mathcal{C}_{\psi\circ g}(\mathcal{B}) = 1$ and so $\mathcal{B}$ satisfies (4).

Finally, we see that we can take $\mathcal{B}$ satisfying (5). Since ϕ and ψ are Mori fiber spaces, we may assume that $\rho(Z) \geq 2$. There is a surjective linear map from $\mathcal{V}(Z)$ to $N^1(Z)$. Then the pullback of the pseudo-effective cone $\overline{\mathrm{Eff}}(Z) \subset N^1(Z)$ via this map is the convex polyhedron $\mathcal{P}$ containing a $(\dim Z)$-dimensional vector subspace V since $\dim \mathcal{V}(Z) = \rho(Z) + \dim Z$. Then possibly replacing H_i by suitable ones, we can take a two-dimensional rational convex polytope $\mathcal{B}$, which does not contan V, since $\operatorname{codim} V = \rho(Z) \geq 2$. Thus, $\mathcal{B}$ satisfies (5) as $\mathcal{E}(\mathcal{B}) = \mathcal{B} \cap \mathcal{P}$. □

Proof of Theorem 1.4. We take a two-dimensional rational convex polytope $\mathcal{B} \subset \mathcal{V}(Z)$ given by Lemma 3.11. We take $D_0 \in \mathcal{A}_{\phi\circ f}$ and $D_1 \in \mathcal{A}_{\psi\circ g}$ belonging to the interior of $\mathcal{B}$. As $\mathcal{B}$ is two-dimensional, removing two points D_0 and D_1, the boundary of $\mathcal{E}(\mathcal{B})$ separates into two parts. Then one of the two parts of the boundary of $\mathcal{E}(\mathcal{B})$ is contained in L by Lemma 3.11 (5). Tracing this part from D_0 to D_1, we obtain finitely many points D_i $(2 \leq i \leq k)$, which are contained in rational polytopes of two-dimensional. By Theorem 3.10, each of D_i gives a Sarkisov link and σ is connected by these links. □

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