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DISCRETE APPROXIMATION TO BROWNIAN MOTION WITH DARNING

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Abstract

Brownian motion with darning (BMD in abbreviation) is introduced and studied in [5] and [6, Chapter 7]. Roughly speaking, BMD travels across the "darning area" at infinite speed, while it behaves like a regular BM outside of this area. In this paper we show that starting from a single point in its state space, BMD is the weak limit of a family of continuous-time simple random walks on square lattices with diminishing mesh sizes. From any vertex in their state spaces, the approximating random walks jump to its nearest neighbors with equal probability after an exponential holding time.

1. Introduction

Brownian motion with darning has been introduced and discussed in [5] and [6, Chapter 7]. Its definition can be found in, e.g., [5, Definition 1.1]. In this paper, let $K \subset \mathbb{R}^d$ be a compact connected subset with Lipschitz-continuous boundary. At every $x \in \partial K$, K satisfies the "cone condition" (see, e.g., [8, Proposition 1.22]), it is thus clear that every point on ∂K is regular for K in the sense that $\mathbb{P}^x[\sigma_K = 0] = 1$. This allows us to define BMD by identifying K as a singleton a^* and equipping $E := (\mathbb{R}^d \backslash K) \cup \{a^*\}$ with the topology induced from $\mathbb{R}^d$ (see, e.g., [5, §1.1]). In other words, the distribution of the process on $\mathbb{R}^d \backslash K$ is the same as regular Brownian motion on $\mathbb{R}^d$, but the "darning area" K offers zero resistance to the process. Diffusion processes with darning can be nicely characterized via Dirichlet forms and have been studied with depth in recent literatures, for example, [5, 6, 7]. In particular, we equip E with a measure m which is the same as the Lebesgue measure on $\mathbb{R}^d \backslash K$, and does not charge a^*, Then the BMD on E described above can be characterized by the following Dirichlet form on $L^2(E, m(dx))$:

$$
(1.1) \qquad \begin{cases} \mathcal{D}(\mathcal{E}) = \left\{ f : f \in W^{1,2}(\mathbb{R}^d \backslash K),\ f \text{ is continuous on } E \right\} \\ \mathcal{E}(f, g) = \dfrac{1}{2} \displaystyle\int_E \nabla f(x) \cdot \nabla g(x) m(dx). \end{cases}
$$

In the classic work [13], the authors studied Markov chain approximation to a wide class of diffusions corresponding to divergence form operators. The approximating Markov chains live on square lattices $\alpha\mathbb{Z}^d$ with mesh-size α tending to zero. However, in that article, the distribution of the approximating Markov chains was only given in terms of the transition density functions of the limiting diffusion process. In other words, without knowing the exact distribution of the limiting diffusion process, it is unclear what the explicit distribution

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of the approximating Markov chains is.

More recently, it was studied in [11, 12] how BMD can be approximated by continuous-time random walks on square lattices. The method used in [11] was adapted from [3], in which the authors showed that reflected Brownian motions in bounded domains can be approximated by both continuous-time and discrete-time simple random walks, and the transition functions of the approximating random walks were given explicitly. The method in [11, 3], however, only works for bounded domains, and the limiting continuous process has to start with its invariant measure as the initial distribution. [12] adopted a different approach, which established the C-tightness of the approximating random walks by proving some sort of "equi-continuity" for their transition density functions through heat kernel estimates. This method allows the discrete approximation to take place on an unbounded state space, and it allows the limiting continuous process to start from an arbitrary single point. However, the discussion in [11, 12] was only limited to a toy model of "Brownian motion with varying dimension". Roughly speaking, the state space of this "toy model" has to be $\mathbb{R}^2 \cup \mathbb{R}_+$, and the "darning point" results from identifying a disc on $\mathbb{R}^2$. This is a very special case in the sense that (a) the dimension of the state space is low; (b) a disc is a symmetric convex domain with C^∞-smooth boundary. This motivates us to ask whether we can establish such discrete approximations to Brownian motion with darning in the general case. In this paper, using the method in [12], we describe how BMD on $\mathbb{R}^d$ with a darning area K satisfying Lipschitz boundary condition can be approximated by random walks on square lattices. The results in this paper provide an intuition for the behavior of BMD upon hitting the "darning point", and how it is affected by the geometric properties (or intuitively, the "shape") of the boundary of the darning area.

Since in the state space E, $K \subset \mathbb{R}^d$ is identified with a singleton a^* with zero diameter, we equip E with the geodesic distance ρ. Namely, for $x, y \in E$, $\rho(x, y)$ is the shortest geodesic path distance (induced from the Euclidean space) in E between x and y. For notation simplicity, we write $|x|_\rho$ for $\rho(x, a^*)$, which equals the shortest Euclidean distance between x and K in $\mathbb{R}^d$. We use $|\cdot|$ to denote the usual Euclidean norm.

We now introduce the state spaces of the approximating random walks. For every $j \in \mathbb{N}$, let $K_j := K \cap 2^{-j}\mathbb{Z}^d$. We identify all the vertices of $2^{-j}\mathbb{Z}^d$ that are contained in the compact set K as a singleton a^*_j. Let $E^j := (2^{-j}\mathbb{Z}^d \cap (\mathbb{R}^d \backslash K)) \cup \{a^*_j\}$.

Recall that in general, a graph G can be written as "$G = \{G_v, G_e\}$", where G_v is its collection of vertices, and G_e is its connection of edges. Given any two vertices in $a, b \in G$, if there is unoriented edge with endpoints a and b, we say a and b are adjacent to each other in G, written " $a \leftrightarrow b$ in G". One can always assume that given two vertices a, b on a graph, there is at most one such unoriented edge connecting these two points (otherwise edges with same endpoints can be removed and replaced with one single edge). This unoriented edge is denoted by e_{ab} or e_{ba} (e_{ab} and e_{ba} are viewed as the same elelment in G_e). In this paper, for notational convenience, we denote by $\mathcal{G}_j := \{2^{-j}\mathbb{Z}^d, \mathcal{V}_j\}$, where $\mathcal{V}_j$ is the collection of the edges of $2^{-j}\mathbb{Z}^d$.

Next we introduce the graph structure on E^j. Denote by $G^j = \{G^j_v, G^j_e\}$ the graph where $G^j_v = E^j$ is the collection of vertices and G^j_e is the collection of unoriented edges over E^j defined as follows:

$$G_e^j := \{e_{xy} : \exists\, x, y \in 2^{-j}\mathbb{Z}^d \cap (\mathbb{R}^d \backslash K), |x-y| = 2^{-j},\ e_{xy} \in \mathcal{V}_j,\ e_{xy} \cap K = \emptyset\}$$

$$\cup\{e_{xa_j^*} : x \in 2^{-j}\mathbb{Z}^d \cap (\mathbb{R}^d \backslash K),\ \exists \text{ at least one edge } e_{xy} \in \mathcal{V}_j \text{ such that } e_{xy} \cap K \neq \emptyset\}. \tag{1.2}$$

Note that $G^j = \{G_v^j, G_e^j\}$ is a connected graph. We emphasize that given any $x \in G_v^j$, $x \neq a_j^*$, there is at most one unoriented edge in G_e^j connecting x and a_j^*. Denote by $v_j(x) = \#\{e_{xy} \in G_e^j\}$, i.e., the number of vertices in G_v^j adjacent to x.

In order to give definition to the approximating random walks for BMD, for every $j \geq 1$, we equip E^j with the measure:

$$m_j(x) := \frac{2^{-jd}}{2d} \cdot v_j(x), \quad x \in E^j. \tag{1.3}$$

Consider the following Dirichlet form on $L^2(E^j, m_j)$:

$$\begin{cases} \mathcal{D}(\mathcal{E}^j) = L^2(E^j, m_j) \\ \mathcal{E}^j(f,f) = \dfrac{2^{-(d-2)j}}{4d} \displaystyle\sum_{e_{xy}^o:\, e_{xy} \in G_e^j} (f(x) - f(y))^2, \end{cases} \tag{1.4}$$

where e_{xy}^o denotes an *oriented edge* from x to y. In other words, given any pair of adjacent vertices $x, y \in G_v^j$, the edge with endpoints x and y is represented twice in the sum: e_{xy}^o and e_{yx}^o. One can verify that $(\mathcal{E}^j, \mathcal{D}(\mathcal{E}^j))$ is a regular symmetric Dirichlet form on $L^2(E^j, m_j)$, therefore there is symmetric strong Markov process associated with it. We denote this process by X^j. In §2, we show that each X^j is a continuous-time random walk whose tragectories of X^j stay at each vertex of E^j for an exponentially distributed holding time with parameter 2^{-2j} before jumping to one of its neighbors with equal probability. The main result of this paper is Theoerem 4.13, which states that starting from a single point, the distributions of $\{X^j,\ j \geq 1\}$ converge weakly to the BMD characterized by (1.1).

The rest of this paper is organized as follows. In §2, we first describe the behavior of X^j by showing their roadmaps. Then we give a brief review on isoperimetric inequalities for weighted graphs, especially the isoperimetric inequalities for $\mathbb{Z}^d$ equipped with equal weights. Using these results, in Proposition 2.7 we prove an isoperimetric inequality for X^j. With the isoperimetric inequality obtained in §2, in §3 we derive a Nash-type inequality for the family of random walks $\{X^j,\ j \geq 1\}$, from which we establish heat kernel upper bounds, first on-diagonal then off-diagonal, for the entire family of $\{X^j,\ j \geq 1\}$. In §4, we use the well-known criterion of tightness presented in [9, Chapter VI, Proposition 3.21] to prove the tightness of $\{X^j,\ j \geq 1\}$. The tightness criterion is verified in Propositions 4.7-4.8, which are proved using the heat kernel upper bounds obtained in §3. Finally, the main result of convergence is given by Theorem 4.13.

In this paper we follow the convention that in the statements of the theorems or propositions, the capital letters $C_1, C_2, \cdots$ or $N_1, N_2, \cdots$ denote positive constants or positive integers, whereas in their proofs, the lower letters $c, c_1, \cdots$ or $n_1, n_2, \cdots$ denote positive constants or positive integers whose exact value is unimportant and may change from line to line.

2. Preliminaries

2.1. Roadmap of the approximating random walks. Suppose E is a locally compact separable metric space and $\{\mathsf{Q}(x,dy)\}$ is a probability kernel on $(\mathsf{E},\mathcal{B}(\mathsf{E}))$ with $\mathsf{Q}(x,\{x\})=0$ for every $x\in\mathsf{E}$. Let $\lambda=\lambda(x)>0,\ x\in\mathsf{E}$ be a positive function, we can construct a pure jump Markov process X as follows: Starting from $x_0\in\mathsf{E}$, X remains at x_0 for an exponentially distributed holding time T_1 with parameter $\lambda(x_0)$ (i.e., $\mathbb{E}[T_1]=1/\lambda(x_0)$), then it jumps to some $x_1\in\mathsf{E}$ according to distribution $\mathsf{Q}(x_0,dy)$; it remains at x_1 for another exponentially distributed holding time T_2 also with parameter $\lambda(x_1)$ before jumping to x_2 according to distribution $\mathsf{Q}(x_1,dy)$. T_2 is independent of T_1. X then continues. The probability kernel $\mathsf{Q}(x,dy)$ is called the *roadmap*, i.e., the one-step distribution of X, and the $\lambda(x)$ is its *speed function*. If there is a σ-finite measure m_0 on E with $\mathrm{supp}[\mathsf{m}_0]=\mathsf{E}$ such that

$$\mathsf{Q}(x,dy)\mathsf{m}_0(dx)=\mathsf{Q}(y,dx)\mathsf{m}_0(dy), \tag{2.1}$$

m_0 is called a *symmetrizing measure* of the roadmap Q. The following theorem is a restatement of [6, Theorem 2.2.2].

Theorem 2.1 ([6])**.** *Given a speed function $\lambda>0$. Suppose* (2.1) *holds, then the reversible pure jump process X described above can be characterized by the following Dirichlet form $(\mathfrak{E},\mathfrak{F})$ on $L^2(\mathsf{E},\mathsf{m}(dx))$ where the underlying reference measure is $\mathsf{m}(dx)=\lambda(x)^{-1}\mathsf{m}_0(dx)$ and*

$$\begin{cases}\mathfrak{F}=L^2(\mathsf{E},\ \mathsf{m}(dx)),\\ \mathfrak{E}(f,g)=\dfrac{1}{2}\displaystyle\int_{\mathsf{E}\times\mathsf{E}}(f(x)-f(y))(g(x)-g(y))\mathsf{Q}(x,dy)\mathsf{m}_0(dx).\end{cases} \tag{2.2}$$

With the theorem above, we present the following proposition which states that at every vertex of E^j, X^j holds for an exponential amount of time with mean 2^{-2j} before jumping to each of its nearest neighbors with equal probability.

Proposition 2.2. *For every $j\geq 1$, X^j has constant speed function $\lambda_j=2^{2j}$ and a roadmap*

$$Q_j(x,dy)=\sum_{\substack{z\in E^j\\ z\leftrightarrow x\ in\ G^j}} q_j(x,z)\delta_{\{z\}}(dy),$$

where $q_j(x,y)=v_j(x)^{-1}$, for all $x,y\in E^j$.

Proof. Define a measure $m_j^0(x):=\lambda_j m_j(x)=2^{-(d-2)j}v_j(x)/(2d)$. The conclusion follows immediately from (1.4) and Theorem 2.1. □

2.2. Isoperimetric inequalities for weighted graphs. In this section we summarize some results on isoperimetric inequalities for weighted graphs in [1]. In general, let Γ be a locally finite connected graph, and let the collection of vertices of Γ be denoted by $\mathbb{V}$. If two vertices $x,y\in\mathbb{V}$ are adjacent to each other, then the the unoriented edge connecting x and y is assigned a unique weight $\mu_{xy}>0$. Set $\mu_{xy}=0$ if x and y are not adjacent in Γ. Denote by $\mu:=\{\mu_{xy}:x,y \text{ connected in } \Gamma\}$ the assignment of the weights on all the unoriented edges. (Γ,μ) is called a locally finite connected *weighted graph*. A weighted graph (Γ,μ) can be equipped with the following intrinsic measure ν on $\mathbb{V}$:

(2.3)
$$v(x) := \sum_{y\in\mathbb{V}: y\leftrightarrow x \text{ in } \Gamma} \mu_{xy}, \quad x \in \mathbb{V}.$$

Given two sets of vertices A, B in $\mathbb{V}$, we define

(2.4)
$$\mu_\Gamma(A,B) := \sum_{x\in A}\sum_{y\in B} \mu_{xy}.$$

The following definition of isoperimetric inequality is taken from [1, Definition 3.1].

Definition 2.3. For $\alpha \in [1,\infty)$, we say that a weighted graph (Γ,μ) satisfies α-isoperimetric inequality (I_α) if there exists $C_0 > 0$ such that

$$\frac{\mu_\Gamma(A, \mathbb{V}\backslash A)}{v(A)^{1-1/\alpha}} \geq C_0, \quad \text{for every finite non-empty } A \subset \mathbb{V}.$$

The following proposition follows from the combination of [1, Theorem 3.7, Lemma 3.9, Theorem 3.14] and the proofs therein. It gives the relationship between Nash-type inequalities and isoperimetric inequalities for weighted graphs.

Proposition 2.4 ([1])**.** *Let (Γ,μ) be a locally finite connected weighted graph satisfying α-isoperimetric inequality with constant C_0. Let v be the measure defined in* (2.3)*. Then (Γ,μ) satisfies the following Nash-type inequality:*

$$\frac{1}{2}\sum_{x\in\mathbb{V}}\sum_{y\in\mathbb{V}, y\leftrightarrow x} (f(x)-f(y))^2\mu_{xy} \geq 4^{-(2+\alpha/2)}C_0^2 \|f\|_{L^2(v)}^{2+4/\alpha}\|f\|_{L^1(v)}^{-4/\alpha}, \quad f\in L^1(v)\cap L^2(v)$$

The next proposition follows immediately from [1, Theorem 3.26]. As a notation in [1], given a weighted graph (Γ,μ) with collection of vertices $\mathbb{V}$. We denote the counting measure times 2^{-jd} on $2^{-j}\mathbb{Z}^d$ by μ_j, which can be viewed as the measure "v" in (2.3) corresponding to weighted $2^{-j}\mathbb{Z}^d$ with all edges weighing $2^{-jd}/2d$.

Proposition 2.5 ([1])**.** *For $j\in\mathbb{N}$, let all edges of $2^{-j}\mathbb{Z}^d$ be assigned with a weight $2^{-jd}/2d$. There exists a constant $C_1 > 0$ independent of j such that for any finite subset A of $2^{-j}\mathbb{Z}^d$,*

(2.5)
$$\mu_{2^{-j}\mathbb{Z}^d}(A, 2^{-j}\mathbb{Z}^d\backslash A) \geq C_1\cdot 2^{-j}\mu_j(A)^{(d-1)/d}.$$

Before establishing the isoperimetric inequality for X_j, we need the following proposition which will be used throughout this article.

Proposition 2.6. *There exist $C_2 > 0$ and $N_0\in\mathbb{N}$ only depending on the darning region K such that for all $j \geq N_0$,*

(2.6)
$$v_j(a_j^*) \leq C_2\cdot 2^{j(d-1)}.$$

Proof. In the following we denote the d- and $(d-1)$-dimensional Lebesgue measures by $m^{(d)}$ and $m^{(d-1)}$, respectively. For any two distinct $x, y\in E^j$ both adjacent to a_j^*, the two Euclidean balls $B_{|\cdot|}(x, \frac{2^{-j}}{2})$ and $B_{|\cdot|}(y, \frac{2^{-j}}{2})$ are disjoint. Also, for any $x\leftrightarrow a_j^*$, the Euclidean ball $B_{|\cdot|}(x, \frac{2^{-j}}{2})$ must be contained in the set $\{x\in\mathbb{R}^d : d_{|\cdot|}(x,\partial K) \leq 2\cdot 2^{-j}\}$. Therefore,

(2.7)
$$v_j(a_j^*)\cdot m^{(d)}\left(B_{|\cdot|}\left(x, \frac{2^{-j}}{2}\right)\right) \leq m^{(d)}\left(\left\{x\in\mathbb{R}^d : d_{|\cdot|}(x,\partial K)\leq 2\cdot 2^{-j}\right\}\right).$$

Since K has Lipschitz-continuous boundary in $\mathbb{R}^d$, ∂K is $(d-1)$-dimensional in the sense of both topological and Minkowski box dimension. This means that there exists $j_0 \in \mathbb{N}$ and a constant $c > 0$ such that for all $j \geq j_0$, ∂K can be covered by $c \cdot 2^{j(d-1)}$ many boxes with side lenght 2^{-j}. This further implies that set $\{x \in \mathbb{R}^d : d_{|\cdot|}(x, \partial K) \leq 2 \cdot 2^{-j}\}$ can be covered by boxes with the same centers but side length $16 \cdot 2^{-j}$, which further implies that for some $c > 0$,

$$m^{(d)}\left(\left\{x \in \mathbb{R}^d : d_{|\cdot|}(x, \partial K) \leq 2 \cdot 2^{-j}\right\}\right) \leq c \cdot 2^{j(d-1)} \cdot 2^{-jd} = c \cdot 2^{-j}, \quad j \geq j_0.$$

The conclusion thus follows on account of (2.7) and the fact that $m^{(d)}(B_{|\cdot|}(x, 2^{-j}/2)) = s(d-1) \cdot 2^{-(j+1)d}$ for all $x \in \mathbb{R}^d$, where $s(d-1) > 0$ is a constant equal to the $(d-1)$-dimensional surface measure. □

Recall the graph structure on E^j defined in (1.2). In the next proposition we establish an isoperimetric inequality for X^j on the weighted graph E^j, where all the edges in G^j_e are equipped with an equal weight of $2^{-jd}/(2d)$.

Proposition 2.7. *For every $j \in \mathbb{N}$, let all edges of E^j be equipped with an equal weight $2^{-j}/(2d)$, which is consistent with the definition of m_j in the sense that*

$$m_j(x) = \frac{2^{-jd}}{2d} \cdot \#\left\{e_{xy} \in G^j\right\}.$$

For the N_0 specified in Proposition 2.6, there exist an integer $N_1 \geq N_0$ and a constant $C_2 > 0$ independent of j such that for all $j \geq N_1$,

$$\mu_{E^j}(A, E^j \backslash A) \geq 2^{-j} C_2 m_j(A)^{(d-1)/d}, \quad \textit{for any finite set } A \subset E^j. \tag{2.8}$$

Proof. Let A be any finite subset of E^j. Recall that in Section 1 we set $K_j := 2^{-j}\mathbb{Z}^d \cap K$ and $\mathcal{G}_j := \{2^{-j}\mathbb{Z}^d, \mathcal{V}_j\}$, where $\mathcal{V}_j$ is the collection of the edges of $2^{-j}\mathbb{Z}^d$. Also recall that we use "e_{xy}" to denote an edge connecting x and y, including these two endpoints. In the following we establish (2.8) by dividing our discussion into two cases depending on whether a^*_j is in A or not.
Case (i). $a^*_j \notin A$. Thus $a^*_j \in E^j \backslash A$ and $A \subset 2^{-j}\mathbb{Z}^d$. In view of the definition of the graph structure G^j in (1.2),

$$\begin{aligned}
\mu_{E^j}(A,\ E^j \backslash A) \quad &= \quad \frac{2^{-jd}}{2d} \sum_{x \in A} \#\left\{y \in E^j \backslash A : y \leftrightarrow x \text{ in } G^j\right\} \\
&= \quad \frac{2^{-jd}}{2d} \sum_{x \in A} \#\left\{y \in \left(E^j \backslash \left(A \cup \{a^*_j\}\right)\right) : y \leftrightarrow x\right\} + \frac{2^{-jd}}{2d} \#\left\{x \in A : x \leftrightarrow a^*_j\right\} \\
&= \quad \frac{2^{-jd}}{2d} \sum_{x \in A} \#\left\{y \in (2^{-j}\mathbb{Z}^d \backslash A) : y \leftrightarrow x \text{ in } 2^{-j}\mathbb{Z}^d,\ e_{xy} \cap K = \emptyset\right\} \\
&+ \quad \frac{2^{-jd}}{2d} \#\left\{x \in A : \exists\, e_{xy} \in \mathcal{V}_j \text{ such that } e_{xy} \cap K \neq \emptyset\right\} \\
&\geq \quad \frac{2^{-jd}}{2d} \sum_{x \in A} \#\left\{y \in (2^{-j}\mathbb{Z}^d \backslash A) :\ \exists\, e_{xy} \in \mathcal{V}_j \text{ such that } e_{xy} \cap K = \emptyset\right\} \\
&+ \quad \frac{2^{-jd}}{2d} \cdot \frac{1}{2d} \sum_{x \in A} \#\left\{y \in 2^{-j}\mathbb{Z}^d :\ \exists\, e_{xy} \in \mathcal{V}_j \text{ such that } e_{xy} \cap K \neq \emptyset\right\}
\end{aligned}$$

$$\geq \quad \frac{2^{-jd}}{4d^2}\sum_{x\in A}\#\left\{y\in 2^{-j}\mathbb{Z}^d\backslash A:\ y\leftrightarrow x \text{ in } 2^{-j}\mathbb{Z}^d\right\},$$

where the first inequality above is due to the fact that for every $x \in A$ such that there is at least one edge in $\mathcal{V}_j$ with an endpoint x intersecting K, there are at most $2d$ many such egdes with different other endpoint y. Now in view of (2.5) and the definition of μ_j and $\mu_{2^{-j}\mathbb{Z}^d}$ earlier in this section, we have

$$\begin{aligned}
(2.9)\qquad & \mu_{E^j}(A,\ E^j\backslash A)\\
\geq\ & \frac{2^{-jd}}{4d^2}\sum_{x\in A}\#\left\{y\in 2^{-j}\mathbb{Z}^d\backslash A:\ y\leftrightarrow x \text{ in } 2^{-j}\mathbb{Z}^d\right\}\\
\geq\ & \frac{1}{2d}\cdot\mu_{2^{-j}\mathbb{Z}^d}(A,2^{-j}\mathbb{Z}^d)\overset{(2.5)}{\geq}\frac{C_1}{2d}\cdot 2^{-j}\mu_j(A)^{(d-1)/d}=\frac{C_1}{2d}\cdot 2^{-j}m_j(A)^{(d-1)/d},
\end{aligned}$$

which the C_1 above is the same as in (2.5). This establishes the desired inequality for the current case. Before continuing to the other case, we note that by the definition of Lipschitz-continuity, $\mu_j(K_j)$ is bounded from below by a positive constant for sufficiently large j. Thus there exists an integer $j_1 \geq N_0$ and a constant $c_1 > 0$ such that

$$(2.10)\qquad \mu_j(K_j)\geq c_1,\quad \text{for all } j\geq j_1.$$

Now in view of Proposition 2.6, there exists an integer $j_2 \geq N_0$ such that

$$(2.11)\qquad \mu_j(K_j)\geq c_1\geq \frac{2^{-jd}}{2d}\cdot C_2\cdot 2^{j(d-1)}\geq m_j(a_j^*),\quad \text{for all } j\geq j_2.$$

Case (ii). $a_j^* \in A$. In this case $E^j\backslash A = 2^{-j}\mathbb{Z}^d\backslash A$. Recall that we let $K_j = K\cap 2^{-j}\mathbb{Z}^d$. Thus for all $j \geq j_2$ given in (2.11),

$$\begin{aligned}
(2.12)\qquad & \mu_{E^j}(A,\ E^j\backslash A)\\
=\ & \frac{2^{-jd}}{2d}\sum_{x\in A}\#\left\{y\in E^j\backslash A: y\leftrightarrow x \text{ in } G^j\right\}\\
=\ & \frac{2^{-jd}}{2d}\sum_{\substack{x\in A\\ x\neq a_j^*}}\#\left\{y\in E^j\backslash A: y\leftrightarrow x \text{ in } G^j\right\}+\frac{2^{-jd}}{2d}\#\left\{y\in E^j\backslash A: y\leftrightarrow a_j^*\right\}\\
\geq\ & \frac{2^{-jd}}{2d}\sum_{\substack{x\in A\\ x\neq a_j^*}}\#\left\{y\in (2^{-j}\mathbb{Z}^d\backslash A):\ \exists\, e_{xy}\in\mathcal{V}_j \text{ s.t. } e_{xy}\cap K=\emptyset\right\}\\
+\ & \frac{2^{-jd}}{4d^2}\sum_{x\in 2^{-j}\mathbb{Z}^d}\#\left\{y\in (2^{-j}\mathbb{Z}^d\backslash A):\ \exists\, e_{xy}\in\mathcal{V}_j \text{ s.t. } e_{xy}\cap K\neq\emptyset\right\}\\
\geq\ & \frac{2^{-jd}}{2d}\sum_{x\in A\backslash\{a_j^*\}}\#\left\{y\in (2^{-j}\mathbb{Z}^d\backslash A):\ \exists\, e_{xy}\in\mathcal{V}_j \text{ s.t. } e_{xy}\cap K=\emptyset\right\}\\
+\ & \frac{2^{-jd}}{4d^2}\sum_{x\in A\backslash\{a_j^*\}}\#\left\{y\in (2^{-j}\mathbb{Z}^d\backslash A):\ \exists\, e_{xy}\in\mathcal{V}_j \text{ s.t. } e_{xy}\cap K\neq\emptyset\right\}\\
+\ & \frac{2^{-jd}}{4d^2}\sum_{x\in K_j}\#\left\{y\in (2^{-j}\mathbb{Z}^d\backslash A):\ \exists\, e_{xy}\in\mathcal{V}_j\right\}
\end{aligned}$$

$$\geq \frac{2^{-jd}}{4d^2} \sum_{x\in(A\backslash\{a_j^*\})\cup K_j} \#\left\{y\in 2^{-j}\mathbb{Z}^d\backslash A:\ y\leftrightarrow x \text{ in } 2^{-j}\mathbb{Z}^d\right\}$$

$$= \frac{1}{2d}\cdot \mu_{2^{-j}\mathbb{Z}^d}\left((A\backslash\{a_j^*\})\cup K_j,\ 2^{-j}\mathbb{Z}^d\backslash A\right)$$

$$\geq \frac{C_1}{2d}\cdot \mu_j\left((A\backslash\{a_j^*\})\cup K_j\right)^{(d-1)/d} \overset{(2.11)}{\geq} \frac{C_1}{2d}\cdot m_j\left((A\backslash\{a_j^*\})\cup K_j\right)^{(d-1)/d}, \tag{2.5}$$

where the first inequality is due to the fact that for every $y \in 2^{-j}\mathbb{Z}^d\backslash A$ such that there is at least one edge in $\mathcal{V}_j$ with an endpoint y intersecting K, there are at most $2d$ many such edges in $\mathcal{V}_j$ with differenet other endpoint x. The proof is complete in view of (2.9) and (2.12). □

3. Nash-type inequality and heat kernel upper bound for random walks on lattices with darning

In this section, using the isoperimetric inequality obtained in Proposition 2.7, we establish first a Nash-type inequality and then heat kernel upper bound for X_j.

Proposition 3.1. *For every $j \in \mathbb{N}$, let $(P_t^j)_{t\geq 0}$ be the transition semigroup of X^j with respect to m_j. There exists a constant $C_3 > 0$ independent of j such that for all $j \geq N_1$ specified in* Proposition 2.7*,*

$$\|P_t^j\|_{1\to\infty} \leq \frac{C_3}{t^{d/2}}, \qquad \forall t\in(0,+\infty]. \tag{3.1}$$

Proof. It follows from (2.8) and Proposition 2.4 that for all $f \in L^1(E^j, m_j)\cap L^2(E^j, m_j)$, it holds

$$\frac{1}{2}\sum_{x\in E^j}\sum_{y\in E^j, y\leftrightarrow x} (f(x)-f(y))^2\,\frac{2^{-jd}}{2d} \geq C_2^2\cdot 2^{-2j}\cdot 4^{-2-\frac{2}{d}}\|f\|_{L^2(m_j)}^{2+\frac{4}{d}}\|f\|_{L^1(m_j)}^{-\frac{4}{d}}.$$

In view of the definition of $\mathcal{E}^j$, this implies that

$$\mathcal{E}^j(f,f) \geq C_2^2\cdot 4^{-2-\frac{2}{d}}\cdot \|f\|_{L^2(m_j)}^{2+4/d}\|f\|_{L^1(m_j)}^{-4/d}, \qquad f\in L^1(E^j,m_j)\cap L^2(E^j,m_j). \tag{3.2}$$

The conclusion now follows from [4, Theorem 2.9]. □

Now for every $j \in \mathbb{N}$, we define a metric $d_j(\cdot,\cdot)$ on E^j as follows:

$$d_j(x,y) := 2^{-j}\times \text{smallest number of edges between } x \text{ and } y \text{ in } G^j. \tag{3.3}$$

With the above on-diagonal heat kernel estimate, using the standard Davies's method, we next derive an off-diagonal heat kernel upper bound estimate for X^j. Since there is a Nash-type inequality holds for each X^j, the family of transition density function of $(P_t^j)_{t\geq 0}$ with respect to m_j exists for every $j \in \mathbb{N}$. We denote this by $\{p_j(t,x,y),\ t>0,\ x,y\in E^j\}$.

Proposition 3.2. *For every $j \geq 1$, fix a sequence of $\{\alpha_j\}_{j\geq 1}$ satisfying $\alpha_j \leq 2^{j-1}$. There exists $C_4 > 0$ independent of j, such that for all $j \geq N_1$ specified in* Proposition 2.7*,*

$$p_j(t,x,y) \leq \frac{C_4}{t^{d/2}}\exp\left(-\alpha_j d_j(x,y) + 4t\alpha_j^2\right), \qquad 0<t<\infty,\ x,y\in E^j. \tag{3.4}$$

Proof. We prove this result using [4, Corollary 3.28]. For each j, we set

$$\widehat{\mathcal{F}}^j := \{h + c : \ h \in \mathcal{D}(\mathcal{E}^j),\ h \text{ bounded, and } c \in \mathbb{R}\}.$$

It is known that the regular symmetric Dirichlet form $(\mathcal{E}^j, \mathcal{D}(\mathcal{E}^j))$ is associated with the following energy measure Γ^j:

$$\mathcal{E}^j(u,u) = \int_{E^j} \Gamma^j(u,u)(dx), \quad u \in \widehat{\mathcal{F}}^j.$$

Now we define $\widehat{\mathcal{F}}^j_\infty$ as a subset of $\psi \in \widehat{\mathcal{F}}^j$ satisfying the following conditions:

(i) Both $e^{-2\psi}\Gamma^j(e^{\psi}, e^{\psi})$ and $e^{2\psi}\Gamma^j(e^{-\psi}, e^{-\psi})$ as measures are absolutely continuous with respect to m_j on E^j.

(ii) Furthermore,

$$\Gamma^j(\psi) := \left(\left\| \frac{de^{-2\psi}\ \Gamma^j(e^{\psi}, e^{\psi})}{dm_j} \right\|_\infty \vee \left\| \frac{de^{2\psi}\ \Gamma^k(e^{-\psi}, e^{-\psi})}{dm_j} \right\|_\infty \right)^{1/2} < \infty. \tag{3.5}$$

For a fixed constant $\alpha_j \le 2^{j-1}$, we denote by

$$\psi_{j,n}(x) := \alpha_j \cdot \left(d_j(x, a_j^*) \wedge n \right). \tag{3.6}$$

In order to apply [4, Corollary 3.28], we need to verify that $\psi_{j,n} \in \widehat{\mathcal{F}}^j_\infty$ for every n. Notice that $\psi_{j,n}$ is a constant outside of a bounded domain of E^j, therefore it is in $\widehat{\mathcal{F}}^j$. We first note that

$$|1 - e^x| \le |2x|, \quad \text{for } -\frac{1}{2} < x < \frac{1}{2}. \tag{3.7}$$

We now first verify conditions (i) and (ii) above for the function $\psi_{j,n}$. Viewing $e^{-2\psi_{j,n}}\Gamma^j(e^{\psi_{j,n}}, e^{\psi_{j,n}})$ as a measure on E^j, given any subset $A \subset E^j$, we have

$$\begin{aligned}
& e^{-2\psi_{j,n}}\Gamma^j(e^{\psi_{j,n}}, e^{\psi_{j,n}})(A) \\
= \ & \frac{2^{-(d-2)j}}{4d} \sum_{x \in E^j \cap A} e^{-2\psi_{j,n}(x)} \left[\sum_{y \leftrightarrow x \text{ in } E^j} \left(e^{\psi_{j,n}(y)} - e^{\psi_{j,n}(x)} \right)^2 \right] \\
\le \ & \frac{2^{-(d-2)j}}{4d} \sum_{x \in E^j \cap A} \sum_{y \leftrightarrow x \text{ in } E^j} \left[\left(1 - e^{\alpha_j \left(d_j(y, a_j^*) \wedge n - d_j(x, a_j^*) \wedge n \right)} \right)^2 \right] \\
= \ & \frac{2^{-(d-2)j}}{4d} \sum_{\substack{x \in E^j \cap A \\ x \ne a_j^*}} \sum_{y \leftrightarrow x \text{ in } E^j} \left[\left(1 - e^{\alpha_j \left(d_j(y, a_j^*) \wedge n - d_j(x, a_j^*) \wedge n \right)} \right)^2 \right] \\
+ \ & \frac{2^{-(d-2)j}}{4d} \cdot \mathbf{1}_{\{a_j^* \in A\}} \cdot \sum_{y \leftrightarrow a_j^*} \left[\left(1 - e^{\alpha_j (d_j(y, a_j^*) \wedge n)} \right)^2 \right] \\
\overset{(3.7)}{\le} \ & \frac{2^{-(d-2)j}}{4d} \cdot \# \left\{ x \in E^j \cap A, x \ne a_j^* \right\} \cdot 2d \cdot (2 \cdot \alpha_j \cdot 2^{-j})^2 \\
+ \ & \frac{2^{-(d-2)j}}{4d} \cdot \mathbf{1}_{\{a_j^* \in A\}} \cdot v_j(a_j^*) \cdot (2 \cdot \alpha_j \cdot 2^{-j})^2 \\
\le \ & 2 \cdot 2^{-jd} \cdot \# \left\{ x \in E^j \cap A, x \ne a_j^* \right\} \cdot \alpha_j^2 + \frac{2^{-jd}}{d} \cdot \mathbf{1}_{\{a_j^* \in A\}} \cdot v_j(a_j^*) \cdot \alpha_j^2.
\end{aligned}$$

Recall that $m_j(x) = \frac{2^{-jd}}{2d} \cdot v_j(x)$. We conclude that for some $c > 0$ independent of j, it holds

$$\left\| \frac{de^{-2\psi_{j,n}}\,\Gamma^j(e^{\psi_{j,n}}, e^{\psi_{j,n}})}{dm_j} \right\|_\infty \le \sqrt{2}\alpha_j, \quad \text{for all } j \ge 1. \tag{3.8}$$

Similarly, it can be computed that

$$\begin{aligned}
& e^{2\psi_{j,n}}\Gamma^j(e^{-\psi_{j,n}}, e^{-\psi_{j,n}})(A) \\
\le\ & \frac{2^{-(d-2)j}}{4d} \sum_{\substack{x\in E^j\cap A\\ x\neq a_j^*}} \sum_{y\leftrightarrow x \text{ in } E^j} \left[\left(1 - e^{\alpha_j\left(d_j(x,a_j^*)\wedge n - d_j(y,a_j^*)\wedge n\right)}\right)^2\right] \\
+\ & \frac{2^{-(d-2)j}}{4d} \cdot \mathbf{1}_{\{a_j^*\in A\}} \cdot \sum_{y\leftrightarrow a_j^*} \left[\left(1 - e^{-\alpha_j(d_j(y,a_j^*)\wedge n)}\right)^2\right] \\
\overset{(3.7)}{\le}\ & \frac{2^{-(d-2)j}}{4d} \cdot \#\left\{x \in E^j \cap A, x \neq a_j^*\right\} \cdot (2d) \cdot (2\cdot\alpha_j\cdot 2^{-j})^2 \\
+\ & \frac{2^{-(d-2)j}}{4d} \cdot \mathbf{1}_{\{a_j^*\in A\}} \cdot v_j(a_j^*) \cdot (2\cdot\alpha_j\cdot 2^{-j})^2 \\
\le\ & 2\cdot 2^{-jd} \cdot \#\left\{x \in E^j \cap A, x \neq a_j^*\right\} \cdot \alpha_j^2 + \frac{2^{-jd}}{d} \cdot \mathbf{1}_{\{a_j^*\in A\}} \cdot v_j(a_j^*) \cdot \alpha_j^2.
\end{aligned}$$

Similar to (3.8), this shows that

$$\left\| \frac{de^{2\psi_{j,n}}\,\Gamma^j(e^{\psi_{j,n}}, e^{\psi_{j,n}})}{dm_j} \right\|_\infty \le \sqrt{2}\alpha_j, \quad \text{for all } j \ge 1,$$

and thus (4.7) is verified. The desired conclusion follows immediately from [4, Theorem 3.25, Corollary 3.28]. □

Corollary 3.3. *There exist $C_5 > 0$ independent of j such that for all $j \ge N_1$ specified in* Proposition 2.7*, all $x, y \in E^j$ and all $t \ge 0$, it holds*

$$p_j(t,x,y) \le \begin{cases} \dfrac{C_5}{t^{d/2}} e^{-d_j(x,y)^2/(64t)}, & \text{when } d_j(x,y) \le 16\cdot 2^j t; \\ \dfrac{C_5}{t^{d/2}} e^{-2^j d_j(x,y)/4}, & \text{when } d_j(x,y) \ge 16\cdot 2^j t. \end{cases}$$

In particular, given any $T > 0$, there exists $C_5 > 0$ such that

$$p_j(t,x,y) \le \frac{C_5}{t^{d/2}} \left(e^{-d_j(x,y)^2/(64t)} + e^{-2^j d_j(x,y)/4}\right), \textit{ for all } (t,x,y) \in (0,T] \times E^j \times E^j.$$

Proof. To prove this, in Proposition 3.2, given any $j \ge N_1$, for any fixed $t_0 > 0$ and any pair of $x_0, y_0 \in E^j$, we take

$$\alpha_j := \frac{d_j(x_0,y_0)}{32t_0} \wedge 2^{j-1}.$$

Then Proposition 3.2 yields that for all $t > 0$ and $x, y \in E^j$,

$$p_j(t_0,x,y) \le \frac{c}{t_0^{d/2}} \exp\left[-\left(\frac{d_j(x_0,y_0)}{32t_0} \wedge 2^{j-1}\right) d_j(x,y) + 4t_0\left(\frac{d_j(x_0,y_0)}{16t_0} \wedge 2^{j-1}\right)^2\right].$$

The desired result follows from first taking $x = x_0$ and $y = y_0$, then simplying the right hand

side above by dividing it into two cases: $d_j(x_0, y_0) \geq 32t_0 \cdot 2^{j-1}$ and $d_j(x_0, y_0) \leq 32t_0 \cdot 2^{j-1}$.
□

4. Tightness of the approximating random walks

The next proposition taken from [10] is a well-known criterion for tightness for càdlàg processes. As a standard notation, given a metric $d(\cdot, \cdot)$, we denote by

$$w_d(x, \theta, T) := \inf_{\{t_i\}_{1\le i\le n}\in\Pi} \max_{1\le i\le n} \sup_{s,t\in[t_i,t_{i-1}]} d(x(s), x(t)),$$

where Π is the collection of all possible partitions of the form $0 = t_0 < t_1 < \cdots < t_{n-1} < T \leq t_n$ with $\min_{1\le i\le n}(t_i - t_{i-1}) \geq \theta$ and $n \geq 1$. Recall the definition of the metric ρ equipped on E in the third paragraph of §1.

Proposition 4.1 (Chapter VI, Theorem 3.21 in [10]). *Let $\{Y_k, \mathbb{P}^y\}_{k\geq 1}$ be a a sequence of càdlàg processes on state space E. Given $y \in E$, the laws of $\{Y_k, \mathbb{P}^y\}_{k\geq 1}$ are tight in the Skorokhod space $D([0, T], E, \rho)$ if and only if*

(i). *For any $T > 0$, $\delta > 0$, there exist $K_1 \in \mathbb{N}$ and $M > 0$ such that for all $k \geq K_1$,*

$$\mathbb{P}^y\left[\sup_{t\in[0,T]} \left|Y^k_t\right|_\rho > M\right] < \delta. \tag{4.1}$$

(ii). *For any $T > 0$, $\delta_1, \delta_2 > 0$, there exist $\delta_3 > 0$ and $K_2 > 0$ such that for all $k \geq K_2$,*

$$\mathbb{P}^y\left[w_\rho\left(Y^k, \delta_3, T\right) > \delta_1\right] < \delta_2. \tag{4.2}$$

In this section, we use the heat kernel upper bounds obtained in Corollary 3.3 to verify the two conditions in Proposition 4.1. Since this section is long and technical, we briefly go through the structure of the rest of this section: Condition (i) in Proposition 4.1 is established via Lemmas 4.2 - 4.7. In Lemma 4.2, we break the left hand side of the inequality in condition (i) into the sum of a few terms. Lemmas 4.3 - 4.6 are some delicate computations as preparations for Lemma 4.7. Finally in Lemma 4.7, we consolidate all the computations in Lemmas 4.3 - 4.6 and establish condition (i) using the inequality in Lemma 4.2. Condition (ii) in Proposition 4.1 is verified in Proposition 4.8.

We begin with the following Lemma 4.2 that can be proved in the same manner as [12, Lemma 4.2]. Indeed, this is a standard result due to the strong Markov property of X^j. We skip the proof to it.

Lemma 4.2. *Given any $T, M > 0$, for any sufficiently large $j \in \mathbb{N}$ such that $2^{-j} < T$, it holds for all $x \in E^j$ that*

$$\begin{aligned}\mathbb{P}^x\left[\sup_{t\in[0,T]} |X^j_t|_\rho \geq M\right] \leq \mathbb{P}^x\left[\sup_{t\in[0,8^{-j}]} |X^j_t|_\rho \geq M\right] + \mathbb{P}^x\left[\left|X^j_T\right|_\rho \geq \frac{M}{2}\right]\\ + \mathbb{P}^x\left[T - 8^{-j} \leq \tau_M \leq T, \left|X^j_T\right|_\rho \leq \frac{M}{2}\right]\\ + \mathbb{P}^x\left[8^{-j} \leq \tau_M \leq T - 8^{-j}, \left|X^j_T\right|_\rho \leq \frac{M}{2}\right],\end{aligned}$$

where $\tau_M := \inf\{t > 0 : \ |X^j_t|_\rho \geq M\}$.

The following lemma can be proved in the same manner as that in [12, Proposition 4.3]. Essentially it results from the fact that X^j makes jumps at a rate of 2^{-2j}, for every $j \geq 1$. The proof is also skipped here.

Lemma 4.3. *For any $\delta > 0$, any $T > 0$, there exists $M_1 > 0$ such that for all $j \geq 1$:*

$$\sup_{y \in E^j} \mathbb{P}^y \left[\sup_{t \in [0, 8^{-j}]} \rho\left(X_0^j, X_t^j\right) \geq M_1 \right] < \delta.$$

Before presenting the next few lemmas, given any $r > 0$, we denote the boundary of a "cube" in $\mathbb{R}^d$ centered at the origin with side length $2r$ by

$$S_r := \left\{ (x_1, \dots, x_d) \in \mathbb{R}^d : \max_{1 \leq i \leq d} |x_i| = r \right\}. \tag{4.3}$$

For the remaining of this paper, we fix a starting point $x_0 \in \cap_{j \geq 1} E^j$ and a $k_0 \in \mathbb{N}$, such that both K and x_0 are contained in the set

$$\left\{ (x_1, \dots, x_d) \in \mathbb{R}^d : \max_{1 \leq i \leq d} |x_i| \leq k_0 \right\}. \tag{4.4}$$

By elementary geometry, it can be told that for all $r \geq 2k_0,\ j \geq 1$,

$$\#\left(S_r \cap E^j\right) \leq (2d) \cdot (2r \cdot 2^j)^{d-1}. \tag{4.5}$$

When $k \geq 2k_0$, for all $j \geq 1$, the definitions of d_j and ρ imply that

$$d_j(x_0, y) \geq \rho(x_0, y) \geq \frac{k}{2}, \quad \text{for } y \in S_k \cap E^j. \tag{4.6}$$

Lemma 4.4. *Fix $x_0 \in \bigcap_{j \geq 1} E^j$ and $T > 0$. For any $\delta > 0$, there exists $M_2 > 0$ such that for all $j \geq N_1$ specified in* Proposition 2.7 *such that $8^{-j} < T$:*

$$\sup_{8^{-j} \leq t \leq T} \mathbb{P}^{x_0} \left[d_j(X_t^j, x_0) \geq M_2 \right] < \delta.$$

Proof. We use Proposition 3.3 to prove this. We first note that the sequence of metrics $\{d_j\}_{j \geq 1}$ is non-increasing in j, in particular, given a fixed $x_0 \in \cap_{j \geq 1} E^j$, $\{d_j(a_j^*, x_0)\}_{j \geq 1}$ is an non-increasing sequence of numbers. Therefore, for the k_0 specified in (4.4), one can choose $M > 2k_0$ sufficiently large such that

$$d_j(a_j^*, x_0) \leq M, \quad \text{for all } j \geq 1. \tag{4.7}$$

For M satisfying (4.7), in view of the definition of m_j and the heat kernel upper bound in Corollary 3.3, there exists some $c_1 > 0$ independent of j such that for all $j \geq N_1$,

$$\begin{aligned} &\mathbb{P}^{x_0} \left[d_j(X_t^j, x_0) \geq M \right] \\ \leq &\sum_{d_j(y, x_0) \geq M} \frac{c_1}{t^{d/2}} \left(e^{-\frac{d_j(x_0, y)^2}{64t}} + e^{-\frac{2^j d_j(x_0, y)}{4}} \right) m_j(y). \\ \leq &\sum_{d_j(y, x_0) \geq M} \frac{c_1}{t^{d/2}} e^{-\frac{d_j(x_0, y)^2}{64t}} \cdot 2^{-jd} + \sum_{d_j(y, x_0) \geq M} \frac{c_1}{t^{d/2}} e^{-\frac{2^j d_j(x_0, y)}{4}} \cdot 2^{-jd}. \end{aligned} \tag{4.8}$$

To give an upper bound for each of the two terms on the right hand side above, we first record the following computation for a generic $k \geq 0$:

$$
\begin{aligned}
(4.9)\qquad & \sum_{l=k\cdot 2^j}^{\infty}\ \sum_{y\in E^j\cap S_{2k_0+l\cdot 2^{-j}}} \frac{1}{t^{d/2}}2^{-jd}\cdot e^{-\frac{d_j(x_0,y)^2}{64t}} \\
\overset{(4.6)}{\le}\ & \sum_{l=k\cdot 2^j}^{\infty}\ \sum_{y\in E^j\cap S_{2k_0+l\cdot 2^{-j}}} \frac{1}{t^{d/2}}2^{-jd}\cdot e^{-\frac{(2k_0+l\cdot 2^{-j})^2}{256t}} \\
\overset{(4.5)}{\le}\ & \sum_{l=k\cdot 2^j}^{\infty} \frac{1}{t^{d/2}}2^{-jd}\cdot e^{-\frac{(2k_0+l\cdot 2^{-j})^2}{256t}}\cdot (2d)\cdot (4k_0+2l\cdot 2^{-j})^{d-1}\cdot 2^{j(d-1)} \\
\le\ & \sum_{l=k\cdot 2^j}^{\infty} \frac{2d\cdot 2^{-j}}{t^{d/2}}\cdot (4k_0+2l\cdot 2^{-j})^{d-1}\cdot e^{-\frac{(2k_0+l\cdot 2^{-j})^2}{256t}} \\
\le\ & \sum_{l=k\cdot 2^j}^{\infty} 2d\cdot 2^{-j}\cdot (4k_0+2l\cdot 2^{-j})^{d-1}\cdot e^{-\frac{(2k_0+l\cdot 2^{-j})^2}{512T}}\cdot\left(\sup_{0<t\le T}\frac{1}{t^{d/2}}e^{-\frac{(2k_0)^2}{512t}}\right) \\
\le\ & c_2\sum_{l=k\cdot 2^j}^{\infty} 2^{-j}\cdot (4k_0+2l\cdot 2^{-j})^{d-1}\cdot e^{-\frac{(2k_0+l\cdot 2^{-j})^2}{512T}} \\
\le\ & c_2\sum_{u=k}^{\infty}\ \sum_{l=2^j\cdot u}^{2^j(u+1)-1} 2^{-j}\cdot (4k_0+2l\cdot 2^{-j})^{d-1}\cdot e^{-\frac{(2k_0+l\cdot 2^{-j})^2}{512T}} \\
\le\ & c_2\sum_{u=k}^{\infty}(4k_0+2u+2)^{d-1}e^{-\frac{(2k_0+u)^2}{512T}} \\
=\ & c_2\sum_{u=k}^{\infty}(4k_0+2u+2)^{d-1}e^{-\frac{(2k_0+u)^2}{1024T}}\cdot e^{-\frac{(2k_0+u)^2}{1024T}} \\
\le\ & c_2\left(\sup_{x\ge 2k_0}(2x+2)^{d-1}e^{-\frac{x^2}{1024T}}\right)\sum_{u=k}^{\infty}e^{-\frac{(2k_0+u)^2}{1024T}}\le c_2\sum_{u=k}^{\infty}e^{-\frac{(2k_0+u)^2}{1024T}}.
\end{aligned}
$$

We note that the last display of (4.9) can be made arbitrarily small by selecting sufficiently large k. It now follows that for any $\delta>0$, there exists $k_1\in\mathbb{N}$ such that for all $k\ge k_1$, it holds for the first display in (4.9) that

$$
(4.10)\qquad \sum_{l=k\cdot 2^j}^{\infty}\ \sum_{y\in E^j\cap S_{2k_0+l\cdot 2^{-j}}} \frac{1}{t^{d/2}}2^{-jd}\cdot e^{-\frac{d_j(x_0,y)^2}{64t}}\le c_2\sum_{u=k}^{\infty}e^{-\frac{(2k_0+u)^2}{1024T}}<\delta/2.
$$

In order to handle the second term on the right hand side of (4.8), similarly, we also first record the following computation for a generic $k\ge 0$:

$$
\begin{aligned}
(4.11)\qquad & \sum_{l=k\cdot 2^j}^{\infty}\ \sum_{y\in E^j\cap S_{2k_0+l\cdot 2^{-j}}} \frac{1}{t^{d/2}}e^{-\frac{2^j d_j(x_0,y)}{4}}\cdot 2^{-jd} \\
\overset{(4.6)}{\le}\ & \sum_{l=k\cdot 2^j}^{\infty}\ \sum_{y\in E^j\cap S_{2k_0+l\cdot 2^{-j}}} \frac{1}{t^{d/2}}e^{-\frac{2^j(2k_0+l\cdot 2^{-j})}{8}}\cdot 2^{-jd} \\
\overset{(4.5)}{\le}\ & \sum_{l=k\cdot 2^j}^{\infty} \frac{1}{t^{d/2}}e^{-\frac{2^j(2k_0+l\cdot 2^{-j})}{8}}(2d)\cdot\left(\left(4k_0+2l\cdot 2^{-j}\right)2^j\right)^{d-1}\cdot 2^{-jd}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{l=k\cdot 2^j}^{\infty} \frac{1}{t^{d/2}} e^{-\frac{2^j(2k_0+l\cdot 2^{-j})}{8}} (2d)\cdot(4k_0+2l\cdot 2^{-j})^{d-1}\cdot 2^{-j}\\
&= 2d\sum_{l=k\cdot 2^j}^{\infty} 2^{-j}(4k_0+2l\cdot 2^{-j})^{d-1} e^{-\frac{2^j(2k_0+l\cdot 2^{-j})}{16}}\left(\frac{1}{t^{d/2}} e^{-\frac{2^j(2k_0+l\cdot 2^{-j})}{16}}\right)\\
&\le 2d\sum_{l=k\cdot 2^j}^{\infty} 2^{-j}(4k_0+2l\cdot 2^{-j})^{d-1} e^{-\frac{2^j(2k_0+l\cdot 2^{-j})}{16}}\left(\frac{1}{t^{d/2}} e^{-\frac{2^j}{16}}\right)\\
(t\ge 8^{-j})\quad &\le 2d\sum_{l=k\cdot 2^j}^{\infty} 2^{-j}(4k_0+2l\cdot 2^{-j})^{d-1} e^{-\frac{2^j(2k_0+l\cdot 2^{-j})}{16}}\left(\sup_{j\ge 1} 8^{\frac{jd}{2}} e^{-\frac{2^j}{16}}\right)\\
&\le c_3\sum_{l=k\cdot 2^j}^{\infty} 2^{-j}(4k_0+2l\cdot 2^{-j})^{d-1} e^{-\frac{2^j(2k_0+l\cdot 2^{-j})}{16}}\\
&\le c_3\sum_{u=k}^{\infty}\sum_{l=2^j\cdot u}^{2^j(u+1)-1} 2^{-j}(4k_0+2l\cdot 2^{-j})^{d-1} e^{-\frac{2^j(2k_0+l\cdot 2^{-j})}{16}}\\
&\le c_3\sum_{u=k}^{\infty}(4k_0+2u+2)^{d-1} e^{-\frac{2^j(2k_0+u)}{16}}\\
&= c_3\sum_{u=k}^{\infty}(4k_0+2u+2)^{d-1} e^{-\frac{2^j(2k_0+u)}{32}}\cdot e^{-\frac{2^j(2k_0+u)}{32}}\\
&\le c_3\left(\sup_{x\ge 2k_0}(2x+2)^{d-1} e^{-\frac{x}{32}}\right)\sum_{u=k}^{\infty} e^{-\frac{2^j(2k_0+u)}{32}} \le c_4\sum_{u=k}^{\infty} e^{-\frac{2k_0+u}{32}}.
\end{aligned}
$$

The last display of (4.11) can be made arbitrarily small by selecting sufficiently large k. Therefore, for any $\delta > 0$, there exists $k_2 \in \mathbb{N}$ such that for all $k \ge k_2$,

$$
\sum_{l=k\cdot 2^j}^{\infty}\ \sum_{y\in E^j\cap S_{2k_0+l\cdot 2^{-j}}} \frac{1}{t^{d/2}} e^{-\frac{2^j d_j(x_0,y)}{2}}\cdot 2^{-jd} \le c_4\sum_{u=k}^{\infty} e^{-\frac{2k_0+u}{16}} < \delta/2. \tag{4.12}
$$

Finally, to claim that the right hand side of (4.8) can be made arbitrarily small by selecting sufficiently large M, we note that for any $j \ge 1$, $(E^j\backslash\{a_j^*\}) \subset \bigcup_{l=0}^{\infty} S_{l\cdot 2^{-j}}$. Therefore, given the x_0, k_0 fixed in (4.4), as well as the k_1, k_2 specified in (4.10) and (4.12), there exists $M > 0$ sufficiently large such that

$$
\left\{y\in E^j : d_j(x_0,y)\ge M\right\} \subset \bigcup_{l=(k_1\vee k_2)\cdot 2^j}^{\infty}\left(E^j\cap S_{2k_0+l\cdot 2^{-j}}\right)\quad \text{for all } j\ge 1. \tag{4.13}
$$

With such chosen M, we can rewrite (4.8) as

$$
\begin{aligned}
&\mathbb{P}^{x_0}\left[d_j(X_t^j, x_0)\ge M\right]\\
(4.8)\quad \le\ &\sum_{d_j(y,x_0)\ge M}\frac{c_1}{t^{d/2}} e^{-\frac{d_j(x_0,y)^2}{64t}}\cdot 2^{-jd} + \sum_{d_j(y,x_0)\ge M}\frac{c_1}{t^{d/2}} e^{-\frac{2^j d_j(x_0,y)}{4}}\cdot 2^{-jd}.
\end{aligned}
$$

$$
\begin{aligned}
(4.13)\quad &\leq \sum_{l=k_1\cdot 2^j}^{\infty} \sum_{y\in E^j\cap S_{2k_0+l\cdot 2^{-j}}} \frac{1}{t^{d/2}} 2^{-jd}\cdot e^{-\frac{d_j(x_0,y)^2}{64t}} \\
&+ \sum_{l=k_2\cdot 2^j}^{\infty} \sum_{y\in E^j\cap S_{2k_0+l\cdot 2^{-j}}} \frac{1}{t^{d/2}} e^{-\frac{2^j d_j(x_0,y)}{2}}\cdot 2^{-jd} \\
(4.10),(4.12)\quad &\leq \frac{\delta}{2}+\frac{\delta}{2}=\delta, \quad \text{for all } j\geq N_1.
\end{aligned}
$$

□

The following lemma, roughly speaking, states that starting from a position y such that $\rho(y,a^*)>M$, for sufficiently large M, the probability that $\rho(X^j_t,a^*)\leq M/2$ is small, for all t before a fixed time T.

Lemma 4.5. *For any fixed $T>0$, $\delta>0$, there exist $M_3>0$ and an integer $N_2\geq N_1$ satisfying $T>2\cdot 8^{-N_2}$, such that for all $j\geq N_2$ and $M>M_3$, it holds that*

$$
\mathbb{P}^y\left[\left|X^j_t\right|_\rho \leq \frac{M}{2}\right] < \delta, \quad \textit{for all } |y|_\rho > M,\ t\in[8^{-j}, T-8^{-j}].
$$

Proof. For a generic $M>0$, given $|y|_\rho>M$ and $t\in[8^{-j},T-8^{-j}]$,

$$
(4.14)\qquad \mathbb{P}^y\left[\left|X^j_t\right|_\rho \leq \frac{M}{2}\right] = \sum_{\substack{\rho(x,a^*_j)\leq M/2\\ x\neq a^*_j}} p_j(t,y,x)m_j(x) + p_j(t,y,a^*_j)m_j(a^*_j)
$$

We first handle the second term on the right hand side above. In view of Proposition 2.6 as well as the definition of m_j in (1.3), we have for some $c_1>0$ (depending on d) that

$$
(4.15)\qquad m_j(a^*_j) \leq \frac{2^{-j}}{2d}\cdot c_1\cdot 2^{j(d-1)} = c_1\cdot 2^{-j}.
$$

In view of Corollary 3.3, since $d_j(y,a^*_j)\geq \rho(y,a^*_j)\geq M$ and $t\in[8^{-j},T-8^{-j}]$,

$$
\begin{aligned}
(4.16)\qquad p_j(t,y,a^*_j)m_j(a^*_j) &\leq \frac{c_1 2^{-j}}{t^{d/2}}\left(e^{-\frac{M^2}{64t}}+e^{-\frac{2^jM}{4}}\right) \\
&\overset{t\geq 8^{-j}}{\leq} c_1\left(\frac{1}{t^{d/2}}e^{-\frac{M^2}{64t}}+\frac{2^{-j}}{8^{-jd/2}}e^{-\frac{2^jM}{4}}\right) \\
&= \frac{c_1}{t^{d/2}}e^{-\frac{M^2}{64t}} + c_1(2^j)^{\frac{3d}{2}-1}e^{-\frac{M\cdot 2^j}{2}}
\end{aligned}
$$

Now we claim that the right hand side above can be made arbitrarily small by selecting sufficiently large M. Indeed, given any $\delta>0$, there exists $c_2>0$ such that when $M\geq c_2$,

$$
(4.17)\qquad \sup_{0<t\leq T}\frac{c_1}{t^{d/2}}e^{-\frac{M^2}{64t}} < \frac{\delta}{4}.
$$

For the second term on the right hand side of (4.16), given any $\delta>0$ and the $c_2>0$ chosen in (4.17), there exist $k_1\in\mathbb{N}$ such that for all $j\geq k_1$ and all $M>c_2$,

$$
(4.18)\qquad c_1(2^j)^{\frac{3d}{2}-1}e^{-\frac{M\cdot 2^j}{2}} < c_1\left(\sup_{x>2^j} x^{\frac{3d}{2}-1}e^{-\frac{c_2x}{2}}\right) < \frac{\delta}{4}.
$$

Combining (4.17), (4.18) with (4.16), we have showed that given any $\delta > 0$, there exist $c_2 > 0$ and $k_1 \in \mathbb{N}$ such that for all $M > c_2$ and all $j \geq k_1$ that

$$p_j(t, y, a_j^*)m_j(a_j^*) < \frac{\delta}{2}. \tag{4.19}$$

Now we take care of the first term on the right hand side of (4.14). First we note that $K \subset \{(x_1, \ldots, x_d) \in \mathbb{R}^d : \max_{1\leq i\leq d} |x_i| \leq k_0\}$ per the choice of k_0 indicated in (4.4). Thus

$$\left\{x \in E^j : \rho(x, a_j^*) \leq \frac{M}{2}\right\} \subset \left\{(x_1, \ldots, x_d) \in \mathbb{R}^d : \max_{1\leq i\leq d} |x_i| \leq k_0 + 2M\right\}.$$

This implies that there exists some $c_3 > 0$ such that for $M \geq 2k_0$,

$$\#\left\{x \in E^j : \rho(x, a_j^*) \leq \frac{M}{2}\right\} \leq \#\left\{(x_1, \ldots, x_d) \in \mathbb{R}^d : \max_{1\leq i\leq d} |x_i| \leq 5M\right\} \leq c_3 \cdot M^d \cdot 2^{jd}. \tag{4.20}$$

In view of the assumption that $\rho(y, a^*) > M$, for x such that $\rho(x, a_j^*) \leq M/2$, triangle inequality implies

$$d_j(x, y) \geq \rho(x, y) \geq \frac{M}{2}, \quad \text{all } j \geq 1. \tag{4.21}$$

Now by Corollary 3.3 as well as the fact that $m_j(x) \leq 2^{-jd}$ for $x \neq a^*$,

$$\begin{aligned}
\text{(4.22)} \quad \sum_{\substack{\rho(x,a_j^*)\leq M/2\\ x\neq a_j^*}} p_j(t, y, x)m_j(x) &\leq \sum_{\substack{\rho(x,a_j^*)\leq M/2\\ x\neq a_j^*}} c_4\left(\frac{1}{t^{d/2}}e^{-\frac{d_j(x,y)^2}{64t}} + \frac{1}{t^{-d/2}}e^{-\frac{2^j d_j(x,y)}{4}}\right)m_j(x) \\
\text{(4.21)} \quad &\leq \sum_{\substack{\rho(x,a_j^*)\leq M/2\\ x\neq a_j^*}} c_4\left(\frac{1}{t^{d/2}}e^{-\frac{M^2}{256t}} + \frac{1}{t^{d/2}}e^{-\frac{2^j M}{8}}\right)2^{-jd} \\
(t \geq 8^{-j}) \quad &\leq \sum_{\substack{\rho(x,a_j^*)\leq M/2\\ x\neq a_j^*}} c_4\left(\frac{1}{t^{d/2}}e^{-\frac{M^2}{256t}} + \frac{1}{8^{-jd/2}}e^{-\frac{2^j M}{8}}\right)2^{-jd} \\
\text{(4.20)} \quad &\leq c_6 \cdot M^d \cdot 2^{jd}\left(\frac{1}{t^{d/2}}e^{-\frac{M^2}{256t}} + 8^{jd/2}e^{-\frac{2^j M}{8}}\right) \cdot 2^{-jd} \\
&= c_4 \cdot M^d\left(\frac{1}{t^{d/2}}e^{-\frac{M^2}{256t}} + 8^{jd/2}e^{-\frac{2^j M}{8}}\right) \\
(t \leq T,\ M \geq 2k_0) \quad &\leq c_4 M^d e^{-\frac{M^2}{512T}}\left(\sup_{0<t\leq T} \frac{1}{t^{d/2}}e^{-\frac{4k_0^2}{512t}}\right) + c_6 \cdot M^d \cdot 2^{\frac{3jd}{2}}e^{-\frac{2^j M}{8}} \\
&\leq c_4 M^d e^{-\frac{M^2}{512T}} + c_4 \cdot M^d \cdot 2^{\frac{3jd}{2}}e^{-\frac{2^j M}{8}}.
\end{aligned}$$

Now we want to claim that given any $\delta > 0$, the last display above can be made arbitrarily small by selecting sufficient large M and j. Indeed, for the first term in the last display of (4.22), there exists $c_5 \geq 2k_0$ such that for all $M \geq c_5$, $c_4 \cdot M^d \cdot e^{-\frac{M^2}{512T}} < \delta/4$. For this chosen c_5, we first denote by $c_6 := \sup_{M\geq c_5} M^d e^{-M/16}$. Now for the second term on the right hand side of (4.22), for all $M \geq c_5$ and all $j \geq 1$,

$$c_4 \cdot M^d \cdot 2^{\frac{3jd}{2}}e^{-\frac{2^j M}{8}} \leq c_4 \cdot M^d \cdot 2^{\frac{3jd}{2}}e^{-\frac{2^j M}{16}} \cdot e^{-\frac{M}{16}} \leq c_4 c_6 \cdot 2^{\frac{3jd}{2}}e^{-\frac{2^j c_5}{16}}. \tag{4.23}$$

From the last display above, one can tell that there exists $k_2 \in \mathbb{N}$ such that for all $j \geq k_2$ and all $M \geq c_5$,

$$c_6 \cdot M^d \cdot 2^{\frac{3jd}{2}} e^{-\frac{2^j M}{8}} < c_4 c_6 \cdot 2^{\frac{3jd}{2}} e^{-\frac{2^j c_5}{16}} < \frac{\delta}{4}. \tag{4.24}$$

Combining (4.24) with the previous discussion regarding the choice of c_5, we have claimed that given any $\delta > 0$, there exist $c_5 > 0$ and $k_2 \in \mathbb{N}$ such that

$$\sum_{\substack{\rho(x,a_j^*)\leq M/2 \\ x\neq a_j^*}} p_j(t,y,x) m_j(x) < \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2}, \quad \text{for all } M \geq c_5,\ j \geq k_2. \tag{4.25}$$

Finally, combining (4.14), (4.19), and (4.25), it has been shown that for all $M \geq \max\{c_2, c_5\}$ and all $j \geq \max\{k_1, k_2\}$,

$$\mathbb{P}^y \left[\left|X_t^j\right|_\rho \leq \frac{M}{2} \right] < \delta, \quad \text{for all } |y|_\rho > M,\ t \in [8^{-j}, T - 8^{-j}].$$

□

Recall the definition of τ_M in Lemma 4.2.

Lemma 4.6. *Fix $x_0 \in \bigcap_{j\geq 1} E^j$. For every $T > 0$ and every $\delta > 0$, for the N_2 and M_3 given in* Lemma 4.5*, it holds for all $j \geq N_2$ and all $M > M_3$ that*

$$\mathbb{P}^{x_0} \left[8^{-j} \leq \tau_M \leq T - 8^{-j},\ \left|X_T^j\right|_\rho \leq \frac{M}{2} \right] < \delta.$$

Proof.

$$\begin{aligned} &\mathbb{P}^{x_0} \left[8^{-j} \leq \tau_M \leq T - 8^{-j},\ \left|X_T^j\right|_\rho \leq \frac{M}{2} \right] \\ = &\int_{8^{-j}}^{T-8^{-j}} \mathbb{E}^{x_0} \left[\mathbb{P}^{X_s^j} \left[\left|X_{T-s}^j\right| \leq \frac{M}{2} \right]; \tau_M \in ds \right] \\ \leq &\int_{8^{-j}}^{T-8^{-j}} \mathbb{E}^{x_0} \left[\sup_{\substack{y:|y|_\rho \geq M \\ t\in[8^{-j},T-8^{-j}]}} \mathbb{P}^y \left[|X_t^j|_\rho \leq \frac{M}{2} \right]; \tau_M \in ds \right] \\ \leq &\sup_{\substack{|y|_\rho \geq M \\ t\in[8^{-j},T-8^{-j}]}} \mathbb{P}^y \left[|X_t^j|_\rho \leq \frac{M}{2} \right]. \end{aligned} \tag{4.26}$$

The conclusion then follows from Lemma 4.5. □

Proposition 4.7. *Given $x_0 \in \cap_{j\geq 1} E^j$, given any $T > 0$, $\delta > 0$, there exist $M_4 > 0$ and an integer $N_3 \geq N_1$ specified in* Proposition 2.7*, such that for all $j \geq N_3$,*

$$\mathbb{P}^{x_0} \left[\sup_{t\in[0,T]} \left|X_t^j\right|_\rho > M_4 \right] < \delta. \tag{4.27}$$

Proof. On account of Lemma 4.2, it suffices to show that given any $\delta > 0$, $T > 0$, there exist $c_1 > 0$ and $n_1 \in \mathbb{N}$ with $8^{-n_1} < T/2$, such that for all $M > c_1$ and all $j \geq n_1$, the following hold:

(i) $\mathbb{P}^{x_0}\left[\sup_{t\in[0,8^{-j}]}|X_t^j|_\rho \geq M\right] < \delta$;

(ii) $\mathbb{P}^{x_0}\left[\left|X_T^j\right|_\rho \geq \frac{M}{2}\right] < \delta$.

(iii) $\mathbb{P}^{x_0}\left[T - 8^{-k} \leq \tau_M \leq T,\ \left|X_T^j\right|_\rho \leq \frac{M}{2}\right] < \delta$;

(iv) $\mathbb{P}^{x_0}\left[8^{-j} \leq \tau_M \leq T - 8^{-j},\ \left|X_T^j\right|_\rho \leq \frac{M}{2}\right] < \delta$.

We claim that all (i)-(iv) are satisfied for all $M \geq |x_0|_\rho + 2 \cdot \max_{1\leq i\leq 4} M_i$ and $j \geq \max_{1\leq i\leq 3} N_i$, where the M_i's and the N_i's are given in Lemmas 4.3, 4.4 and 4.6. Actually it is evident that (i) and (ii) hold on account of Lemma 4.3 and 4.4, respectively, together with triangle inequality for distances. (iv) holds in view of Lemma 4.6. To justify (iii), by the same argument as that for [12, (4.18)] we have

$$\mathbb{P}^{x_0}\left[T - 8^{-j} \leq \tau_M \leq T,\ \left|X_T^j\right|_\rho \leq \frac{M}{2}\right] \leq \sup_{|y|_\rho \geq M} \mathbb{P}^y\left[\sup_{t\in[0,8^{-j}]} \rho\left(X_t^j, X_0^j\right) \geq \frac{M}{2}\right]. \tag{4.28}$$

Thus (iii) follows from Lemma 4.5. □

As a standard notation, given a metric $d(\cdot,\ \cdot)$, we denote by

$$w_d(x,\ \theta,\ T) := \inf_{\{t_i\}_{1\leq i\leq n}\in\Pi}\ \max_{1\leq i\leq n}\ \sup_{s,t\in[t_i,t_{i-1}]} d(x(s), x(t)),$$

where Π is the collection of all possible partitions of the form $0 = t_0 < t_1 < \cdots < t_{n-1} < T \leq t_n$ with $\min_{1\leq i\leq n}(t_i - t_{i-1}) \geq \theta$ and $n \geq 1$. It is clear from the definition that as θ decreases, w_d is nonincreasing.

Proposition 4.8. *Fix any $x_0 \in \cap_{j\geq 1} E^j$. For any $T > 0$, $\delta_1, \delta_2 > 0$, there exist $\delta_3 > 0$ and an integer $N_4 \geq N_1$ such that for all $j \geq N_4$,*

$$\mathbb{P}^{x_0}\left[w_\rho\left(X^j, \delta_3, T\right) > \delta_1\right] < \delta_2, \tag{4.29}$$

where

$$w_\rho(x,\ \delta_3,\ T) := \inf_{\{t_i\}}\max_i\ \sup_{s,t\in[t_i,t_{i-1}]} \rho(x(s), x(t)),$$

where $\{t_i\}$ ranges over all possible partitions of the form $0 = t_0 < t_1 < \cdots < t_{n-1} < T \leq t_n$ with $\min_{1\leq i\leq n}(t_i - t_{i-1}) \geq \delta_3$ and $n \geq 1$.

Proof. Using the same argument as that for [12, (4.52)-(4.54)], one can get for $j \in \mathbb{N}$ and any $\delta_1, \delta_3 > 0$,

$$\mathbb{P}^{x_0}\left[w_\rho\left(X^j, \delta_3, T\right) > \delta_1\right] \leq 2\left(\left[\frac{T}{\delta_3}\right] + 1\right) \sup_{\substack{y\in E\\ 0\leq s\leq\delta_3}} \mathbb{P}^y\left[\rho\left(X_0^j, X_s^j\right) \geq \frac{\delta_1}{4}\right]. \tag{4.30}$$

For the right hand side of (4.30), we further have that for $\delta_3 > 0$ and $j \in \mathbb{N}$ satisfying $8^{-j} < \delta_3$,

$$\sup_{0<s\leq\delta_3} \mathbb{P}^y\left[\rho\left(X_0^j, X_s^j\right) \geq \frac{\delta_1}{4}\right] \leq \sup_{0<s\leq 8^{-j}} \mathbb{P}^y\left[\rho\left(X_0^j, X_s^j\right) \geq \frac{\delta_1}{4}\right] + \sup_{8^{-j}\leq s\leq\delta_3} \mathbb{P}^y\left[\rho\left(X_0^j, X_s^j\right) \geq \frac{\delta_1}{4}\right]. \tag{4.31}$$

We first handle the second term on the right hand side above. For any $\delta_3 > 0$, $t > 0$, $j \in \mathbb{N}$

such that $8^{-j} < t < \delta_3$, for any $y \in E^j$, there exists some $c_1 > 0$ such that

$$\begin{aligned}
\mathbb{P}^y\left[\rho(y, X_t^j) \ge \frac{\delta_1}{4}\right] &\le \mathbb{P}^y\left[d_j(y, X_t^j) \ge \frac{\delta_1}{4}\right] \\
&\le \sum_{\substack{x\in E^j: d_j(x,y)\ge \frac{\delta_1}{4} \\ x\neq a_j^*}} \frac{c_1}{t^{d/2}}\left(e^{-\frac{d_j(x,y)^2}{64t}} + e^{-\frac{2^j d_j(x,y)}{4}}\right) m_j(x) \\
&+ \frac{c_1}{t^{d/2}}\left(e^{-\frac{\delta_1^2}{256t}} + e^{-\frac{2^j\delta_1}{16}}\right) m_j(a_j^*).
\end{aligned} \tag{4.32}$$

For the first term on the right hand side of (4.32), we first note that for a given $y \in E^j$ we have

$$\left\{x \in E^j : d_j(x, y) \ge \frac{\delta_1}{4},\ x \neq a_j^*\right\} \subset \bigcup_{k=1}^{\infty} \left\{x \in E^j : d_j(x, y) \le \frac{\delta_1}{4} + k,\ x \neq a_j^*\right\}.$$

In view of the definition of k_0 in (4.4), the diameter of K under Euclidean distance is at most $2k_0$. Thus for all $j \ge 1$ and all $x, y \in E^j$, it holds

$$d_j(x, y) \ge \rho(x, y), \quad \rho(x, y) + 2k_0 \ge |x - y|.$$

Therefore,

$$\left\{x \in E^j : d_j(x, y) \le \frac{\delta_1}{4} + k,\ x \neq a_j^*\right\} \subset \left\{x \in 2^{-j}\mathbb{Z}^d : |x - y| \le \frac{\delta_1}{4} + k + 2k_0\right\}.$$

It then follows that for any $k \in \mathbb{N}$,

$$\#\left\{x \in E^j : d_j(x, y) \le \frac{\delta_1}{4} + k,\ x \neq a_j^*\right\} \le \left(\frac{\delta_1}{4} + k + 2k_0\right)^{2d} 2^{jd}. \tag{4.33}$$

Now for the first term on the right hand side of (4.32), noticing that $m_j(x) \le 2^{-jd}$ for $x \neq a_j^*$, we have

$$\begin{aligned}
&\sum_{\substack{x\in E^j: d_j(x,y)\ge \frac{\delta_1}{4} \\ x\neq a_j^*}} \frac{c_1}{t^{d/2}}\left(e^{-\frac{d_j(x,y)^2}{64t}} + e^{-\frac{2^j d_j(x,y)}{4}}\right) m_j(x) \\
\le\ & \sum_{k=0}^{\infty} \sum_{\substack{x\in E^j: d_j(x,y)\le \frac{\delta_1}{4}+k \\ x\neq a_j^*}} \frac{c_1}{t^{d/2}}\left(e^{-\frac{d_j(x,y)^2}{64t}} + e^{-\frac{2^j d_j(x,y)}{4}}\right) m_j(x) \\
\overset{(4.33)}{\le}\ & \sum_{k=0}^{\infty} \frac{c_1}{t^{d/2}}\left(e^{-\frac{(\delta+k)^2}{1024t}} + e^{-\frac{2^j(\delta+k)}{16}}\right) \cdot (\delta_1 + k + 2k_0)^{2d} 2^{jd} 2^{-jd} \\
\le\ & \sum_{k=0}^{\infty} \frac{c_1(\delta_1 + k + 2k_0)^{2d}}{t^{d/2}} e^{-\frac{(\delta_1+k)^2}{1024t}} + \sum_{k=0}^{\infty} \frac{c_1(\delta_1 + k + 2k_0)^{2d}}{t^{d/2}} e^{-\frac{2^j(\delta_1+k)}{16}}.
\end{aligned} \tag{4.34}$$

Now, for the first term on the right hand side of (4.34), for any $8^{-j} < t < \delta_3 \le T$ where δ_3 is a generic constant,

$$\sum_{k=0}^{\infty} \frac{c_1(\delta_1 + k + 2k_0)^{2d}}{t^{d/2}} e^{-\frac{(\delta_1+k)^2}{1024t}} \le e^{-\frac{\delta_1^2}{2048\delta_3}} \left(\sup_{t\in(0,T]} \frac{c_1}{t^{d/2}} e^{-\frac{\delta_1^2}{4096t}}\right) \sum_{k=0}^{\infty} (\delta_1 + k + 2k_0)^{2d} e^{-\frac{(\delta_1+k)^2}{4096T}}$$

$$\leq \quad c_2(\delta_1, T, k_0) e^{-\frac{\delta_1^2}{2048\delta_3}}. \tag{4.35}$$

For the second term on the right hand side of (4.34), noticing that $t \geq 8^{-j}$,

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{c_1(\delta_1 + k + 2k_0)^{2d}}{t^{d/2}} e^{-\frac{2^j(\delta_1+k)}{16}} &\leq c_1 \left(\frac{1}{t^{d/2}} e^{-\frac{2^j\delta_1}{32}} \right) \sum_{k=0}^{\infty} (\delta_1 + k + 2k_0)^{2d} e^{-\frac{2^j(\delta_1+k)}{32}} \\
(t \geq 8^{-j}) \quad &\leq c_1 \left((2^j)^{\frac{3d}{2}} e^{-\frac{2^j\delta_1}{32}} \right) \sum_{k=0}^{\infty} (\delta_1 + k + 2k_0)^{2d} e^{-\frac{\delta_1+k}{32}} \\
&\leq \left((2^j)^{\frac{3d}{2}} e^{-\frac{2^j\delta_1}{32}} \right) \cdot c_3(\delta_1, k_0).
\end{aligned} \tag{4.36}$$

Before we plugging the computation above back into the right hand side of (4.32), we record the following computation for the second term on the right hand side of (4.32). For any $8^{-j} < t < \delta_3 \leq T$, noticing the upper bound for $m_j(a_j^*)$ given in (4.15), we have

$$\begin{aligned}
&\frac{c_1}{t^{d/2}} \left(e^{-\frac{\delta_1^2}{256t}} + e^{-\frac{2^j\delta_1}{16}} \right) m_j(a_j^*) \\
(4.15) \quad &\leq \frac{c_4}{t^{d/2}} \left(e^{-\frac{\delta_1^2}{256t}} + e^{-\frac{2^j\delta_1}{16}} \right) 2^{-j} \\
(8^{-j} < t < \delta_3) \quad &\leq c_4 \cdot e^{-\frac{\delta_1^2}{512\delta_3}} \left(\sup_{t \in (0,T]} \frac{1}{t^{d/2}} e^{-\frac{\delta_1^2}{512t}} \right) + c_4 \cdot (2^j)^{\frac{3d}{2}-1} e^{-\frac{2^j\delta_1}{16}} \\
&\leq c_5(\delta_1, T) e^{-\frac{\delta_1^2}{512\delta_3}} + c_4 \cdot (2^j)^{\frac{3d}{2}-1} e^{-\frac{2^j\delta_1}{16}}.
\end{aligned} \tag{4.37}$$

From here, first we replace the two terms on the right hand side of (4.34) with (4.35) and (4.36), then plug the resulting expression together with (4.37) back into the right hand side of (4.32). Consolidating the common terms gives us that for any pair of $\delta_3 > 0,\ j \in \mathbb{N}$ such that $8^{-j} < \delta_3$,

$$\sup_{\substack{y \in E^j \\ t \in [8^{-j}, \delta_3]}} \mathbb{P}^y \left[\rho(y, X_t^j) \geq \frac{\delta_1}{4} \right] \leq c_6(\delta_1, T, k_0) e^{-\frac{\delta_1^2}{2048\delta_3}} + c_7(\delta_1, k_0) \left((2^j)^{\frac{3d}{2}} e^{-\frac{2^j\delta_1}{32}} \right). \tag{4.38}$$

From the above, given any pair of $\delta_1, \delta_2 > 0$, we can first choose $\delta_3 > 0$ sufficiently small such that

$$c_6(\delta_1, T, k_0) e^{-\frac{\delta_1^2}{2048\delta_3}} < \frac{\delta_2\delta_3}{4(T + 2\delta_3)}. \tag{4.39}$$

With this δ_3 chosen above, then we choose $n_1 \in \mathbb{N}$ satisfying $8^{-n_1} < \delta_3$ such that for all $j \geq n_1$,

$$c_7(\delta_1, k_0) \left((2^j)^{\frac{3d}{2}} e^{-\frac{2^j\delta_1}{32}} \right) < \frac{\delta_2\delta_3}{4(T + 2\delta_3)}.$$

Thus given any pair of $\delta_1, \delta_2 > 0$, there exists $\delta_3 > 0$ and $n_1 \in \mathbb{N}$ with $8^{-n_1} < \delta_3$ such that for all $j \geq n_1$,

$$\sup_{\substack{y \in E^j, \\ 8^{-j} \leq t \leq \delta_3}} \mathbb{P}^y \left[\rho\left(X_0^j, X_t^j\right) \geq \frac{\delta_1}{4} \right] < \frac{\delta_2\delta_3}{2(T + 2\delta_3)}. \tag{4.40}$$

The same argument as that for [12, (4.50)] yields that given any $\delta_1, \delta_2 > 0$ and the $\delta_3 > 0$ accordingly selected in (4.39), there exists $n_2 \in \mathbb{N}$ sufficiently large such that for all $j \geq n_2$,

$$\sup_{\substack{y \in E^j, \\ 0 < s \leq 8^{-j}}} \mathbb{P}^y \left[\rho\left(X_0^j, X_s^j\right) \geq \frac{\delta_1}{4} \right] < \frac{\delta_2 \delta_3}{2(T + 2\delta_3)}. \tag{4.41}$$

Finally, plugging both (4.40) and (4.41) into the right hand side of (4.30) yields that given $\delta_1, \delta_2 > 0$, there exist $\delta_3 > 0$ and $n_1, n_2 \in \mathbb{N}$ such that for all $j \geq \max\{n_1, n_2\}$,

$$\mathbb{P}^{x_0}\left[w_\rho\left(X^j, \delta_3, T\right) > \delta_1 \right] \leq 2\left(\left[\frac{T}{\delta_3} \right] + 1 \right) \cdot \frac{\delta_2 \delta_3}{T + 2\delta_3} \leq 2 \cdot \frac{T + 2\delta_3}{\delta_3} \cdot \frac{\delta_2 \delta_3}{T + 2\delta_3} < 2\delta_2. \qquad \square$$

Theorem 4.9. *Fix $x_0 \in \bigcap_{j \geq 1} E^j$. For every $T > 0$, the laws of $\{X^j, \mathbb{P}^{x_0}, j \geq 1\}$ are C-tight in the Skorokhod space $\mathbf{D}([0, T], E, \rho)$ equipped with the Skorokhod topology.*

Proof. This follows immediately from [10, Chapter VI, Proposition 3.21]. In view of Propositions 4.7-4.8. $\square$

Remark 4.10. By the same proof as that to Theorem 4.9, it can be shown that given any $T > 0$, the laws of $\{X^j, \mathbb{P}^{a_j^*}, j \geq 1\}$ are C-tight in the Skorokhod space $\mathbf{D}([0, T], E, \rho)$ equipped with the Skorokhod topology.

Before proving the next lemma, we define the following class of functions:

$$\mathcal{G} := \{ f \in C_c^3(\mathbb{R}^d),\ f = \text{constant on } K. \}. \tag{4.42}$$

For $f \in \mathcal{G}$, we define

$$\mathcal{L}^j f(x) := 2^{2j} \sum_{y \leftrightarrow x \text{ in } E^j} (f(y) - f(x)) \frac{1}{v_j(x)}, \qquad \text{for } x \in E_0^k. \tag{4.43}$$

We also set

$$S^j := \{ x \in E^j : \ x \neq a_j^*,\ v_j(x) = 2d \text{ in } G^j \}, \tag{4.44}$$

where G^j has been defined in §1.

Lemma 4.11. *For every $\delta > 0$ and every $f \in \mathcal{G}$, there exists some $n_{\delta,f} \in \mathbb{N}$, such that for all $j \geq n_{\delta,f}$:*

(i) *$m_j(E^j \backslash S^j) < \delta$;*
(ii) *As $j \to \infty$, $\mathcal{L}^j f$ converges uniformly to*

$$\mathcal{L} f(x) := \frac{1}{2d} \Delta f(x) + O(1) 2^{-j} \quad \text{on } S^{n_{\delta,f}}. \tag{4.45}$$

Also, there exists some constant $C_7 > 0$ independent of j such that for all $j \geq 1$,

$$\mathcal{L}^j f(x) \leq C_7, \quad \text{for all } x \in E^j.$$

Proof. To claim (i), we notice that $\{E^j \backslash S^j\} \subset \{x = a_j^* \text{ or } x \leftrightarrow a_j^*\}$. Thus by Proposition 2.6, there exists $c_1 > 0$ such that for all $j \geq j_0$ specified in Proposition 2.6,

$$m_j(E^j \backslash S^j) \leq m_j(\{x = a_j^* \text{ or } x \leftrightarrow a_j^*\}) \leq m_j(a_j^*) + v_j(a_j^*) \cdot 2^{-jd} \leq c_1 \cdot 2^{-j}.$$

Using Taylor's expansion we have for any $f \in \mathcal{G}$ and any $j \geq j_0$,

$$\widetilde{\mathcal{L}}_j f(x) = 2^{2j} \sum_{y \leftrightarrow x} \left[\sum_{i=1}^{d} \frac{\partial f(x)}{\partial x_i}(y_i - x_i) + \frac{1}{2} \sum_{i,l=1}^{d} \frac{\partial^2 f(x)}{\partial x_i \partial x_l}(y_i - x_i)(y_l - x_l) + O(1)|y - x|^3 \right] \frac{1}{v_j(x)}.$$

Hence

$$\mathcal{L}^j f(x) = \frac{1}{2d} \Delta f(x) + O(1) 2^{-j}, \quad \text{for } x \in S^j. \tag{4.46}$$

Since $\{S^j\}_{j \geq 1}$ is an increasing sequence of sets in j, both (i) and (ii) have been justified. To justify the last claim, we first note that by (4.46) and the fact that $f \in C_c^3(\mathbb{R}^d)$, there exists $c_2 > 0$ independent of j such that

$$\mathcal{L}^j f(x) \leq c_2, \quad \text{for all } x \in \bigcup_{j \geq 1} S^j. \tag{4.47}$$

We denote by $c_K := f|_K$. Given any $x \leftrightarrow a_j^*$ in E^j, by the definition of G^j in (1.2), there must exist a point $a \in K$ such that $|x - a| \leq 2^{-j}$. Since $f \in C_c^3(\mathbb{R}^d)$ and is constant on K, the first order derivatives of f vanish on K. By Taylor expansion, there exists some constant $c_3 > 0$ only depending on f but not j such that

$$(f(x) - c_K) = |f(x) - f(a)| \leq \sum_{i=1}^{d} \left| \frac{\partial f}{\partial x_i}(a) \right| \cdot |x - a| + c_2 \cdot |x - a|^2 \leq c_3 \cdot 2^{-2j}. \tag{4.48}$$

Thus by the definition of $\mathcal{L}^j$ and Proposition 2.6,

$$\mathcal{L}^j f(a_j^*) = 2^{2j} \sum_{x \leftrightarrow a_j^*} (f(x) - c_K) \cdot v_j(a_j^*)^{-1} \leq 2^{2j} \cdot c_3 \cdot 2^{-2j} \leq c_3. \tag{4.49}$$

To bound $\mathcal{L}^j f(x)$ for $x \leftrightarrow a_j^*$, we first note in this case,

$$\mathcal{L}^j f(x) = 2^{2j} \sum_{y \leftrightarrow x} (f(y) - f(x)) \cdot v_j(x)^{-1} \leq 2^{2j} \max_{x: x \leftrightarrow a_j^*} \max_{y: y \leftrightarrow x} |f(y) - f(x)|. \tag{4.50}$$

For each y in the second "max" in (4.50), notice that there exists $a \in K$ such that $|y - a| \leq 2 \cdot 2^{-j}$. Thus by similar reasoning for (4.48), $|f(y) - c_K| \leq c_3 \cdot (2 \cdot 2^{-j})^2$. Since for all x in the first "max" in (4.50), $|f(x) - c_K| \leq c_3 \cdot 2^{-2j}$, by triangle inequality it follows that for some $c_4 > 0$ independent of j, for all $j \geq 1$, $\mathcal{L}^j f(x) \leq c_4$ for all x such that $x \leftrightarrow a_j^*$. This together with (4.49) shows

$$\mathcal{L}^j f(x) \leq c_3 \vee c_4, \quad \text{for all } x \in \bigcup_{j \geq 1} (E^j \backslash S^j).$$

This combined with (4.47) proves the last claim of this lemma. The proof is complete. □

The following lemma is used in the proof of the main result: Theorem 4.13.

Lemma 4.12. *Fix $T > 0$. Given any $\delta > 0$, there exist $C_8 > 0$ and an integer $N_\delta \geq N_1$, such that for all $j \geq N_\delta$,*

$$\sup_{t \in [(2^j \delta)^{-2/d}, T]} \mathbb{P}^{x_0} \left[X_t^j \notin S^j \right] \leq C_8 \delta,$$

Proof. Given any $\delta > 0$, choose $j_\delta \in \mathbb{N}$ large enough such that $(2^{j_\delta}\delta)^{-2/d} < T$. For any $t \in [(2^j\delta)^{-2/d}, T]$ with $j \geq j_\delta$, by Corollary 3.3 and Proposition 2.6, there exists $c_1, c_2 > 0$ such that

$$\begin{aligned}
\mathbb{P}^{x_0}\left[X_t^j \notin S^j\right] &\leq \sum_{y \notin S^j} \frac{c_1}{t^{d/2}} m_j(y) \\
&\leq \sum_{x \leftrightarrow a_j^*} \left(\frac{c_1}{t^{d/2}} \cdot m_j(x)\right) + \frac{c_1}{t^{d/2}} \cdot m_j(a_j^*) \\
\text{(Proposition 2.6)} \quad &\leq c_2 \cdot 2^{j(d-1)} \cdot c_1 t^{-d/2} \cdot 2^{-jd} + c_1 t^{-d/2} \cdot c_2 \cdot 2^{j(d-1)} \cdot 2^{-jd} \\
(t > (2^j\delta)^{-2/d}) \quad &\leq 2c_1c_2 \cdot \delta.
\end{aligned}$$

This proves the desired result by letting N_δ be the selected j_δ. □

Theorem 4.13. *Given* $x_0 \in \cap_{j\geq 1} E^j$, $\{X^j, \mathbb{P}^{x_0}, j \geq 1\}$ *converges weakly to the BMD described in* (1.1) *starting from* x_0.

Proof. Since the laws of $\{X^j\}_{j\geq 1}$ are C-tight in $\mathbf{D}([0,T], E, \rho)$, any sequence has a weakly convergent subsequence supported on the set of continuous paths. Denote by $\{X^{j_l} : l \geq 1\}$ any such weakly convergent subsequence, and denote by Y its weak limit which is continuous. By Skorokhod representation theorem (see, e.g., [9, Chapter 3, Theorem 1.8]), we may assume that $\{X^{j_l}, l \geq 1\}$ as well as Y are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, so that $\{X^{j_l}, l \geq 1\}$ converges almost surely to Y in the Skorokhod topology.

For every $t \in [0.T]$, we set $\mathcal{M}_t^{j_l} := \sigma(X_s^{j_l}, s \leq t)$ and $\mathcal{M}_t := \sigma(Y_s, s \leq t)$. It is obvious that $\mathcal{M}_t \subset \sigma\{\mathcal{M}_t^{k_j} : j \geq 1\}$. By the same argument as that for [12, Theorem 5.3] with the use of Lemma 5.2 in [12] being replaced with Lemma 4.12, it can be shown that $(Y, \mathbb{P}^{x_0})$ is indeed a solution to the $\mathbf{D}([0,T], E, \rho)$ martingale problem $(\mathcal{L}, \mathcal{G})$ with respect to the filtration $\{\mathcal{M}_t\}_{t\geq 0}$.

Next we claim that the BMD associated with the Dirichlet form described by (1.1) is a strong Feller process. To see this, we denote by $\{G^\alpha\}_{\alpha>0}$ the resolvent operators of X, and denote by $\{G^\alpha_{E\setminus\{a^*\}}\}_{\alpha>0}$ the resolvent operators of $X^{E\setminus\{a^*\}}$, the part process of X killed hitting a^*, which has the same distribution as regular Brownian motion on $\mathbb{R}^d$ killed upon hitting K. By strong Markov property, for $x \in E$, for every bounded measurable function $f : E \to \mathbb{R}$,

$$G^\alpha f(x) \quad = \quad G^\alpha_{E\setminus\{a^*\}} + G^\alpha f(a^*) \cdot \int_0^\infty e^{-\alpha s} \, \mathbb{P}^x[\sigma_{\{a^*\}} \in ds] \tag{4.51}$$

In the right hand side above, the map $x \mapsto \mathbb{P}^x[\sigma_{\{a^*\}} \in ds]$ is continuous because every point on the boundary of $K \subset \mathbb{R}^d$ is regular for K. Therefore we concludee that X is a strong Feller process because $G^\alpha(b\mathcal{B}(E)) \subset C(E)$. In view of [5, §1.5], the infinitesimal generator of X can be described by $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$, where $u \in \mathcal{F}$ is in $\mathcal{D}(\mathcal{L})$ if there exists $f \in L^2(E)$ such that

$$\frac{1}{2}\int_{E\setminus\{a^*\}} \nabla u(x) \nabla v(x) dm = \int_{E\setminus\{a^*\}} f(x) \cdot v(x) dm, \quad \text{for all } v \in \mathcal{F}. \tag{4.52}$$

It also holds that $\mathcal{L}u = f = \frac{1}{2}\Delta u$ for $u \in \mathcal{D}(\mathcal{L})$. It then is clear that the bp-closure of the graph of $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is contained in the bp-closure of the graph of $(\mathcal{L}, \mathcal{G})$. By [2, Lemema 3.4.18], any solution to the martingale problem $(\mathcal{L}, \mathcal{G})$ must be a solution to the martingale problem $(\mathcal{L}, \mathcal{G})$. Since X is a strong Feller process, the martingale problem $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ must be well-posed with its unique solution being X. Therefore the martingale problem $(\mathcal{L}, \mathcal{G})$ must be well-posed with its unique solution being X. This means that X is the sequencial limit of any weakly convergent subsequence of $\{X^j\}_{j\geq 1}$, the proof is complete. □

Remark 4.14. In view of Remark 4.10, the same argument can show that $\{X^j, \mathbb{P}^{a^*_j}, \ j \geq 1\}$ converges weakly to the BMD given by (1.1) starting from a^*.

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