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THE INTERSECTION POLYNOMIALS OF A VIRTUAL KNOT III: CHARACTERIZATION

Ryuji HIGA, Takuji NAKAMURA, Yasutaka NAKANISHI and Shin SATOH

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Abstract

We introduced three kinds of invariants of a virtual knot called the first, second, and third intersection polynomials in the first paper [5]. We also gave the connected sum formulae of the intersection polynomials in the second paper [6]. In this paper, we give characterizations of intersection polynomials.

1. Introduction

This paper is a continuation of [5, 6]. In the first paper [5], we defined three kinds of invariants of a virtual knot, which are called the first, second, and third intersection polynomials. We also studied the symmetry of a virtual knot and calculated the intersection polynomials of virtual knots with crossing number four or less. In the second paper [6], we gave a precise definition of a connected sum of virtual knots and the connected sum formulae of the intersection polynomials. As a corollary, we showed that there are infinitely many connected sums of any pair of virtual knots.

It is known that the writhe polynomial $W_K(t)$ of a virtual knot K satisfies $W_K(1) = W'_K(1) = 0$. Conversely, for any Laurent polynomial $f(t)$ with $f(1) = f'(1) = 0$, there is a virtual knot K with $W_K(t) = f(t)$ [13].

In this paper, we study fundamental properties of the intersection polynomials and characterizations of the polynomials. For the first intersection polynomial $I_K(t)$, we will prove $I_K(1) = I'_K(1) = 0$. This property characterizes the first intersection polynomial as follows.

Theorem 1.1. *For $f(t) \in \mathbb{Z}[t, t^{-1}]$, there is a virtual knot K with $I_K(t) = f(t)$ if and only if $f(1) = f'(1) = 0$.*

For the second intersection polynomial $I\!I_K(t)$, it holds that $I\!I_K(t) = I\!I_K(t^{-1})$, $I\!I_K(1) = 0$, and $I\!I''_K(1) \equiv 0 \pmod 4$. This property characterizes the second intersection polynomial as follows.

Theorem 1.2. *For $f(t) \in \mathbb{Z}[t, t^{-1}]$, there is a virtual knot K with $I\!I_K(t) = f(t)$ if and only if $f(t) = f(t^{-1})$, $f(1) = 0$, and $f''(1) \equiv 0 \pmod 4$.*

A characterization of the third intersection polynomial $I\!I\!I_K(t)$ is similarly obtained.

This paper is organized as follows. In Section 2, we review the definitions of the intersection polynomials, the connected sum formulae, and the calculation of intersection numbers

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by using a Gauss diagram. In Sections 3 and 4, we study the fundamental properties of the first intersection polynomial (Theorem 3.2), the second intersection polynomial (Theorem 3.3), and the third intersection polynomial (Theorem 4.1). Section 5 is devoted to giving characterizations of the intersection polynomials (Theorems 5.1, 5.5, and 5.9). Theorem 1.1 is a combination of Theorems 3.2 and 5.1, and Theorem 1.2 is a combination of Theorems 3.3 and 5.5. In Section 6, we characterize the intersection polynomials of a connected sum of trivial knots.

2. Preliminaries

In this section, we review the definitions of the writhe polynomial and the three kinds of intersection polynomials of a virtual knot. They are defined by using the intersection number of curves on a surface.

We consider the set of pairs of a closed, connected, oriented surface Σ and a knot diagram D on Σ with classical crossings. Two knot diagrams D on Σ and D' on Σ' are said to be *equivalent* if D and D' are related to each other by a finite sequence of an orientation preserving homeomorphism of the underlying surface, a (de)stabilization which changes the genus of the surface by ± 1, and a Reidemeister move R1, R2, or R3 on the surface. Such an equivalence class of knot diagrams on surfaces is called a *virtual knot* (cf. [1, 7]). Here, a stabilization/destabilization is a 1-handle addition/deletion on a surface missing a knot diagram as shown in Fig.1. We remark that the notion of a virtual knot was originally introduced by Kauffman [9] as an equivalence class of knot diagrams in a plane with classical and virtual crossings under seven kinds of generalized Reidemeister moves.

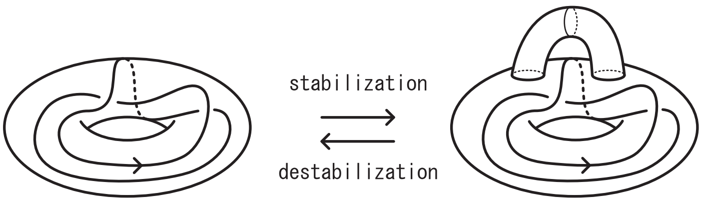

Fig. 1

Let D be a diagram of a virtual knot K on Σ, and $c_1, c_2, \ldots, c_n$ the crossings of D. For each i, we perform smoothing at c_i to obtain a pair of cycles γ_i and $\overline{\gamma}_i$ on Σ such that γ_i is oriented from the over to the under at c_i, and $\overline{\gamma}_i$ the other one. The sum $\gamma_i + \overline{\gamma}_i$ coincides with the cycle γ_D presented by D. See Fig.2.

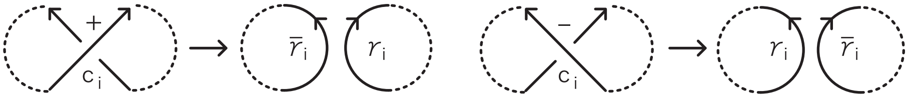

Fig. 2

The *writhe polynomial* of K is defined by

$$W_K(t) = \sum_{i=1}^{n} \varepsilon_i (t^{\gamma_i \cdot \overline{\gamma}_i} - 1) = \sum_{i=1}^{n} \varepsilon_i (t^{\gamma_i \cdot \gamma_D} - 1) \in \mathbb{Z}[t, t^{-1}],$$

where ε_i is the sign of c_i, and $\gamma_i \cdot \overline{\gamma}_i$ is the intersection number between γ_i and $\overline{\gamma}_i$ on Σ (cf. [2, 3, 11, 13]). Similarly, we consider the polynomials

$$f_{01}(D;t) = \sum_{1\leq i,j\leq n} \varepsilon_i\varepsilon_j(t^{\gamma_i\cdot\overline{\gamma}_j} - 1),$$
$$f_{00}(D;t) = \sum_{1\leq i,j\leq n} \varepsilon_i\varepsilon_j(t^{\gamma_i\cdot\gamma_j} - 1), \text{ and}$$
$$f_{11}(D;t) = \sum_{1\leq i,j\leq n} \varepsilon_i\varepsilon_j(t^{\overline{\gamma}_i\cdot\overline{\gamma}_j} - 1).$$

The *first, second*, and *third intersection polynomials* of K are defined by

$$\begin{aligned} I_K(t) &= f_{01}(D;t) - \omega_D W_K(t), \\ II_K(t) &= f_{00}(D;t) + f_{11}(D;t) - \omega_D \overline{W}_K(t), \text{ and} \\ III_K(t) &\equiv f_{00}(D;t) \pmod{\overline{W}_K(t)}. \end{aligned}$$

Here $\omega_D = \sum_{i=1}^{n} \varepsilon_i$ is the writhe of D, $\overline{W}_K(t) = W_K(t) + W_K(t^{-1})$, and $f(t) \equiv g(t) \pmod{h(t)}$ means $f(t) = g(t) + mh(t)$ for some $m \in \mathbb{Z}$. These polynomials do not depend on a particular choice of D of K [5].

Let $-K$, $K^{\#}$, and K^* denote the reverse, the vertical mirror image, and the horizontal mirror image of a virtual knot K, respectively.

Lemma 2.1 (cf. [2, 5, 11, 13])**.** *For a virtual knot K, we have the following.*

(i) $W_{-K}(t) = W_K(t^{-1})$ *and* $W_{K^{\#}}(t) = W_{K^*}(t) = -W_K(t^{-1})$.
(ii) $I_{-K}(t) = I_{K^{\#}}(t) = I_{K^*}(t) = I_K(t^{-1})$.
(iii) $II_{-K}(t) = II_{K^{\#}}(t) = II_{K^*}(t) = II_K(t)$.
(iv) $III_{-K}(t) = III_{K^{\#}}(t) = II_K(t) - III_K(t)$ *and* $III_{K^*}(t) = III_K(t)$.

We review a connected sum of virtual knots and its intersection polynomials. Refer to [6] for more details. A *dotted virtual knot* T is a virtual knot equipped with a base point p. Let (D, p) be a diagram of T, and $c_1, c_2, \dots, c_n$ the crossings of D. The set of indices $1, 2, \dots, n$ is divided into

$$M_0(D) = \{i \mid p \text{ lies on } \overline{\gamma}_i\} \text{ and } M_1(D) = \{i \mid p \text{ lies on } \gamma_i\}.$$

Then we define two polynomials

$$W_T^0(t) = \sum_{i\in M_0(D)} \varepsilon_i(t^{\gamma_i\cdot\overline{\gamma}_i} - 1) \text{ and } W_T^1(t) = \sum_{i\in M_1(D)} \varepsilon_i(t^{\gamma_i\cdot\overline{\gamma}_i} - 1),$$

which do not depend on a particular choice of (D, p). The *closure* $\widehat{T}$ of T is the virtual knot by forgetting p of T. By definition, it holds that

$$W_T^0(t) + W_T^1(t) = W_{\widehat{T}}(t) \text{ and } W_T^0(1) = W_T^1(1) = 0.$$

The following proposition plays an important role in this paper.

Proposition 2.2 ([6])**.** *Let K be a virtual knot, and* $f(t)$ *a Laurent polynomial with* $f(1) = 0$*. Then there is a dotted virtual knot T such that*

(i) $\widehat{T} = K$,
(ii) $W_T^0(t) = f(t)$, *and*

(iii) $W_T^1(t) = W_K(t) - f(t)$.

For a pair of dotted virtual knots T and T', we denote by $T + T'$ the one obtained by connecting T and T' at their base points as shown in Fig.3. A *connected sum* of virtual knots K and K' is a virtual knot in the set

$$C(K, K') = \{\widehat{T + T'} \mid T, T'\text{: dotted virtual knots with } \widehat{T} = K \text{ and } \widehat{T'} = K'\}.$$

We remark that there are infinitely many connected sums of any pair (K, K') [6].

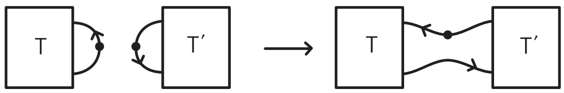

Fig.3

The writhe polynomial is additive under a connected sum; that is, $W_{K''}(t) = W_K(t)+W_{K'}(t)$ holds for any $K'' \in C(K, K')$ (cf. [2, 3, 13]). On the other hand, the intersection polynomials are not additive in general.

Theorem 2.3 ([6])**.** *For $K'' \in C(K, K')$, let T and T' be dotted virtual knots such that $\widehat{T} = K$, $\widehat{T'} = K'$, and $\widehat{T + T'} = K''$.*

(i) *The first intersection polynomial satisfies*

$$I_{K''}(t) = I_K(t) + I_{K'}(t) + W_T^0(t)W_{T'}^1(t) + W_T^1(t)W_{T'}^0(t).$$

(ii) *The second intersection polynomial satisfies*

$$\begin{aligned} II_{K''}(t) &= II_K(t) + II_{K'}(t) \\ &\quad + W_T^0(t)W_{T'}^0(t^{-1}) + W_T^1(t)W_{T'}^1(t^{-1}) \\ &\quad + W_T^0(t^{-1})W_{T'}^0(t) + W_T^1(t^{-1})W_{T'}^1(t). \end{aligned}$$

(iii) *Suppose that $\overline{W}_K(t) = 0$ or $\overline{W}_{K'}(t) = 0$. The third intersection polynomial satisfies*

$$III_{K''}(t) \equiv III_K(t) + III_{K'}(t) + W_T^1(t)W_{T'}^1(t^{-1}) + W_T^1(t^{-1})W_{T'}^1(t) \pmod{\overline{W}_{K''}(t)}.$$

A diagram D of a virtual knot is presented by a *Gauss diagram*: it consists of an oriented circle and a finite number of oriented and signed chords corresponding to the crossings of D. The orientation of a chord is from the over to the under, and the sign comes from that of the crossing. For a chord, the terminal endpoint is given the same sign as that of the chord, and the initial is given the opposite.

The intersection number of a pair of cycles on D is interpreted by using a Gauss diagram. The endpoints of a chord divide the circle of a Gauss diagram into two arcs. Let α and β be such arcs obtained from two chords c and c', respectively, and $S(\alpha,\beta)$ the sum of signs of endpoints of chords on intα whose opposite endpoints lie on intβ. Then the intersection number of cycles corresponding to the ordered pair (α,β) is equal to $S(\alpha,\beta) \pm 1$ in the four cases in Fig.4, and otherwise $S(\alpha,\beta)$. Therefore the intersection polynomials are calculated from a Gauss diagram [5, 6].

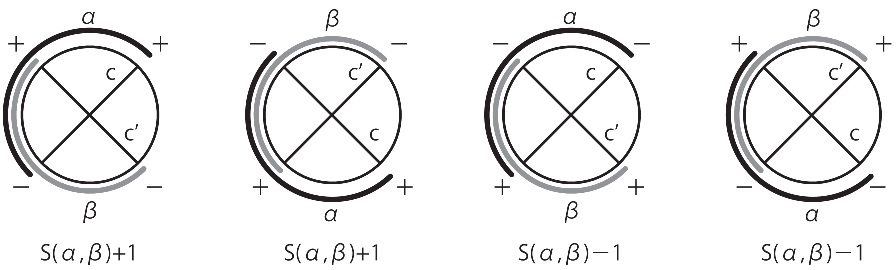

Fig. 4

3. Fundamental properties of $I_K(t)$ and $I\!I_K(t)$

The writhe polynomial $W_K(t)$ is characterized by the following property.

Theorem 3.1 ([13]). *Any virtual knot K satisfies $W_K(1) = W'_K(1) = 0$. Conversely, if a Laurent polynomial $f(t) \in \mathbb{Z}[t, t^{-1}]$ satisfies $f(1) = f'(1) = 0$, then there is a virtual knot K with $f(t) = W_K(t)$.*

We remark that the equation $W'_K(1) = 0$ is equivalent to

$$\sum_{i=1}^{n} \varepsilon_i(\gamma_i \cdot \overline{\gamma}_i) = \sum_{i=1}^{n} \varepsilon_i(\gamma_i \cdot \gamma_D) = 0$$

by definition.

The first intersection polynomial $I_K(t)$ satisfies the same property as $W_K(t)$, which characterizes a Laurent polynomial to be coincident with the first intersection polynomial of some virtual knot. The characterization will be given in Section 5.

Theorem 3.2. *Any virtual knot K satisfies $I_K(1) = I'_K(1) = 0$.*

Proof. We have $I_K(1) = 0$ by definition. Since $W'_K(1) = 0$ holds by Theorem 3.1, we obtain

$$\begin{aligned}
I'_K(1) &= f'_{01}(D;1) - \omega_D W'_K(1) = f'_{01}(D;1) \\
&= \sum_{1 \le i,j \le n} \varepsilon_i\varepsilon_j(\gamma_i \cdot \overline{\gamma}_j) = \left(\sum_{i=1}^{n} \varepsilon_i\gamma_i\right) \cdot \left(\sum_{j=1}^{n} \varepsilon_j\overline{\gamma}_j\right) \\
&= \left(\sum_{i=1}^{n} \varepsilon_i\gamma_i\right) \cdot \left(\sum_{j=1}^{n} \varepsilon_j(\gamma_D - \gamma_j)\right) \\
&= \left(\sum_{i=1}^{n} \varepsilon_i\gamma_i\right) \cdot \left(\omega_D\gamma_D - \sum_{j=1}^{n} \varepsilon_j\gamma_j\right) \\
&= \omega_D \sum_{i=1}^{n} \varepsilon_i(\gamma_i \cdot \gamma_D) - \left(\sum_{i=1}^{n} \varepsilon_i\gamma_i\right) \cdot \left(\sum_{j=1}^{n} \varepsilon_j\gamma_j\right) \\
&= \omega_D W'_K(1) - 0 = 0.
\end{aligned}$$

□

A Laurent polynomial $f(t)$ is *reciprocal* if it satisfies $f(t^{-1}) = f(t)$. The second intersection polynomial $I\!I_K(t)$ satisfies different properties from $I_K(t)$ as follows. These properties characterize a Laurent polynomial to be coincident with the second intersection polynomial of some virtual knot. The characterization will be given in Section 5.

Theorem 3.3. *For any virtual knot K, the second intersection polynomial $I\!I_K(t)$ is reciprocal with*

$$I\!I_K(1) = 0 \text{ and } I\!I''_K(1) \equiv 0 \pmod 4.$$

To prove this theorem, we prepare Lemmas 3.4 and 3.5 as follows.

Lemma 3.4. *Let $f(t) = \sum_{k\in\mathbb{Z}} a_k t^k$ be a Laurent polynomial in $\mathbb{Z}[t, t^{-1}]$.*

(i) *If $f'(1) = 0$, then $f''(1) \equiv \sum_{k:\text{odd}} a_k \pmod 4$.*

(ii) *If $f(t)$ is reciprocal, then $f''(1) \equiv \sum_{k:\text{odd}} a_k \pmod 4$.*

Proof. (i) It holds that

$$f'(t) = \sum_{k\in\mathbb{Z}} k a_k t^{k-1} \text{ and } f''(t) = \sum_{k\in\mathbb{Z}} k(k-1) a_k t^{k-2}.$$

Then we have

$$f''(1) = \sum_{k\in\mathbb{Z}} k(k-1)a_k = \sum_{k\in\mathbb{Z}} k^2 a_k - f'(1) = \sum_{k\in\mathbb{Z}} k^2 a_k \equiv \sum_{k:\text{odd}} a_k \pmod 4.$$

(ii) We may take $f(t) = \sum_{k\geq 1} a_k(t^k + t^{-k}) + a_0$. Since $f'(t) = \sum_{k\geq 1} k a_k(t^{k-1} - t^{-k-1})$ holds, we have $f'(1) = 0$. By (i), we have the conclusion. □

Lemma 3.5. *For a Laurent polynomial $f(t)$, the following are equivalent.*

(i) *$f(t)$ is reciprocal, $f(1) = 0$, and $f''(1) \equiv 0 \pmod 4$.*

(ii) *$f(t) = \sum_{k\geq 1} a_k(t^k + t^{-k} - 2)$ for some $a_k \in \mathbb{Z}$ $(k \geq 1)$ with $\sum_{k:\text{odd}\geq 1} a_k \equiv 0 \pmod 2$.*

(iii) *There is a Laurent polynomial $g(t) \in \mathbb{Z}[t, t^{-1}]$ such that $g(1) = g'(1) = 0$ and $f(t) = g(t) + g(t^{-1})$.*

Proof. (i)⇒(ii). Since $f(t)$ is reciprocal, we may take $f(t) = \sum_{k\geq 1} a_k(t^k + t^{-k}) + a_0$. Since $f(1) = 0$, we have $a_0 = -2\sum_{k\geq 1} a_k$ to obtain $f(t) = \sum_{k\geq 1} a_k(t^k + t^{-k} - 2)$. Furthermore, the sum of the coefficients of odd terms of $f(t)$ is equal to $2\sum_{k:\text{odd}\geq 1} a_k$, it follows by Lemma 3.4 (ii) that

$$2\sum_{k:\text{odd}\geq 1} a_k \equiv f''(1) \equiv 0 \pmod 4.$$

(ii)⇒(iii). We have

$$\begin{aligned} f(t) &= \sum_{k\geq 1} a_k(t^k + t^{-k} - 2) \\ &= \sum_{k\geq 2} a_k(t^k - kt + k - 1) + \sum_{k\geq 2} a_k(t^{-k} - kt^{-1} + k - 1) \\ &\quad + \sum_{k\geq 1} k a_k(t + t^{-1} - 2). \end{aligned}$$

By assumption, we may put $\sum_{k\geq 1} k a_k = 2m$ for some $m \in \mathbb{Z}$. Consider the Laurent polynomial

$$g(t) = \sum_{k\geq 2} a_k(t^k - kt + k - 1) + m(t + t^{-1} - 2).$$

Then it satisfies that $g(1) = g'(1) = 0$ and $f(t) = g(t) + g(t^{-1})$.

(iii)⇒(i). Since $g(1) = g'(1) = 0$, we can take $g(t) = (t-1)^2 h(t)$ for some $h(t) \in \mathbb{Z}[t, t^{-1}]$. Then the reciprocal polynomial $f(t) = g(t) + g(t^{-1})$ satisfies

$$f(1) = 2g(1) = 0 \text{ and } f''(1) = 2g''(1) = 4h(1) \equiv 0 \pmod 4. \qquad \square$$

Proof of Theorem 3.3. Since $\gamma_i \cdot \gamma_j = -\gamma_j \cdot \gamma_i$ and $\overline{\gamma}_i \cdot \overline{\gamma}_j = -\overline{\gamma}_j \cdot \overline{\gamma}_i$ hold, the Laurent polynomials $f_{00}(D; t)$, $f_{11}(D; t)$, and $\overline{W}_K(t)$ are reciprocal by definition. Therefore $I\!I_K(t)$ is also reciprocal. We have $I\!I_K(1) = 0$ by definition.

We will prove $I\!I''_K(1) \equiv 0 \pmod 4$. Since $\overline{W}_K(t) = W_K(t) + W_K(t^{-1})$ with $W_K(1) = W'_K(1) = 0$, we have $\overline{W}''_K(1) \equiv 0 \pmod 4$ by Lemma 3.5. Let S be the sum of the coefficients of odd terms of $f_{00}(D; t) + f_{11}(D; t)$. Since $f_{00}(D; t) + f_{11}(D; t)$ is reciprocal, it is sufficient to prove that $S \equiv 0 \pmod 4$ by Lemma 3.4 (ii).

By definition, we have

$$\begin{aligned} S &= \sum_{\gamma_i\cdot\gamma_j:\text{odd}} \varepsilon_i\varepsilon_j + \sum_{\overline{\gamma}_i\cdot\overline{\gamma}_j:\text{odd}} \varepsilon_i\varepsilon_j = 2\Bigg(\sum_{\gamma_i\cdot\gamma_j:\text{odd}, i<j} \varepsilon_i\varepsilon_j + \sum_{\overline{\gamma}_i\cdot\overline{\gamma}_j:\text{odd}, i<j} \varepsilon_i\varepsilon_j\Bigg) \\ &\equiv 2\Bigg(\sum_{1\leq i<j\leq n} \gamma_i\cdot\gamma_j + \sum_{1\leq i<j\leq n} \overline{\gamma}_i\cdot\overline{\gamma}_j\Bigg) \pmod 4. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \sum_{1\leq i<j\leq n} \gamma_i\cdot\gamma_j + \sum_{1\leq i<j\leq n} \overline{\gamma}_i\cdot\overline{\gamma}_j &= \sum_{1\leq i<j\leq n} \Big(\gamma_i\cdot\gamma_j + (\gamma_D - \gamma_i)\cdot(\gamma_D - \gamma_j)\Big) \\ &= \sum_{1\leq i<j\leq n} (2\gamma_i\cdot\gamma_j - \gamma_i\cdot\gamma_D - \gamma_D\cdot\gamma_j) \\ &\equiv \sum_{1\leq i<j\leq n} (\gamma_i\cdot\gamma_D + \gamma_j\cdot\gamma_D) \pmod 2 \\ &= \sum_{1\leq i<j\leq n} (\gamma_i\cdot\gamma_D) + \sum_{1\leq i<j\leq n} (\gamma_j\cdot\gamma_D) \\ &= \sum_{i=1}^{n} (n-i)(\gamma_i\cdot\gamma_D) + \sum_{j=1}^{n} (j-1)(\gamma_j\cdot\gamma_D) \\ &= (n-1)\sum_{i=1}^{n} \gamma_i\cdot\gamma_D \\ &\equiv (n-1)\sum_{i=1}^{n} \varepsilon_i(\gamma_i\cdot\gamma_D) \pmod 2 \\ &= (n-1)W'_K(1) = 0. \end{aligned}$$

Therefore we have $S \equiv 0 \pmod 4$. $\square$

4. Fundamental properties of $\mathit{III}_K(t)$

As seen in the proof of Theorem 3.3, we have $\overline{W}''_K(1) \equiv 0 \pmod 4$. Therefore if two Laurent polynomials $f(t)$ and $g(t)$ satisfy $f(t) \equiv g(t) \pmod{\overline{W}_K(t)}$, then it holds that $f''(1) \equiv g''(1) \pmod 4$. This induces the well-definedness of $\mathit{III}''_K(1) \pmod 4$. Then the third intersection polynomial $\mathit{III}_K(t)$ satisfies the following properties. The characterization of $\mathit{III}_K(t)$ will be given in Section 5.

Theorem 4.1. *For any virtual knot K, the third intersection polynomial $\mathit{III}_K(t)$ is reciprocal with*

$$\mathit{III}_K(1) = 0 \text{ and } \mathit{III}''_K(1) \equiv W''_K(1) \pmod 4.$$

Recall that an *upper* (or *lower*) *forbidden move* changes the positions of consecutive over-crossings (or under-crossings) which is known as an unknotting operation [8, 12]. In a Gauss diagram, an upper (or lower) forbidden move changes the positions of consecutive initial (or terminal) endpoints of chords.

In this section, we will use a Gauss diagram to calculate intersection numbers. Let $c_1, c_2, \ldots, c_n$ be the chords of a Gauss diagram. The endpoints of c_i divide the circle of the Gauss diagram into two arcs. The arc from the initial endpoint of c_i to the terminal is corresponding to the cycle γ_i, and the other $\overline{\gamma}_i$. We denote the arcs also by γ_i and $\overline{\gamma}_i$, respectively.

Our proof of the congruence in Theorem 4.1 is divided into two steps. First we will prove

$$\mathit{III}''_K(1) - W''_K(1) \equiv \sum_{\gamma_i \cdot \gamma_j:\text{odd}} \varepsilon_i\varepsilon_j - \sum_{\gamma_i \cdot \overline{\gamma}_i:\text{odd}} \varepsilon_i \pmod 4$$

in Lemma 4.2. Next we will show that the right hand side in this congruence is invariant under a forbidden move in Proposition 4.3. Since the forbidden move is an unknotting operation, we see that the right hand side is congruent to zero.

Lemma 4.2. $\mathit{III}''_K(1) - W''_K(1) \equiv \sum_{\gamma_i \cdot \gamma_j:\text{odd}} \varepsilon_i\varepsilon_j - \sum_{\gamma_i \cdot \overline{\gamma}_i:\text{odd}} \varepsilon_i \pmod 4$.

Proof. Since we have $\mathit{III}_K(t) = f_{00}(D;t) + m\overline{W}_K(t)$ $(m \in \mathbb{Z})$, $\overline{W}''_K(1) \equiv 0 \pmod 4$, and $f_{00}(D;t) = \sum_{1 \le i,j \le n} \varepsilon_i\varepsilon_j(t^{\gamma_i \cdot \gamma_j} - 1)$, it holds that

$$\begin{aligned}
\mathit{III}''_K(1) &\equiv f''_{00}(D;1) \pmod 4 \\
&= \sum_{1 \le i,j \le n} \varepsilon_i\varepsilon_j(\gamma_i \cdot \gamma_j)(\gamma_i \cdot \gamma_j - 1) \\
&= \sum_{1 \le i,j \le n} \varepsilon_i\varepsilon_j(\gamma_i \cdot \gamma_j)^2 - \sum_{1 \le i,j \le n} \varepsilon_i\varepsilon_j(\gamma_i \cdot \gamma_j) \\
&\equiv \sum_{\gamma_i \cdot \gamma_j:\text{odd}} \varepsilon_i\varepsilon_j \pmod 4.
\end{aligned}$$

We remark that $(\gamma_i \cdot \gamma_j)^2 \equiv 0 \pmod 4$ if $\gamma_i \cdot \gamma_j$ is even, and $\gamma_i \cdot \gamma_j = -\gamma_j \cdot \gamma_i$ holds.

On the other hand, since we have $W_K(t) = \sum_{i=1}^n \varepsilon_i(t^{\gamma_i \cdot \overline{\gamma}_i} - 1)$, it holds that

$$W''_K(1) = \sum_{i=1}^n \varepsilon_i(\gamma_i \cdot \overline{\gamma}_i)(\gamma_i \cdot \overline{\gamma}_i - 1)$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \varepsilon_i(\gamma_i \cdot \overline{\gamma}_i)^2 - \sum_{i=1}^{n} \varepsilon_i(\gamma_i \cdot \overline{\gamma}_i) \\
&= \sum_{i=1}^{n} \varepsilon_i(\gamma_i \cdot \overline{\gamma}_i)^2 - W'_K(1) \\
&\equiv \sum_{\gamma_i \cdot \overline{\gamma}_i:\text{odd}} \varepsilon_i \quad (\text{mod } 4).
\end{aligned}
$$

Therefore we have the conclusion. □

Proposition 4.3. $\sum_{\gamma_i\cdot\gamma_j:\text{odd}} \varepsilon_i\varepsilon_j - \sum_{\gamma_i\cdot\overline{\gamma}_i:\text{odd}} \varepsilon_i \pmod 4$ *is invariant under a forbidden move.*

We will prove this proposition for an upper forbidden move only. The invariance under a lower forbidden move can be proved similarly.

Assume that a Gauss diagram G' is obtained from G by an upper forbidden move involving a pair of chords c_1 and c_2 of G as shown in Fig.5. For $1 \le i \le n$, let c'_i be the chord of G' corresponding to c_i, ε'_i the sign of c'_i, and γ'_i the cycle at c'_i. Let x_1 and x_2 be the terminal endpoints of c_1 and c_2, respectively. We classify the chords $c_3, \dots, c_n$ of G into four sets such that

$$
\begin{aligned}
P &= \{c_i \mid \text{both } x_1 \text{ and } x_2 \text{ lie on } \overline{\gamma}_i\}, \\
Q &= \{c_i \mid \text{both } x_1 \text{ and } x_2 \text{ lie on } \gamma_i\}, \\
R &= \{c_i \mid x_1 \text{ lies on } \overline{\gamma}_i \text{ and } x_2 \text{ lies on } \gamma_i\}, \text{ and} \\
S &= \{c_i \mid x_1 \text{ lies on } \gamma_i \text{ and } x_2 \text{ lies on } \overline{\gamma}_i\}.
\end{aligned}
$$

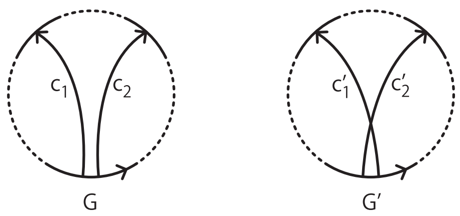

Fig.5

Lemma 4.4. (i) $\sum_{1\le i<j\le n}(\gamma'_i \cdot \gamma'_j - \gamma_i \cdot \gamma_j) \equiv \#R + \#S + (\varepsilon_1 + \varepsilon_2)/2 \pmod 2$.
(ii) $\#R + \#S \equiv \gamma_1 \cdot \overline{\gamma}_1 + \gamma_2 \cdot \overline{\gamma}_2 \pmod 2$.
(iii) $\sum_{\gamma'_i\cdot\gamma'_j:\text{odd}} \varepsilon'_i\varepsilon'_j - \sum_{\gamma_i\cdot\gamma_j:\text{odd}} \varepsilon_i\varepsilon_j \equiv 2\gamma_1 \cdot \overline{\gamma}_1 + 2\gamma_2 \cdot \overline{\gamma}_2 + \varepsilon_1 + \varepsilon_2 \pmod 4$.

Proof. (i) Since it holds that $\gamma'_i \cdot \gamma'_j = \gamma_i \cdot \gamma_j$ $(3 \le i < j \le n)$, we have

$$
\begin{aligned}
\sum_{1\le i<j\le n} (\gamma'_i \cdot \gamma'_j - \gamma_i \cdot \gamma_j) &= \sum_{3\le j\le n} (\gamma'_1 \cdot \gamma'_j - \gamma_1 \cdot \gamma_j) + \sum_{3\le j\le n} (\gamma'_2 \cdot \gamma'_j - \gamma_2 \cdot \gamma_j) \\
&\quad +(\gamma'_1 \cdot \gamma'_2 - \gamma_1 \cdot \gamma_2).
\end{aligned}
$$

The first sum in the right hand side have the same parity as $\#Q + \#R$. In fact, if $c_j \in P \cup S$, then we have $\gamma'_1 \cdot \gamma'_j = \gamma_1 \cdot \gamma_j$. On the other hand, if $c_j \in Q \cup R$, it holds that $\gamma'_1 \cdot \gamma'_j = \gamma_1 \cdot \gamma_j + \varepsilon_2$.

Similarly, the second sum have the same parity as $\#Q + \#S$. In fact, if $c_j \in P \cup R$, then we have $\gamma'_2 \cdot \gamma'_j = \gamma_2 \cdot \gamma_j$. On the other hand, if $c_j \in Q \cup S$, it holds that $\gamma'_2 \cdot \gamma'_j = \gamma_2 \cdot \gamma_j - \varepsilon_1$.

Finally it holds that $\gamma'_1 \cdot \gamma'_2 = \gamma_1 \cdot \gamma_2 - (\varepsilon_1 + \varepsilon_2)/2$.

(ii) Let m_1, m_2, and m_3 be the numbers of endpoints of chords as shown in Fig.6. Since it holds that

$$\gamma_1 \cdot \overline{\gamma}_1 \equiv m_1,\ \gamma_2 \cdot \overline{\gamma}_2 \equiv m_2,\ \text{and } m_1 + m_2 + m_3 \equiv 0 \pmod 2,$$

we see that $\gamma_1 \cdot \overline{\gamma}_1 + \gamma_2 \cdot \overline{\gamma}_2$ has the same parity as m_3.

On the other hand, a chord belongs to $R \cup S$ if and only if it is linked with exactly one of c_1 and c_2. Since the number of such chords has the same parity as m_3, we have the conclusion.

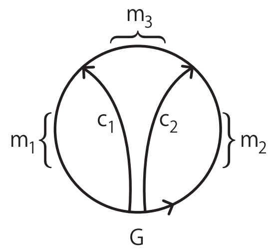

Fig.6

(iii) It holds that

$$\sum_{\gamma_i \cdot \gamma_j:\text{odd}} \varepsilon_i \varepsilon_j = 2 \sum_{\gamma_i \cdot \gamma_j:\text{odd}, i<j} \varepsilon_i \varepsilon_j \equiv 2 \sum_{1 \le i < j \le n} \gamma_i \cdot \gamma_j \pmod 4.$$

Then we have the conclusion by (i) and (ii). □

Lemma 4.5. $\sum_{\gamma'_i \cdot \overline{\gamma}'_i:\text{odd}} \varepsilon'_i - \sum_{\gamma_i \cdot \overline{\gamma}_i:\text{odd}} \varepsilon_i = (-1)^{\gamma_1 \cdot \overline{\gamma}_1} \varepsilon_1 + (-1)^{\gamma_2 \cdot \overline{\gamma}_2} \varepsilon_2$.

Proof. We see that $\gamma_i \cdot \overline{\gamma}_i$ and $\gamma'_i \cdot \overline{\gamma}'_i$ have opposite parity for $i = 1, 2$ and coincide for $3 \le i \le n$. Since $\varepsilon_i = \varepsilon'_i$ holds, we have

$$\sum_{\gamma'_i \cdot \overline{\gamma}'_i:\text{odd}} \varepsilon'_i - \sum_{\gamma_i \cdot \overline{\gamma}_i:\text{odd}} \varepsilon_i = \begin{cases} \varepsilon_1 + \varepsilon_2 & \text{for } \gamma_1 \cdot \overline{\gamma}_1 \equiv \gamma_2 \cdot \overline{\gamma}_2 \equiv 0 \pmod 2, \\ \varepsilon_1 - \varepsilon_2 & \text{for } \gamma_1 \cdot \overline{\gamma}_1 \equiv 0, \gamma_2 \cdot \overline{\gamma}_2 \equiv 1 \pmod 2, \\ -\varepsilon_1 + \varepsilon_2 & \text{for } \gamma_1 \cdot \overline{\gamma}_1 \equiv 1, \gamma_2 \cdot \overline{\gamma}_2 \equiv 0 \pmod 2, \text{ and} \\ -\varepsilon_1 - \varepsilon_2 & \text{for } \gamma_1 \cdot \overline{\gamma}_1 \equiv \gamma_2 \cdot \overline{\gamma}_2 \equiv 1 \pmod 2. \end{cases}$$ □

Proof of Proposition 4.3. Assume that a Gauss diagram G' is obtained from G by an upper forbidden move. Then it follows by Lemmas 4.4 (iii) and 4.5 that

$$\begin{aligned} & \left(\sum_{\gamma'_i \cdot \gamma'_j:\text{odd}} \varepsilon'_i \varepsilon'_j - \sum_{\gamma'_i \cdot \overline{\gamma}'_i:\text{odd}} \varepsilon'_i \right) - \left(\sum_{\gamma_i \cdot \gamma_j:\text{odd}} \varepsilon_i \varepsilon_j - \sum_{\gamma_i \cdot \overline{\gamma}_i:\text{odd}} \varepsilon_i \right) \\ \equiv\ & 2\gamma_1 \cdot \overline{\gamma}_1 + 2\gamma_2 \cdot \overline{\gamma}_2 + \varepsilon_1 + \varepsilon_2 - (-1)^{\gamma_1 \cdot \overline{\gamma}_1} \varepsilon_1 - (-1)^{\gamma_2 \cdot \overline{\gamma}_2} \varepsilon_2 \\ \equiv\ & 0 \pmod 4. \end{aligned}$$

In other words, $\sum_{\gamma_i \cdot \gamma_j:\text{odd}} \varepsilon_i \varepsilon_j - \sum_{\gamma_i \cdot \overline{\gamma}_i:\text{odd}} \varepsilon_i \pmod 4$ is invariant under an upper forbidden move. □

Proof of Theorem 4.1. Since $f_{00}(D;t)$ and $\overline{W}_K(t)$ are reciprocal, so is $I\!I\!I_K(t)$. We have $I\!I\!I_K(1) = 0$ by $f_{00}(D;1) = \overline{W}_K(1) = 0$. The congruence follows by Lemma 4.2 and Proposition 4.3 immediately. □

REMARK 4.6. The *odd writhe* [10] of a virtual knot K is the sum of the coefficients of odd terms of $W_K(t)$, and denoted by $J(K) \in \mathbb{Z}$. We have $W''_K(1) \equiv J(K) \pmod 4$ by Lemma 3.4. Therefore the congruence in Theorem 4.1 is equivalent to $I\!I\!I''_K(1) \equiv J(K) \pmod 4$.

5. Characterizations of intersection polynomials

We first give a characterization of the first intersection polynomials. Let $\mathcal{P}_1$ denote the set of Laurent polynomials defined by $\mathcal{P}_1 = \{I_K(t) \mid K : \text{virtual knots}\}$.

Theorem 5.1. *For $f(t) \in \mathbb{Z}[t, t^{-1}]$, the following are equivalent.*
(i) $f(t) \in \mathcal{P}_1$.
(ii) $f(1) = f'(1) = 0$.
(iii) $f(t) = \sum_{k\neq 0,1} a_k(t^k - kt + k - 1)$ *for some* $a_k \in \mathbb{Z}$ $(k \neq 0, 1)$.

This characterization is exactly the same as that of the writhe polynomial given in Theorem 3.1. To prove Theorem 5.1, we prepare Lemmas 5.2–5.4 as follows.

Lemma 5.2. *For any $f(t), g(t) \in \mathcal{P}_1$, we have the following.*
(i) $f(t^{-1}) \in \mathcal{P}_1$.
(ii) $f(t) + g(t) \in \mathcal{P}_1$.

Proof. Let K and K' be virtual knots with $I_K(t) = f(t)$ and $I_{K'}(t) = g(t)$.
(i) It holds that $I_{-K}(t) = f(t^{-1}) \in \mathcal{P}_1$ by Lemma 2.1.
(ii) By Proposition 2.2, there are dotted virtual knots T and T' such that

$$\begin{cases} \widehat{T} = K, & W^0_T(t) = W_K(t), \quad W^1_T(t) = 0, \text{ and} \\ \widehat{T'} = K', & W^0_{T'}(t) = W_{K'}(t), \quad W^1_{T'}(t) = 0. \end{cases}$$

Let K'' be the virtual knot $\widehat{T + T'}$. By Theorem 2.3 (i), we have

$$\begin{aligned} I_{K''}(t) &= I_K(t) + I_{K'}(t) + W^0_T(t)W^1_{T'}(t) + W^1_T(t)W^0_{T'}(t) \\ &= I_K(t) + I_{K'}(t) \\ &= f(t) + g(t) \in \mathcal{P}_1. \end{aligned}$$

□

The table of virtual knots up to crossing number four are given by Green [4]. In what follows, we denote by $K(n.k)$ the virtual knot labeled $n.k$ in his table. The calculations of the intersection polynomials of these virtual knots are given in [5].

Lemma 5.3. *Let $n \geq 2$ be an integer.*
(i) *There are integers a_k $(0 \leq k \leq n-1)$ such that $t^n + \sum_{k=0}^{n-1} a_k t^k \in \mathcal{P}_1$.*
(ii) *There are integers a'_k $(0 \leq k \leq n-1)$ such that $-t^n + \sum_{k=0}^{n-1} a'_k t^k \in \mathcal{P}_1$.*

Proof. (i) For $n = 2$, we have $I_{K(4.44)}(t) = (t-1)^2 \in \mathcal{P}_1$.
For $n \geq 3$, we consider the trivial virtual knot O and the virtual knot $K(3.4)$ with $W_{K(3.4)}(t)$

$= (t-1)^2$ and $I_{K(3.4)}(t) = 0$. By Proposition 2.2, there are dotted virtual knots T and T' such that

$$\begin{cases} \widehat{T} = O, & W_T^0(t) = -(t-1)t^{n-3}, \quad W_T^1(t) = (t-1)t^{n-3}, \text{ and} \\ \widehat{T'} = K(3.4), & W_{T'}^0(t) = (t-1)^2, \qquad W_{T'}^1(t) = 0. \end{cases}$$

Let K'' be the virtual knot $\widehat{T+T'}$. By Theorem 2.3 (i), we have

$$I_{K''}(t) = I_K(t) + I_{K'}(t) + W_T^0(t)W_{T'}^1(t) + W_T^1(t)W_{T'}^0(t) = (t-1)^3 t^{n-3} \in \mathcal{P}_1.$$

(ii) For $n = 2$, we have $I_{K(3.1)}(t) = -(t-1)^2 \in \mathcal{P}_1$. For $n \geq 3$, we consider the trivial knot O and the virtual knot $-K(3.4)^{\#}$. We remark that $W_{-K(3.4)^{\#}}(t) = -W_{K(3.4)}(t) = -(t-1)^2$ and $I_{-K(3.4)^{\#}}(t) = I_{K(3.4)}(t) = 0$. Then we have $-(t-1)^3 t^{n-3} \in \mathcal{P}_1$ similarly to the proof of (i). □

Lemma 5.4. *Let $n \leq -1$ be an integer.*

(i) *There are integers a_k $(n+1 \leq k \leq 1)$ such that $t^n + \sum_{k=n+1}^{1} a_k t^k \in \mathcal{P}_1$.*
(ii) *There are integers a'_k $(n+1 \leq k \leq 1)$ such that $-t^n + \sum_{k=n+1}^{1} a'_k t^k \in \mathcal{P}_1$.*

Proof. For $n = -1$, we have

$$I_{K(4.9)}(t) = t^{-1} - 2 + t \in \mathcal{P}_1 \text{ and } I_{K(2.1)}(t) = -t^{-1} + 2 - t \in \mathcal{P}_1.$$

Assume that $n \leq -2$. By Lemmas 5.2 (i) and 5.3, we have

$$t^n + \sum_{k=n+1}^{0} a_k t^k \in \mathcal{P}_1 \text{ and } -t^n + \sum_{k=n+1}^{0} a'_k t^k \in \mathcal{P}_1$$

for some $a_k, a'_k \in \mathbb{Z}$ $(n+1 \leq k \leq 0)$. □

Proof of Theorem 5.1. (i)⇒(ii). This follows by Theorem 3.2.
(ii)⇒(iii). Assume that $f(t) = \sum_{k \in \mathbb{Z}} a_k t^k$ satisfies $f(1) = f'(1) = 0$. Then we have

$$a_0 = \sum_{k \neq 0,1} (k-1)a_k \text{ and } a_1 = -\sum_{k \neq 0,1} k a_k$$

to obtain

$$f(t) = \sum_{k \neq 0,1} a_k t^k + a_1 t + a_0 = \sum_{k \neq 0,1} a_k (t^k - kt + k - 1).$$

(iii)⇒(i). For the coefficients a_k $(k \neq 0, 1)$ of $f(t)$, there are integers a'_0 and a'_1 such that

$$\sum_{k \neq 0,1} a_k t^k + a'_1 t + a'_0 \in \mathcal{P}_1$$

by Lemmas 5.2 (ii), 5.3, and 5.4. Put this polynomial by $g(t)$. Since $g(t) \in \mathcal{P}_1$, we have $g(1) = g'(1) = 0$ by Theorem 3.2. Then it holds that

$$a'_0 = \sum_{k \neq 0,1} (k-1)a_k \text{ and } a'_1 = -\sum_{k \neq 0,1} k a_k.$$

Therefore we have $f(t) = g(t) \in \mathcal{P}_1$. □

Next we give a characterization of the second intersection polynomials. Let $\mathcal{P}_2$ denote the set of Laurent polynomials defined by $\mathcal{P}_2 = \{I\!I_K(t) \mid K : \text{virtual knots}\}$.

Theorem 5.5. *For $f(t) \in \mathbb{Z}[t, t^{-1}]$, the following are equivalent.*

(i) $f(t) \in \mathcal{P}_2$.

(ii) *$f(t)$ is reciprocal, $f(1) = 0$, and $f''(1) \equiv 0 \pmod 4$.*

To prove this theorem, we prepare Lemmas 5.6–5.8 as follows.

Lemma 5.6. *For any $f(t), g(t) \in \mathcal{P}_2$, we have $f(t) + g(t) \in \mathcal{P}_2$.*

Proof. Let K and K' be virtual knots with $I\!I_K(t) = f(t)$ and $I\!I_{K'}(t) = g(t)$. By Proposition 2.2, there are dotted virtual knots T and T' such that

$$\begin{cases} \widehat{T} = K, & W_T^0(t) = 0, & W_T^1(t) = W_K(t), \text{ and} \\ \widehat{T'} = K', & W_{T'}^0(t) = W_{K'}(t), & W_{T'}^1(t) = 0. \end{cases}$$

Let K'' be the virtual knot $\widehat{T + T'}$. By Theorem 2.3 (ii), we have

$$\begin{aligned} I\!I_{K''}(t) &= I\!I_K(t) + I\!I_{K'}(t) + W_T^0(t)W_{T'}^0(t^{-1}) + W_T^1(t)W_{T'}^1(t^{-1}) \\ &\quad + W_T^0(t^{-1})W_{T'}^0(t) + W_T^1(t^{-1})W_{T'}^1(t) \\ &= I\!I_K(t) + I\!I_{K'}(t) \\ &= f(t) + g(t) \in \mathcal{P}_2. \end{aligned}$$

□

Lemma 5.7. *Let $n \geq 2$ be an integer.*

(i) *There are integers a_k $(0 \leq k \leq n-1)$ such that*

$$(t^n + t^{-n}) + \sum_{k=1}^{n-1} a_k(t^k + t^{-k}) + a_0 \in \mathcal{P}_2.$$

(ii) *There are integers a'_k $(0 \leq k \leq n-1)$ such that*

$$-(t^n + t^{-n}) + \sum_{k=1}^{n-1} a'_k(t^k + t^{-k}) + a'_0 \in \mathcal{P}_2.$$

Proof. (i) We consider the trivial virtual knot O and the virtual knot $K(4.20)$ with $W_{K(4.20)}(t) = (t-1)^2$ and $I\!I_{K(4.20)}(t) = 0$. By Proposition 2.2, there are dotted virtual knots T and T' such that

$$\begin{cases} \widehat{T} = O, & W_T^0(t) = (t-1)t^{n-1}, & W_T^1(t) = -(t-1)t^{n-1}, \text{ and} \\ \widehat{T'} = K(4.20), & W_{T'}^0(t) = (t-1)^2, & W_{T'}^1(t) = 0. \end{cases}$$

Let K'' be the virtual knot $\widehat{T + T'}$. By Theorem 2.3 (ii), we have

$$\begin{aligned} I\!I_{K''}(t) &= (t-1)(t^{-1}-1)^2 t^{n-1} + (t-1)^2(t^{-1}-1)t^{-n+1} \\ &= (t-1)^3 t^{n-3} + (t^{-1}-1)^3 t^{-n+3} \in \mathcal{P}_2. \end{aligned}$$

(ii) We consider the trivial knot O and the virtual knot $-K(4.20)^\#$. We remark that $W_{-K(4.20)^\#}(t) = -W_{K(4.20)}(t) = -(t-1)^2$ and $I\!I_{-K(4.20)^\#}(t) = I\!I_{K(4.20)}(t) = 0$. Then we have $-(t-1)^3 t^{n-3} - (t^{-1}-1)^3 t^{-n+3} \in \mathcal{P}_2$ similarly to the proof of (i). □

Lemma 5.8. $2t - 4 + 2t^{-1} \in \mathcal{P}_2$ *and* $-2t + 4 - 2t^{-1} \in \mathcal{P}_2$.

Proof. We have $I\!I_{K(2.1)}(t) = -2t+4-2t^{-1} \in \mathcal{P}_2$. Furthermore, since $I\!I_{K(4.56)} = 4t-8+4t^{-1} \in \mathcal{P}_2$, we have

$$2t - 4 + 2t^{-1} = (-2t + 4 - 2t^{-1}) + (4t - 8 + 4t^{-1}) \in \mathcal{P}_2$$

by Lemma 5.6. □

Proof of Theorem 5.5. (i)⇒(ii). This follows by Theorem 3.3.

(ii)⇒(i). By Lemma 3.5, we may take $f(t) = \sum_{k\geq 1} a_k(t^k + t^{-k} - 2)$ for some $a_k \in \mathbb{Z}$ $(k \geq 1)$ with $\sum_{k:\mathrm{odd}\geq 1} a_k \equiv 0 \pmod 2$. For the coefficients a_k $(k \geq 2)$ of $f(t)$, there are integers a'_0 and a'_1 such that

$$\sum_{k\geq 2} a_k(t^k + t^{-k}) + a'_1(t + t^{-1}) + a'_0 \in \mathcal{P}_2$$

by Lemmas 5.6 and 5.7. We denote this polynomial by $g(t)$. Since $g(t) \in \mathcal{P}_2$, we have $g''(1) = \sum_{k\geq 2} 2k^2 a_k + 2a'_1 \equiv 0 \pmod 4$ by Theorem 3.3. Then it holds that $\sum_{k:\mathrm{odd}\geq 3} a_k + a'_1 \equiv 0 \pmod 2$ and hence $a'_1 \equiv a_1 \pmod 2$. Therefore for the coefficients a_k $(k \geq 1)$ of $f(t)$, there is an integer a''_0 such that

$$\sum_{k\geq 1} a_k(t^k + t^{-k}) + a''_0 \in \mathcal{P}_2$$

by Lemmas 5.6 and 5.8. We denote this polynomial by $h(t)$. Since $h(t) \in \mathcal{P}_2$, we have $h(1) = 0$ by Theorem 3.3. Then it holds that $a''_0 = -2\sum_{k\geq 1} a_k$ and $f(t) = h(t) \in \mathcal{P}_2$. □

Let $\mathcal{P}_3$ denote the set of pairs of Laurent polynomials defined by

$$\mathcal{P}_3 = \left\{ (f(t), g(t)) \;\middle|\; \begin{array}{l} f(t) = W_K(t) \text{ and} \\ g(t) \equiv I\!I\!I_K(t) \ (\mathrm{mod}\ \overline{W}_K(t)) \text{ for some virtual knot } K \end{array} \right\}.$$

A pair of Laurent polynomials in the set $\mathcal{P}_3$ is characterized as follows.

Theorem 5.9. *For $f(t)$ and $g(t) \in \mathbb{Z}[t, t^{-1}]$, the following are equivalent.*

(i) $(f(t), g(t)) \in \mathcal{P}_3$.

(ii) *$g(t)$ is reciprocal, $f(1) = f'(1) = g(1) = 0$, and $f''(1) \equiv g''(1) \pmod 4$.*

Proof. (i)⇒(ii). This follows by Theorems 3.1 and 4.1.

(ii)⇒(i). By Theorem 3.1, there is a virtual knot K with $W_K(t) = f(t)$. We take a Laurent polynomial $h(t)$ with $I\!I\!I_K(t) \equiv h(t) \ (\mathrm{mod}\ \overline{W}_K(t))$. By Theorem 4.1, $h(t)$ is reciprocal, $h(1) = 0$, and $h''(1) \equiv f''(1) \pmod 4$.

Consider the Laurent polynomial $p(t) = g(t) - h(t)$. Then $p(t)$ is reciprocal, $p(1) = g(1) - h(1) = 0$, and $p''(1) = g''(1) - h''(1) \equiv 0 \pmod 4$. By Lemma 3.5, there is a Laurent polynomial $q(t)$ such that

$$p(t) = (t - 1)(t^{-1} - 1)q(t) + (t^{-1} - 1)(t - 1)q(t^{-1}).$$

It follows by Proposition 2.2 that there are dotted virtual knots T and T' such that

$$\begin{cases} \widehat{T} = K, & W^0_T(t) = W_K(t) - (t - 1)q(t), \quad W^1_T(t) = (t - 1)q(t), \text{ and} \\ \widehat{T'} = O, & W^0_{T'}(t) = -(t - 1), \qquad\qquad\qquad W^1_{T'}(t) = t - 1. \end{cases}$$

Then we have $p(t) = W^1_T(t)W^1_{T'}(t^{-1}) + W^1_T(t^{-1})W^1_{T'}(t)$. Let K'' be the virtual knot $\widehat{T+T'}$. By Theorem 2.3 (iii), it holds that

$$\begin{aligned} W_{K''}(t) &= W_K(t) + W_{K'}(t) = f(t) \text{ and} \\ I\!I\!I_{K''}(t) &\equiv h(t) + p(t) = g(t) \pmod{\overline{W}_{K''}(t)}. \end{aligned}$$

□

6. Connected sums of trivial knots

In [6], we prove that there are infinitely many connected sums of any pair of virtual knots. In particular, the intersection polynomials of a connected sum of two trivial virtual knots are characterized as shown in Propositions 6.1–6.3. Here, we use the notations $2\mathcal{P}_1 = \{2f(t) \mid f(t) \in \mathcal{P}_1\}$ and $2\mathcal{P}_2 = \{2f(t) \mid f(t) \in \mathcal{P}_2\}$.

Proposition 6.1. *For $f(t) \in \mathbb{Z}[t, t^{-1}]$, the following are equivalent.*

(i) *There is a virtual knot $K \in \mathcal{C}(O, O)$ with $f(t) = I_K(t)$.*

(ii) $f(t) \in 2\mathcal{P}_1$.

Proof. (i)⇒(ii). Let T and T' be dotted virtual knots with $K = \widehat{T+T'}$ and $\widehat{T} = \widehat{T'} = O$. By Theorem 3.2, we have $f(1) = f'(1) = 0$. Furthermore, since

$$W^1_T(t) = -W^0_T(t) \text{ and } W^1_{T'}(t) = -W^0_{T'}(t),$$

we have $I_K(t) = -2W^0_T(t)W^0_{T'}(t)$ by Theorem 2.3 (i). Therefore all the coefficients of $f(t)$ are even.

(ii)⇒(i). By Theorem 5.1, there is a Laurent polynomial $g(t) \in \mathbb{Z}[t, t^{-1}]$ with $f(t) = 2(t-1)^2 g(t)$. By Proposition 2.2, there are dotted virtual knots T and T' such that

$$\begin{cases} \widehat{T} = O, & W^0_T(t) = -(t-1), \quad W^1_T(t) = t-1, \text{ and} \\ \widehat{T'} = O, & W^0_{T'}(t) = (t-1)g(t), \quad W^1_{T'}(t) = -(t-1)g(t). \end{cases}$$

Then the connected sum $K = \widehat{T+T'} \in \mathcal{C}(O, O)$ satisfies

$$I_K(t) = -2W^0_T(t)W^0_{T'}(t) = 2(t-1)^2 g(t) = f(t).$$

□

Proposition 6.2. *For $f(t) \in \mathbb{Z}[t, t^{-1}]$, the following are equivalent.*

(i) *There is a virtual knot $K \in \mathcal{C}(O, O)$ with $f(t) = I\!I_K(t)$.*

(ii) $f(t) \in 2\mathcal{P}_2$.

Proof. (i)⇒(ii). Let T and T' be dotted virtual knots with $K = \widehat{T+T'}$ and $\widehat{T} = \widehat{T'} = O$. By Theorem 3.3, $f(t)$ is reciprocal, $f(1) = 0$, and $f''(1) \equiv 0 \pmod 4$. Furthermore, since we may take

$$W^1_T(t) = -W^0_T(t) = (t-1)p(t) \text{ and } W^1_{T'}(t) = -W^0_{T'}(t) = (t-1)q(t)$$

for some $p(t), q(t) \in \mathbb{Z}[t, t^{-1}]$, we have

$$\begin{aligned} f(t) = I\!I_K(t) &= 2(W^0_T(t)W^0_{T'}(t^{-1}) + W^0_T(t^{-1})W^0_{T'}(t)) \\ &= 2(t-1)(t^{-1}-1)(p(t)q(t^{-1}) + p(t^{-1})q(t)). \end{aligned}$$

Therefore all the coefficients of $f(t)$ are even.

(ii)⇒(i). Put $\widetilde{f}(t) = f(t)/2 \in \mathbb{Z}[t, t^{-1}]$. By Theorem 3.3, it satisfies that $\widetilde{f}(t)$ is reciprocal, $\widetilde{f}(1) = 0$, and $\widetilde{f}''(1) \equiv 0 \pmod 4$. It follows by Lemma 3.5 that there is a Laurent polynomial $\widetilde{g}(t)$ such that

$$\widetilde{f}(t) = (t-1)(t^{-1}-1)\widetilde{g}(t) + (t^{-1}-1)(t-1)\widetilde{g}(t^{-1}).$$

By Proposition 2.2, there are dotted virtual knots T and T' such that

$$\begin{cases} \widehat{T} = O, & W_T^0(t) = (t-1)\widetilde{g}(t), \quad W_T^1(t) = -(t-1)\widetilde{g}(t), \text{ and} \\ \widehat{T'} = O, & W_{T'}^0(t) = t-1, \qquad\quad W_{T'}^1(t) = -(t-1). \end{cases}$$

Then the connected sum $K = \widehat{T + T'} \in \mathcal{C}(O, O)$ satisfies

$$I\!I_K(t) = 2(t-1)(t^{-1}-1)(\widetilde{g}(t) + \widetilde{g}(t^{-1})) = f(t).$$ □

Proposition 6.3. *For $f(t) \in \mathbb{Z}[t, t^{-1}]$, the following are equivalent.*

(i) *There is a virtual knot $K \in \mathcal{C}(O, O)$ with $f(t) = I\!I\!I_K(t)$.*

(ii) *$f(t) \in \mathcal{P}_2$.*

Proof. By Theorem 2.3 (iii) and Proposition 2.2, the condition (i) is equivalent to

$$f(t) \in \{p(t)q(t) + p(t^{-1})q(t^{-1}) \mid p(t), q(t) \in \mathbb{Z}[t, t^{-1}], p(1) = q(1) = 0\}.$$

This set is coincident with

$$\{g(t) + g(t^{-1}) \mid g(t) \in \mathbb{Z}[t, t^{-1}], g(1) = g'(1) = 0\}.$$

Therefore (i) is equivalent to (ii) by Lemma 3.5 and Theorem 5.5. □

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Ryuji Higa
Department of Mathematics, Kobe University
Rokkodai-cho 1–1, Nada-ku, Kobe 657–8501
Japan
e-mail: higa@math.kobe-u.ac.jp

Takuji Nakamura
Faculty of Education, University of Yamanashi
Takeda 4–4–37, Kofu, Yamanashi, 400–8510
Japan
e-mail: takunakamura@yamanashi.ac.jp

Yasutaka Nakanishi
Department of Mathematics, Kobe University
Rokkodai-cho 1–1, Nada-ku, Kobe 657–8501
Japan
e-mail: nakanisi@math.kobe-u.ac.jp

Shin Satoh
Department of Mathematics, Kobe University
Rokkodai-cho 1–1, Nada-ku, Kobe 657–8501
Japan
e-mail: shin@math.kobe-u.ac.jp