

STABILITY OF UNIQUENESS AND COEXISTENCE OF EQUILIBRIUM STATES OF THE ISING MODEL UNDER LONG RANGE PERTURBATIONS

SHUNSUKE USUKI

(Received May 31, 2022, revised August 19, 2022)

Abstract

In this paper, we study perturbations of the d -dimensional Ising model for $d \geq 2$, including long range ones to which the Pirogov-Sinai theory is not applicable. We show that the uniqueness of the equilibrium state of the Ising model at high temperature and the coexistence of equilibrium states at low temperature are preserved by spin-flip symmetric perturbations.

1. Introduction

In this paper, we deal with \mathbb{Z}^d -lattice systems for $d \geq 2$. We begin with the definition of them. Let F be a finite set and $\Omega = F^{\mathbb{Z}^d}$. Then Ω is a compact metrizable space with respect to the product topology. For $\Lambda \subset \mathbb{Z}^d$, we write $\Omega_\Lambda = F^\Lambda$ and $\omega = (\omega(x))_{x \in \Lambda}$ for an element of Ω_Λ . For each $a \in \mathbb{Z}^d$, the **translation** by a is canonically defined, that is, for each $\Lambda \subset \mathbb{Z}^d$, we define $\tau^a : \Omega_\Lambda \rightarrow \Omega_{\Lambda-a}$ by

$$(\tau^a \omega)(x) = \omega(x + a), \quad x \in \Lambda - a$$

for $\omega \in \Omega_\Lambda$. When $\Lambda = \mathbb{Z}^d$, these translations define the \mathbb{Z}^d -action on Ω . In this paper, we write $\Lambda \Subset \mathbb{Z}^d$ when $\Lambda \subset \mathbb{Z}^d$ and Λ is a finite set. We say Φ is an **interaction** if Φ is a family of functions $\Phi_\Lambda : \Omega_\Lambda \rightarrow \mathbb{R}$ for each $\Lambda \Subset \mathbb{Z}^d$ and write $\Phi = \{\Phi_\Lambda\}_{\Lambda \Subset \mathbb{Z}^d}$. We always assume in this paper that an interaction Φ is **translation invariant**, that is, for any $a \in \mathbb{Z}^d$ and $\Lambda \Subset \mathbb{Z}^d$,

$$\Phi_\Lambda(\omega) = \Phi_{\Lambda-a}(\tau^a \omega).$$

We also assume that Φ is **absolutely summable**, that is, the norm

$$\|\Phi\| = \sum_{0 \in \Lambda \Subset \mathbb{Z}^d} \sup_{\xi \in \Omega_\Lambda} |\Phi_\Lambda(\xi)|$$

is finite. Under such an interaction Φ , for each $\Lambda \Subset \mathbb{Z}^d$ and $\omega \in \Omega$, the **Hamiltonian** is defined by

$$H_{\Phi, \Lambda}(\omega) = \sum_{\Delta \in \mathcal{F}_\Lambda} \Phi_\Delta(\omega),$$

where $\mathcal{F}_\Lambda = \{\Delta \Subset \mathbb{Z}^d \mid \Delta \cap \Lambda \neq \emptyset\}$ and $\Phi_\Delta(\omega) = \Phi_\Delta((\omega(x))_{x \in \Delta})$. By the translation invariance and the absolute summability of Φ , it is easily seen that the infinite sum converges.

Then Ω is interpreted as the **configuration space** and $H_{\Phi, \Lambda}$ ($\Lambda \in \mathbb{Z}^d$) define the energy for each configuration $\omega \in \Omega$. Then we regard Ω with Φ as a \mathbb{Z}^d -lattice system. We say that an interaction Φ is **finite range** if

$$r(\Phi) = \inf \left\{ R > 0 \mid \Phi_{\Lambda} = 0 \text{ for any } \Lambda \in \mathbb{Z}^d \text{ with } \text{diam}(\Lambda) > R \right\} < \infty.$$

In the above, for $\Lambda \subset \mathbb{Z}^d$, we write $\text{diam}(\Lambda) = \sup \{ \|x - y\|_{\infty} \mid x, y \in \Lambda \}$, where $\|x\|_{\infty} = \max \{ |x_1|, \dots, |x_d| \}$ for $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$. We emphasize that, in this paper, we deal with interactions which are **long range**, that is, $r(\Phi) = \infty$ in general.

The **Ising model** is a famous and important example of a \mathbb{Z}^d -lattice system, which we mainly deal with in this paper. This is a simple mathematical model of ferromagnetism.

Definition 1.1 (The Ising model). Let $\beta > 0$. The d -dimensional Ising model at the inverse temperature β is a \mathbb{Z}^d -lattice system on $\Omega = \{1, -1\}^{\mathbb{Z}^d}$ with a finite range interaction Φ^{β} defined by

$$\Phi_{\Lambda}^{\beta}(\omega) = \begin{cases} -\beta\omega(x)\omega(y), & \text{if } \Lambda = \{x, y\} \text{ with } \|x - y\|_1 = 1 \\ 0, & \text{otherwise,} \end{cases}$$

for each $\Lambda \in \mathbb{Z}^d$ and $\omega \in \Omega_{\Lambda} = \{1, -1\}^{\Lambda}$, where $\|x\|_1 = \sum_{i=1}^d |x_i|$ for $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$. Here, β is the parameter representing the inverse of the temperature of the system.

For a \mathbb{Z}^d -lattice system, Borel probability measures on Ω which represent statistical states of the system stable under the interaction are determined and called **Gibbs states**. Translation invariant probability measures representing stable statistical states are called **equilibrium states**. We give the rigorous definition in Section 2.1. It is an important problem to determine whether there is more than one equilibrium state for the system or not. For the case of $d = 1$, it is known that there exists the only one equilibrium state for any interaction under a reasonable condition. For example, finite range interactions satisfy this condition (see [27, Chapter 5]). On the other hand, the system might have more than one equilibrium state when $d \geq 2$. Concerning this problem, the Ising model defined in Definition 1.1 shows the remarkable property below.

Theorem 1.1 (Phase transition on the Ising model). *Let $d \geq 2$. Then there exists $\beta_c > 0$ depending on d such that the followings hold.*

- (1) *If $\beta < \beta_c$, then Φ^{β} has a unique Gibbs state and, in particular, a unique equilibrium state.*
- (2) *If $\beta > \beta_c$, then Φ^{β} has more than one equilibrium state.*

This phenomenon is called **phase transition** on the Ising model and β_c as above the critical inverse temperature. The coexistence of equilibrium states at low temperature was shown in [22]. For an intelligible explanation of Theorem 1.1, see [13, Chapter 3].

- Remark 1.1.**
- (i) For the Ising model with $\beta > \beta_c$, the concrete structure of the set of equilibrium states is known. See [1] or [16] for $d = 2$ and [6] for $d \geq 3$. For the result on the Ising model on a general amenable graph, see [25]. We will mention this structure in detail later (Section 4).
 - (ii) It seems difficult to study the Ising model at the critical temperature. However, it

is shown that, at $\beta = \beta_c$, Φ^{β_c} has a unique Gibbs state. This was shown in [29] for $d = 2$, [3] for $d \geq 4$ and [2] for $d = 3$.

The central question of this paper is to determine for the Ising model whether the uniqueness or coexistence of equilibrium states is preserved if we perturb the interaction Φ^β . Here a perturbation of Φ^β means an interaction $\Phi^\beta + \Psi$ defined by

$$(\Phi^\beta + \Psi)_\Lambda = \Phi^\beta_\Lambda + \Psi_\Lambda$$

for each $\Lambda \in \mathbb{Z}^d$ for some small perturbation interaction Ψ . Perturbations of the interaction may correspond to some influence from the surroundings to the system or some noise in the interaction. Hence, it is interesting to study what happens under perturbations of the interaction. We remark that the ‘upper semicontinuity’ of equilibrium states in perturbations of interactions holds in the following sense.

Proposition 1.2. *Let Φ_0 be an interaction, I_{Φ_0} be the set of equilibrium states for Φ_0 and I be the set of translation invariant Borel probability measures on Ω . Then, for any neighborhood U of I_{Φ_0} in I with respect to the weak*-topology, there exists $0 < \delta < 1$ such that, for any interaction Ψ with $\|\Psi\| < \delta$, we have*

$$I_{\Phi_0+\Psi} \subset U.$$

We give the proof in the appendix. However, Proposition 1.2 does not ensure the stability of the uniqueness or coexistence of equilibrium states.

If the perturbations are finite range, then this perturbation problem can be studied by the known theory, called the **Pirogov-Sinai theory** and it is known that the coexistence of equilibrium states at low temperature is stable under finite range perturbations which preserve some symmetry of the Ising model, called the spin-flip symmetry.

Definition 1.2. An interaction Ψ on $\Omega = \{1, -1\}^{\mathbb{Z}^d}$ is **spin-flip symmetric** if

$$\Psi_\Lambda(\omega) = \Psi_\Lambda(-\omega)$$

for $\Lambda \in \mathbb{Z}^d$ and $\omega \in \Omega_\Lambda$.

Our main theorem says that the same statement holds under long range perturbations to which the Pirogov-Sinai theory is not applicable. We remark in detail the relation between our main theorem and the Pirogov-Sinai theory in Section 2.4. We consider perturbations by small perturbation interactions with respect to the norm $\|\cdot\|$, which we call the **d -th order decaying norm**. We give its definition in Section 2.2. The following is our main theorem.

Theorem 1.3 (Main theorem). *Let $d \geq 2$ and, for $\beta > 0$, Φ^β be the interaction of the d -dimensional Ising model at the inverse temperature β . We write β_c for the critical inverse temperature. Then we have the following.*

- (1) *There exist $0 < \beta_h < \beta_c$ and $\delta > 0$ such that, for any $0 < \beta \leq \beta_h$ and any translation invariant interaction Ψ with $\|\Psi\| < \delta$, $\Phi = \Phi^\beta + \Psi$ has a unique Gibbs state and, in particular, a unique equilibrium state.*
- (2) *There exist $\beta_c < \beta_l < \infty$ and $\delta > 0$ such that, for any $\beta \geq \beta_l$ and any translation invariant interaction Ψ with $\|\Psi\| < \delta$ which is spin-flip symmetric, $\Phi = \Phi^\beta + \Psi$ has more than one equilibrium state.*

We notice that this norm is stronger than $\|\cdot\|$ and weaker than the exponentially decaying norm.

Remark 1.2. (i) Actually, Theorem 1.3 (1) is a direct corollary of **Dobrushin’s criterion**, which gives a condition for an interaction to have a unique Gibbs state. We remark this in detail in Section 2.3.

(ii) Theorem 1.3 (2) follows from stronger results, Theorems 3.1 and 3.2 below.

(iii) In Theorem 1.3, the d -th order decaying condition is crucial. See Remark 3.2.

We notice that Theorem 1.3 (2) fails if we drop the spin-flip symmetry condition. To see this, we see the phase diagram of the **Ising model with external fields**. For $\beta, h \in \mathbb{R}$, we define the interaction $\{\Phi_{\Lambda}^{\beta,h}\}_{\Lambda \in \mathbb{Z}^d}$ by

$$\Phi_{\Lambda}^{\beta,h}(\omega) = \begin{cases} -\beta\omega(x)\omega(y), & \Lambda = \{x, y\} \text{ with } \|x - y\|_1 = 1 \\ -h\omega(x), & \Lambda = \{x\} \\ 0, & \text{otherwise,} \end{cases}$$

for each $\Lambda \in \mathbb{Z}^d$ and $\omega \in \Omega_{\Lambda}$. When $h = 0$, then $\Phi^{\beta,0} = \Phi^{\beta}$ and, by taking $h \neq 0$ arbitrarily small, we can think $\Phi^{\beta,h}$ as a perturbation of Φ^{β} . The following was shown in [19] (see [13, Chapter 3] for an intelligible explanation).

Theorem 1.4. *For any $\beta, h \in \mathbb{R}$ with $h \neq 0$, $\Phi^{\beta,h}$ has a unique Gibbs state.*

We notice that the perturbations $\Phi^{\beta,h}$ ($h \neq 0$) are not spin-flip symmetric. Hence, Theorem 1.4 says that, for sufficiently large β , non-spin-flip symmetric perturbations of Φ^{β} can break the structure of the equilibrium states. Moreover, it is known that the uniqueness of the equilibrium state is a ‘generic’ property in the space of interactions (with appropriate norms). This fact is discussed in [8], [12], [28] and [17]. Our Main Theorem 1.3 says that spin-flip symmetric perturbations are ‘not generic’ and preserve the coexistence.

In Section 2.1, we give the rigorous definition of Gibbs states and equilibrium states. We give the definition of the d -th order decaying norm $\|\cdot\|$ in Section 2.2. In Section 2.3, we give Dobrushin’s criterion and deduce Theorem 1.3 (1) from it. We remark the relation between our main theorem and the Pirogov-Sinai theory in Section 2.4. We prove the main theorem at low temperature in Section 3. We use in the proof some extension of Peierls’ argument. We notice further questions in Section 4.

2. Preliminaries

2.1. Gibbs states and equilibrium states. In this section, for a \mathbb{Z}^d -lattice system on $\Omega = F^{\mathbb{Z}^d}$ with an interaction Φ , we give the rigorous definition of Gibbs states and equilibrium states. We first define Gibbs states by the thermodynamical condition, called the DLR condition¹. We next define equilibrium states by the variational principle in the theory of dynamical systems and notice that these different definitions are equivalent under the translation invariant condition.

For $\Lambda \in \mathbb{Z}^d$, $\eta \in \Omega_{\mathbb{Z}^d \setminus \Lambda}$ and $\xi_{\Lambda} \in \Omega_{\Lambda}$, we write $\xi_{\Lambda} \vee \eta$ for the element of Ω defined by $\xi_{\Lambda} \vee \eta|_{\Lambda} = \xi_{\Lambda}$ and $\xi_{\Lambda} \vee \eta|_{\mathbb{Z}^d \setminus \Lambda} = \eta$. We define the **finite Gibbs state $\mu_{\Phi,\Lambda}^{\eta}$ in Λ with the**

¹This is named after R. L. Dobrushin, O. E. Lanford and D. Ruelle.

boundary condition η as the Borel probability measure on Ω_Λ defined by

$$(2.1) \quad \mu_{\Phi,\Lambda}^\eta(\{\omega_\Lambda\}) = \frac{1}{Z_{\Phi,\Lambda}^\eta} \exp(-H_{\Phi,\Lambda}(\omega_\Lambda \vee \eta))$$

for each $\omega_\Lambda \in \Omega_\Lambda$, where

$$Z_{\Phi,\Lambda}^\eta = \sum_{\xi_\Lambda \in \Omega_\Lambda} \exp(-H_{\Phi,\Lambda}(\xi_\Lambda \vee \eta)).$$

We call $Z_{\Phi,\Lambda}^\eta$ the **partition function in Λ with the boundary condition η** . Let $C(\Omega)$ be the real Banach space of continuous real-valued functions on Ω with the supremum norm and $M(\Omega)$ be the set of Borel probability measures on Ω . Then $M(\Omega)$ can be viewed as a subset of the dual space of the Banach space $C(\Omega)$ and is a compact, convex and metrizable space with respect to the weak*-topology.

Definition 2.1 (Gibbs state). A probability measure $\sigma \in M(\Omega)$ is a Gibbs state for an interaction Φ if for each $\Lambda \Subset \mathbb{Z}^d$, there exists a Borel probability measure $\sigma_{\mathbb{Z}^d \setminus \Lambda}$ on $\Omega_{\mathbb{Z}^d \setminus \Lambda}$ such that for any $\omega_\Lambda \in \Omega_\Lambda$,

$$\sigma(\{\xi \in \Omega \mid \xi|_\Lambda = \omega_\Lambda\}) = \int_{\Omega_{\mathbb{Z}^d \setminus \Lambda}} \mu_{\Phi,\Lambda}^\eta(\{\omega_\Lambda\}) d\sigma_{\mathbb{Z}^d \setminus \Lambda}(\eta).$$

Definition 2.1 is equivalent to the following: for each $\Lambda \Subset \mathbb{Z}^d$ and $\eta \in \Omega_{\mathbb{Z}^d \setminus \Lambda}$, the conditional probability that $\xi|_\Lambda = \omega_\Lambda$ under $\xi|_{\mathbb{Z}^d \setminus \Lambda} = \eta$ is $\mu_{\Phi,\Lambda}^\eta(\{\omega_\Lambda\})$. This condition is called the DLR condition. Let K_Φ be the set of Gibbs states for Φ . This is a nonempty, compact and convex subset of $M(\Omega)$ (see [27, Chapter 1]). We notice that $K_{\Phi+C} = K_\Phi$ for any constant interaction C (that is, an interaction such that C_Λ is constant for each $\Lambda \Subset \mathbb{Z}^d$).

Next, we state the variational principle in the theory of dynamical systems and give the definition of equilibrium states. Let I be the set of Borel probability measures μ on Ω which are translation invariant, that is, $\tau_*^a \mu = \mu$ for each $a \in \mathbb{Z}^d$. This is a nonempty, compact and convex subset of $M(\Omega)$. For a sequence $\{\Lambda_n\}_{n=1}^\infty$ of finite subsets of \mathbb{Z}^d , we write $\Lambda_n \nearrow \infty$ (limit in the sense of van Hove) when $|\Lambda_n| \rightarrow \infty$ and, for any $a \in \mathbb{Z}^d$, $|(\Lambda_n + a) \setminus \Lambda_n|/|\Lambda_n| \rightarrow 0$. An example of such a sequence is $\{B(n)\}_{n=1}^\infty$, where $B(n) = \{-n, \dots, n\}^d$.

Definition 2.2. For each $\mu \in I$, the limit

$$h(\mu) = \lim_{n \rightarrow \infty} -\frac{1}{|\Lambda_n|} \sum_{\omega \in \Omega_{\Lambda_n}} \mu_{\Lambda_n}(\{\omega\}) \log \mu_{\Lambda_n}(\{\omega\}),$$

where $\mu_{\Lambda_n}(\{\omega\}) = \mu(\{\xi \in \Omega \mid \xi|_{\Lambda_n} = \omega\})$, exists and is independent of the choice of sequence $\{\Lambda_n\}_{n=1}^\infty$ such that $\Lambda_n \nearrow \infty$ (see [27, Chapter 3]). $h(\mu)$ is called the **measure-theoretic entropy** of μ .

Definition 2.3. For each interaction Φ and $\Lambda \Subset \mathbb{Z}^d$, we define the **partition function in Λ with free boundary condition** by

$$Z_{\Phi,\Lambda} = \sum_{\omega \in \Omega_\Lambda} \exp\left(-\sum_{\Delta \subset \Lambda} \Phi_\Delta(\omega)\right).$$

Then, the limit

$$P^\Phi = \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \log Z_{\Phi, \Lambda_n}$$

exists and is independent of the choice of sequence $\{\Lambda_n\}_{n=1}^\infty$ such that $\Lambda_n \nearrow \infty$ (see [27, Chapter 3]). We call P^Φ the **pressure** of Φ .

For an interaction Φ , we define $A_\Phi : \Omega \rightarrow \mathbb{R}$ by

$$A_\Phi(\omega) = - \sum_{\Lambda} \Phi_\Lambda(\omega),$$

where the sum runs over those $\Lambda \in \mathbb{Z}^d$ such that Λ contains 0 in \mathbb{Z}^d and 0 is the middle element of Λ (that is, the $\lfloor (|\Lambda| + 1) / 2 \rfloor$ -th² element) with respect to the lexicographic order on Λ . Since Φ is absolutely summable, A_Φ is continuous. Here, $-A_\Phi(\omega)$ represents the contribution of $0 \in \mathbb{Z}^d$ to the energy in the configuration ω . The **variational principle** is the statement which connects entropy, pressure and A_Φ .

Proposition 2.1 (Variational principle). *For each interaction Φ ,*

$$P^\Phi = \sup_{\mu \in I} \left(h(\mu) + \int_{\Omega} A_\Phi d\mu \right).$$

Moreover, there exists some $\mu \in I$ such that μ achieves the supremum (see [27, Theorem 3.12]).

Definition 2.4 (Equilibrium state). An element $\mu \in I$ is an equilibrium state for an interaction Φ if μ achieves the supremum of Proposition 2.1.

Let I_Φ be the set of the equilibrium states for Φ . This is a nonempty, compact and convex subset of $M(\Omega)$. The definition of the equilibrium states turns out to be a characterization of the translation invariant Gibbs states, that is, the following proposition holds (see [27, Theorem 4.2]).

Proposition 2.2. *For each interaction Φ ,*

$$I_\Phi = K_\Phi \cap I.$$

2.2. d -th order decaying interactions. In this section, we introduce d -th order decaying interactions on $\Omega = \{-1, 1\}^{\mathbb{Z}^d}$ and give the definition of the d -th order decaying norm $\|\cdot\|$ in Theorem 1.3.

Definition 2.5. An interaction Ψ is **d -th order decaying** if

$$\|\Psi\| = \sum_{0 \in \Lambda \in \mathbb{Z}^d} (\text{diam}(\Lambda) + 1)^d \sup_{\omega \in \Omega_\Lambda} |\Psi_\Lambda(\omega)| < \infty.$$

We call the norm $\|\cdot\|$ the d -th order decaying norm.

It is obvious that

$$\|\Psi\| \leq \|\Psi\|$$

for any d -th order decaying interaction Ψ on Ω . Let us see some examples of d -th order

²For $a \in \mathbb{R}$, $\lfloor a \rfloor$ denotes the largest integer that is not larger than a .

decaying interactions. Clearly, we have many d -th order decaying interactions other than these examples.

- Example 2.1. (1) Finite range interactions are clearly d -th order decaying.
 (2) (Two body interactions.) Let Ψ be a translation invariant interaction on Ω which satisfies

$$\Psi_\Lambda = 0$$

unless Λ consists of distinct two points. We emphasize that these two body interactions are not finite range in general. We assume that Ψ is $(2d + \varepsilon)$ -th order decaying, that is, there exists $\varepsilon > 0$ such that

$$\|\Psi\|_\varepsilon = \sup \left\{ \|x - y\|_\infty^{2d+\varepsilon} |\Psi_{\{x,y\}}(\omega)| \mid x, y \in \mathbb{Z}^d, x \neq y, \omega \in \Omega_{\{x,y\}} \right\} < \infty.$$

We have

$$\begin{aligned} \|\Psi\| &= \sum_{x \in \mathbb{Z}^d \setminus \{0\}} (\|x\|_\infty + 1)^d \sup_{\omega \in \Omega_{\{0,x\}}} |\Psi_{\{0,x\}}(\omega)| \\ &\leq \sum_{x \in \mathbb{Z}^d \setminus \{0\}} (\|x\|_\infty + 1)^d \|x\|_\infty^{-(2d+\varepsilon)} \|\Psi\|_\varepsilon \\ &= M_\varepsilon \|\Psi\|_\varepsilon, \end{aligned}$$

where $M_\varepsilon = \sum_{n=1}^\infty ((2n + 1)^d - (2n - 1)^d)(n + 1)^d n^{-(2d+\varepsilon)} < \infty$. Hence, Ψ is d -th order decaying.

Here, we give a remark on two body interactions. For a translation invariant two body interaction Ψ which has a form $\Psi_{\{x,y\}}(\omega) = K_{x,y}\omega(x)\omega(y)$, $K_{x,y} \in \mathbb{R}$, the following fact was shown in [15]. If Ψ satisfies that $K_{0,e_i} + \sum_{x \in \mathbb{Z}^d, \|x\|_1 > 1} |x_i| K_{0,x} < 0$ for each $i = 1, \dots, d$, then, for sufficiently large $\beta > 0$, $\beta\Psi$ has more than one equilibrium state. Here $e_i \in \mathbb{Z}^d$ is the element such that the i -th component is 1 and the other components are 0. From this, we can see that, if a perturbation interaction Ψ has a form as above, we obtain the coexistence of equilibrium states under the weaker condition than the d -th order decaying condition.

2.3. High temperature cases and Dobrushin’s criterion. In this section, we give Dobrushin’s criterion, which gives a condition for an interaction to have a unique Gibbs state. This was first proved in [11]. We also see that Theorem 1.3 (1) follows from Dobrushin’s criterion. Let Φ be a translation invariant interaction. For any $x \in \mathbb{Z}^d \setminus \{0\}$, we define

$$\rho_\Phi(x) = \sup \left\{ \left| \mu_{\Phi,\{0\}}^\eta(\{1\}) - \mu_{\Phi,\{0\}}^\zeta(\{1\}) \right| \mid \eta, \zeta \in \Omega_{\mathbb{Z}^d \setminus \{0\}}, \eta|_{\mathbb{Z}^d \setminus \{0,x\}} = \zeta|_{\mathbb{Z}^d \setminus \{0,x\}} \right\}.$$

Proposition 2.3 (Dobrushin’s criterion). *If*

$$\sum_{x \in \mathbb{Z}^d \setminus \{0\}} \rho_\Phi(x) < 1,$$

then Φ has a unique Gibbs state.

Proposition 2.4 below is a convenient version of Dobrushin's criterion. We define

$$\text{var}_\Lambda \Phi = \sup_{\omega, \omega' \in \Omega_\Lambda} |\Phi_\Lambda(\omega) - \Phi_\Lambda(\omega')|$$

for each $\Lambda \in \mathbb{Z}^d$ and

$$\|\Phi\|_{\text{var}} = \sum_{0 \in \Lambda \in \mathbb{Z}^d} (|\Lambda| - 1) \text{var}_\Lambda \Phi,$$

where $|\Lambda|$ be the cardinality of Λ . It is obvious that

$$(2.2) \quad \|\Phi\|_{\text{var}} \leq 2\|\Phi\|$$

for any d -th order decaying interaction Φ .

Proposition 2.4. *If*

$$\|\Phi\|_{\text{var}} < 2,$$

then Φ satisfies Dobrushin's criterion. Hence, it has a unique Gibbs state.

We refer [14, Chapter 8] for proofs of the above propositions. It is easy to see that $\|\Phi^\beta\|_{\text{var}} = 4d\beta$. From this, Inequality (2.2) and Proposition 2.4, we have Theorem 1.3 (1).

2.4. Low temperature cases and Pirogov-Sinai theory. We notice the relation between Theorem 1.3 (2) and the Pirogov-Sinai theory. If β is sufficiently large and **a perturbation interaction Ψ is finite range**, then the equilibrium states of $\Phi = \Phi^\beta + \Psi$ can be studied by an application of the Pirogov-Sinai theory. It was introduced by S. A. Pirogov and Ya. G. Sinai in [23] and [24], and sophisticated in [30] to study phase diagrams of more general finite range interactions at low temperature. We refer [13, Chapter 7] for an intelligible introduction to this theory. The periodic ground states of Φ^β , that is, the states which have the lowest energy under Φ^β , are exactly the constant configurations η^+ and $\eta^- \in \Omega$, which are constant 1 and -1 on \mathbb{Z}^d , respectively, and it can be seen that sufficiently small and spin-flip symmetric perturbations of Φ^β preserves η^+ and η^- as periodic ground states. Hence, by the Pirogov-Sinai theory, if Ψ is sufficiently small, spin-flip symmetric and finite range interaction, then $\Phi = \Phi^\beta + \Psi$ has equilibrium states corresponding to two periodic ground states, η^+ and η^- .

The assumption that interactions are finite range is crucial in the Pirogov-Sinai theory. Hence, Theorem 1.3 (2), including results in the case of long range perturbations, does not follows from it. The Pirogov-Sinai theory was extended to some extent in [20] and [21], but this extension is possible only for sum of finite range interactions and two body interactions satisfying some exponentially decaying condition. It was also extended to quantum cases in [7], or a series of papers [9] and [10]. In these cases, perturbation interactions can be long range but must exponentially decay. Hence, Theorem 1.3 (2), allowing d -th order decaying perturbations, also does not follow from these extended versions. On the other hand, we can not study spin-flip symmetric perturbations (which are long range in general) in detail using the techniques in this paper as well as finite range perturbations using the Pirogov-Sinai theory. We will mention this later (Section 4).

3. Perturbations of the Ising model at low temperature

3.1. The main theorem in low temperature cases and rearrangement of perturbation interactions. In Section 3, we give a proof of Theorem 1.3 (2). As we mentioned in Remark 1.2, Theorem 1.3 (2) follows from stronger results: Theorems 3.1 and 3.2 below. We first state these results.

We introduce a norm $\|\cdot\|'$, which is weaker than the d -th order decaying norm $\|\cdot\|$. For a translation invariant interaction Ψ , we define

$$\|\Psi\|' = \sum_{0 \in \Lambda \in \mathbb{Z}^d} |\Lambda|^{-1} (\text{diam}(\Lambda) + 1)^d \sup_{\omega \in \Omega_\Lambda} |\Psi_\Lambda(\omega)|.$$

It is clear that $\|\Psi\|' \leq \|\Psi\|$. Then we have the following theorem.

Theorem 3.1. *Let $d \geq 2$. Then there exists $0 < L < \infty$ satisfying the following. If $\beta, \delta > 0$ satisfy $\beta - \delta > L$, then, for any translation invariant interaction Ψ with $\|\Psi\|' < \delta$ which is spin-flip symmetric, $\Phi = \Phi^\beta + \Psi$ has more than one equilibrium state.*

Theorem 3.1 follows from Theorem 3.2 below. We say that $\Lambda \subset \mathbb{Z}^d$ is $\|\cdot\|_1$ -connected if, for each x and $y \in \Lambda$, there exist finite points $x_0, x_1, \dots, x_{n-1}, x_n \in \Lambda$ such that $x_0 = x, x_n = y, \|x_{i+1} - x_i\|_1 = 1$ for $i = 0, \dots, n - 1$. We call such a finite sequence $x_0, x_1, \dots, x_{n-1}, x_n$ a $\|\cdot\|_1$ -path from x to y . For an interaction Ψ , we say that Ψ is **zero on non- $\|\cdot\|_1$ -connected sets** if $\Psi_\Lambda \equiv 0$ whenever $\Lambda \Subset \mathbb{Z}^d$ is not $\|\cdot\|_1$ -connected.

Theorem 3.2. *Let $d \geq 2$. Then there exists $0 < L < \infty$ satisfying the following. If $\beta, \delta > 0$ satisfy $\beta - \delta > L$, then, for any translation invariant interaction Ψ with $\|\Psi\| < \delta$ which is spin-flip symmetric and zero on non- $\|\cdot\|_1$ -connected sets, $\Phi = \Phi^\beta + \Psi$ has more than one equilibrium state.*

We see that how Theorem 3.1 reduces to Theorem 3.2. For this purpose, we rearrange a perturbation interaction. Let Ψ be a translation invariant interaction with $\|\Psi\|' < \infty$. For $R \Subset \mathbb{Z}^d$, we say R is a **rectangle** if $R = [a_1, b_1] \times \dots \times [a_d, b_d] \cap \mathbb{Z}^d$ for some $a_1 \leq b_1, \dots, a_d \leq b_d \in \mathbb{Z}$. For each finite $\Lambda \Subset \mathbb{Z}^d$, we write $R(\Lambda)$ for the minimal rectangle which contains Λ with respect to the partial order given by inclusion. We notice that $\text{diam}(\Lambda) = \text{diam}(R(\Lambda))$. For each rectangle R , we define $\mathcal{S}(R) = \{\Lambda \Subset \mathbb{Z}^d \mid R(\Lambda) = R\}$. For Ψ as above, we define an interaction $\tilde{\Psi}$ as

$$\tilde{\Psi}_\Lambda \equiv 0$$

if Λ is not a rectangle and

$$\tilde{\Psi}_R(\omega) = \sum_{\Lambda \in \mathcal{S}(R)} \Psi_\Lambda(\omega|_\Lambda)$$

for any rectangle $R \Subset \mathbb{Z}^d$ and $\omega \in \Omega_R$. This interaction $\tilde{\Psi}$ is made by putting interactions Ψ_Λ on $\Lambda \in \mathcal{S}(R)$ together into the rectangle R . Obviously $\tilde{\Psi}$ is a translation invariant interaction. We write \mathcal{S}_0 for the set of $\Lambda \Subset \mathbb{Z}^d$ such that Λ contains 0 in \mathbb{Z}^d and 0 is the middle element of Λ (that is, the $[(|\Lambda| + 1) / 2]$ -th element) with respect to the lexicographic order on Λ . Then, by the translation invariance of $\tilde{\Psi}$, we have

$$(3.1) \quad \|\tilde{\Psi}\| = \sum_{0 \in R \Subset \mathbb{Z}^d, R: \text{rectangle}} \sup_{\omega \in \Omega_R} |\tilde{\Psi}_R(\omega)|$$

$$\begin{aligned}
 &= \sum_{\substack{0 \in R \in \mathbb{Z}^d, R: \text{rectangle}, \\ R \in \mathcal{S}_0}} |R| \sup_{\omega \in \Omega_R} |\tilde{\Psi}_R(\omega)| \\
 &\leq \sum_{\substack{0 \in R \in \mathbb{Z}^d, R: \text{rectangle}, \\ R \in \mathcal{S}_0}} |R| \sum_{\Lambda \in \mathcal{S}(R)} \sup_{\omega \in \Omega_\Lambda} |\Psi_\Lambda(\omega)| \\
 &= \sum_{\substack{\Lambda \in \mathbb{Z}^d, \\ R(\Lambda) \in \mathcal{S}_0}} |R(\Lambda)| \sup_{\omega \in \Omega_\Lambda} |\Psi_\Lambda(\omega)| \\
 &\stackrel{(*)}{=} \sum_{0 \in \Lambda \in \mathbb{Z}^d} |\Lambda|^{-1} |R(\Lambda)| \sup_{\omega \in \Omega_\Lambda} |\Psi_\Lambda(\omega)| \\
 &\leq \sum_{0 \in \Lambda \in \mathbb{Z}^d} |\Lambda|^{-1} (\text{diam}(\Lambda) + 1)^d \sup_{\omega \in \Omega_\Lambda} |\Psi_\Lambda(\omega)| \\
 &= \|\Psi\|'.
 \end{aligned}$$

Here the equation (*) holds since Ψ is translation invariant and, for each $\Lambda \in \mathbb{Z}^d$, there is exactly one translation of Λ that appears in the summation $\sum_{\Lambda \in \mathbb{Z}^d, R(\Lambda) \in \mathcal{S}_0}$. The following lemma shows that $\tilde{\Psi}$ and Ψ induce the same statistical system.

Lemma 3.3. *Let Ψ and $\tilde{\Psi}$ be the interactions as above. For any interaction Φ_0 , interactions $\Phi = \Phi_0 + \Psi$ and $\tilde{\Phi} = \Phi_0 + \tilde{\Psi}$ have the same Gibbs states.*

In particular, by Proposition 2.2, Φ and $\tilde{\Phi}$ have the same equilibrium states.

Proof. We take arbitrary $\Lambda \in \mathbb{Z}^d$ and $\eta \in \Omega_{\mathbb{Z}^d \setminus \Lambda}$. For any $\omega_\Lambda \in \Omega_\Lambda$, we have, by (2.1),

$$\begin{aligned}
 &\mu_{\tilde{\Phi}, \Lambda}^\eta(\{\omega_\Lambda\}) \\
 &= \frac{\exp\left(-\sum_{\Delta \in \mathcal{F}_\Lambda} (\Phi_\Delta(\omega_\Lambda \vee \eta) + \tilde{\Psi}_\Delta(\omega_\Lambda \vee \eta))\right)}{\sum_{\xi_\Lambda \in \Omega_\Lambda} \exp\left(-\sum_{\Delta \in \mathcal{F}_\Lambda} (\Phi_\Delta(\xi_\Lambda \vee \eta) + \tilde{\Psi}_\Delta(\xi_\Lambda \vee \eta))\right)} \\
 &= \frac{\exp\left(-\sum_{\Delta \in \mathcal{F}_\Lambda} \Phi_\Delta(\omega_\Lambda \vee \eta) - \sum_{R \in \mathcal{R}_\Lambda} \sum_{S \in \mathcal{S}(R)} \Psi_S(\omega_\Lambda \vee \eta)\right)}{\sum_{\xi_\Lambda \in \Omega_\Lambda} \exp\left(-\sum_{\Delta \in \mathcal{F}_\Lambda} \Phi_\Delta(\xi_\Lambda \vee \eta) - \sum_{R \in \mathcal{R}_\Lambda} \sum_{S \in \mathcal{S}(R)} \Psi_S(\xi_\Lambda \vee \eta)\right)},
 \end{aligned}$$

where $\mathcal{R}_\Lambda = \{R \in \mathbb{Z}^d \mid R \text{ is a rectangle and } R \cap \Lambda \neq \emptyset\}$. Here, for any $\xi_\Lambda \in \Omega_\Lambda$,

$$\begin{aligned}
 &\sum_{R \in \mathcal{R}_\Lambda} \sum_{S \in \mathcal{S}(R)} \Psi_S(\xi_\Lambda \vee \eta) \\
 &= \sum_{R \in \mathcal{R}_\Lambda} \sum_{S \in \mathcal{S}(R) \cap \mathcal{F}_\Lambda} \Psi_S(\xi_\Lambda \vee \eta) + \sum_{R \in \mathcal{R}_\Lambda} \sum_{S \in \mathcal{S}(R) \setminus \mathcal{F}_\Lambda} \Psi_S(\eta) \\
 &= \sum_{\Delta \in \mathcal{F}_\Lambda} \Psi_\Delta(\xi_\Lambda \vee \eta) + \sum_{R \in \mathcal{R}_\Lambda} \sum_{S \in \mathcal{S}(R) \setminus \mathcal{F}_\Lambda} \Psi_S(\eta)
 \end{aligned}$$

and the second term is independent of ξ_Λ . Hence, we have

$$\begin{aligned} & \mu_{\Phi, \Lambda}^\eta(\{\omega_\Lambda\}) \\ &= \frac{\exp\left(-\sum_{\Delta \in \mathcal{F}_\Lambda} \Phi_\Delta(\omega_\Lambda \vee \eta) - \sum_{R \in \mathcal{R}_\Lambda} \sum_{S \in \mathcal{S}(R)} \Psi_S(\omega_\Lambda \vee \eta)\right)}{\sum_{\xi_\Lambda \in \Omega_\Lambda} \exp\left(-\sum_{\Delta \in \mathcal{F}_\Lambda} \Phi_\Delta(\xi_\Lambda \vee \eta) - \sum_{R \in \mathcal{R}_\Lambda} \sum_{S \in \mathcal{S}(R)} \Psi_S(\xi_\Lambda \vee \eta)\right)} \\ &= \frac{\exp\left(-\sum_{\Delta \in \mathcal{F}_\Lambda} (\Phi_\Delta(\omega_\Lambda \vee \eta) + \Psi_\Delta(\omega_\Lambda \vee \eta))\right)}{\sum_{\xi_\Lambda \in \Omega_\Lambda} \exp\left(-\sum_{\Delta \in \mathcal{F}_\Lambda} (\Phi_\Delta(\xi_\Lambda \vee \eta) + \Psi_\Delta(\xi_\Lambda \vee \eta))\right)} \\ &= \mu_{\Phi, \Lambda}^\eta(\{\omega_\Lambda\}). \end{aligned}$$

This implies that each finite Gibbs state coincides for Φ and $\tilde{\Phi}$. Hence, the statement holds. □

Obviously $\tilde{\Psi}$ is zero on non- $\|\cdot\|_1$ -connected sets. Hence, by replacing Ψ with $\tilde{\Psi}$ in Theorem 3.1 and using (3.1) and Lemma 3.3, we see that Theorem 3.1 follows from Theorem 3.2.

3.2. Gibbs states with the +, --boundary condition. We will give a proof of Theorem 3.2. Let Φ be a translation invariant interaction such that $\|\Phi\| < \infty$. For $\Lambda \Subset \mathbb{Z}^d$, we define the **finite Gibbs state with the + and --boundary condition**: $\mu_{\Phi, \Lambda}^+$ and $\mu_{\Phi, \Lambda}^-$ as the Borel probability measures on Ω_Λ defined by (2.1) for $\eta = \eta_{\mathbb{Z}^d \setminus \Lambda}^+$ and $\eta_{\mathbb{Z}^d \setminus \Lambda}^-$: the constant 1 and -1 configurations on $\mathbb{Z}^d \setminus \Lambda$, respectively. That is,

$$(3.2) \quad \begin{aligned} \mu_{\Phi, \Lambda}^+(\{\omega\}) &= \frac{1}{Z_{\Phi, \Lambda}^+} \exp\left(-\sum_{\Delta \in \mathcal{F}_\Lambda} \Phi_\Delta(\omega \vee \eta_{\mathbb{Z}^d \setminus \Lambda}^+)\right) \text{ and} \\ \mu_{\Phi, \Lambda}^-(\{\omega\}) &= \frac{1}{Z_{\Phi, \Lambda}^-} \exp\left(-\sum_{\Delta \in \mathcal{F}_\Lambda} \Phi_\Delta(\omega \vee \eta_{\mathbb{Z}^d \setminus \Lambda}^-)\right) \end{aligned}$$

for each $\omega \in \Omega_\Lambda$, where

$$\begin{aligned} Z_{\Phi, \Lambda}^+ &= \sum_{\omega \in \Omega_\Lambda} \exp\left(-\sum_{\Delta \in \mathcal{F}_\Lambda} \Phi_\Delta(\omega \vee \eta_{\mathbb{Z}^d \setminus \Lambda}^+)\right) \text{ and} \\ Z_{\Phi, \Lambda}^- &= \sum_{\omega \in \Omega_\Lambda} \exp\left(-\sum_{\Delta \in \mathcal{F}_\Lambda} \Phi_\Delta(\omega \vee \eta_{\mathbb{Z}^d \setminus \Lambda}^-)\right). \end{aligned}$$

Here, we state the Key Proposition 3.4 and see that Theorem 3.2 follows from it. We say that $\Lambda \subset \mathbb{Z}^d$ is $\|\cdot\|_\infty$ -**connected** if, for each x and $y \in \Lambda$, there exist finite points $x_0, x_1, \dots, x_{n-1}, x_n \in \Lambda$ such that $x_0 = x, x_n = y, \|x_{i+1} - x_i\|_\infty = 1$ for $i = 0, \dots, n-1$. We call such a finite sequence $x_0, x_1, \dots, x_{n-1}, x_n$ a $\|\cdot\|_\infty$ -**path** from x to y . Moreover, we say that Λ is **c-connected** if Λ and $\mathbb{Z}^d \setminus \Lambda$ are $\|\cdot\|_\infty$ -connected.

Proposition 3.4. *Let $d \geq 2$. Then there exists sufficiently large $0 < L < \infty$ and $0 < \varepsilon(L) < 1/2$ with $\varepsilon(L) \rightarrow 0$ as $L \rightarrow \infty$ satisfying the following. Let $\beta, \delta > 0$ and $\beta - \delta > L$ and Ψ be a translation invariant interaction with $\|\Psi\| < \delta$ which is spin-flip symmetric and zero on non- $\|\cdot\|_1$ -connected sets. We write $\Phi = \Phi^\beta + \Psi$. Then, for any c -connected $\Lambda \Subset \mathbb{Z}^d$ and $x \in \Lambda$, we have*

$$\mu_{\Phi, \Lambda}^+ (\{\omega \in \Omega_\Lambda \mid \omega(x) = 1\}) > 1 - \varepsilon(L) \quad \text{and} \quad \mu_{\Phi, \Lambda}^- (\{\omega \in \Omega_\Lambda \mid \omega(x) = -1\}) > 1 - \varepsilon(L).$$

This proposition means that, with respect to $\mu_{\Phi, \Lambda}^+$ and $\mu_{\Phi, \Lambda}^-$, the spin at x is magnetized for each $x \in \Lambda$. Let us see that Theorem 3.2 follows from Proposition 3.4. For a sequence $\{\Lambda_n\}_{n=1}^\infty$ of finite subsets of \mathbb{Z}^d , we write $\Lambda_n \uparrow \mathbb{Z}^d$ when $\Lambda_n \subset \Lambda_{n+1}$ for each n and $\bigcup_{n=1}^\infty \Lambda_n = \mathbb{Z}^d$. For example, $\Lambda_n = B(n)$ satisfies these conditions. For each $\Lambda \Subset \mathbb{Z}^d$, we canonically regard $\mu_{\Phi, \Lambda}^+$ and $\mu_{\Phi, \Lambda}^-$ as Borel probability measures on Ω with $\mu_{\Phi, \Lambda}^+(\Omega_\Lambda^+) = 1$ and $\mu_{\Phi, \Lambda}^-(\Omega_\Lambda^-) = 1$, where $\Omega_\Lambda^+ = \{\omega \in \Omega \mid \omega(x) = 1, x \in \mathbb{Z}^d \setminus \Lambda\}$ and $\Omega_\Lambda^- = \{\omega \in \Omega \mid \omega(x) = -1, x \in \mathbb{Z}^d \setminus \Lambda\}$, respectively. It is known that if μ_{Φ, Λ_n}^+ (resp. μ_{Φ, Λ_n}^-) converges to some μ^+ (resp. μ^-) in $M(\Omega)$ as $n \rightarrow \infty$, then $\mu^+ \in K_\Phi$ (resp. $\mu^- \in K_\Phi$) (see [27, Chapter 1]).

Proof of Theorem 3.2. Assume that Proposition 3.4 holds. We take L, β, δ, Ψ and Φ as in Proposition 3.4. We take a sequence $\{\Lambda_n\}_{n=1}^\infty$ of finite and c -connected subsets of \mathbb{Z}^d such that $\Lambda_n \uparrow \mathbb{Z}^d$. Then we have divergent subsequences $\{n_k\}_{k=1}^\infty$ and $\{m_k\}_{k=1}^\infty$ of \mathbb{N} such that $\mu_{\Phi, \Lambda_{n_k}}^+$ and $\mu_{\Phi, \Lambda_{m_k}}^-$ converge to some $\widehat{\mu}_\Phi^+$ and $\widehat{\mu}_\Phi^-$ in $M(\Omega)$, respectively. Then, as we mentioned above, $\widehat{\mu}_\Phi^+$ and $\widehat{\mu}_\Phi^-$ are in K_Φ . By Proposition 3.4, $\widehat{\mu}_\Phi^+$ satisfies

$$(3.3) \quad \begin{aligned} \widehat{\mu}_\Phi^+ (\{\omega \in \Omega \mid \omega(x) = 1\}) &= \lim_{k \rightarrow \infty} \mu_{\Phi, \Lambda_{n_k}}^+ (\{\omega \in \Omega_{\Lambda_{n_k}}^+ \mid \omega(x) = 1\}) \\ &\geq 1 - \varepsilon(L) \end{aligned}$$

for each $x \in \mathbb{Z}^d$. For each $N \in \mathbb{N}$, we define $\nu_{\Phi, N}^+ \in M(\Omega)$ by

$$\nu_{\Phi, N}^+ = \frac{1}{|B(N)|} \sum_{x \in B(N)} \tau_x^* \widehat{\mu}_\Phi^+.$$

Since Φ is translation invariant and $\widehat{\mu}_\Phi^+ \in K_\Phi$, $\tau_x^* \widehat{\mu}_\Phi^+ \in K_\Phi$ for each $x \in B(N)$. Hence, by the convexity of K_Φ , $\nu_{\Phi, N}^+ \in K_\Phi$. Moreover, from Inequality (3.3), we have

$$(3.4) \quad \begin{aligned} \nu_{\Phi, N}^+ (\{\omega \in \Omega \mid \omega(0) = 1\}) &= \frac{1}{|B(N)|} \sum_{x \in B(N)} \widehat{\mu}_\Phi^+ (\{\omega \in \Omega \mid \omega(x) = 1\}) \\ &\geq 1 - \varepsilon(L). \end{aligned}$$

We can take a divergent subsequence $\{N_l\}_{l=1}^\infty \subset \mathbb{N}$ such that ν_{Φ, N_l}^+ converges to some ν_Φ^+ in $M(\Omega)$. It is seen that $\nu_\Phi^+ \in I$ and, since $K_\Phi \subset M(\Omega)$ is closed, $\nu_\Phi^+ \in K_\Phi$. Hence, by Proposition 2.2 we have $\nu_\Phi^+ \in I_\Phi$. Moreover, from Inequality (3.4), we have

$$\begin{aligned} \nu_\Phi^+ (\{\omega \in \Omega \mid \omega(0) = 1\}) &= \lim_{l \rightarrow \infty} \nu_{\Phi, N_l}^+ (\{\omega \in \Omega \mid \omega(0) = 1\}) \\ &\geq 1 - \varepsilon(L). \end{aligned}$$

By doing the same argument to $\widehat{\mu}_\Phi^-$, we have $\nu_\Phi^- \in I_\Phi$ such that $\nu_\Phi^- (\{\omega \in \Omega \mid \omega(0) = -1\}) \geq 1 - \varepsilon(L)$. Since $\varepsilon(L) < 1/2$, we have $\nu_\Phi^+ \neq \nu_\Phi^-$ and complete the proof. \square

3.3. Contours for a configuration and Peierls’ argument. We will give a proof of Proposition 3.4. We notice that the following argument is based on **Peierls’ argument**, which was used in [22] to show that the nonperturbed Ising model at low temperature shows ferromagnetism.

We introduce the notion of a **contour** on \mathbb{Z}^d . We follow the notion in [13, Chapter 5]. For each $x \in \mathbb{Z}^d$, we write \mathcal{I}_x for the closed unit cube in \mathbb{R}^d centered at x :

$$\mathcal{I}_x = x + \left[-\frac{1}{2}, \frac{1}{2} \right]^d,$$

and, for each $\Lambda \subset \mathbb{Z}^d$, define

$$\mathcal{M}(\Lambda) = \bigcup_{x \in \Lambda} \mathcal{I}_x \subset \mathbb{R}^d.$$

Definition 3.1. A set $\gamma \subset \mathbb{R}^d$ is a **contour** if

$$\gamma = \partial \mathcal{M}(\Lambda)$$

for some c -connected $\Lambda \Subset \mathbb{Z}^d$ (where ∂ denotes the boundary with respect to the usual topology of \mathbb{R}^d). It is seen that, for each contour γ , $\Lambda \Subset \mathbb{Z}^d$ such that $\gamma = \partial \mathcal{M}(\Lambda)$ is uniquely determined. We call such Λ the **interior** of γ and write $\text{int}\gamma$.

It is seen that a contour γ is a connected sum of **plaquettes**, which are $(d-1)$ -dimensional faces of $\mathcal{I}_x, x \in \mathbb{Z}^d$, such that γ divides \mathbb{Z}^d into two $\|\cdot\|_\infty$ -connected sets: the interior and exterior of γ (for example, see [13, Appendix B]). We write $|\gamma|$ for the number of plaquettes which are contained in γ . We write $\mathcal{E} = \{ \{x, y\} \subset \mathbb{Z}^d \mid \|x - y\|_1 = 1 \}$ and call an element of \mathcal{E} an **edge** (considering the graph $(\mathbb{Z}^d, \mathcal{E})$). There is the one-to-one mapping between plaquettes and \mathcal{E} , associating a plaquette with the unique edge crossing it. By this mapping, the plaquettes contained in γ correspond to elements of $\{ \{x, y\} \in \mathcal{E} \mid x \in \text{int}\gamma, y \in \mathbb{Z}^d \setminus \text{int}\gamma \}$.

For each $\Lambda \Subset \mathbb{Z}^d$, let $\Omega_\Lambda^+ = \{ \omega \in \Omega \mid \omega(x) = 1 \text{ on } \mathbb{Z}^d \setminus \Lambda \}$. We write $\Omega^+ = \bigcup_{\Lambda \Subset \mathbb{Z}^d} \Omega_\Lambda^+$. For each $\omega \in \Omega^+$, we define $\Lambda^-(\omega) = \{ x \in \mathbb{Z}^d \mid \omega(x) = -1 \} \Subset \mathbb{Z}^d$ and

$$\mathcal{M}(\omega) = \mathcal{M}(\Lambda^-(\omega)).$$

We decompose $\partial \mathcal{M}(\omega)$ into connected components:

$$\partial \mathcal{M}(\omega) = \bigsqcup_{i=1}^n \gamma_i,$$

then it is shown that each $\gamma_i (i = 1, \dots, n)$ is a contour and, by the mapping mentioned above, a plaquette contained in some γ_i is associated with an edge $\{x, y\} \in \mathcal{E}$ such that $\omega(x) = 1$ and $\omega(y) = -1$ (see [13, Section 5.7.4]). We write

$$\Gamma(\omega) = \{ \gamma_1, \dots, \gamma_n \}.$$

Let us consider the Ising model with inverse temperature $\beta > 0$. For $\Lambda \Subset \mathbb{Z}^d$, we define $\mathcal{E}_\Lambda = \{ \{x, y\} \in \mathcal{E} \mid \{x, y\} \cap \Lambda \neq \emptyset \}$. Then for $\omega \in \Omega_\Lambda^+$, we have

$$\begin{aligned} \exp\left(-\sum_{\Delta \in \mathcal{F}_\Lambda} \Phi_\Delta^\beta(\omega)\right) &= \exp\left(\beta \sum_{\{x,y\} \in \mathcal{E}_\Lambda} \omega(x)\omega(y)\right) \\ &= e^{\beta|\mathcal{E}_\Lambda|} e^{-2\beta|\{(x,y) \in \mathcal{E}_\Lambda | \omega(x)=1, \omega(y)=-1\}|} \\ &= e^{\beta|\mathcal{E}_\Lambda|} \prod_{\gamma \in \Gamma(\omega)} e^{-2\beta|\gamma|}. \end{aligned}$$

Suppose Λ is c-connected. For $\omega \in \Omega_\Lambda^+$ and $\gamma \in \Gamma(\omega)$, we define $\omega_\gamma \in \Omega$ by

$$\omega_\gamma(x) = \begin{cases} -\omega(x), & x \in \text{int}\gamma \\ \omega(x), & x \in \mathbb{Z}^d \setminus \text{int}\gamma. \end{cases}$$

Then, by the assumption that Λ is c-connected, $\omega_\gamma \in \Omega_\Lambda^+$ and

$$(3.5) \quad \Gamma(\omega_\gamma) = \Gamma(\omega) \setminus \{\gamma\}.$$

Hence, we have

$$\exp\left(-\sum_{\Delta \in \mathcal{F}_\Lambda} \Phi_\Delta^\beta(\omega)\right) = e^{-2\beta|\gamma|} \exp\left(-\sum_{\Delta \in \mathcal{F}_\Lambda} \Phi_\Delta^\beta(\omega_\gamma)\right).$$

This is a key equation in Peierls' argument.

Let us consider a perturbation of Φ^β by a spin-flip symmetric interaction Ψ such that $\|\Psi\| < \delta$ and it takes zero on non- $\|\cdot\|_1$ -connected sets. Then, since $\omega_\gamma = -\omega$ on $\text{int}\gamma$ and $\omega_\gamma = \omega$ on $\mathbb{Z}^d \setminus \text{int}\gamma$, by the spin-flip symmetry of Ψ , we have

$$\begin{aligned} \sum_{\Delta \in \mathcal{F}_\Lambda} \Psi_\Delta(\omega) &= \sum_{\substack{\Delta \in \mathcal{F}_\Lambda, \\ \Delta \subset \text{int}\gamma \text{ or } \Delta \cap \text{int}\gamma = \emptyset}} \Psi_\Delta(\omega) + \sum_{\substack{\Delta \in \mathcal{F}_\Lambda, \\ \Delta \not\subset \text{int}\gamma, \Delta \cap \text{int}\gamma \neq \emptyset}} \Psi_\Delta(\omega) \\ &= \sum_{\substack{\Delta \in \mathcal{F}_\Lambda, \\ \Delta \subset \text{int}\gamma \text{ or } \Delta \cap \text{int}\gamma = \emptyset}} \Psi_\Delta(\omega_\gamma) + \sum_{\substack{\Delta \in \mathcal{F}_\Lambda, \\ \Delta \not\subset \text{int}\gamma, \Delta \cap \text{int}\gamma \neq \emptyset}} \Psi_\Delta(\omega). \end{aligned}$$

If $\Delta \in \mathcal{F}_\Lambda$ is $\|\cdot\|_1$ -connected, $\Delta \not\subset \text{int}\gamma$ and $\Delta \cap \text{int}\gamma \neq \emptyset$, then Δ contains some edge $\{x, y\} \in \mathcal{E}$ associated with some plaquette contained in γ . Hence, we have

$$\left| \sum_{\substack{\Delta \in \mathcal{F}_\Lambda, \\ \Delta \not\subset \text{int}\gamma, \Delta \cap \text{int}\gamma \neq \emptyset}} \Psi_\Delta(\omega) \right| \leq |\gamma| \|\Psi\| < |\gamma| \delta \quad \text{and} \quad \left| \sum_{\substack{\Delta \in \mathcal{F}_\Lambda, \\ \Delta \not\subset \text{int}\gamma, \Delta \cap \text{int}\gamma \neq \emptyset}} \Psi_\Delta(\omega_\gamma) \right| < |\gamma| \delta.$$

From these, we have

$$\begin{aligned} (3.6) \quad & \exp\left(-\sum_{\Delta \in \mathcal{F}_\Lambda} (\Phi^\beta + \Psi)_\Delta(\omega)\right) \\ &= e^{-2\beta|\gamma|} \exp\left(-\sum_{\Delta \in \mathcal{F}_\Lambda} \Phi_\Delta^\beta(\omega_\gamma)\right) \exp\left(-\sum_{\Delta \in \mathcal{F}_\Lambda} \Psi_\Delta(\omega)\right) \\ &= e^{-2\beta|\gamma|} \exp\left(-\sum_{\Delta \in \mathcal{F}_\Lambda} (\Phi^\beta + \Psi)_\Delta(\omega_\gamma)\right) \exp\left(-\sum_{\substack{\Delta \in \mathcal{F}_\Lambda, \\ \Delta \not\subset \text{int}\gamma, \Delta \cap \text{int}\gamma \neq \emptyset}} (\Psi_\Delta(\omega) - \Psi_\Delta(\omega_\gamma))\right) \end{aligned}$$

$$= e^{-2\beta|\gamma|+\delta(\Lambda,\omega,\gamma)} \exp\left(-\sum_{\Delta \in \mathcal{F}_\Lambda} (\Phi^\beta + \Psi)_\Delta(\omega_\gamma)\right)$$

with

$$(3.7) \quad |\delta(\Lambda, \omega, \gamma)| < 2|\gamma|\delta.$$

3.4. Proof of Proposition 3.4. First, we give the following lemma about the number of contours surrounding a fixed point. The similar statement for a little different definition of contours is in [26]. We give the proof for the completeness.

Lemma 3.5. *For any $n \in \mathbb{N}$, let*

$$\Gamma_{n,0} = \{\gamma : \text{contour } \|\gamma\| = n, 0 \in \text{int}\gamma\}.$$

Then we have

$$|\Gamma_{n,0}| \leq (n + 1)C_d^{2n},$$

where $C_d \in \mathbb{N}$ is the number of $(d - 1)$ -dimensional faces of $\mathcal{I}_x, x \in \mathbb{Z}^d$ which are not $\{1/2\} \times [-1/2, 1/2]^{d-1}$ and intersect with it.

Proof. Let $\gamma \in \Gamma_{n,0}$. Then it is easily seen that, if we write $\mathcal{H}_i = \{1/2 + i\} \times [-1/2, 1/2]^{d-1}$: the $(d - 1)$ -dimensional faces for $i \in \mathbb{Z}$, then $\mathcal{H}_i \subset \gamma$ for some $i = 0, \dots, n$. We define the graph $G = (V, E)$, where the set of vertices V is the set of all $(d - 1)$ -dimensional faces of $\mathcal{I}_x, x \in \mathbb{Z}^d$ and the set of edges is the set of all pairs $\{\mathcal{H}, \mathcal{H}'\}$ of elements of V such that $\mathcal{H} \cap \mathcal{H}' \neq \emptyset$. For any $\mathcal{H} \in V$, the number of edges which have \mathcal{H} as an end point is C_d . By elementary arguments of graph theory, it is seen that, for any finite connected subset $S \subset V$ and any $\mathcal{H} \in S$, there exists a walk p in G such that p starts from and ends at \mathcal{H} , passes through every vertex in S and does not pass through any other vertices and the length of p is bounded by $2|S|$. We fix $i = 0, \dots, n$. Then, by regarding a contour as a finite connected subset in V , we can associate each $\gamma \in \Gamma_{n,0}$ such that $\mathcal{H}_i \subset \gamma$ with a walk p in G starting from and ends at \mathcal{H}_i which satisfies the conditions for γ as above. Since the number of walks starting from \mathcal{H}_i the length of which is bounded by $2n$ is bounded by C_d^{2n} , we obtain $|\Gamma_{n,0}| \leq (n + 1)C_d^{2n}$. \square

Remark 3.1. The proof above is simple. However, the estimate in Lemma 3.5 is not the best. The estimate has been improved in [18] and [4] and, as the recent result, it was shown in [4] that the number of contours of size n is bounded by $(Cd)^{2n/d}$, where C is a constant independent of n and d .

The next lemma is a key to prove Proposition 3.4.

Lemma 3.6. *Let $\beta > 0, 0 < \delta < 1$ and Ψ be a translation invariant and spin-flip symmetric interaction which is zero on non- $\|\cdot\|_1$ -connected sets and $\|\Psi\| < \delta$. We write $\Phi = \Phi^\beta + \Psi$. Let $\Lambda \Subset \mathbb{Z}^d$ be c -connected and $\gamma_1, \dots, \gamma_n$ be contours such that $\text{int}\gamma_1, \dots, \text{int}\gamma_n \subset \Lambda$ and they are pairwise disjoint. Then we have*

$$\mu_{\Phi,\Lambda}^+(\{\omega \in \Omega_\Lambda^+ \mid \gamma_1, \dots, \gamma_n \in \Gamma(\omega)\}) \leq \exp(-2(\beta - \delta)(|\gamma_1| + \dots + |\gamma_n|))$$

(where we canonically regard $\mu_{\Phi,\Lambda}^+$ as a measure on Ω_Λ^+).

Proof. For each $i = 1, \dots, n$, we define a map $\theta_{\gamma_i} : \{\omega \in \Omega_\Lambda^+ \mid \gamma_i \in \Gamma(\omega)\} \ni \omega \mapsto \omega_{\gamma_i} \in \{\omega \in \Omega_\Lambda^+ \mid \gamma_i \notin \Gamma(\omega)\}$. It is clear that θ_{γ_i} is injective. From Equation (3.5), we can define the composition $\theta = \theta_{\gamma_n} \circ \dots \circ \theta_{\gamma_1} : \{\omega \in \Omega_\Lambda^+ \mid \gamma_1, \dots, \gamma_n \in \Gamma(\omega)\} \rightarrow \Omega_\Lambda^+$ and it is injective. Moreover, for each $\omega \in \Omega_\Lambda^+$ such that $\gamma_1, \dots, \gamma_n \in \Gamma(\omega)$, from Equation (3.6) and Inequality (3.7), we have

$$\begin{aligned} \exp\left(-\sum_{\Delta \in \mathcal{F}_\Lambda} \Phi_\Delta(\omega)\right) &\leq e^{-2(\beta-\delta)|\gamma_1|} \exp\left(-\sum_{\Delta \in \mathcal{F}_\Lambda} \Phi_\Delta(\theta_{\gamma_1}(\omega))\right) \\ &\leq e^{-2(\beta-\delta)|\gamma_1|} e^{-2(\beta-\delta)|\gamma_2|} \exp\left(-\sum_{\Delta \in \mathcal{F}_\Lambda} \Phi_\Delta(\theta_{\gamma_2}(\theta_{\gamma_1}(\omega)))\right) \\ &\quad \vdots \\ &\leq \exp(-2(\beta-\delta)(|\gamma_1| + \dots + |\gamma_n|)) \exp\left(-\sum_{\Delta \in \mathcal{F}_\Lambda} \Phi_\Delta(\theta(\omega))\right). \end{aligned}$$

Hence, we have

$$\begin{aligned} &\mu_{\Phi, \Lambda}^+(\{\omega \in \Omega_\Lambda^+ \mid \gamma_1, \dots, \gamma_n \in \Gamma(\omega)\}) \\ &= \frac{1}{Z_{\Phi, \Lambda}^+} \sum_{\substack{\omega \in \Omega_\Lambda^+ \\ \gamma_1, \dots, \gamma_n \in \Gamma(\omega)}} \exp\left(-\sum_{\Delta \in \mathcal{F}_\Lambda} \Phi_\Delta(\omega)\right) \\ &\leq \exp(-2(\beta-\delta)(|\gamma_1| + \dots + |\gamma_n|)) \cdot \frac{1}{Z_{\Phi, \Lambda}^+} \sum_{\substack{\omega \in \Omega_\Lambda^+ \\ \gamma_1, \dots, \gamma_n \in \Gamma(\omega)}} \exp\left(-\sum_{\Delta \in \mathcal{F}_\Lambda} \Phi_\Delta(\theta(\omega))\right) \\ &\leq \exp(-2(\beta-\delta)(|\gamma_1| + \dots + |\gamma_n|)) \end{aligned}$$

and obtain the statement. □

We prove Proposition 3.4 using these lemmas.

Proof of Proposition 3.4. Let $\beta, \delta > 0$ and Ψ be a translation invariant and spin-flip symmetric interaction which is zero on non- $\|\cdot\|_1$ -connected sets and $\|\Psi\| < \delta$. We write $\Phi = \Phi^\beta + \Psi$. We take arbitrary $\Lambda \Subset \mathbb{Z}^d$ and $x \in \Lambda$ and write $\Omega_{\Lambda, x}^{+, -} = \{\omega \in \Omega_\Lambda^+ \mid \omega(x) = -1\}$. Let $\Gamma_x = \{\gamma : \text{contour } |x \in \text{int}\gamma\}$. For each $\omega \in \Omega_{\Lambda, x}^{+, -}$, it is seen that $\Gamma(\omega) \cap \Gamma_x \neq \emptyset$. Hence, by Lemmas 3.6 and 3.5, we have

$$\begin{aligned} \mu_{\Phi, \Lambda}^+(\Omega_{\Lambda, x}^{+, -}) &\leq \sum_{\gamma \in \Gamma_x} \mu_{\Phi, \Lambda}^+(\{\omega \in \Omega_\Lambda^+ \mid \gamma \in \Gamma(\omega)\}) \\ &\leq \sum_{\gamma \in \Gamma_x} e^{-2(\beta-\delta)|\gamma|} \\ &= \sum_{k=1}^\infty \sum_{\gamma \in \Gamma_{k,0}} e^{-2(\beta-\delta)k} \end{aligned}$$

$$\leq \sum_{k=1}^{\infty} (k+1) C_d^{2k} e^{-2(\beta-\delta)k}.$$

uniformly in Λ and x . Hence, if we take $\log C_d < L < \infty$ sufficiently large and $\beta - \delta > L$ we have

$$\begin{aligned} \mu_{\Phi, \Lambda}^+ (\Omega_{\Lambda, x}^{+, -}) &\leq \sum_{k=1}^{\infty} (k+1) C_d^{2k} e^{-2(\beta-\delta)k} \\ &\leq \sum_{k=1}^{\infty} (k+1) e^{-2(L-\log C_d)k} \\ &= \varepsilon(L) < \frac{1}{2} \end{aligned}$$

and $\varepsilon(L) \rightarrow 0$ as $L \rightarrow \infty$. We obtain the statement in the case of $+$. By the same argument in the case of $-$, we obtain Proposition 3.4. \square

Remark 3.2. In the proof of Theorem 1.3 (2) above, we showed that there exist equilibrium states ν_{Φ}^+ and ν_{Φ}^- for $\Phi = \Phi^\beta + \Psi$ such that the magnetization is positive and negative, respectively, that is,

$$\int_{\Omega} \omega(0) d\nu_{\Phi}^+(\omega) > 0 \quad \text{and} \quad \int_{\Omega} \omega(0) d\nu_{\Phi}^-(\omega) < 0.$$

It can be seen that some condition on order of decay of Ψ , like d -th order decaying condition, is necessary for Φ to have non-magnetization-vanishing equilibrium states. In [12], a translation invariant and spin-flip symmetric two body interaction Φ is constructed such that every equilibrium state μ for $\Phi = \Phi^\beta + \Psi$ vanishes magnetization. This interaction Ψ is antiferromagnetic type (that is, $\Psi_{\{x,y\}}(\omega) = K_{x,y}\omega(x)\omega(y)$, $K_{x,y} \geq 0$) and small with respect to $\|\cdot\|$, but not d -th order decaying. Moreover, it is shown in [5] that, if $d < s \leq d + 1$, then, for the antiferromagnetic interaction Ψ of the form $K_{x,y} = \delta/|x - y|^s$, $\delta > 0$ (where $|\cdot|$ is the Euclidean distance), every equilibrium state for $\Phi = \Phi^\beta + \Psi$ vanishes magnetization. (It seems unknown whether the uniqueness of the equilibrium state holds for these perturbations.)

4. Further questions

Here we notice some further questions next to our results. We showed that there is more than one equilibrium state for $\Phi = \Phi^\beta + \Psi$, where $\beta > 0$ is sufficiently large and Ψ is a translation invariant and spin-flip symmetric interaction such that $\|\Psi\|$ is sufficiently small. When $\Psi = 0$, that is, $\Phi = \Phi^\beta$, the nonperturbed Ising model at sufficiently low temperature, the structure of I_{Φ^β} is completely known. Remember $\mu_{\Phi^\beta, \Lambda}^+$ and $\mu_{\Phi^\beta, \Lambda}^-$ defined for each $\Lambda \Subset \mathbb{Z}^d$ in (3.2).

Theorem 4.1 (Completeness of phase diagram). *Let $\beta > \beta_c$. We take a sequence $\{\Lambda_n\}_{n=1}^\infty$ of finite and c -connected subsets of \mathbb{Z}^d such that $\Lambda_n \uparrow \mathbb{Z}^d$. Then $\{\mu_{\Phi^\beta, \Lambda_n}^+\}_{n=1}^\infty$ and $\{\mu_{\Phi^\beta, \Lambda_n}^-\}_{n=1}^\infty$ converge to some $\mu_{\Phi^\beta}^+$ and $\mu_{\Phi^\beta}^-$ in $M(\Omega)$, respectively and they are independent of the choice of $\{\Lambda_n\}_{n=1}^\infty$. Here, $\mu_{\Phi^\beta}^+$ and $\mu_{\Phi^\beta}^-$ are translation invariant Gibbs states for Φ^β and, then $\mu_{\Phi^\beta}^+, \mu_{\Phi^\beta}^- \in I_{\Phi^\beta}$. Furthermore, $\mu_{\Phi^\beta}^+$ and $\mu_{\Phi^\beta}^-$ span I_{Φ^β} , that is,*

$$I_{\Phi^\beta} = \left\{ p\mu_{\Phi^\beta}^+ + (1-p)\mu_{\Phi^\beta}^- \mid 0 \leq p \leq 1 \right\}.$$

For a proof of the first half of the statement, see [13, Chapter 3] and for that of the second statement, see [1] or [16] for $d = 2$ and [6] for $d \geq 3$. These results are due to the FKG inequality, the crucial property of ferromagnetic interactions. If $\beta > 0$ is sufficiently large and a perturbation interaction Ψ is finite range, then, as we saw in Section 2.4, we can apply the Pirogov-Sinai theory to $\Phi = \Phi^\beta + \Psi$ and the same statement holds (see [30]).

Problem 4.2. *For sufficiently large $\beta > 0$, does the same statement as Theorem 4.1 hold in general when Ψ is a translation invariant and spin-flip symmetric interaction such that $\|\Psi\|$ is sufficiently small?*

We notice that these general cases include many cases to which the FKG inequality and existing extensions of the Pirogov-Sinai theory can not be applicable.

We notice other questions. In Theorem 1.3, we saw the stability of uniqueness of the equilibrium state for sufficiently small β and coexistence for sufficiently large β . Then what about the case when $\beta > 0$ is intermediate? At $\beta = \beta_c$, it is clear that the stability is broken. Hence, we have a problem as follows.

- Problem 4.3.** (1) *Determine for each $0 < \beta < \beta_c$ whether the stability of uniqueness of the equilibrium state as Theorem 1.3 (1) holds or not.*
 (2) *Determine for each $\beta_c < \beta < \infty$ whether the stability of coexistence of equilibrium states as Theorem 1.3 (2) holds or not.*

The Ising model can be considered not only on \mathbb{Z}^d , but also amenable graphs and it has similar properties to the Ising model on \mathbb{Z}^d . It can be expected that the same result as Theorem 1.3 holds for the Ising model on amenable graphs. Furthermore, it is interesting to determine whether the same result holds for the Ising model on non-amenable graphs, such as Cayley trees.

- Problem 4.4.** (1) *Does the same statement as Theorem 1.3 hold for the Ising model on any amenable graph?*
 (2) *What about the Ising model on non-amenable graphs, such as Cayley trees?*

Appendix

Here we give a proof of Proposition 1.2.

Proof of Proposition 1.2. Let Φ_0 be an interaction. We take an open set $I_{\Phi_0} \subset U \subset I$. For an interaction Φ , we define the function $F_\Phi : I \rightarrow \mathbb{R}$ as

$$F_\Phi(\mu) = h(\mu) + \int_{\Omega} A_\Phi d\mu, \quad \mu \in I.$$

Then F_Φ takes its maximal value $P^\Phi = \sup_{\mu \in I} F_\Phi(\mu)$ exactly on I_Φ . Moreover, since $I \ni \mu \mapsto \int_{\Omega} A_\Phi d\mu \in \mathbb{R}$ is continuous and $I \ni \mu \mapsto h(\mu) \in \mathbb{R}$ is upper semicontinuous, $F_\Phi(\mu)$ is upper semicontinuous in $\mu \in I$. Hence, for $\Phi = \Phi_0$, we have

$$I \setminus U \subset \left\{ \mu \in I \mid F_{\Phi_0}(\mu) < P^{\Phi_0} \right\}$$

and the left-hand side is compact and the right-hand side is open in I . Then, using the upper semicontinuity of F_{Φ_0} again, there exists $0 < \varepsilon < 1$ such that

$$I \setminus U \subset \left\{ \mu \in I \mid F_{\Phi_0}(\mu) < P^{\Phi_0} - \varepsilon \right\}.$$

We set $\delta = \varepsilon/4$ and take an arbitrary interaction Ψ with $\|\Psi\| < \delta$. We write $\Phi = \Phi_0 + \Psi$. Then, for any $\mu \in I$, we have

$$\begin{aligned} |F_{\Phi}(\mu) - F_{\Phi_0}(\mu)| &= \left| \int_{\Omega} A_{\Phi} d\mu - \int_{\Omega} A_{\Phi_0} d\mu \right| \\ &\leq \int_{\Omega} |A_{\Phi} - A_{\Phi_0}| d\mu \\ &\leq \|\Psi\| \\ &< \delta \end{aligned}$$

and

$$|P^{\Phi} - P^{\Phi_0}| = \left| \sup_{\mu \in I} F_{\Phi}(\mu) - \sup_{\mu \in I} F_{\Phi_0}(\mu) \right| \leq \delta.$$

If $\mu \in I \setminus U$, then $F_{\Phi_0}(\mu) < P^{\Phi_0} - \varepsilon$. Hence, we have

$$\begin{aligned} F_{\Phi}(\mu) &\leq F_{\Phi_0}(\mu) + \delta \\ &< P^{\Phi_0} - \varepsilon + \delta \\ &\leq P^{\Phi} - \varepsilon + 2\delta \\ &= P^{\Phi} - \frac{\varepsilon}{2} \end{aligned}$$

and μ does not achieve the supremum. Hence, $\mu \notin I_{\Phi}$. □

ACKNOWLEDGEMENTS. The author is grateful to Masayuki Asaoka and Mitsuhiro Shishikura for their helpful advice, and Takehiko Morita, Masaki Tsukamoto and Tom Meyerovitch for their useful comments.

References

- [1] M. Aizenman: *Translation invariance and instability of phase coexistence in the two dimensional Ising system*, Comm. Math. Phys. **73** (1980), 83–94.
- [2] M. Aizenman, H. Duminil-Copin and V. Sidoravicius: *Random currents and continuity of Ising model's spontaneous magnetization*, Comm. Math. Phys. **334** (2015), 719–742.
- [3] M. Aizenman and R. Fernández: *On the critical behavior of the magnetization in high-dimensional Ising models*, J. Stat. Phys. **44** (1986), 393–454.
- [4] P.N. Balister and B. Bollobás: *Counting regions with bounded surface area*, Comm. Math. Phys. **273** (2007), 305–315.
- [5] M. Biskup, L. Chayes and S.A. Kivelson: *On the absence of ferromagnetism in typical 2D ferromagnets*, Comm. Math. Phys. **274** (2007), 217–231.
- [6] T. Bodineau: *Translation invariant Gibbs states for the Ising model*, Probab. Theory Related Fields **135** (2006), 153–168.

- [7] C. Borgs, R. Kotecký and D. Ueltschi: *Low temperature phase diagrams for quantum perturbations of classical spin systems*, Comm. Math. Phys. **181** (1996), 409–446.
- [8] H.A.M. Daniëls and A.C.D. van Enter: *Differentiability properties of the pressure in lattice systems*, Comm. Math. Phys. **71** (1980), 65–76.
- [9] N. Datta, R. Fernández and J. Fröhlich: *Low-temperature phase diagrams of quantum lattice systems. I. Stability for quantum perturbations of classical systems with finitely-many ground states*, J. Stat. Phys. **84** (3–4) (1996), 455–534.
- [10] N. Datta, J. Fröhlich, L. Rey-Bellet and R. Fernandez: *Low-temperature phase diagrams of quantum lattice systems. II. Convergent perturbation expansions and stability in systems with infinite degeneracy*, Helv. Phys. Acta **69** (1996), 752–820.
- [11] R.L. Dobrushin: *Description of a random field by means of conditional probabilities and conditions for its regularity*, Teor. Veroyatnost. i Primenen **13** (1968), 201–229, in Russian.
- [12] A.C.D. van Enter: *A note on the stability of phase diagrams in lattice systems*, Comm. Math. Phys. **79** (1981), 25–32.
- [13] S. Friedli and Y. Velenik: *Statistical mechanics of lattice systems: a concrete mathematical introduction*, Cambridge University Press, Cambridge, 2017.
- [14] H.O. Georgii: *Gibbs measures and phase transitions*, second edition, Walter De Gruyter, Berlin, 2011.
- [15] J. Ginibre, A. Grossmann and D. Ruelle: *Condensation of lattice gases*, Comm. Math. Phys. **3** (1966), 187–193.
- [16] Y. Higuchi: *On the absence of non-translation invariant Gibbs states for the two-dimensional Ising model*; in Random fields, Vol. I, II (Esztergom, 1979), Colloq. Math. Soc. János Bolyai **27**, North-Holland, Amsterdam-New York, 1981, 517–534.
- [17] R.B. Israel: *Generic triviality of phase diagrams in spaces of long-range interactions*, Comm. Math. Phys. **106** (1986), 459–466.
- [18] J.L. Lebowitz and A.E. Mazel: *Improved Peierls argument for high-dimensional Ising models*, J. Stat. Phys. **90** (1998), 1051–1059.
- [19] T.D. Lee and C.N. Yang: *Statistical theory of equations of state and phase transitions. II. Lattice gas and Ising model*, Phys. Rev. (2) **87** (1952), 410–419.
- [20] Y.M. Park: *Extension of Pirogov-Sinai theory of phase transitions to infinite range interactions. I. Cluster expansion*, Comm. Math. Phys. **114** (1988), 187–218.
- [21] Y.M. Park: *Extension of Pirogov-Sinai theory of phase transitions to infinite range interactions. II. Phase diagram*, Comm. Math. Phys. **114** (1988), 219–241.
- [22] R. Peierls: *On Ising's model of ferromagnetism*, Math. Proc. Cambridge Philos. Soc. **32** (1936), 477–481.
- [23] S.A. Pirogov and Y.G. Sinai: *Phase diagrams of classical lattice systems*, Teoret. Mat. Fiz **25** (1975), 358–369, in Russian. English translation: Theoret. and Math. Phys. **25** (1975), 1185–1192.
- [24] S.A. Pirogov and Y.G. Sinai: *Phase diagrams of classical lattice systems (Continuation)*, Teoret. Mat. Fiz **26** (1976), 61–76, in Russian. English translation: Theoret. and Math. Phys. **26** (1976), 39–49.
- [25] A. Raoufi: *Translation-invariant Gibbs states of the Ising model: general setting*, Ann. Probab. **48** (2020), 760–777.
- [26] D. Ruelle: *Statistical mechanics: Rigorous results*, W.A. Benjamin, inc., New York-Amsterdam, 1969.
- [27] D. Ruelle: *Thermodynamic formalism: The mathematical structures of equilibrium statistical mechanics*, second edition, Cambridge University Press, Cambridge, 2004.
- [28] A.D. Sokal: *More surprises in the general theory of lattice systems*, Comm. Math. Phys. **86** (1982), 327–336.
- [29] C.N. Yang: *The spontaneous magnetization of a two-dimensional Ising model*, Phys. Rev. (2) **85** (1952), 808–816.
- [30] M. Zahradník: *An alternate version of Pirogov-Sinai Theory*, Comm. Math. Phys. **93** (1984), 559–581.

Department of Mathematics
 Faculty of Science
 Kyoto University
 Kitashirakawa Oiwake-cho, Sakyo-ku
 Kyoto 606–8502
 Japan
 e-mail: usuki.shunsuke.72r@st.kyoto-u.ac.jp