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LONG TIME BEHAVIOR OF JUMP-DIFFUSION PROCESSES ON RIEMANNIAN MANIFOLDS

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Abstract

In this paper, we will show that if the sectional curvature of a Hadamard manifold M is pinched by two negative constants, then M-valued jump-diffusion process $\{X_t\ ; 0 \le t < e\}$ satisfying suitable conditions on the Lévy measure is irreducible, transient and conservative. In order to show such properties of paths, we need the upper and lower estimates of the radial part of the jump-diffusion process.

1. Introduction

It is a classical task to construct Markov processes in various spaces and to study their long time behavior. In this paper, we study the jump-diffusion process whose infinitesimal generator is similar to that of Lévy processes. There are previous studies about the construction of such processes: Hunt [8] studied the Lévy process on Lie groups in terms of the semigroup and its corresponding generator. Elles-Elworthy-Malliavin constructed the Brownian motion on manifolds by projecting the solution of the certain stochastic differential equation in the orthonormal frame bundle onto the base space. Later, Applebaum [2], [3] extended such method to the case of the jump-diffusion process on Riemannian manifolds. Remark that an integral curve on a homogeneous space with a two-sided invariant metric is a geodesic. Therefore, the exponential map as the Lie group coincides with that as the Riemannian manifold. From this fact, the jump-diffusion process constructed by Applebaum [2], [3] coincides with the one constructed by Hunt [8].

This paper deals with irreducibility, recurrence, transience, and conservativeness as the long time behavior. Lévy processes, a kind of Markov processes on Euclidean space, are irreducible and conservative. Previous works about long time behavior of the jump-diffusion process are as follows: Recurrence and transience of the Lévy process on Euclidian space are characterized by Chung-Fuchs [4] in terms of the characteristic functions. Applebaum [1] studied some properties of the process on symmetric spaces through the Fourier transforms. He found an analogy with the Chung-Fuchs result [4] on Lévy processes on Euclidean space. On the other hand, Ichihara [9], [10] showed that recurrence, transience, and conservativeness of the Brownian motion on general manifolds can be investigated by evaluating its radial part. Grigor'yan-Huang-Masamune [6] and Masamune-Uemura-Wang [15] studied the long time behavior of symmetric jump-diffusion processes on Riemannian manifolds via Dirichlet forms. These works reveal that the symmetric jump-diffusion process is

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conservative if the volume of the geodesic ball satisfies a certain growth rate. If the sectional curvature is bounded from below by a negative constant, then the volume of the geodesic ball satisfies the growth rate described in [15]. Therefore, our work is regarded as a kind of criterion of the conservativeness of the jump-diffusion process that cannot guarantee a symmetric Markov process. More details are presented at the end of this paper.

In this paper, the properties of paths are studied by evaluating the radial part of the jump-diffusion process. Such method clearly shows how the curvature of the manifold affects their paths, and that the jump-diffusion process on the simply connected Riemannian manifold whose sectional curvature is pinched by negative constants is irreducible, transient, and conservative. Since the sectional curvature of a homogenous space is pinched by two constants, the results of this paper can be applied to the jump-diffusion process on homogeneous spaces as well. This is a kind of extension of Ichihara's works [9], [10] on the global properties of the Brownian motion on manifolds.

The organization of this paper is as follows: In Section 2, we will prepare for the differential geometry and the probabilistic setting. See Sakai [14] for the differential geometry, and Kai-Takeuchi [11] and Hsu [7] for the probabilistic setting. We will construct the jump-diffusion process by projecting the solution of the Marcus-type stochastic differential equation. By Applebaum-Estrade [2], the rotational invariance of the Lévy measure enables us to see that the jump-diffusion process is Markovian, and that the generator of the jump-diffusion process on the manifold is well-defined. Irreducibility, recurrence, transience and conservativeness are defined by the first hitting time, the last exit time and the explosion time, respectively. In Section 3, we summarize the main results obtained in this paper. The conditions under which the jump-diffusion process is irreducible, transient, and conservative are mentioned. In Section 4, we shall provide the proofs for each claims. First, we shall prove the irreducibility of the jump-diffusion process. Our strategy to attack this problem is a functional analysis approach. Next, we shall prove transience. The lower estimate of the radial part is helpful since it indicates that the jump-diffusion process diverges to infinity at the rate stronger than its randomness. Since our target manifold is non-compact, it is enough to check that the radial part of the jump-diffusion process diverges to infinity. Finally, we shall prove the conservativeness of the jump-diffusion process. To prove this, we shall study the property of the explosion time, and prove that there exists the upper estimate of the radial part of the jump-diffusion process which shows that it does not diverge rapidly to infinity. The comparison theorem of the Hessian will play an important role to find the nice estimates of the radial part of the jump-diffusion process.

The results of this paper are argued for both the pure-jump process and the jump-diffusion process. Therefore, the discussion will be divided into the cases of each process.

2. Preliminaries

We first prepare the notions from the differential geometry that we will use throughout this paper. The setting of this paper is based on Kai-Takeuchi [11]. Let (M, g) be a complete, orientable, connected and smooth Riemannian manifold of dimension m (≥ 2), and ∇ the Levi-Civita connection. The one-point compactification of the manifold M by an infinite-point ∂_M is written as $\widehat{M} = M \cup \{\partial_M\}$. Denote the bundle of orthonormal frames by $O(M)$, and let

$$\pi : O(M) \to M$$

be the canonical projection. For $u \in O(M)$, we write $u = ((v_1)_{\pi u}, \ldots, (v_m)_{\pi u})$, where $\{(v_i)_{\pi u};\ 1 \le i \le m\}$ is an orthonormal basis for $T_{\pi u}M$. From now on, we will regard $u \in O(M)$ as a linear operator from $\mathbb{R}^m$ to $T_{\pi u}M$ through the following action

$$\mathbb{R}^m \ni z \mapsto uz = \sum_{i=1}^{m} z^i (v_i)_{\pi u} \in T_{\pi u}M.$$

Denote by $Hz(u)$ the horizontal lift of uz. Now, for given $z \in \mathbb{R}^m$, the horizontal vector field on $O(M)$ is given by

$$O(M) \ni u \mapsto Hz(u) \in T_u O(M).$$

When $\{e_i;\ 1 \le i \le m\}$ is the standard orthonormal basis on $\mathbb{R}^m$, the family

$$\{H_i = He_i;\ i = 1, \ldots, m\}$$

of the horizontal vector fields is called fundamental. For any $z \in \mathbb{R}^m$, the horizontal vector field Hz has the following property

$$Hz(f \circ \pi)(u) = (uz) f(\pi u),$$

where $f \in C^\infty(M)$. Here $C^\infty(M)$ denotes the space of smooth functions on M. Write $\mathbb{R}^m_0 = \mathbb{R}^m \backslash \{0\}$. For $z \in \mathbb{R}^m_0$ and $u \in O(M)$, let $\{\Xi^z_s(u); -\infty < s < \infty\}$ be the unique solution to the ordinary differential equation on $O(M)$ of the form:

$$\frac{d}{ds}\Xi^z_s(u) = Hz(\Xi^z_s(u)), \quad \Xi^z_0 = u.$$

Remark that for given $z \in \mathbb{R}^m_0$, the curve $\{\pi\Xi^z_s(u);\ -\infty < s < \infty\}$ is a geodesic. See Kobayashi-Nomizu [12]. Denote the exponential map at $x \in M$ by $\exp_x$, and so we have

$$\pi\Xi^z_s(u) = \exp_{\pi u}(suz), \quad -\infty < s < \infty.$$

Let $\mathrm{dist}(\cdot,\cdot) : M \times M \to [0,\infty)$ be the distance function on M induced by the Riemannian metric g. Denote the inner product and the norm on T_xM by $\langle \cdot,\cdot \rangle_x = g_x(\cdot,\cdot)$ and $|\cdot|_x = g_x(\cdot,\cdot)^{1/2}$, respectively. Notice that if $u \in O(M)$, then

$$\langle Z_1, Z_2 \rangle_{\pi u} = \langle u^{-1}Z_1, u^{-1}Z_2 \rangle$$

holds for all $Z_1, Z_2 \in T_{\pi u}M$. Here $\langle \cdot,\cdot \rangle$ is the inner product on $\mathbb{R}^m$. Remark that

$$\mathrm{dist}(x, \exp_x Z) = |Z|_x$$

holds for all $Z \in T_xM$ within the cut-locus of $x \in M$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let ν be a Lévy measure over $\mathbb{R}^m_0$, that is, $\nu(dz)$ satisfies

$$\int_{\mathbb{R}^m_0} (|z|^2 \wedge 1)\nu(dz) < \infty.$$

Here, we shall summarize conditions for the measure $\nu(dz)$ used throughout this paper.

Assumption 1. The measure $\nu(dz)$ is rotationally invariant.

Assumption 2. The measure $\nu(dz)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_0^m$, and its Radon-Nikodým derivative

$$h(z) := \frac{\nu(dz)}{dz}$$

is continuous and strictly positive.

Assumption 3. The measure $\nu(dz)$ satisfies that

$$\int_{|z|>1} |z|^2 \nu(dz) < \infty.$$

Assumption 3 is not necessary to construct a jump-diffusion process on M. This Assumption is necessary to justify Lemma 4.

Remark 1. If the measure $\nu(dz)$ satisfies Assumptions 1 and 2, then the function

$$h(z) = \frac{\nu(dz)}{dz}$$

is also rotationally invariant. Thus, $h(z)$ depends only on $|z|$, which can be expressed by $h(|z|)$.

Let $B = \{B_t = (B_t^1, \ldots, B_t^m);\ t \geq 0\}$ be an m-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. A Poisson random measure and its compensated Poisson random measure over $\mathbb{R}_0^m \times [0, \infty)$ with intensity measure $\hat{n}(dz, ds) = \nu(dz)ds$ are given by $N(dz, ds)$ and $\widetilde{N}(dz, ds)$, respectively.

Now, let us introduce the Marcus-type stochastic differential equation on $O(M)$ of the form:

$$\begin{aligned} dU_t = \sigma \sum_{i=1}^m H_i(U_{t-}) \circ dB_t^i &+ \eta \int_{|z|\leq 1} \Big(\Xi_1^z(U_{t-}) - U_{t-}\Big) \widetilde{N}(dz, ds) \\ &+ \kappa \int_{|z|>1} \Big(\Xi_1^z(U_{t-}) - U_{t-}\Big) N(dz, ds), \end{aligned} \tag{2.1}$$

where $\circ dB_t^i$ $(i = 1, \ldots, m)$ is the Stratonovich stochastic integral, and σ, η and κ are constants in $\{0, 1\}$. A stochastic process

$$\{U_t;\ 0 \leq t < e\}$$

is called the solution to the stochastic differential equation (2.1), if for any $F \in C^\infty(O(M))$ with a compact support,

$$\begin{aligned} F(U_t) - F(U_0) = {} & \sigma \sum_{i=1}^m \int_0^t H_i F(U_{s-}) \circ dB_s^i \\ & + \eta \int_0^t \int_{|z|\leq 1} \Big(F \circ \Xi_1^z(U_{s-}) - F(U_{s-})\Big) \widetilde{N}(dz, ds) \\ & + \kappa \int_0^t \int_{|z|>1} \Big(F \circ \Xi_1^z(U_{s-}) - F(U_{s-})\Big) N(dz, ds) \end{aligned}$$

$$+ \eta \int_0^t \int_{|z|\le 1} \Big(F \circ \Xi_1^z(U_{s-}) - F(U_{s-}) - (HzF)(U_{s-})\Big)\nu(dz)ds$$

holds for all $t \geq 0$, where e is an explosion time. This stochastic differential equation has the strong unique càdlàg solution up to the explosion time. See Kunita [13, Theorem 7.1.1] for details. Define

$$\{X_t = \pi U_t;\ 0 \le t < e\}, \tag{2.2}$$

and let us consider

$$X_t = \partial_M$$

for all $t \geq e$. Denote the filtrations generated by $\{U_t;\ 0 \le t < e\}$ and $\{X_t;\ 0 \le t < \infty\}$ by $\mathcal{F}_*^U = \{\mathcal{F}_t^U;\ 0 \le t < \infty\}$ and $\mathcal{F}_*^X = \{\mathcal{F}_t^X;\ 0 \le t < \infty\}$ respectively.

In this paper, we consider the following five cases.

- $(\sigma, \eta, \kappa) = (0, 1, 0)$; the pure jump process without large jumps.
- $(\sigma, \eta, \kappa) = (0, 1, 1)$; the pure jump process.
- $(\sigma, \eta, \kappa) = (1, 1, 0)$; the jump-diffusion process without large jumps.
- $(\sigma, \eta, \kappa) = (1, 1, 1)$; the jump-diffusion process.
- $(\sigma, \eta, \kappa) = (1, 0, 0)$; the Brownian motion.

Let us denote the family of probability measures $\{\mathbb{P}_u[\ \cdot\ ];\ u \in O(M)\}$ by

$$\mathbb{P}_u[\ \cdot\ ] = \mathbb{P}[\ \cdot \mid U_0 = u].$$

Denote the space of bounded measurable functions on $O(M)$ by $\mathcal{M}_b(O(M))$. The semigroup $\{S_t;\ 0 \le t < \infty\}$ on $\mathcal{M}_b(O(M))$ is given by

$$S_t F(u) = \mathbb{E}_u[F(U_t)1_{\{t<e\}}].$$

We shall define the linear operator $\mathcal{H}$ on $C_c^\infty(O(M))$ by

$$\mathcal{H}F(u) = \sigma \frac{1}{2}\sum_{i=1}^m H_i^2 F(u) + \eta \int_{|z|\le 1} \Big(F \circ \Xi_1^z(u) - F(u) - (Hz)F(u)\Big)\nu(dz)$$
$$+ \kappa \int_{|z|>1} \Big(F \circ \Xi_1^z(u) - F(u)\Big)\nu(dz)$$

for $F \in C_c^\infty(O(M))$, where $C_c^\infty(O(M))$ is the space of smooth functions on $O(M)$ with a compact support. Remark that if $\{U_t;\ 0 \le t < e\}$ is a Feller process, then $\mathcal{H}$ is the expression on $C_c^\infty(O(M))$ of the infinitesimal generator of $\{U_t;\ 0 \le t < e\}$.

Remark 2. Let

$$\{U_t;\ 0 \le t < e\}$$

be the stochastic process determined by (2.1). In general, the M-valued process $\{X_t = \pi U_t;\ 0 \le t < e\}$ is not always Markov process because the law of πU_t depends on the choice of the frame of the initial point $X_0 = x \in M$. Suppose that the Lévy measure $\nu(dz)$ satisfies the condition of Assumption 1. Then, the law of πU_t is independent of the choice of the frame of the initial point, and the stochastic process

$$\{X_t = \pi U_t;\ 0 \le t < e\}$$

is Markov process. See Applebaum-Estrade [2] and Kai-Takeuchi [11] for details.

Moreover, $\{X_t;\ 0 \le t < e\}$ has the strong Markov property with respect to the filtration $\mathcal{F}_*^X$. This can be seen from the following discussion: Let τ be a $\mathcal{F}_*^X$-stopping time. The stopping time τ is also $\mathcal{F}_*^U$-stopping time because $\mathcal{F}_t^X \subset \mathcal{F}_t^U$ holds for all $t \ge 0$. Since $\{U_t;\ 0 \le t < e\}$ is the strong unique solution to the stochastic differential equation (2.1), $\{U_t;\ 0 \le t < e\}$ has the strong Markov property with respect to $\mathcal{F}_*^U$. For any nonnegative $f \in \mathcal{M}(M)$, the strong Markov property implies

$$\mathbb{E}[f(X_{t+\tau})|\mathcal{F}_\tau^X] = \mathbb{E}[(f \circ \pi)(U_{t+\tau})|\mathcal{F}_\tau^X] = \mathbb{E}_{U_\tau}[(f \circ \pi)(U_t)] = \mathbb{E}_{U_\tau}[f(X_t)].$$

Since the law of X_t is independent of the choice of the initial frame, it holds that

$$\mathbb{E}_{U_\tau}[(f \circ \pi)(U_t)] = \mathbb{E}_{X_\tau}[f(X_t)].$$

Thus, we see that

$$\mathbb{E}[f(X_{t+\tau})|\mathcal{F}_\tau^X] = \mathbb{E}_{X_\tau}[f(X_t)],$$

which implies that $\{X_t;\ 0 \le t < e\}$ is the strong Markov process.

Next, we shall study the generator of the stochastic process

$$\{X_t = \pi U_t;\ 0 \le t < e\}$$

on M under Assumption 1. Let $\{T_t;\ 0 \le t < \infty\}$ be the family of the linear operators on $\mathcal{M}_b(M)$ given by

$$T_t f(x) = \mathbb{E}_x[f(X_t)1_{\{t<e\}}].$$

If we define $f(\partial_M) = 0$ and extend the domain of the function f on M to $\widehat{M}$, then we get

$$T_t f(x) = \mathbb{E}_x[f(X_t)].$$

From now on, any functions on M will be extended to $\widehat{M}$ in such a way. Since $\{X_t;\ 0 \le t < e\}$ is Markovian under Assumption 1, the family of linear operators $\{T_t;\ 0 \le t < \infty\}$ is a semigroup. Since the orthonormal frame bundle $O(M)$ has the orthogonal group as its structural group, which is compact, we have $f \circ \pi \in C_c^\infty(O(M))$ for any $f \in C_c^\infty(M)$. Thus, for $f \in C_c^\infty(M)$, we have

$$S_t(f \circ \pi)(u) = \mathbb{E}_u[(f \circ \pi)(U_t)]$$

and

$$\begin{aligned}\mathcal{H}(f \circ \pi)(u) = \sigma\frac{1}{2}\Delta_M f(\pi u) &+ \eta \int_{|z|\le 1}\Big(f \circ \exp_{\pi u} uz - f(\pi u) - \langle \nabla f(\pi u), uz\rangle_{\pi u}\Big)\nu(dz)\\ &+ \kappa \int_{\mathbb{R}_0^m}\Big(f \circ \exp_{\pi u} uz - f(\pi u)\Big)\nu(dz).\end{aligned}$$

Define the measure on $T_{\pi u}M$ as $\nu \circ u^{-1}$, which is independent of the choice of the frame $u \in \pi^{-1}(\{x\})$ under Assumption 1. So, we can write $\nu_x = \nu \circ u^{-1}$, where u is any frame of $\pi^{-1}(\{x\})$. Moreover, we define the linear operator L on $C_c^\infty(M)$ by

$$Lf(x) = \sigma\frac{1}{2}\Delta_M f(x) + \eta\int_{T_xM_0}\Big(f\circ\exp_x Z - f(x) - \langle\nabla f(x), Z\rangle_x\Big)1_{\{|Z|_x\le 1\}}\nu_x(dZ)$$
$$+\kappa\int_{T_xM_0}\Big(f\circ\exp_x Z - f(x)\Big)1_{\{|Z|_x>1\}}\nu_x(dZ)$$

where $T_xM_0 = T_xM\backslash\{0\}$. The jump-diffusion process related to the (infinitesimal) generator L is studied in [2].

Now, let us introduce some properties of the paths of a Markov process on M. Let $\mathcal{D}$ be the family of relatively compact and non-empty open domains on M. For given $D \in \mathcal{D}$, define the first hitting time of $\{X_t;\ 0 \le t < e\}$ to the set D by

$$T_D = \inf\{t > 0;\ X_t \in D\}$$

and the last exit time σ_D by

$$\sigma_D = \sup\{t > 0;\ X_t \in D\}.$$

Definition 1. A càdlàg Markov process $\{X_t;\ 0 \le t < e\}$ is irreducible, if for any $D \in \mathcal{D}$,

$$\mathbb{P}_x[T_D < \infty] > 0$$

holds for all $x \in M$.

Definition 2. The recurrence and transience of a càdlàg Markov process

$$\{X_t;\ 0 \le t < e\}$$

on M are defined as follows.

- The Markov process $\{X_t;\ 0 \le t < e\}$ is recurrent, if for any $D \in \mathcal{D}$,

$$\mathbb{P}_x[\sigma_D = \infty] = 1$$

  holds for all $x \in M$.
- The Markov process $\{X_t;\ 0 \le t < e\}$ is transient, if for any $D \in \mathcal{D}$,

$$\mathbb{P}_x[\sigma_D < \infty] = 1$$

  holds for all $x \in M$.

Remark 3. If a càdlàg Markov process

$$\{X_t;\ 0 \le t < e\}$$

is irreducible, then $\{X_t;\ 0 \le t < e\}$ is recurrent or transient. Details can be seen in Tweedie [16, Theorem 2.3].

Remark 4. If a càdlàg Markov process

$$\{X_t;\ 0 \le t < e\}$$

is irreducible and recurrent, then

$$\mathbb{P}_x[e = \infty] = 1$$

holds for all $x \in M$. See Getoor [5, Lemma 3.4]. Therefore, by Remark 3, if there exists a point x such that $\mathbb{P}_x[e < \infty] > 0$, then the M-valued process $\{X_t;\ 0 \leq t < e\}$ is transient.

Definition 3. A càdlàg Markov process $\{X_t;\ 0 \leq t < e\}$ is called conservative, if

$$\mathbb{P}_x[e = \infty] = 1$$

holds for all $x \in M$.

3. Main results

In this section, we shall introduce our main results in this paper. Those proofs will be given in the next section. Recall that the M-valued process $\{X_t;\ 0 \leq t < e\}$ is determined by (2.1) and (2.2). Let K be the sectional curvature tensor of M. We shall add the following conditions:

Assumption 4. Suppose that M is simply connected, and that there is a negative constant β such that

$$K \leq \beta < 0.$$

Assumption 5. Suppose that M is simply connected, and that there are negative constants α, β such that

$$\alpha \leq K \leq \beta < 0.$$

Remark that when the manifold M is simply connected and $K \leq 0$, M is a diffeomorphic to the Euclidean space (cf. Sakai [14, Chapter V, Theorem 4.1]). Thus, M is non-compact. The Poincaré half-plane model is a typical example of a manifold satisfying Assumption 5.

Assumption 6. There exists the density function $p(t, x, y)$ of the probability law of X_t with respect to the volume element $\mathrm{Vol}(dy)$.

Assumption 7. The density function $p(t, x, y)$ described in Assumption 6 is of C^2-class for $x \in M$, and there exist functions

$$G_1 : [0, \infty) \times M \to [0, \infty)$$

and

$$G_2 : [0, \infty) \times M \to [0, \infty)$$

such that

$$\left|\nabla_x \log\, p(t, x, y)\right|_x \leq G_1(t, y),$$

$$\left|\nabla_x \nabla_x \log\, p(t, x, y)\right|_x \leq G_2(t, y),$$

$$\int_M G_1(t, y) \mathrm{Vol}(dy) < \infty$$

and

$$\int_M G_2(t,y)\mathrm{Vol}(dy) < \infty$$

hold for all $x \in M$, $y \in M$ and $t \in [0,\infty)$.

Lemma 1. *Assuming that the conditions of* Assumption 7 *hold, then $T_t f$ is of C^2-class for any $f \in \mathcal{M}_b(M)$.*

Proof. Computing the logarithmic derivative of $p(t,x,y)$, $\nabla_x p(t,x,y)$ and $\nabla_x\nabla_x p(t,x,y)$ are calculated by

$$\nabla_x p(t,x,y) = p(t,x,y)\nabla_x \log\, p(t,x,y)$$

and

$$\begin{aligned}\nabla_x\nabla_x p(t,x,y) &= \nabla_x p(t,x,y) \otimes \nabla_x \log\, p(t,x,y) + p(t,x,y)\nabla_x\nabla_x \log\, p(t,x,y)\\ &= p(t,x,y)\Big(\nabla_x \log\, p(t,x,y) \otimes \nabla_x \log\, p(t,x,y) + \nabla_x\nabla_x \log\, p(t,x,y)\Big),\end{aligned}$$

where $\otimes$ is the tensor product. Thus, we see that

$$|\nabla_x p(t,x,y)|_x \le p(t,x,y)G_1(t,y) \tag{3.1}$$

and

$$|\nabla_x\nabla_x p(t,x,y)|_x \le p(t,x,y)\Big((m^2-m+1)G_1(t,y) + G_2(t,y)\Big) \tag{3.2}$$

hold for all $x \in M$, $y \in M$ and $t \in [0,\infty)$. On the other hand, it follows that

$$T_t f(x) = \int_M f(y)p(t,x,y)\mathrm{Vol}(dy).$$

Since $|\nabla_x p(t,x,y)|_x$ and $|\nabla_x\nabla_x p(t,x,y)|_x$ are evaluated by (3.1) and (3.2) respectively, we see by Fubini's theorem that $T_t f$ is of C^2-class. □

Theorem 1. $\{X_t;\ 0 \le t < e\}$ *is irreducible under* Assumptions 1 *and* 2.

Theorem 2. *Suppose that* Assumptions 1, 2, 4 *and* 6 *are satisfied. (When $\kappa = 1$, we additionally assume* Assumption 3.*) Then,*

$$\{X_t;\ 0 \le t < e\}$$

is transient.

Furthermore, if the manifold M satisfies Assumption 5, the conservativeness of $\{X_t;\ 0 \le t < e\}$ can be shown.

Theorem 3. *Suppose that* Assumptions 1, 2, 5, 6 *and* 7 *are satisfied. (When $\kappa = 1$, we additionally assume* Assumption 3.*) Then,*

$$\{X_t;\ 0 \le t < e\}$$

is conservative.

Remark 5. Since M is diffeomorphic to the Euclidean space under Assumption 5, the density function $p(t,x,y)$ of the probability law of X_t with respect to the volume element $\mathrm{Vol}(dy)$ will be C^2-class for both $x \in M$ and $y \in M$ under suitable condition. See Kunita

[13] for details.

4. Proofs

4.1. Proof of Theorem 1. We shall give the proof of Theorem 1. Suppose that the Lévy measure $\nu(dz)$ satisfies Assumption 1 . Then, as pointed out before, the M-valued process $\{X_t;\ 0 \le t < e\}$ is Markovian.

Proof of Theorem 1. We will begin with the proof of Theorem 1 in case of $(\sigma, \eta, \kappa) = (0, 1, 0)$. Let $D \in \mathcal{D}$ and $x \in M$. At the beginning, we shall show that

$$\mathbb{P}_x[T_D < \infty] > 0$$

holds for any $x \in M$ with $\mathrm{dist}(x, D) \le 1/2$. It is clear that

$$\mathbb{P}_x[T_D < \infty] = 1$$

holds for all $x \in D$, since D is finely open and $\{X_t;\ 0 \le t < e\}$ is càdlàg. We shall show that

$$\liminf_{t \searrow 0} \frac{\mathbb{P}_x[X_t \in D]}{t} = \liminf_{t \searrow 0} \frac{\mathbb{P}_x[X_t \in D] - 1_D(x)}{t} > 0 \tag{4.1}$$

holds for any $x \in D^c$ with $\mathrm{dist}(x, D) \le 1/2$. Remark that (4.1) implies that there exists $t > 0$ such that

$$\frac{\mathbb{P}_x[X_t \in D]}{t} > 0,$$

which indicates $\mathbb{P}_x[T_D < \infty] > 0$. Recall that $\{T_t;\ 0 \le t < \infty\}$ is the semigroup corresponding to $\{X_t;\ 0 \le t < e\}$. Then, (4.1) is equivalent to

$$\liminf_{t \searrow 0} \frac{T_t 1_D(x) - 1_D(x)}{t} > 0. \tag{4.2}$$

Let $\widehat{D} \in \mathcal{D}$ be a set such that

$$\mathrm{dist}((\bar{D})^c, \widehat{D}) < \frac{1}{4}.$$

and $\{f_{\epsilon,\widehat{D}} \in C_c^\infty(M);\ \epsilon > 0\}$ the family of the cutoff functions satisfying $f_{\epsilon,\widehat{D}}(x) = 1$ for all $x \in \widehat{D}$ and $f_{\epsilon,\widehat{D}}(x) = 0$ for all $x \in M$ with $\mathrm{dist}(x, \widehat{D}) \ge \epsilon$. To show (4.2), we shall prove that

$$\liminf_{t \searrow 0} \frac{T_t 1_D(x)}{t} \ge \lim_{\epsilon \searrow 0} \lim_{t \searrow 0} \frac{T_t f_{\epsilon,\widehat{D}}(x) - f_{\epsilon,\widehat{D}}(x)}{t} = \int_{|Z|_x \le 1} 1_{\widehat{D}}(\exp_x Z) \nu_x(dZ)$$

holds.

If $\epsilon < 1/4$, then

$$f_{\epsilon,\widehat{D}}(x) \le 1_D(x)$$

holds for all $x \in M$. Since the semigroup $\{T_t;\ 0 \le t \le \infty\}$ is positive preserving, it holds that

$$T_t f_{\epsilon,\widehat{D}}(x) \le T_t 1_D(x)$$

for any $\epsilon < 1/4$, $t \in [0, \infty)$ and $x \in M$. Thus, we see that

$$\frac{T_t 1_D(x) - f_{\epsilon,\widehat{D}}(x)}{t} \geq \frac{T_t f_{\epsilon,\widehat{D}}(x) - f_{\epsilon,\widehat{D}}(x)}{t} \tag{4.3}$$

for any $\epsilon < 1/4$, $t \in [0,\infty)$ and $x \in M$. If $x \in D^c$, then (4.3) is equivalent to

$$\frac{T_t 1_D(x)}{t} \geq \frac{T_t f_{\epsilon,\widehat{D}}(x) - f_{\epsilon,\widehat{D}}(x)}{t}.$$

Since $f_{\epsilon,\widehat{D}} = 0$ in the neighborhood of $x \in D^c$ when ϵ is sufficiently small, it follows that

$$f_{\epsilon,\widehat{D}}(\exp_x Z) - f_{\epsilon,\widehat{D}}(x) - \langle \nabla f_{\epsilon,\widehat{D}}(x), Z\rangle_x = f_{\epsilon,\widehat{D}}(\exp_x Z).$$

Thus, we get

$$\begin{aligned}\lim_{t\searrow 0} \frac{T_t f_{\epsilon,\widehat{D}}(x) - f_{\epsilon,\widehat{D}}(x)}{t} &= \int_{|Z|_x\leq 1} \Big(f_{\epsilon,\widehat{D}}(\exp_x Z) - f_{\epsilon,\widehat{D}}(x) - \langle \nabla f_{\epsilon,\widehat{D}}, Z\rangle_x\Big)\nu_x(dZ) \\ &\xrightarrow[\epsilon\searrow 0]{} \int_{|Z|_x\leq 1} 1_{\widehat{D}}(\exp_x Z)\nu_x(dZ).\end{aligned}$$

Take a point $x \in D^c$ such that $\mathrm{dist}(x, D) \leq 1/2$. Denote the unit geodesic ball centered at $x \in M$ by $B(x,1)$. Since $\mathrm{dist}(x,D) \leq 1/2$ and $\mathrm{dist}((\bar{D})^c, \widehat{D}) < 1/4$, we see that $\widehat{D} \cap B(x,1) \neq \emptyset$. From Assumption 2, we have

$$\int_{|Z|_x\leq 1} 1_{\widehat{D}}(\exp_x Z)\nu_x(dZ) > 0$$

for any $x \in D^c$ with $\mathrm{dist}(x,D) \leq 1/2$. Define $D_0 = D$ and

$$D_n = \left\{x \in M;\ \mathrm{dist}(x, D_{n-1}) \leq \frac{1}{2}\right\}.$$

If we take a point x from D_2, we get $\mathbb{P}_x[T_{D_1} < \infty] > 0$ by the same argument mentioned above. Since $\mathbb{P}_x[T_{D_1} < \infty] > 0$ for any $x \in D_2$ and $\mathbb{P}_x[T_D < \infty] > 0$ for any $x \in D_1$, the strong Markov property of $\{X_t;\ 0 \leq t < e\}$ implies that

$$\mathbb{P}_x[T_D < \infty] \geq \mathbb{E}_x\Big[\mathbb{P}_{X_{T_{D_1}}}[T_D < \infty] 1_{\{T_{D_1}<\infty\}}\Big] > 0$$

holds for all $x \in D_2$. Inductively, we get

$$\mathbb{P}_x[T_D < \infty] > 0$$

for all $x \in M$.

The proof in case of $(\sigma,\eta,\kappa) = (0,1,1)$ is almost the same as that of $(\sigma,\eta,\kappa) = (0,1,0)$. In fact, we have

$$\liminf_{t\searrow 0} \frac{\mathbb{P}_x[X_t \in D]}{t} \geq \int_{T_xM_0} 1_{\widehat{D}}(\exp_x Z)\nu_x(dZ)$$

for any $x \in D^c$.

Next, we shall give a proof in case of $(\sigma,\eta,\kappa) = (1,1,0)$. For any $x \in D^c$ and $\epsilon < 1/4$, it holds that

$$\liminf_{t\searrow 0} \frac{\mathbb{P}_x[X_t \in D]}{t}$$

$$\geq \lim_{t\searrow 0} \frac{T_t f_{\epsilon,\widehat{D}}(x) - f_{\epsilon,\widehat{D}}(x)}{t}$$
$$= \frac{1}{2}\Delta_M f_{\epsilon,\widehat{D}}(x) + \int_{|Z|_x\leq 1} \Big(f_{\epsilon,\widehat{D}}(\exp_x(Z)) - f_{\epsilon,\widehat{D}}(x) - \langle \nabla f_{\epsilon,\widehat{D}}(x), Z\rangle_x\Big)\nu_x(dZ).$$

Take ϵ sufficiently small so that $f_{\epsilon,D} = 0$ in a neighborhood of x. Then, we have $\Delta_M f_{\epsilon,D}(x) = 0$. Hence, we have

$$\liminf_{t\searrow 0} \frac{\mathbb{P}_x[X_t \in D]}{t}$$
$$\geq \lim_{t\searrow 0} \frac{T_t f_{\epsilon,\widehat{D}}(x) - f_{\epsilon,\widehat{D}}(x)}{t}$$
$$= \frac{1}{2}\Delta_M f_{\epsilon,\widehat{D}}(x) + \int_{|Z|_x\leq 1} \Big(f_{\epsilon,\widehat{D}}(\exp_x(Z)) - f_{\epsilon,\widehat{D}}(x) - \langle \nabla f_{\epsilon,\widehat{D}}(x), Z\rangle_x\Big)\nu_x(dZ)$$
$$\xrightarrow[\epsilon\searrow 0]{} \int_{|Z|_x\leq 1} 1_{\widehat{D}}(\exp_x(Z))\nu_x(dZ) > 0$$

for all $x \in D^c$ with $\mathrm{dist}(x, D) \leq 1/2$. By the same argument in the proof in case of $(\sigma, \eta, \kappa) = (0, 1, 0)$, we see that the M-valued process $\{X_t;\ 0 \leq t < e\}$ is irreducible. The case $(\sigma, \eta, \kappa) = (1, 1, 1)$ can also be proved in such a way.

The proof in case of $(\sigma, \eta, \kappa) = (1, 0, 0)$ is described in Hsu [7, Proposition 4.4.4]. □

4.2. Proof of Theorem 2. First, we shall evaluate the radial part of the jump-diffusion process on M. Fix the base point $o \in M$. Define $r(\cdot) = \mathrm{dist}(o, \cdot)$, and write $\xi_Z(x) = \exp_x Z$. Remark that if M is a Hadamard manifold, there are no cut-locus. Therefore, the radial function r is smooth on $M\backslash\{o\}$. In order to find the nice lower estimate of the radial part of the M-valued process $\{X_t;\ 0 \leq t < e\}$, we have to evaluate Lr on $M\backslash\{o\}$, which is computed as follows:

$$Lr(y) = \sigma\frac{1}{2}\Delta_M r(y) + \eta\int_{|Z|_y\leq 1}\Big(r\circ\xi_Z(y) - r(y) - \langle\nabla r(y), Z\rangle_y\Big)\nu_y(dz)$$
$$+ \kappa\int_{|Z|_y>1}\Big(r\circ\xi_Z(y) - r(y)\Big)\nu_y(dz),\quad y \in M\backslash\{o\}.$$

Now, for any $y \in M$, we represent $Z \in T_yM$ by $Z = \rho\Theta$, where $\rho \in [0,\infty)$ and $\Theta \in U_yM = \{Z \in T_yM;\ |Z|_y = 1\}$. Let us define $Q = Q(\rho, \Theta, y)$ by

$$Q(\rho) = Q(\rho, \Theta, y) = r\circ\xi_{\rho\Theta}(y) - r(y) - \langle\nabla r(y), \Theta\rangle_y\rho. \tag{4.4}$$

Here, we summarize the properties of Q.

Lemma 2. *For given $y \in M\backslash\{o\}$ and $\Theta \in U_yM$ with $\langle\nabla r(y), \Theta\rangle_y < 0$, let us define $\rho_0 = \rho(\Theta, y)$ by*

$$\rho_0 = \sup\{\rho > 0;\ \frac{d}{d\rho}Q(\rho) \leq -\langle\nabla r(y), \Theta\rangle_y\}.$$

Then, Q satisfies the following conditions under Assumption 4*:*

- *For any $y \in M\backslash\{o\}$ and $\Theta \in U_yM$, the function*

$$[0,\infty) \ni \rho \mapsto Q(\rho) \in \mathbb{R}$$

is convex, and $Q(\rho) \geq 0$ for all $\rho \geq 0$.

- *If $\langle \nabla r(y), \Theta \rangle_y < 0$, then the following inequality*

$$Q(\rho) \geq \frac{1}{2}\sqrt{|\beta|}(1 - \langle \nabla r(y), \Theta \rangle_y^2)\rho^2$$

holds for all $\rho \leq \rho_0$.

- *If $\langle \nabla r(y), \Theta \rangle_y < 0$ and $\rho_0 < \infty$, then the following inequality*

$$Q(\rho) \geq -\langle \nabla r(y), \Theta \rangle_y (\rho - \rho_0)$$

holds for all $\rho \geq 0$.

Proof. Since the sectional curvature satisfies $K < 0$, the second variation formula enables us to see that

$$\frac{d^2}{d\rho^2} Q(\rho) \geq 0.$$

See Sakai [14, Chapter III, Remark 2.6] for details. Thus, the function

$$\rho \mapsto Q(\rho)$$

is convex. Furthermore, a simple calculation reveals

$$Q(0) = 0$$

and

$$\left.\frac{d}{d\rho} Q(\rho)\right|_{\rho=0} = 0.$$

Therefore, we see that

$$Q(\rho) \geq 0$$

holds for all $\rho \geq 0$.

Next, we shall show that if $\langle \nabla r(y), \Theta \rangle_y < 0$, then the following inequality

$$Q(\rho) \geq \frac{1}{2}\sqrt{|\beta|}(1 - \langle \nabla r(y), \Theta \rangle_y^2)\rho^2$$

holds for all $\rho \leq \rho_0$. By applying Taylor's theorem to the function

$$\rho \mapsto Q(\rho),$$

there exists $\theta \in (0, \rho)$ such that

$$Q(\rho) = Q(\rho) - \left.\frac{d}{d\rho} Q(\rho)\right|_{\rho=0} \rho - Q(0) = \left.\frac{1}{2}\frac{d^2}{d\rho^2} Q(\rho)\right|_{\rho=\theta} \rho^2.$$

On the other hand, $\frac{d^2}{d\rho^2} Q(\rho)$ is computed as follows:

$$\frac{d^2}{d\rho^2} Q(\rho) = \frac{d^2}{d\rho^2}\Big(r \circ \xi_{\rho\Theta}(y)\Big) = \nabla^2 r(\xi_{\rho\Theta}(y))((d\exp_y \rho\Theta)(\Theta), (d\exp_y \rho\Theta)(\Theta)) \tag{4.5}$$

where ∇^2 denotes the Hessian of M. For given $Z \in T_y M$, we define

$$Z^{\perp} = Z - \langle \nabla r(y), Z\rangle_y \nabla r(y).$$

Applying the comparison theorem on the Hessian (cf. Sakai [14, Chapter IV, Lemma 2.9]) implies that

$$\nabla^2 r(y)(\Theta,\Theta) \geq \frac{\sqrt{|\beta|}|\Theta^{\perp}|_y^2}{\tanh\big(\sqrt{|\beta|}r(y)\big)} = \frac{\sqrt{|\beta|}(1-\langle \nabla r(y),\Theta\rangle_y^2)}{\tanh\big(\sqrt{|\beta|}r(y)\big)} \tag{4.6}$$

holds for any $y \in M\backslash\{o\}$ and $\Theta \in U_yM$. From the Gauss lemma (cf. Sakai [14, Chapter II, Proposition 2.3]), we have

$$|(d\exp_y \rho Z)Z|_{\exp_y \rho Z} = |Z|_y \tag{4.7}$$

for any $\rho \geq 0$, $y \in M$ and $Z \in T_yM$. Thus, we see by (4.5), (4.6) and (4.7) that

$$\begin{aligned} \frac{d^2}{d\rho^2}Q(\rho) &\geq \frac{\sqrt{|\beta|}(1-\langle \nabla r(\exp_y\rho\Theta), (d\exp_y\rho\Theta)(\Theta)\rangle^2_{\exp_y\rho\Theta})}{\tanh\big(\sqrt{|\beta|}r(\exp_y\rho\Theta)\big)} \\ &\geq \sqrt{|\beta|}(1-\langle \nabla r(\exp_y\rho\Theta), (d\exp_y\rho\Theta)(\Theta)\rangle^2_{\exp_y\rho\Theta}) \end{aligned} \tag{4.8}$$

holds for all $\rho \geq 0$. Since the function

$$[0,\infty) \ni \rho \mapsto \frac{d}{d\rho}Q(\rho) \in [0,\infty]$$

is monotone increasing, we have

$$0 \leq \frac{d}{d\rho}Q(\rho) = \langle \nabla r(\exp_y\rho\Theta), (d\exp_y\rho\Theta)(\Theta)\rangle_{\exp_y\theta\Theta} - \langle \nabla r(y),\Theta\rangle_y \leq -\langle \nabla r(y),\Theta\rangle_y$$

for all $\rho \leq \rho_0$. Clearly, this implies that

$$\langle \nabla r(y),\Theta\rangle_y \leq \langle \nabla r(\exp_y\rho\Theta), (d\exp_y\rho\Theta)(\Theta)\rangle_{\exp_y\rho\Theta} \leq 0.$$

Therefore, we see that if $\langle \nabla r(y),\Theta\rangle_y < 0$, then

$$\big|\langle \nabla r(\exp_y\rho\Theta), (d\exp_y\rho\Theta)(\Theta)\rangle_{\exp_y\rho\Theta}\big| \leq \big|\langle \nabla r(y),\Theta\rangle_y\big| \tag{4.9}$$

holds for all $\rho \leq \rho_0$. Thus, if $\langle \nabla r(y),\Theta\rangle_y < 0$, then we see by (4.8) and (4.9) that

$$\frac{d^2}{d\rho^2}Q(\rho) \geq \sqrt{|\beta|}(1-\langle \nabla r(y),\Theta\rangle_y^2)$$

holds for all $\rho \leq \rho_0$.

Finally, we shall show that if $\langle \nabla r(y),\Theta\rangle_y < 0$ and $\rho_0 < \infty$, then the following inequality

$$Q(\rho) \geq -\langle \nabla r(y),\Theta\rangle_y(\rho-\rho_0)$$

holds for all $\rho \geq 0$. Since the function

$$\rho \to Q(\rho)$$

is convex and

$$\frac{d}{d\rho}Q(\rho)\bigg|_{\rho=\rho_0} = -\langle \nabla r(y),\Theta\rangle_y,$$

we have

$$Q(\rho) \geq Q(\rho_0) - \langle \nabla r(y), \Theta \rangle_y (\rho - \rho_0) \geq -\langle \nabla r(y), \Theta \rangle_y (\rho - \rho_0)$$

for all $\rho \geq 0$. □

Lemma 3. *Let* $\{U_t;\ 0 \leq t < e\}$ *be the* $O(M)$*-valued process determined by the stochastic differential equation* (2.1) *and* $\pi U_t = X_t$. *Suppose that*

$$\mathbb{P}_x[e = \infty] = 1$$

holds for all $x \in M$, *and that* Assumptions 1, 2, 4 *and* 6 *are satisfied. (When* $\kappa = 1$, *we additionally assume* Assumption 3.*) Then, there exists a constant* $C > 0$ *such that the following inequality*

$$\begin{aligned} r(X_t) \geq r(x) + Ct + \sigma \int_0^t \langle U_{s-}^{-1} \nabla r(X_{s-}), dB_s \rangle \\ + \int_0^t \int_{\mathbb{R}_0^m} \Big(r \circ \xi_{U_{s-}z}(X_{s-}) - r(X_{s-}) \Big) \Big(\eta 1_{\{|z| \leq 1\}} + \kappa 1_{\{|z|>1\}} \Big) \widetilde{N}(dz, ds) \end{aligned}$$

holds for all $t \geq 0$.

Proof. Assumption 6 implies that

$$\mathbb{P}_x\Big[X_t = o\Big] = 0 \tag{4.10}$$

holds for all $x \in M$ and $t \geq 0$. Hence, we see by Fubini's theorem that

$$\mathbb{E}_x\Big[\int_0^\infty 1_{\{X_s = o\}} ds\Big] = \int_0^\infty \mathbb{P}_x[X_s = o] ds = 0,$$

which implies

$$\mathbb{P}_x\Big[\int_0^\infty 1_{\{X_s = o\}} ds = 0\Big] = 1.$$

Thus, from the Itô formula and (4.10), the following equality holds under Assumptions 1, 4 and 6:

$$\begin{aligned} r(X_t) &= r(x) + \sigma \int_0^t \langle U_{s-}^{-1} \nabla r(X_{s-}), dB_s \rangle + \sigma \int_0^t \frac{1}{2} \Delta_M r(X_s) ds \\ &\quad + \eta \Big\{ \int_0^t \int_{|z| \leq 1} \Big(r \circ \xi_{U_{s-}z}(X_{s-}) - r(X_{s-}) \Big) \widetilde{N}(dz, ds) \\ &\quad + \int_0^t \int_{|z| \leq 1} \Big(r \circ \xi_{U_{s-}z}(X_{s-}) - r(X_{s-}) - \langle U_{s-}^{-1} \nabla r(X_{s-}), z \rangle \Big) \nu(dz) ds \Big\} \\ &\quad + \kappa \Big\{ \int_0^t \int_{|z|>1} \Big(r \circ \xi_{U_{s-}z}(X_{s-}) - r(X_{s-}) \Big) \widetilde{N}(dz, ds) \\ &\quad + \int_0^t \int_{|z|>1} \Big(r \circ \xi_{U_{s-}z}(X_{s-}) - r(X_{s-}) \Big) \nu(dz) ds \Big\} \\ &= r(x) + \sigma \int_0^t \langle U_{s-}^{-1} \nabla r(X_{s-}), dB_s \rangle + \sigma \int_0^t \frac{1}{2} \Delta_M r(X_s) ds \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \int_{\mathbb{R}_0^m} \Big(r\circ\xi_{U_{s-}z}(X_{s-}) - r(X_{s-})\Big)\Big(\eta 1_{\{|z|\le 1\}} + \kappa 1_{\{|z|>1\}}\Big)\widetilde{N}(dz,ds)\\
&+\eta\int_0^t \int_{|z|\le 1} \Big(r\circ\xi_{U_{s-}z}(X_{s-}) - r(X_{s-}) - \langle U_{s-}^{-1}\nabla r(X_{s-}), z\rangle\Big)\nu(dz)ds\\
&+\kappa\int_0^t \int_{|z|>1} \Big(r\circ\xi_{U_{s-}z}(X_{s-}) - r(X_{s-})\Big)\nu(dz)ds.
\end{aligned}
$$

For any $y \in M\backslash\{o\}$, write

$$
\begin{aligned}
\Gamma_1(y) &= \int_{|Z|_y\le 1} \Big(r\circ\xi_Z(y) - r(y) - \langle\nabla r(y), Z\rangle_y\Big)\nu_y(dZ),\\
\Gamma_2(y) &= \int_{|Z|_y> 1} \Big(r\circ\xi_Z(y) - r(y)\Big)\nu_y(dZ),\\
\Gamma_3(y) &= \frac{1}{2}\Delta_M r(y).
\end{aligned}
\tag{4.11}
$$

Our strategy is to evaluate Γ_1, Γ_2 and Γ_3. First, we shall prove that there exists a constant $C_1 > 0$ satisfying

$$
\Gamma_1(y) = \int_{|Z|_y\le 1} \Big(r\circ\xi_Z(y) - r(y) - \langle\nabla r(y), Z\rangle_y\Big)\nu_y(dZ) \ge C_1
$$

for all $y \in M\backslash\{o\}$. Recall that Q is defined by (4.4). By Lemma 2, we get

$$
\begin{aligned}
\Gamma_1(y) &= \int_0^1 \int_{U_yM} Q(\rho)h(\rho)d\Theta d\rho\\
&= \int_{U_yM} \int_0^1 Q(\rho)h(\rho)d\rho d\Theta\\
&\ge \int_{\langle\nabla r(y),\Theta\rangle_y<0} \Big(\int_0^{\rho_0\wedge 1} Q(\rho)h(\rho)d\rho + \int_{\rho_0\wedge 1}^1 Q(\rho)h(\rho)d\rho\Big)d\Theta\\
&\ge \int_{\langle\nabla r(y),\Theta\rangle_y<0} \Big\{\int_0^{\rho_0\wedge 1} \frac{1}{2}\sqrt{|\beta|}(1-\langle\nabla r(y),\Theta\rangle_y^2)\rho^2 h(\rho)d\rho\\
&\qquad\qquad + \int_{\rho_0\wedge 1}^1 -\langle\nabla r(y),\Theta\rangle_y(\rho-\rho_0)h(\rho)d\rho\Big\}d\Theta,
\end{aligned}
$$

where $h(\rho)$ is the density introduced in Assumption 2 and Remark 1. Let us define $C_0 = C_0(\langle\nabla r(y),\Theta\rangle)$ by

$$
C_0 = \inf_{0<s\le 1}\Big(\frac{1}{2}\int_0^s \sqrt{|\beta|}(1-\langle\nabla r(y),\Theta\rangle_y^2)\rho^2 h(\rho)d\rho + \int_s^1 -\langle\nabla r(y),\Theta\rangle_y(\rho-s)h(\rho)d\rho\Big).
$$

Since it holds that

$$
\int_0^1 \rho^2 h(\rho)d\rho > 0,
$$

we have

$$
\lim_{s\searrow 0}\int_0^{1-s} \rho h(\rho+s)d\rho > 0.
$$

Thus, if $-1 < \langle \nabla r(y), \Theta \rangle_y < 0$, then $C_0 > 0$. Therefore, we obtain

$$\Gamma_1(y) \geq \int_{\langle \nabla r(y), \Theta \rangle_y < 0} C_0(\langle \nabla r(y), \Theta \rangle_y) d\Theta > 0.$$

Now, we shall choose the constant C_1 as follows:

$$C_1 = \int_{\langle \nabla r(y), \Theta \rangle_y < 0} C_0(\langle \nabla r(y), \Theta \rangle_y) d\Theta.$$

The rotational invariance of the Lebesgue measure on $U_y M$ enables us to see that

$$C_1 = \int_{\langle \nabla r(y), \Theta \rangle_y < 0} C_0(\langle \nabla r(y), \Theta \rangle_y) d\Theta = \int_{\mathbb{S}^{m-1} \cap \{z_1 < 0\}} C_0(z_1) dz,$$

which implies that C_1 is independent of $y \in M \backslash \{o\}$.

Next, we shall show that

$$\Gamma_2(y) = \int_{|Z|_y > 1} \Big(r \circ \xi_Z(y) - r(y) \Big) \nu_y(dZ) \geq 0,$$

if the Lévy measure $\nu(dz)$ satisfies Assumption 3. By Taylor's theorem and the second variation formula (cf. Sakai [14, Chapter III, Remark 2.6]), there exists $\theta \in (0, \rho)$ such that

$$\begin{aligned} r \circ \xi_{\rho\Theta}(y) - r(y) &= \langle \nabla r(y), \Theta \rangle_y \rho + \frac{1}{2} \nabla^2 r(\xi_{\theta\Theta}(y))((d\exp_y \theta\Theta)\Theta, (d\exp_y \theta\Theta)\Theta)\rho^2 \\ &\geq \langle \nabla r(y), \Theta \rangle_y \rho. \end{aligned}$$

Since the Lévy measure $\nu(dz)$ satisfies Assumption 3, we have

$$\int_{|z|>1} |z| \nu(dz) < \infty.$$

Hence, we can obtain

$$\begin{aligned} \int_{|Z|_y > 1} \Big(r \circ \xi_Z(y) - r(y) \Big) \nu_y(dZ) &= \int_1^\infty \int_{U_y M} \Big(r \circ \xi_{\rho\Theta}(y) - r(y) \Big) h(\rho) d\Theta d\rho \\ &\geq \int_1^\infty \int_{U_y M} \langle \nabla r(y), \Theta \rangle_y \rho h(\rho) d\Theta d\rho \\ &= \int_1^\infty \Big(\int_{U_y M} \langle \nabla r(y), \Theta \rangle_y d\Theta \Big) \rho h(\rho) d\rho \\ &= \int_{|Z|_y > 1} \langle \nabla r(y), Z \rangle_y \nu_y(dZ). \end{aligned}$$

Moreover, the rotational invariance of $\nu(dz)$ implies that

$$\int_{|Z|_y > 1} \langle \nabla r(y), Z \rangle_y \nu_y(dZ) = \int_{|z|>1} \langle v, z \rangle \nu(dz) = \int_{|z|>1} \langle v, (-z) \rangle \nu(dz)$$

for any unit vector $v \in \mathbb{R}^m$. Then, we have

$$\int_{|Z|_y > 1} \langle \nabla r(y), Z \rangle_y \nu_y(dZ) = 0.$$

By the comparison theorem on the Laplacian (cf. Sakai [14, Chapter V, Lemma 2.9]), it holds that

$$\Gamma_3(y) = \frac{1}{2}\Delta_M r(y) \geq \frac{\sqrt{|\beta|}(m-1)}{2\tanh(\sqrt{|\beta|}r(y))} \geq \frac{\sqrt{|\beta|}(m-1)}{2}.$$

Let us define a constant C by

$$C = \eta\, C_1 + \frac{1}{2}\sigma\sqrt{|\beta|}(m-1).$$

Then, we can see that

$$\int_0^t Lr(X_{s-})ds = \int_0^t \Big(\eta\Gamma_1(X_{s-}) + \kappa\Gamma_2(X_{s-}) + \sigma\Gamma_3(X_{s-})\Big)ds \geq Ct.$$

The proof of the theorem is complete. □

Now, we write

$$M_t = \int_0^t \int_{\mathbb{R}_0^m} \Big(r \circ \xi_{U_{s-}z}(X_{s-}) - r(X_{s-})\Big)\Big(\eta 1_{\{|z|\leq 1\}} + \kappa 1_{\{|z|>1\}}\Big)\widetilde{N}(dz, ds), \tag{4.12}$$

$$\lambda = \int_{\mathbb{R}_0^m} |z|^2\Big(\eta 1_{\{|z|\leq 1\}} + \kappa 1_{\{|z|>1\}}\Big)\nu(dz), \tag{4.13}$$

$$W_t = \int_0^t \langle U_{s-}^{-1}\nabla r(X_{s-}), dB_s\rangle. \tag{4.14}$$

Remark that if the Lévy measure $\nu(dz)$ satisfies Assumption 3, then (4.12) is well-defined and (4.13) is finite in case of $\kappa = 1$. For the proof of Theorem 2, we need the following lemma.

Lemma 4. *Let* $\{M_t;\ 0 \leq t < \infty\}$ *be the martingale defined by* (4.12). *Suppose that*

$$\mathbb{P}_x[e = \infty] = 1$$

holds for all $x \in M$. *Then, we have*

$$\mathbb{P}_x\Big[\lim_{t\to\infty} \frac{M_t}{t} = 0\Big] = 1 \tag{4.15}$$

for all $x \in M$ *in case of* $(\sigma, \eta, \kappa) = (0, 1, 0), (1, 1, 0)$. *Moreover, if the Lévy measure* $\nu(dz)$ *satisfies* Assumption 3, *then* (4.15) *holds in case of* $(\sigma, \eta, \kappa) = (0, 1, 1), (1, 1, 1)$.

Proof. First, we consider the case of $\kappa = 0$. It is clear that $\left|\frac{M_t}{t}\right| \leq \left|\frac{M_t}{s}\right|$ for any $s \leq t$. From Doob's inequality,

$$\mathbb{E}_x\left[\sup_{s\leq t\leq u}\left|\frac{M_t}{t}\right|^2\right] \leq s^{-2}\mathbb{E}_x\left[\sup_{s\leq t\leq u}|M_t|^2\right] \leq 4s^{-2}\mathbb{E}_x[M_u^2] \leq 4\lambda s^{-2}u$$

holds for all $0 \leq s \leq u < \infty$ and $x \in M$. Choose $s = 2^n$ and $u = 2^{n+1}$ in the above inequality. Then, we obtain

$$\mathbb{E}_x\left[\sup_{2^n\leq t\leq 2^{n+1}}\left|\frac{M_t}{t}\right|^2\right] \leq \lambda 2^{-(n-3)}.$$

Then, from Chebyshev's inequality,

$$\mathbb{P}_x\left[\sup_{2^n\le t\le 2^{n+1}}\left|\frac{M_t}{t}\right| > \epsilon\right] \le \epsilon^{-2}\mathbb{E}_x\left[\sup_{2^n\le t\le 2^{n+1}}\left|\frac{M_t}{t}\right|^2\right] \le \epsilon^{-2}\lambda 2^{-(n-3)} \tag{4.16}$$

holds for any $\epsilon > 0$, $n \in \mathbb{N}$ and $x \in M$. From the Borel-Cantelli lemma with (4.16), we have

$$\mathbb{P}_x\Big[\lim_{t\to\infty}\frac{M_t}{t} = 0\Big] = 1.$$

Next, we turn to consider the case of $\kappa = 1$. If the Lévy measure $\nu(dz)$ satisfies Assumption 3, then (4.13) is finite and hence the inequality (4.16) holds, via a similar argument stated above. Thus, we also see that (4.15) holds for $\kappa = 1$. □

Lemma 5. *Let* $\{W_t;\ 0 \le t < \infty\}$ *be the martingale defined by* (4.14). *Suppose that*

$$\mathbb{P}_x[e = \infty] = 1$$

holds for all $x \in M$. *Then, we have*

$$\mathbb{P}_x\Big[\lim_{t\to\infty}\frac{W_t}{t} = 0\Big] = 1$$

for all $x \in M$.

Proof. Since $|\nabla r(x)| = 1$ for all $x \in M\backslash\{o\}$, we have

$$\mathbb{E}_x[|W_t|^2] = t.$$

Hence, by the same argument in the proof of Lemma 4, we have

$$\mathbb{P}_x\Big[\lim_{t\to\infty}\frac{W_t}{t} = 0\Big] = 1.$$

□

Proof of Theorem 2. Assume that the Lévy measure $\nu(dz)$ satisfies Assumptions 1 and 2. Then, we see from Theorem 1 that the M-valued process $\{X_t;\ 0 \le t < e\}$ is irreducible. Let us consider the following cases:

(i) There exists $x \in M$ such that

$$\mathbb{P}_x[e < \infty] > 0.$$

(ii) For any $x \in M$,

$$\mathbb{P}_x[e = \infty] = 1$$

holds.

Remark 4 tells us that $\{X_t;\ 0 \le t < e\}$ is transient in the case (i). So, we only need to consider the case (ii). From Lemmas 4 and 5, it is easy to verify that

$$\frac{r(x) + \sigma W_t + M_t + Ct}{t} \xrightarrow[t\to\infty]{a.s} C > 0,$$

which implies

$$r(x) + \sigma W_t + M_t + Ct \xrightarrow[t\to\infty]{a.s} \infty.$$

Thus, by Lemma 3, we have

$$r(X_t) \geq r(x) + \sigma W_t + M_t + Ct \xrightarrow[t\to\infty]{a.s} \infty.$$

The proof of Theorem 2 is complete. □

4.3. Proof of Theorem 3. In order to prove Theorem 3, let us discuss the properties of the explosion time e defined by

$$e = \inf\{t > 0;\ X_t = \partial_M\}.$$

Lemma 6. *Define a function j on M by*

$$j(x) = \mathbb{P}_x[e = \infty].$$

Suppose that the Lévy measure $\nu(dz)$ satisfies Assumptions 1, 2, 4, 6 *and* 7. *(When $\kappa = 1$, we additionally assume* Assumption 3.*) Then, the function j satisfies one of the following:*

- *For all $x \in M$, $j(x) = 1$.*
- *For all $x \in M$, $j(x) = 0$.*
- *For all $x \in M$, $0 < j(x) < 1$.*

Proof. Since the proof in case of $\kappa = 1$ is similar to the case of $\kappa = 0$, we shall only give the proof for $\kappa = 0$. From the Markov property, we see that

$$\mathbb{P}_x[e = \infty] = \mathbb{E}_x\Big[\mathbb{P}_{X_t}[e = \infty]\Big]$$

holds for any $x \in M$ and $t \in [0, \infty)$. Thus, we get

$$j(x) = T_t j(x) \tag{4.17}$$

for all $x \in M$. From (4.17), j is expressed by

$$j(x) = \int_M j(y)p(t, x, y)\mathrm{Vol}(dy).$$

From Lemma 1, we see that j is of C^2-class. Therefore, j belongs to the domain of L. Moreover, we see by (4.17) that

$$Lj(x) = \sigma\frac{1}{2}\Delta_M j(x) + \int_{|Z|_x\leq 1}\Big(j\circ\exp_x Z - j(x) - \langle\nabla j(x), Z\rangle_x\Big)\nu_x(dZ) = 0 \tag{4.18}$$

holds for all $x \in M$. Let $x_0 \in M$ such that $j(x_0) = 1$. Then, from Assumptions 1, 2 and (4.18), we have

$$\sigma\frac{1}{2}\Delta_M j(x_0) + \int_{|Z|_{x_0}\leq 1}\Big(j\circ\exp_{x_0} Z - 1\Big)h(|Z|_{x_0})dZ = 0. \tag{4.19}$$

Since x_0 is the maximizer of the function j, it holds that

$$\Delta_M j(x_0) \leq 0.$$

Moreover, it is clear that

$$j\circ\exp_{x_0} Z - 1 \leq 0$$

holds for any $Z \in T_{x_0}M$. Thus, we see by (4.19) that

$$\Delta_M j(x_0) = 0$$

and

$$\int_{|Z|_{x_0} \le 1} \Big(j \circ \exp_{x_0} Z - 1 \Big) h(|Z|_{x_0}) dZ = 0.$$

Since the functions

$$j \circ \exp_{x_0} : T_{x_0} M \to [0, 1]$$

and

$$h(|\cdot|) : T_{x_0} M \to (0, \infty)$$

are continuous, we have

$$j \circ \exp_{x_0} Z = 1 \tag{4.20}$$

for all $Z \in T_{x_0} M_0$ with $|Z|_{x_0} \le 1$. Furthermore, (4.20) implies that

$$j(x) = 1 \tag{4.21}$$

holds for any $x \in B(x_0, 1) = \{x \in M;\ \mathrm{dist}(x_0, x) < 1\}$. Applying the same argument to all points in $B(x_0, 1)$, we have (4.21) for all $x \in B(x_0, 2)$. Inductively, (4.21) holds for all $x \in M$, because M is connected.

On the other hand, by the same argument, we see that the existence of the point $x_0 \in M$ such that $j(x_0) = 0$ implies $j(x) = 0$ for all $x \in M$.

Next, we shall prove Lemma 6 in case of $(\sigma, \eta, \kappa) = (1, 0, 0)$. By the same discussion stated above, we see that

$$L j(x) = \frac{1}{2} \Delta_M j(x) = 0$$

holds for any $x \in M$. Therefore, the function j is a bounded harmonic function. The proof in case of $(\sigma, \eta, \kappa) = (1, 0, 0)$ is complete. □

Next, we shall study the upper estimate of the radial part of the jump-diffusion process which plays an important role in the proof of Theorem 3.

Lemma 7. *Let $\delta > 0$ be a positive constant, and recall that Q is defined by* (4.4). *Then, Q satisfies the following conditions under* Assumption 5*:*

- *If $r(y) > \delta$, then the following inequality*

$$Q(\rho) \le \frac{\sqrt{|\alpha|}}{2 \tanh\big(\sqrt{|\alpha|}\delta\big)} \rho^2 \tag{4.22}$$

  holds for all $\rho \in [0, r(y) - \delta]$ and $\Theta \in U_y M$.
- *The following inequality*

$$Q(\rho) \le (1 - \langle \nabla r(y), \Theta \rangle_y) \rho \tag{4.23}$$

  holds for all $\rho \ge 0$, $y \in M \backslash \{o\}$ and $\Theta \in U_y M$.

Proof. First, we shall show that if $r(y) > \delta$, then (4.22) holds for all $\rho \in [0, r(y) - \delta]$. Applying Taylor's theorem enables us to see that there exists $\theta \in (0, \rho)$ such that

$$Q(\rho) = \frac{1}{2}\nabla^2 r(\xi_{\theta\Theta}(y))(d\exp(\theta\Theta)\Theta, d\exp(\theta\Theta)\Theta)\rho^2.$$

So, we see by the comparison theorem on the Hessian (cf. Sakai [14, Chapter IV, Lemma 2.9]) that

$$Q(\rho) \leq \frac{\sqrt{|\alpha|}|\Theta^{\perp}|_y^2}{2\tanh\Big(\sqrt{|\alpha|}(r \circ \xi_{\theta\Theta}(y))\Big)}\rho^2 \leq \frac{\sqrt{|\alpha|}}{2\tanh\Big(\sqrt{|\alpha|}(r \circ \xi_{\theta\Theta}(y))\Big)}\rho^2 \tag{4.24}$$

holds for any $y \in M\backslash\{o\}$, $\Theta \in U_yM$ and $\rho \geq 0$. From the triangle inequality, it holds that

$$r \circ \xi_{\rho\Theta}(y) + \rho \geq r(y).$$

Thus, if $r(y) > \delta$, then we have

$$r \circ \xi_{\rho\Theta}(y) \geq \delta \tag{4.25}$$

for all $\rho \in [0, r(y) - \delta]$ and $\Theta \in U_yM$. Therefore, for any $y \in M\backslash\{o\}$ and $\Theta \in U_yM$, (4.24) and (4.25) enable us to see that

$$Q(\rho) \leq \frac{\sqrt{|\alpha|}}{2\tanh\Big(\sqrt{|\alpha|}\delta\Big)}\rho^2$$

holds for all $\rho \in [0, r(y) - \delta]$.

Next, we shall show that (4.23) holds for all $\rho \geq 0$. From the triangle inequality, we have

$$r \circ \xi_{\rho\Theta}(y) - r(y) \leq \rho,$$

which implies that

$$Q(\rho) = r \circ \xi_{\rho\Theta}(y) - r(y) - \langle \nabla r(y), \Theta \rangle_y \rho \leq \rho - \langle \nabla r(y), \Theta \rangle_y \rho$$

holds for all $\rho \geq 0$. The proof is complete. □

Lemma 8. *Let $\{U_t;\ 0 \leq t < e\}$ be the solution to the stochastic differential equation* (2.1) *and $X_t = \pi U_t$. Fix a positive constant $\delta > 0$, and define the stopping time $\tau = \tau(\delta)$ by*

$$\tau = \inf\{t > 0;\ r(X_t) < 2\delta\}.$$

Suppose that Assumptions 1, 2, 5 *and* 6 *are satisfied. (When $\kappa = 1$, we additionally assume* Assumption 3.*) Then, there exists a positive constant $V = V(\delta) < \infty$ such that*

$$\mathbb{P}_x\Big[r(X_t) \leq r(x) + \sigma W_t + M_t + Vt \text{ holds for all } t < \tau \wedge e\Big] = 1$$

holds for any $x \in M$. Here, M_t and W_t are defined by (4.12) *and* (4.14).

Proof. Our strategy to prove the statement is similar to Lemma 3. First, we shall prove that there exists $V_1 = V_1(\delta) < \infty$ satisfying

$$\int_{|Z|_y \leq 1}\Big(r \circ \xi_Z(y) - r(y) - \langle \nabla r(y), Z \rangle_y\Big)\nu_y(dZ) \leq V_1$$

for $r(y) \geq 2\delta$. For any $y \in M\backslash\{o\}$ with $r(y) > \delta$, the following equation

$$\int_{|Z|_y\le 1}\Big(r\circ\xi_Z(y)-r(y)-\langle\nabla r(y),Z\rangle_y\Big)\nu_y(dZ)=\int_{U_yM}\int_0^1 Q(\rho)h(\rho)d\rho d\Theta \tag{4.26}$$
$$=\int_{U_yM}\int_0^{(r(y)-\delta)\wedge 1} Q(\rho)h(\rho)d\rho d\Theta$$
$$+\int_{U_yM}\int_{(r(y)-\delta)\wedge 1}^1 Q(\rho)h(\rho)d\rho d\Theta$$

holds under Assumptions 1 and 2. By (4.22), we can easily verify that

$$\int_{U_yM}\int_0^{(r(y)-\delta)\wedge 1} Q(\rho)h(\rho)d\rho d\Theta\le\frac{\sqrt{|\alpha|}}{2\tanh\Big(\sqrt{|\alpha|}\delta\Big)}\int_{|Z|_y\le 1}|Z|_y^2\nu_y(dZ) \tag{4.27}$$
$$=\frac{\sqrt{|\alpha|}}{2\tanh\Big(\sqrt{|\alpha|}\delta\Big)}\int_{|z|\le 1}|z|^2\nu(dz)<\infty$$

holds for any $y\in M\backslash\{o\}$ with $r(y)>\delta$. On the other hand, if $r(y)\ge 2\delta$, then we see by (4.23) that

$$\int_{U_yM}\int_{(r(y)-\delta)\wedge 1}^1 Q(\rho)h(\rho)d\rho d\Theta\le\int_{U_yM}\int_{\delta\wedge 1}^1(1-\langle\nabla r(y),\Theta\rangle_y)\rho h(\rho)d\rho d\Theta \tag{4.28}$$
$$=\int_{(\delta\wedge 1)\le|Z|_y\le 1}|Z|_y\nu_y(dZ)$$
$$=\int_{(\delta\wedge 1)\le|z|\le 1}|z|\nu(dz)<\infty$$

holds. Now, let us define $V_1=V_1(\delta)$ by

$$V_1=\frac{\sqrt{|\alpha|}}{2\tanh\Big(\sqrt{|\alpha|}\delta\Big)}\int_{|z|\le 1}|z|^2\nu(dz)+\int_{(\delta\wedge 1)\le|z|\le 1}|z|\nu(dz).$$

Then, by (4.26), (4.27) and (4.28), we have

$$\int_{|Z|_y\le 1}\Big(r\circ\xi_Z(y)-r(y)-\langle\nabla r(y),Z\rangle_y\Big)\nu_y(dZ)\le V_1$$

for all $y\in M$ with $r(y)\ge 2\delta$. Thus, the following inequality

$$\int_0^t\int_{|z|\le 1}\Big(r\circ\xi_{U_{s-}z}(X_{s-})-r(X_{s-})-\langle U_{s-}^{-1}\nabla r(X_{s-}),z\rangle\Big)\nu(dz)ds\le V_1 t$$

holds for all $t<\tau\wedge e$.

If $\nu(dz)$ satisfies Assumption 3, from the triangle inequality, we have

$$\int_{|Z|>1}\Big(r\circ\xi_Z(y)-r(y)\Big)\nu_y(dZ)\le\int_{|z|>1}|z|\nu(dz)<\infty.$$

On the other hand, by the comparison theorem on the Laplacian (cf. Sakai [14, Chapter III, Lemma 2.9]), we see that

$$\Delta_M r(y)\le\frac{\sqrt{|\alpha|}(m-1)}{\tanh(\sqrt{|\alpha|}r(y))}$$

holds for all $y \in M\backslash\{o\}$. Thus, we have

$$\Delta_M r(X_t) \le \frac{\sqrt{|\alpha|}(m-1)}{\tanh(2\sqrt{|\alpha|}\delta)}$$

for all $t < \tau \wedge e$. Define $V = V(\delta)$ by

$$V = \eta\, V_1 + \kappa \int_{|z|>1} |z|\nu(dz) + \sigma \frac{\sqrt{|\alpha|}(m-1)}{2\tanh(2\sqrt{|\alpha|}\delta)}.$$

Then, we see that

$$\int_0^t Lr(X_{s-})ds = \int_0^t \Big(\eta\Gamma_1(X_{s-}) + \kappa\Gamma_2(X_{s-}) + \sigma\Gamma_3(X_{s-})\Big)ds \le Vt$$

holds for $t < \tau \wedge e$, where Γ_1, Γ_2 and Γ_3 are defined by (4.11). The proof is complete. □

In order to prove Theorem 3, we need to study how the martingales $\{M_t;\ 0 \le t < e\}$ and $\{W_t;\ 0 \le t < e\}$ will behave as $t \to e$ when the explosion time is finite. Such kind of problem can be solved by the following lemma.

Lemma 9. *Suppose that* Assumptions 1 *and* 2 *are satisfied. (If* $\kappa = 1$*, then we additionally assume* Assumption 3.*) Then,*

$$\mathbb{P}_x\Big[\limsup_{t\nearrow e} |M_t| < \infty,\ e < \infty\Big] = \mathbb{P}_x\Big[e < \infty\Big]$$

holds for all $x \in M$.

Proof. Define the stopping time τ_n by $\tau_n = \inf\{t > 0;\ r(X_t) \ge 2^n\}$. From Doob's inequality, we see that

$$\mathbb{E}_x\Big[\sup_{2^n \le t < 2^{n+1}} |M_{t\wedge\tau_n}|^2\Big] \le 4\mathbb{E}_x\Big[|M_{2^{n+1}\wedge\tau_n}|^2\Big] \le 2^{n+3}\lambda$$

holds for all $n \in \mathbb{N}$, where λ is defined by (4.13). From Chebyshev's inequality, we have

$$\mathbb{P}_x\Big[\sup_{2^n \le t < 2^{n+1}} |M_{t\wedge\tau_n}| \ge 2^n\Big] \le 2^{-2n}\mathbb{E}_x\Big[\sup_{2^n \le t < 2^{n+1}} |M_{t\wedge\tau_n}|^2\Big]$$

for all $n \in \mathbb{N}$. Thus, we can verify that

$$\mathbb{P}_x\Big[\sup_{2^n \le t < 2^{n+1}} |M_{t\wedge\tau_n}| \ge 2^n\Big] \le 2^{-n+3}\lambda \tag{4.29}$$

holds for all $n \in \mathbb{N}$. Applying the Borel-Cantelli lemma with (4.29) implies

$$\mathbb{P}_x\Big[\liminf_{n\to\infty}\{\sup_{2^n \le t < 2^{n+1}} |M_{2^n\wedge\tau_n}| < 2^n\}\Big] = 1.$$

So, we see that

$$\mathbb{P}_x\Big[\bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty}\{\sup_{2^n \le t < 2^{n+1}} |M_{2^n\wedge\tau_n}| < 2^n\},\ e < 2^l\Big] = \mathbb{P}_x[e < 2^l]$$

holds for any $l \in \mathbb{N}$. From the definition of τ_n, we have

$$\mathbb{P}_x[\tau_n \le e] = 1$$

and

$$\mathbb{P}_x[\lim_{n\to\infty} \tau_n = e] = 1.$$

Therefore, we get

$$\begin{aligned}
\mathbb{P}_x[e < 2^l] &= \mathbb{P}_x\Big[\liminf_{n\to\infty}\{\sup_{2^n\le t<2^{n+1}} |M_{t\wedge\tau_n}| < 2^n\},\ e < 2^l\Big] \\
&= \mathbb{P}_x\Big[\bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty}\{\sup_{2^n\le t<2^{n+1}} |M_{t\wedge\tau_n}| < 2^n\},\ e < 2^l\Big] \\
&= \mathbb{P}_x\Big[\bigcup_{N=l}^{\infty}\bigcap_{n=N}^{\infty}\{\sup_{2^n\le t<2^{n+1}} |M_{t\wedge\tau_n}| < 2^n\},\ e < 2^l\Big] \\
&= \mathbb{P}_x\Big[\bigcup_{N=l}^{\infty}\bigcap_{n=N}^{\infty}\{\sup_{2^n\le t<2^{n+1}} |M_{\tau_n}| < 2^n\},\ e < 2^l\Big] \\
&= \mathbb{P}_x\Big[\bigcup_{N=l}^{\infty}\bigcap_{n=N}^{\infty}\{|M_{\tau_n}| < 2^n\},\ e < 2^l\Big] \\
&= \mathbb{P}_x\Big[\liminf_{n\to\infty}\{|M_{\tau_n}| < 2^n\},\ e < 2^l\Big],
\end{aligned}$$

which implies that

$$\mathbb{P}_x\Big[\limsup_{t\nearrow e} |M_t| < \infty,\ e < 2^l\Big] = \mathbb{P}_x\Big[e < 2^l\Big]$$

holds for all $l \in \mathbb{N}$. The proof is complete. □

We shall omit the proof of Lemma 10, because it is the same discussion as the one of Lemma 9.

Lemma 10. *The following equality holds for any $x \in M$:*

$$\mathbb{P}_x\Big[\limsup_{t\nearrow e} |W_t| < \infty,\ e < \infty\Big] = \mathbb{P}_x\Big[e < \infty\Big].$$

Now, we prove Theorem 3.

Proof of Theorem 3. Under Assumptions 1 and 2, we see from Remark 2 and Theorem 1 that the M-valued process $\{X_t;\ 0 \le t < e\}$ is Markovian and irreducible. It is clear that the following equality holds for any $x \in M$:

$$\begin{aligned}
&\mathbb{P}_x[e < \tau < \infty] + \mathbb{P}_x[\tau < e = \infty] + \mathbb{P}_x[\tau = e = \infty] \\
&\quad + \mathbb{P}_x[e = \tau < \infty] + \mathbb{P}_x[e < \tau = \infty] + \mathbb{P}_x[\tau < e < \infty] = 1,
\end{aligned} \tag{4.30}$$

where $\tau = \tau(\delta)$ is introduced in Lemma 8. We shall show that

$$\mathbb{P}_x[e < \tau < \infty] = \mathbb{P}_x[e < \tau = \infty] = \mathbb{P}_x[\tau < e < \infty] = 0.$$

It is clear that $\mathbb{P}_x[e = \tau < \infty] = 0$. Since $\mathbb{P}_{\partial_M}[\tau < \infty] = 0$, we see that

$$\mathbb{P}_x[e < \tau < \infty] = 0 \tag{4.31}$$

holds for all $x \in M$. By Lemma 8, it holds that

$$\begin{aligned}(4.32)\qquad &\mathbb{P}_x[e<\tau=\infty]\\ &=\mathbb{P}_x\Big[r(X_t)\le r(x)+\sigma W_t+M_t+Vt\ \text{ holds for all } t<e\wedge\tau,\ e<\tau=\infty\Big].\end{aligned}$$

Let us show that the right hand side of (4.32) is 0. From the definition of e,

$$\mathbb{P}_x\Big[\lim_{t\nearrow e} r(X_t)=\infty\Big]=1 \tag{4.33}$$

holds for any $x\in M$. On the other hand, Lemmas 9 and 10 tell us that

$$\mathbb{P}_x\Big[\limsup_{t\nearrow e}\Big(r(x)+\sigma W_t+M_t+Vt\Big)<\infty,\ e<\infty\Big]=\mathbb{P}_x\Big[e<\infty\Big] \tag{4.34}$$

holds for any $x\in M$. Hence, by (4.33) and (4.34), we obtain

$$\mathbb{P}_x\Big[r(X_t)\le r(x)+\sigma W_t+M_t+Vt\ \text{ holds for all } t<e,\ e<\infty\Big]=0. \tag{4.35}$$

Applying (4.35) to (4.32) enables us to see that

$$\mathbb{P}_x[e<\tau=\infty]=0 \tag{4.36}$$

holds for all $x\in M$. Next, we shall show that

$$\mathbb{P}_x[\tau<e<\infty]=0$$

holds for all $x\in M$. From the Markov property, it holds that

$$\mathbb{P}_x[\tau<e<\infty]=\mathbb{E}_x\Big[\mathbb{P}_{X_\tau}[e<\infty]1_{\{\tau<e<\infty\}}\Big].$$

Hence, we see that

$$\mathbb{E}_x\Big[\Big(1-\mathbb{P}_{X_\tau}[e<\infty]\Big)1_{\{\tau<e<\infty\}}\Big]=0 \tag{4.37}$$

holds for any $x\in M$, and that (4.37) implies

$$\mathbb{P}_x\Big[\Big(1-\mathbb{P}_{X_\tau}[e<\infty]\Big)1_{\{\tau<e<\infty\}}=0\Big]=1. \tag{4.38}$$

If there exists $x\in M$ such that $\mathbb{P}_x[\tau<e<\infty]>0$, then, by (4.38), there exists $x_0\in M$ such that

$$\mathbb{P}_{x_0}[e<\infty]=1.$$

From Lemma 6,

$$\mathbb{P}_x[e<\infty]=1 \tag{4.39}$$

holds for all $x\in M$. Therefore, (4.30), (4.31), (4.36) and (4.39) enable us to see that

$$\mathbb{P}_x[\tau<e<\infty]=1$$

holds for all $x\in M$. Thus, we can verify that

$$\mathbb{P}_x[\tau<\infty]=1 \tag{4.40}$$

holds for any $x\in M$. However, since (4.40) holds for all $x\in M$, we see that the geodesic ball $B(o,2\delta)$ is recurrent on the M-valued process $\{X_t;\ 0\le t<e\}$. Moreover, since (4.39) holds for all $x\in M$, we see by Remark 4 that the M-valued process $\{X_t;\ 0\le t<e\}$ is transient. These conclusions contradict our claim in Theorem 1 that the M-valued process

$\{X_t;\ 0 \le t < e\}$ is irreducible. So, it follows that

$$\mathbb{P}_x[\tau < e < \infty] = 0 \tag{4.41}$$

for all $x \in M$. Thus, from (4.30), (4.31), (4.36), (4.39) and (4.41), we conclude that

$$\mathbb{P}_x[e = \infty] = 1.$$

□

Finally, we shall discuss the relationship between Theorem 3 and Masamune-Uemura-Wang [15] in the following remark.

Remark 6. If there exist suitable conditions on the Lévy measure $\nu(dz)$ and the manifold M such that M-valued Markov process $\{X_t,\ 0 \le t < e\}$ is symmetric with respect to the Riemannian volume measure $\mathrm{Vol}(dy)$, then the condition

$$\liminf_{r\to\infty} \frac{\log \mathrm{Vol}(B(o,r))}{r\log r} < \infty \tag{4.42}$$

implies that $\{X_t;\ 0 \le t < e\}$ is conservative. See [15, Theorem 1.1]. In fact, Assumption 5 implies (4.42). We shall prove this fact. The volume of the $m-1$ dimensional unit sphere on Euclidean space is denoted by $\mathrm{Vol}(S^{m-1}(1))$. If the condition of Assumption 5 holds, then from the Bishop-Gromov inequality (cf. Sakai [14, Chapter IV, Corollary 3.2, and Chapter IV, Theorem 3.3]), the function

$$[0,\infty) \ni r \to \frac{\mathrm{Vol}(B(o,r))}{u(r)}$$

is monotone decreasing, where $u(r)$ is defined by

$$u(r) = \mathrm{Vol}(S^{m-1}(1)) \int_0^r \Big(\frac{\sinh(\sqrt{|\alpha|}t)}{\sqrt{|\alpha|}}\Big)^{m-1} dt.$$

Since $\sinh(\sqrt{|\alpha|}t) \le e^{\sqrt{|\alpha|}t}$ holds for any $t \ge 0$, we have

$$u(r) \le \mathrm{Vol}(S^{m-1}(1)) \frac{e^{(m-1)\sqrt{|\alpha|}r}}{(m-1)|\alpha|^{m/2}}.$$

Now, let us define

$$I(o,\alpha) = \frac{\mathrm{Vol}(B(o,1))}{u(1)(m-1)|\alpha|^{m/2}} \mathrm{Vol}(S^{m-1}(1)),$$

then we obtain

$$\mathrm{Vol}(B(o,r)) \le I(o,\alpha) e^{(m-1)\sqrt{|\alpha|}r},$$

which implies

$$\frac{\log \mathrm{Vol}(B(o,r))}{r\log r} \le \frac{\log I(o,\alpha) + (m-1)\sqrt{|\alpha|}r}{r\log r} \xrightarrow[r\to\infty]{} 0.$$

Therefore, the symmetric Markov process $\{X_t;\ 0 \le t < e\}$ is conservative. From this discussion, Masamune-Uemura-Wang's work [15] could be extended to the non-symmetric case if it is confirmed that the sectional curvature is pinched by negative constants.

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