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WASSERSTEIN DISTANCE ON SOLUTIONS TO STOCHASTIC DIFFERENTIAL EQUATIONS WITH JUMPS

Atsushi TAKEUCHI

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Abstract

In this paper, we shall focus on the Wasserstein distance between two jump processes determined by stochastic differential equations in $\mathbb{R}^d$ or the Riemannian manifold M. As an application, the study on the Wasserstein distance implies that the law of the subordinated Brownian motion on M is different from the one of the canonical projected process of the Marcus-type equation with jumps valued in the bundle of orthonormal frames $O(M)$.

1. Introduction

Let T be a positive constant fixed throughout the paper, and denote by $\nu(dz)$ the Lévy measure over $\mathbb{R}^d_0 := \mathbb{R}^d \backslash \{0\}$ such that the function $|z|^2 \wedge 1$ is integrable with respect to the measure $\nu(dz)$. Write $K_\rho = \{z \in \mathbb{R}^d_0\,;\, |z| \leq \rho\}$ for $\rho > 0$. Let $(\Omega,\ \mathcal{F},\ \mathbb{P})$ a complete probability space with the filtration $\{\mathcal{F}_t\,;\, t \in [0,T]\}$ such that the process $\{W_t\,;\, t \in [0,T]\}$ is the d-dimensional Brownian motion starting from the origin in $\mathbb{R}^d$, and that $dJ(\,\equiv\, J(ds,dz))$ is the Poisson random measure over $(0,T] \times \mathbb{R}^d_0$ with the intensity measure $d\hat{J}(\,\equiv\, \hat{J}(ds,dz)) := ds\,\nu(dz)$. Denote by $d\tilde{J}(\,\equiv\, \tilde{J}(ds,dz)) := dJ - d\hat{J}$ the compensated one. From now on, we shall write $d\overline{J}(\,\equiv\, \overline{J}(ds,dz)) = \mathbb{I}_{K_1}(z)\, d\tilde{J} + \mathbb{I}_{K_1^c}(z)\, dJ$, in order to simplify our notations.

Let A_0, A_i, B_i, C_0, $C_i \in C^1(\mathbb{R}^d\,;\,\mathbb{R}^d)$ $(1 \leq i \leq d)$ such that all partial derivatives of any orders greater than 1 are bounded. Let $D \in C^{1,1}(\mathbb{R}^d \times \mathbb{R}^d_0\,;\,\mathbb{R}^d)$, that is, $D(\,\cdot\,, z) \in C^1(\mathbb{R}^d\,;\,\mathbb{R}^d)$ for each $z \in \mathbb{R}^d_0$ and $D(x,\ \cdot\,) \in C^1(\mathbb{R}^d_0\,;\,\mathbb{R}^d)$ for each $x \in \mathbb{R}^d$. We further assume that the first derivatives with respect to the variable in $\mathbb{R}^d$ are bounded. Moreover, suppose that

$$\sup_{x\in\mathbb{R}^d} \left| \int_{K_1} (\partial D)(x,z)\, \nu(dz) \right| + \sup_{x\in\mathbb{R}^d} \int_{K_1} \left|(\partial D)(x,z)\right|^p \nu(dz) < +\infty, \tag{1}$$

$$\sup_{x\in\mathbb{R}^d} \sup_{z\in K_1^c} \left|(\partial_z D)(x,z)\right| < +\infty, \tag{2}$$

$$\lim_{|z|\to 0} \sup_{x\in\mathbb{R}^d} |D(x,z)| = 0 \tag{3}$$

for all $p \geq 2$. Here, the notation "∂" means the gradient with respect to the variable in $\mathbb{R}^d$, while "∂_z" is the one with respect to $z \in \mathbb{R}^d_0$. Write $A = (A_1,\ \ldots,\ A_d)$, $B = (B_1,\ \ldots,\ B_d)$ and $C = (C_1,\ \ldots,\ C_d)$.

For $\xi \in \mathbb{R}^d$, let us consider the $\mathbb{R}^d$-valued processes $\{X_t\,;\, t \in [0,T]\}$ and $\{Y_t\,;\, t \in [0,T]\}$ determined by the equations:

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$$X_t = \xi + \int_0^t A_0(X_s)\,ds + \int_0^t A(X_{s-})\,dW_s + \int_0^t \int_{\mathbb{R}^d_0} B(X_{s-})\,z\,d\tilde{J}, \tag{4}$$

$$Y_t = \xi + \int_0^t C_0(Y_s)\,ds + \int_0^t C(Y_{s-})\,dW_s + \int_0^t \int_{\mathbb{R}^d_0} D(Y_{s-}, z)\,d\tilde{J}. \tag{5}$$

Under the conditions on all of the coefficients of (4) and (5), there exist the unique solutions (cf. [1, 9, 13]). Our main goal is to study the upper and lower estimates on the Wasserstein distance, which can be also interpreted as the Kantrovich-Rubinstein one via the duality formula, between the random variables X_t and Y_t, in terms of the coefficients of (4) and (5). Historical background on the Wasserstein distance can be seen in a standard book [19].

Our interest can be applied to the study on the Riemannian manifold M. Although there are several approaches to construct the jump processes on M, we shall adopt in the present paper the method by the canonical projection of the process valued in the bundle of the orthonormal frames $O(M)$ over M, which is often called the Eells-Elworthy-Malliavin construction. The $O(M)$-valued process is defined by the solution to the Marcus-type stochastic differential equations with jumps such that the process jumps along the exponential mapping along the horizontal vector fields over $O(M)$. See [2, 3, 10, 13]. On the other hand, since the Brownian motion on M can be constructed by the analytic approach via the semigroup approach of the Laplace-Beltrami operator, or by the probabilistic one via the Eells-Elworthy-Malliavin approach in terms of the Stratonovich-type stochastic differential equations without any jumps seen in [7, 9], we can obtain the M-valued jump process by the subordination. Our interest is to study the Wasserstein distance between the subordinated Brownian motion on M and the projected jump process determined by the Marcus-type equation with jumps. Furthermore, we can also revisit the results stated in [5, 16] that the law of the subordinated Brownian motion on M doesn't always coincide with the one of the projected jump process determined by the Marcus-type equation. Since the lower bound of the L^1-Wasserstein distance between those processes is studied in Section 4, our approach it to attack the interest in the paper completely different from [5, 16].

The organization of the paper is as follows: let us study in $\mathbb{R}^d$ the upper estimate of the Wasserstein distance in Section 2, and its lower estimate in Section 3. As one of the applications to our results, consider the problem on the laws of the processes valued in the Riemannian manifolds in Section 4.

2. Upper estimate on Wasserstein distance

Let X and Y be the $\mathbb{R}^d$-valued processes determined by the equations (4) and (5). The Wasserstein distance between the $\mathbb{R}^d$-valued random variables X_t and Y_t is defined by

$$d_W(X_t,\, Y_t) := \sup_{h \in \mathrm{Lip}_1(\mathbb{R}^d)} \left| \mathbb{E}[h(X_t)] - \mathbb{E}[h(Y_t)] \right|, \tag{6}$$

where $\mathrm{Lip}_1(\mathbb{R}^d)$ is the family of the Lipschitz continuous functions h on $\mathbb{R}^d$ such that $|h(x) - h(y)| \le |x - y|$ for $x,\, y \in \mathbb{R}^d$ (see [15] for the Wasserstein distance). Choose $0 \le t \le T$, and define the stopping time σ by

$$\sigma := \inf\{s > 0\,;\ J((0, s] \times K_1^c) \neq 0\} \wedge t. \tag{7}$$

Then, it is a routine work to obtain that

Proposition 1. *For $p \geq 1$, it holds that*

$$\mathbb{E}\left[\sup_{u<\sigma}|X_u - Y_u|^p\right] \leq \sqrt{t}\, C_{1,p,T,A_0,A,B,C_0,C,D,\xi}, \tag{8}$$

where $C_{1,p,T,A_0,A,B,C_0,C,D,\xi}$ is the positive constant.

Proof. First, let us consider the case of $p \geq 2$, and study the estimate on $\mathbb{E}\left[\sup_{u<t\wedge\sigma}|Y_u - \xi|^p\right]$. Let $0 \leq t \leq \tilde{T} \leq T$. Since

$$Y_u = \xi + \int_0^u C_0(Y_s)\,ds + \int_0^u C(Y_{s-})\,dW_s + \int_0^u\int_{K_1} D(Y_{s-}, z)\,d\tilde{J}$$

for $0 \leq u < \sigma$, we see from the Hölder inequality and the Burkholder-Davis-Gundy one (cf. Proposition 7.1.2 in [12] and Proposition 2.6.1 in [13]) that

$$\begin{aligned}\mathbb{E}\left[\sup_{u<t\wedge\sigma}|Y_u - \xi|^p\right] &\leq C_{2,p,\tilde{T}}\,\mathbb{E}\left[\int_0^t |C_0(Y_{s\wedge\sigma})|^p\,ds\right] + C_{3,p,\tilde{T}}\,\mathbb{E}\left[\int_0^t \|C(Y_{s\wedge\sigma})\|^p\,ds\right]\\ &\quad + C_{4,p,\tilde{T}}\,\mathbb{E}\left[\left(\int_0^t\int_{K_1}|D(Y_{s\wedge\sigma}, z)|^2\,d\hat{J}\right)^{p/2}\right]\\ &\quad + C_{5,p,\tilde{T}}\,\mathbb{E}\left[\int_0^t\int_{K_1}|D(Y_{s\wedge\sigma}, z)|^p\,d\hat{J}\right]\\ &\leq \tilde{T}\,C_{6,p,\tilde{T},\xi} + C_{7,p,\tilde{T},C_0,C,D}\int_0^t \mathbb{E}\left[\sup_{u<s\wedge\sigma}|Y_u - \xi|^p\right]ds.\end{aligned}$$

Here, we use the fact that the set $\{s \in [0,t]\,;\, Y_s \neq Y_{s-}\}$ is at most countable a.s. This leads us via the Gronwall inequality to obtain that

$$\mathbb{E}\left[\sup_{u<t\wedge\sigma}|Y_u - \xi|^p\right] \leq \tilde{T}\,C_{6,p,\tilde{T},\xi}\,\exp\left(t\,C_{7,p,\tilde{T},C_0,C,D}\right) \leq \tilde{T}\,C_{8,p,T,C_0,C,D,\xi},$$

where $C_{8,p,T,C_0,C,D,\xi} := \sup_{0\leq t\leq T}\left\{C_{6,p,t,\xi}\,\exp\left(t\,C_{7,p,t,C_0,C,D}\right)\right\}$. In particular, choosing $\tilde{T} = t$ yields that

$$\mathbb{E}\left[\sup_{u<t\wedge\sigma}|Y_u - \xi|^p\right] \leq t\,C_{8,p,T,C_0,C,D,\xi}. \tag{9}$$

Secondly, let us study the estimate on $\mathbb{E}\left[\sup_{u<t\wedge\sigma}|X_u - Y_u|^p\right]$. Since

$$\begin{aligned}X_u &= \xi + \int_0^u A_0(X_s)\,ds + \int_0^u A(X_{s-})\,dW_s + \int_0^u\int_{K_1} B(X_s-)\,z\,d\tilde{J},\\ Y_u &= \xi + \int_0^u C_0(Y_s)\,ds + \int_0^u C(Y_{s-})\,dW_s + \int_0^u\int_{K_1} D(Y_{s-}, z)\,d\tilde{J}\end{aligned}$$

for $0 \leq u < t \wedge \sigma$, we have

$$\begin{aligned}\mathbb{E}\left[\sup_{u<t\wedge\sigma}|X_u - Y_u|^p\right] &\leq C_{9,p}\,\mathbb{E}\left[\sup_{u<t\wedge\sigma}\left|\int_0^u\{A_0(X_s) - C_0(Y_s)\}\,ds\right|^p\right]\\ &\quad + C_{9,p}\,\mathbb{E}\left[\sup_{u<t\wedge\sigma}\left|\int_0^u\{A(X_s) - C(Y_s)\}\,dW_s\right|^p\right]\\ &\quad + C_{9,p}\,\mathbb{E}\left[\sup_{u<t\wedge\sigma}\left|\int_0^u\int_{K_1}\{B(X_{s-})\,z - D(Y_{s-}, z)\}\,d\tilde{J}\right|^p\right]\end{aligned}$$

$$=: I_{1,p,t} + I_{2,p,t} + I_{3,p,t}.$$

Now, we shall study the upper estimate of $I_{1,p,t}$, $I_{2,p,t}$ and $I_{3,p,t}$ via the Burkholder-Davis-Gundy inequality (cf. Proposition 7.1.2 in [12] and Proposition 2.6.1 in [13]) and the Hölder one.

As for $I_{1,p,t}$, since the functions A_0 and C_0 satisfy the linear growth conditions, we see that

$$\begin{aligned}
I_{1,p,t} &\leq C_{10,p,\tilde{T}}\, \mathbb{E}\left[\int_0^t \left|A_0(X_{s\wedge\sigma}) - C_0(Y_{s\wedge\sigma})\right|^p ds\right] \\
&\leq C_{11,p,\tilde{T}}\, \mathbb{E}\left[\int_0^t \left|A_0(X_{s\wedge\sigma}) - A_0(Y_{s\wedge\sigma})\right|^p ds\right] \\
&\quad + C_{11,p,\tilde{T}}\, \mathbb{E}\left[\int_0^t \left|A_0(Y_{s\wedge\sigma}) - A_0(\xi)\right|^p ds\right] \\
&\quad + C_{11,p,\tilde{T}}\, \mathbb{E}\left[\int_0^t \left|C_0(Y_{s\wedge\sigma}) - C_0(\xi)\right|^p ds\right] + t\, C_{11,p,\tilde{T}} \left|A_0(\xi) - C_0(\xi)\right|^p \\
&\leq C_{12,p,\tilde{T},A_0} \int_0^t \mathbb{E}\left[\sup_{u<s\wedge\sigma} |X_u - Y_u|^p\right] ds + \tilde{T}\, C_{13,p,\tilde{T},A_0,C_0,\xi}.
\end{aligned}$$

As for $I_{2,p,t}$, similarly to the estimate of $I_{1,p,t}$, since the functions A_i and C_i $(1 \leq i \leq d)$ satisfy the linear growth conditions, we see that

$$\begin{aligned}
I_{2,p,t} &\leq C_{14,p,\tilde{T}}\, \mathbb{E}\left[\int_0^t \left\|A(X_{s\wedge\sigma}) - C(Y_{s\wedge\sigma})\right\|^p ds\right] \\
&\leq C_{15,p,\tilde{T}}\, \mathbb{E}\left[\int_0^t \left\|A(X_{s\wedge\sigma}) - A(Y_{s\wedge\sigma})\right\|^p ds\right] + C_{15,p,\tilde{T}}\, \mathbb{E}\left[\int_0^t \left\|A(Y_{s\wedge\sigma}) - A(\xi)\right\|^p ds\right] \\
&\quad + C_{15,p,\tilde{T}}\, \mathbb{E}\left[\int_0^t \left\|C(Y_{s\wedge\sigma}) - C(\xi)\right\|^p ds\right] + C_{15,p,\tilde{T}} \left\|A(\xi) - C(\xi)\right\|^p \\
&\leq C_{16,p,\tilde{T},A} \int_0^t \mathbb{E}\left[\sup_{u<s\wedge\sigma} |X_u - Y_u|^p\right] ds + \tilde{T}\, C_{17,p,\tilde{T},A,C,\xi}.
\end{aligned}$$

As for $I_{3,p,t}$, similarly to the estimate of $I_{2,p,t}$, since

$$\begin{aligned}
B(X_s)\, z - D(Y_s, z) &= \{B(X_s)\, z - B(Y_s)\, z\} + \{B(Y_s)\, z - B(\xi)\, z\} \\
&\quad - \{D(Y_s, z) - D(\xi, z)\} + \{B(\xi)\, z - D(\xi, z)\},
\end{aligned}$$

we have

$$\begin{aligned}
I_{3,p,t} &\leq C_{18,p,\tilde{T}}\, \mathbb{E}\left[\left(\int_0^t \int_{K_1} \left|B(X_{s\wedge\sigma})\, z - D(Y_{s\wedge\sigma}, z)\right|^2 d\hat{J}\right)^{p/2}\right] \\
&\quad + C_{18,p,\tilde{T}}\, \mathbb{E}\left[\int_0^t \int_{K_1} \left|B(X_{s\wedge\sigma})\, z - D(Y_{s\wedge\sigma}, z)\right|^p d\hat{J}\right] \\
&\leq C_{19,p,\tilde{T}}\, \mathbb{E}\left[\left(\int_0^t \int_{K_1} \left|B(X_{s\wedge\sigma}) - B(Y_{s\wedge\sigma})\right|^2 |z|^2\, d\hat{J}\right)^{p/2}\right] \\
&\quad + C_{19,p,\tilde{T}}\, \mathbb{E}\left[\left(\int_0^t \int_{K_1} \left|B(Y_{s\wedge\sigma})\, z - B(\xi)\, z\right|^2 d\hat{J}\right)^{p/2}\right]
\end{aligned}$$

$$
\begin{aligned}
&+ C_{19,p,\tilde{T}}\, \mathbb{E}\left[\left(\int_0^t \int_{K_1} \left|D(Y_{s\wedge\sigma}, z) - D(\xi, z)\right|^2 d\hat{J}\right)^{p/2}\right] \\
&+ t\, C_{19,p,\tilde{T}} \left(\int_{K_1} \left|B(\xi)\, z - D(\xi, z)\right|^2 \nu(dz)\right)^{p/2} \\
&+ C_{19,p,\tilde{T}}\, \mathbb{E}\left[\int_0^t \int_{K_1} \left|B(X_{s\wedge\sigma}) - B(Y_{s\wedge\sigma})\right|^p |z|^p\, d\hat{J}\right] \\
&+ C_{19,p,\tilde{T}}\, \mathbb{E}\left[\int_0^t \int_{K_1} \left|B(Y_{s\wedge\sigma})\, z - B(\xi)\, z\right|^p d\hat{J}\right] \\
&+ C_{19,p,\tilde{T}}\, \mathbb{E}\left[\int_0^t \int_{K_1} \left|D(Y_{s\wedge\sigma}, z) - D(\xi, z)\right|^p d\hat{J}\right] \\
&+ t\, C_{19,p,\tilde{T}} \int_{K_1} \left|B(\xi)\, z - D(\xi, z)\right|^p \nu(dz) \\
\le\ & C_{20,p,\tilde{T},B} \int_0^t \mathbb{E}\left[\sup_{u<s\wedge\sigma} |X_u - Y_u|^p\right] ds + \tilde{T}\, C_{21,p,\tilde{T},B,D,\xi}.
\end{aligned}
$$

Hence, we can obtain that

$$
\begin{aligned}
\mathbb{E}\left[\sup_{u<t\wedge\sigma} \left|X_u - Y_u\right|^p\right] \le\ & \tilde{T}\, (C_{13,p,\tilde{T},A_0,C_0,\xi} + C_{17,p,\tilde{T},A,C,\xi} + C_{21,p,\tilde{T},B,D,\xi}) \\
&+ (C_{12,p,\tilde{T},A_0} + C_{16,p,\tilde{T},A} + C_{20,p,\tilde{T},B}) \int_0^t \mathbb{E}\left[\sup_{u<s\wedge\sigma} \left|X_u - Y_u\right|^p\right] ds,
\end{aligned}
$$

which implies from the Gronwall inequality that

$$
\begin{aligned}
\mathbb{E}\left[\sup_{u<t\wedge\sigma} \left|X_u - Y_u\right|^p\right] &\le \tilde{T}\, (C_{13,p,\tilde{T},A_0,C_0,\xi} + C_{17,p,\tilde{T},A,C,\xi} + C_{21,p,\tilde{T},B,D,\xi}) \\
&\qquad \times \exp\left[(C_{12,p,\tilde{T},A_0} + C_{16,p,\tilde{T},A} + C_{20,p,\tilde{T},B})\, t\right] \\
&\le \tilde{T}\, C_{22,p,T,A_0,A,B,C_0,C,D,\xi},
\end{aligned}
$$

where

$$
\begin{aligned}
C_{22,p,T,A_0,A,B,C_0,C,D,\xi} := \sup_{0\le t\le T} \Big\{ &(C_{13,p,t,A_0,C_0,\xi} + C_{17,p,t,A,C,\xi} + C_{21,p,t,B,D,\xi}) \\
&\times \exp\left[(C_{12,p,t,A_0} + C_{16,p,t,A} + C_{20,p,t,B})\, t\right] \Big\}.
\end{aligned}
$$

In particular, choosing $\tilde{T} = t$ implies that

$$
\mathbb{E}\left[\sup_{u<t\wedge\sigma} \left|X_u - Y_u\right|^p\right] \le t\, C_{22,p,T,A_0,A,B,C_0,C,D,\xi},
$$

Finally, the study for $1 \le p < 2$ is just the direct consequence of the Cauchy-Schwarz inequality:

$$
\mathbb{E}\left[\sup_{u<t\wedge\sigma} \left|X_u - Y_u\right|^p\right] \le \mathbb{E}\left[\sup_{u<t\wedge\sigma} \left|X_u - Y_u\right|^{2p}\right]^{1/2} \le \sqrt{t}\, \sqrt{C_{22,2p,T,A_0,A,B,C_0,C,D,\xi}}.
$$

Thus, we can get the assertion by choosing as

$$C_{1,p,T,A_0,A,B,C_0,C,D,\xi} := \begin{cases} \sqrt{T}\, C_{22,p,T,A_0,A,B,C_0,C,D,\xi} & (p \geq 2), \\ \sqrt{C_{22,2p,T,A_0,A,B,C_0,C,D,\xi}} & (1 \leq p < 2). \end{cases} \tag{10}$$

The proof is complete. □

Proposition 2. *Let $0 \leq t \leq T$. Suppose that there exists a constant $p_0 > 1$ satisfying*

$$\int_{K_1^c} |z|^{p_0}\, \nu(dz) < +\infty.$$

Then, for all $1 \leq p < p_0$, it holds that

$$\mathbb{E}\left[\sup_{u \leq t} |X_u - Y_u|^p\right] \leq \left(t^{1/(2\alpha_p)} \vee t^{p/p_0}\right) C_{23,p,p_0,T,A_0,A,B,C_0,C,D,\xi}, \tag{11}$$

where $\alpha_p := p_0/(p_0 - p)$ and $C_{23,p,p_0,T,A_0,A,B,C_0,C,D,\xi}$ is the positive constant.

Proof. Define the sequence of stopping times $\{\sigma_n \,;\, n \in \mathbb{N}\}$ given by

$$\sigma_1 := \sigma, \quad \sigma_{n+1} := \inf\left\{s > \sigma_n \,;\, J((\sigma_n,\, s] \times K_1^c) \neq 0\right\} \wedge t$$

for $n \in \mathbb{N}$, inductively.

(i) Since the upper estimate of $\mathbb{E}\left[\sup_{u<\sigma_1} |X_u - Y_u|^p\right]$ has been already proved in Proposition 1, it is sufficient to study the estimate of $\mathbb{E}[|X_{\sigma_1} - Y_{\sigma_1}|^p]$. Write

$$L_t = \int_0^t \int_{\mathbb{R}_0^d} z\, d\overline{J}.$$

Remark that

$$\begin{aligned} \mathbb{E}[|\Delta L_{\sigma_1}|^{p_0}] &= \mathbb{E}\left[\left|\int_0^t \int_{K_1^c} z\, dJ\right|^{p_0}\right] \\ &\leq C_{24,p_0} \left|\int_0^t \int_{K_1^c} z\, d\hat{J}\right|^{p_0} + C_{24,p_0}\, \mathbb{E}\left[\left|\int_0^t \int_{K_1^c} z\, d\tilde{J}\right|^{p_0}\right] \\ &\leq (C_{25,p_0,T} + C_{26,p_0}) \int_0^t \int_{K_1^c} |z|^{p_0}\, \nu(dz) + C_{26,p_0} \left(\int_0^t \int_{K_1^c} |z|^2\, d\hat{J}\right)^{p_0/2} \\ &\leq t\, C_{27,p_0,T} \int_{K_1^c} |z|^{p_0}\, \nu(dz) \\ &=: t\, C_{28,p_0,T} \end{aligned}$$

from the Hölder inequality and the Burkholder inequality, and

$$\begin{aligned} \left|X_{\sigma_1} - Y_{\sigma_1}\right| &\leq \left|X_{\sigma_1-} - Y_{\sigma_1-}\right| + \left|B(X_{\sigma_1-})\, \Delta L_{\sigma_1} - D(Y_{\sigma_1-}, \Delta L_{\sigma_1})\right| \\ &\leq |X_{\sigma_1-} - Y_{\sigma_1-}|\, (1 + \|\partial B\|_\infty\, |\Delta L_{\sigma_1}|) \\ &\quad + |Y_{\sigma_1-} - \xi|\, (\|\partial B\|_\infty\, |\Delta L_{\sigma_1}| + \|\partial D(\cdot, \Delta L_{\sigma_1})\|_\infty) \\ &\quad + \sup_{z \in K_1^c} |B(\xi) - (\partial_z D)(\xi, z)|\, |\Delta L_{\sigma_1}|. \end{aligned}$$

Here, we have used (3) in the second equality with the help of the mean value theorem for $w \longmapsto D(\xi, w)$ between 0 and ΔL_{σ_1} of ξ. Then, we see that

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{u\le\sigma_1}|X_u-Y_u|^p\right]\\
&\le C_{29,p}\,\mathbb{E}\left[\sup_{u<\sigma_1}|X_u-Y_u|^p\right]+C_{29,p}\,\mathbb{E}\big[\big|X_{\sigma_1}-Y_{\sigma_1}\big|^p\big]\\
&\le C_{30,p}\,\mathbb{E}\left[\left(\sup_{u<\sigma_1}|X_u-Y_u|^p\right)\left(2+\|\partial B\|_\infty^p\,|\Delta L_{\sigma_1}|^p\right)\right]\\
&\quad+C_{30,p}\,\mathbb{E}\left[\left(\sup_{u<\sigma_1}|Y_u-\xi|^p\right)\left(\|\partial B\|_\infty^p\,|\Delta L_{\sigma_1}|^p+\|\partial D(\cdot,\Delta L_{\sigma_1})\|_\infty^p\right)\right]\\
&\quad+C_{30,p}\left(\sup_{z\in K_1^c}\big|B(\xi)-(\partial_z D)(\xi,z)\big|^p\right)\mathbb{E}[|\Delta L_{\sigma_1}|^p]\\
&\quad+C_{30,p}\left(\sup_{z\in K_1^c}\big|B(\xi)-(\partial_z D)(\xi,z)\big|^p\right)\mathbb{E}[|\Delta L_{\sigma_1}|^{p_0}]^{p/p_0}\\
&\le \left(t^{1/(2\alpha_p)}\vee t^{p/p_0}\right)C_{31,p,p_0,T,A_0,A,B,C_0,C,D,\xi}
\end{aligned}
$$

from Proposition 1 and the moment estimate (9). Here, we have used the Hölder inequality in the third inequality.

(ii) Consider the case of $\sigma_1\le u\le\sigma_2$. Since

$$
\begin{aligned}
X_u-Y_u&=X_{\sigma_1}-Y_{\sigma_1}+\int_{\sigma_1}^u\{A_0(X_s)-C_0(Y_s)\}\,ds+\int_{\sigma_1}^u\{A(X_s)-C(Y_s)\}\,dW_s\\
&\quad+\int_{\sigma_1}^u\int_{K_1}\{B(X_{s-})\,z-D(X_{s-},z)\}\,d\tilde{J}
\end{aligned}
$$

for $\sigma_1\le u<\sigma_2$, and

$$
\begin{aligned}
\big|X_{\sigma_2}-Y_{\sigma_2}\big|&\le\big|X_{\sigma_2-}-Y_{\sigma_2-}\big|+\big|B(X_{\sigma_2-})\,\Delta L_{\sigma_2}-D(Y_{\sigma_2-},\Delta L_{\sigma_2})\big|\\
&\le|X_{\sigma_2-}-Y_{\sigma_2-}|\,(1+\|\partial B\|_\infty\,|\Delta L_{\sigma_2}|)\\
&\qquad+|Y_{\sigma_2-}-Y_{\sigma_1}|\,(\|\partial B\|_\infty\,|\Delta L_{\sigma_2}|+\|\partial D(\cdot,\Delta L_{\sigma_2})\|_\infty)\\
&\qquad+\sup_{z\in K_1^c}|B(Y_{\sigma_1})-(\partial_z D)(Y_{\sigma_1},z)|\,|\Delta L_{\sigma_2}|,
\end{aligned}
$$

the estimate $\mathbb{E}\left[\sup_{u\le\sigma_2}|X_u-Y_u|^p\right]$ can be also given via a similar method studied in Proposition 1 and (i) stated above, because

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{u\le\sigma_2}\big|X_u-Y_u\big|^p\right]\\
&\le C_{32,p}\,\mathbb{E}\left[\sup_{u\le\sigma_1}\big|X_u-Y_u\big|^p\right]+C_{32,p}\,\mathbb{E}\left[\sup_{\sigma_1\le u\le\sigma_2}\big|X_u-Y_u\big|^p\right]\\
&\le\left(t^{1/(2\alpha_p)}\vee t^{p/p_0}\right)C_{32,p}\left(C_{31,p,p_0,T,A_0,A,B,C_0,C,D,\xi}+C_{33,p,p_0,T,A_0,A,B,C_0,C,D,\xi}\right)\\
&=:\left(t^{1/(2\alpha_p)}\vee t^{p/p_0}\right)C_{34,p,p_0,T,A_0,A,B,C_0,C,D,\xi}.
\end{aligned}
$$

(iii) The inductive argument as stated in (ii) enables us to obtain that

$$
\mathbb{E}\left[\sup_{u\le\sigma_n}|X_u-Y_u|^p\right]\le\left(t^{1/(2\alpha_p)}\vee t^{p/p_0}\right)C_{35,p,p_0,T,A_0,A,B,C_0,C,D,\xi}.
$$

The proof is complete. □

Theorem 1. *Under the same situation as* Proposition 2*, it holds that*

$$d_W(X_t,\ Y_t) \le (t^{1/(2\alpha_1)} \vee t^{p/p_0})\, C_{23,1,p_0,T,A_0,A,B,C_0,C,D,\xi}. \tag{12}$$

Proof. The assertion is the direct consequence of Proposition 2, because

$$d_W(X_t,\ Y_t) \le \mathbb{E}[|X_t - Y_t|] \le (t^{1/(2\alpha_1)} \vee t^{p/p_0})\, C_{23,1,p_0,T,A_0,A,B,C_0,C,D,\xi}.$$ □

Remark 1. Under the same situation stated in Theorem 1, the Wasserstein distance $d_W(X_t,\ Y_t)$ is small, if the coefficients in each term of the equations (4) and (5) are close, which implies that the characteristic functions of the random variables X_t and Y_t are also close, since the function $\mathbb{R}^d \ni x \longmapsto e^{i\theta\cdot x}$ for $\theta \in \mathbb{R}^d$ is in $\mathrm{Lip}_1(\mathbb{R}^d)$.

3. Lower estimate on Wasserstein distance

In this section, we shall consider the lower estimate of the Wasserstein distance. Assume the condition in Proposition 2 throughout this section again. Since

$$X_t = \xi + \int_0^t A_0(X_s)\, ds + \int_0^t A(X_s)\, dW_s + \int_0^t\!\int_{K_1} B(X_{s-})\, z\, d\tilde{J} + \int_0^t\!\int_{K_1^c} B(X_{s-})\, z\, dJ, \tag{13}$$

$$Y_t = \xi + \int_0^t C_0(Y_s)\, ds + \int_0^t C(Y_s)\, dW_s + \int_0^t\!\int_{K_1} D(Y_{s-}, z)\, d\tilde{J} + \int_0^t\!\int_{K_1^c} D(Y_{s-}, z)\, dJ, \tag{14}$$

we have

$$\mathbb{E}[X_t - Y_t] = \int_0^t \mathbb{E}[A_0(X_s) - C_0(Y_s)]\, ds + \int_0^t\!\int_{K_1^c} \mathbb{E}[B(X_s)\, z - D(Y_s, z)]\, d\hat{J}. \tag{15}$$

For $1 \le k \le d$, let us define

$$C_{36,k,A_0,B,C_0,D} := \inf_{x\in\mathbb{R}^d} \left| \{A_0^k(x) - C_0^k(x)\} + \int_{K_1^c} \{B^k(x)\, z - D^k(x, z)\}\, \nu(dz) \right|. \tag{16}$$

Proposition 3. *Under the same situation as* Proposition 2*, it holds that*

$$\left|\mathbb{E}[X_t - Y_t]\right| \ge \left\{ t \left(C_{36,k,A_0,B,C_0,D} - C_{37,p_0,T,A_0,A,B,C_0,C,D,\xi}\, (t^{1/(2\alpha_1)} \vee t^{1/p_0}) \right) \right\} \vee 0, \tag{17}$$

where $C_{37,p_0,T,A_0,A,B,C_0,C,D,\xi}$ is a positive constant.

Proof. Now, we are in the position that

$$\begin{aligned}\mathbb{E}[X_t - Y_t] &= \int_0^t \mathbb{E}[A_0(X_s) - C_0(Y_s)]\, ds + \int_0^t\!\int_{K_1^c} \mathbb{E}[B(X_s)\, z - D(Y_s, z)]\, d\hat{J} \\ &= \int_0^t (\psi_{0,s} + \psi_s)\, ds + \int_0^t (\varphi_{0,s} + \varphi_s)\, ds,\end{aligned}$$

where

$$\psi_{0,t} := \mathbb{E}[A_0(Y_t) - C_0(Y_t)], \qquad \varphi_{0,t} := \mathbb{E}[A_0(X_t) - A_0(Y_t)],$$

$$\psi_t := \int_{K_1^c} \mathbb{E}[B(Y_t)\, z - D(Y_t, z)]\, \nu(dz), \qquad \varphi_t := \int_{K_1^c} \mathbb{E}[B(X_t)\, z - B(Y_t)\, z]\, \nu(dz).$$

Hence, since the coefficients A_i $(0 \le i \le d)$ and B are Lipschitz continuous, we see that

$$\begin{aligned} \left|\mathbb{E}[X_t - Y_t]\right| &\ge \left|\int_0^t (\psi_{0,s} + \psi_s)\, ds\right| - \left|\int_0^t (\varphi_{0,s} + \varphi_s)\, ds\right| \\ &\ge \left|\int_0^t (\psi_{0,s} + \psi_s)\, ds\right| - C_{38,A_0,B} \int_0^t \mathbb{E}[|X_s - Y_s|]\, ds. \end{aligned}$$

On the other hand, since

$$\int_0^t \mathbb{E}[|X_s - Y_s|]\, ds \le t\,(t^{1/(2\alpha_1)} \vee t^{1/p_0})\, C_{23,1,p_0,T,A_0,A,B,C_0,C,D,\xi}$$

from Proposition 2, and

$$\int_0^t (\psi_{0,s} + \psi_s)\, ds = t\,(\psi_{0,\kappa} + \psi_\kappa)$$

from the mean value theorem, where $0 < \kappa < t$ is a constant, we have

$$\begin{aligned} \left|\mathbb{E}[X_t - Y_t]\right| &\ge t\,\left\{\left|\psi_{0,\kappa} + \psi_\kappa\right| - C_{38,A_0,B}\, C_{23,1,p_0,T,A_0,A,B,C_0,C,D,\xi}\,(t^{1/(2\alpha_1)} \vee t^{1/p_0})\right\} \\ &= t\,\left\{\left|\psi_{0,\kappa} + \psi_\kappa\right| - C_{39,p_0,T,A_0,A,B,C_0,C,D,\xi}\,(t^{1/(2\alpha_1)} \vee t^{1/p_0})\right\}, \end{aligned} \tag{18}$$

where $C_{39,p_0,T,A_0,A,B,C_0,C,D,\xi} := C_{38,A_0,B}\, C_{23,1,p_0,T,A_0,A,B,C_0,C,D,\xi}$. Then, we have

$$\begin{aligned} \left|\psi_{0,\kappa} + \psi_\kappa\right| &= \left|\mathbb{E}[A_0(Y_\kappa) - C_0(Y_\kappa)] + \int_{K_1^c} \mathbb{E}[B(Y_\kappa)\, z - D(Y_\kappa, z)]\, \nu(dz)\right| \\ &\ge \left|\mathbb{E}[A_0^k(Y_\kappa) - C_0^k(Y_\kappa)] + \int_{K_1^c} \mathbb{E}[B^k(Y_\kappa)\, z - D^k(Y_\kappa, z)]\, \nu(dz)\right| \\ &\ge C_{36,k,A_0,B,C_0,D} \end{aligned}$$

from the condition (16). We can derive from (18) that

$$\left|\mathbb{E}[X_t - Y_t]\right| \ge t\,\left\{C_{36,k,A_0,B,C_0,D} - C_{39,p_0,T,A_0,A,B,C_0,C,D,\xi}\,(t^{1/(2\alpha_1)} \vee t^{1/p_0})\right\}.$$

Since it is trivial that $\left|\mathbb{E}[X_t - Y_t]\right| \ge 0$, we can get the conclusion. □

Theorem 2 (Lower estimate)**.** *Under the same condition in* Proposition 2*, it holds that*

$$d_W(X_t,\, Y_t) \ge \left[t\,\left\{C_{36,k,A_0,B,C_0,D} - C_{39,p_0,T,A_0,A,B,C_0,C,D,\xi}\,(t^{1/(2\alpha_1)} \vee t^{1/p_0})\right\}\right] \vee 0. \tag{19}$$

Proof. Since the function h defined by $h(x) = x$ is in $\mathrm{Lip}_1(\mathbb{R}^d)$, Proposition 3 implies that

$$\begin{aligned} d_W(X_t,\, Y_t) &\ge \left|\mathbb{E}[X_t - Y_t]\right| \\ &\ge \left[t\,\left\{C_{36,k,A_0,B,C_0,D} - C_{39,p_0,T,A_0,A,B,C_0,C,D,\xi}\,(t^{1/(2\alpha_1)} \vee t^{1/p_0})\right\}\right] \vee 0, \end{aligned}$$

which completes the proof. □

Corollary 1. *Under the same condition in* Proposition 2*, the laws of the* $\mathbb{R}^d$*-valued processes* X *and* Y *are not always equivalent.*

Proof. Suppose that the laws of the processes X and Y are equivalent. Then, those any finite-dimensional distributions are also equivalent, which implies that $d_W(X_t,\, Y_t) = 0$ for all

$0 < t \leq T$. Now, let us choose $t \in [0, T]$ such that

$$0 < t < \left(\frac{C_{36,k,A_0,B,C_0,D}}{C_{39,p_0,T,A_0,A,B,C_0,C,D,\xi}} \right)^{2\alpha_1} \wedge \left(\frac{C_{36,k,A_0,B,C_0,D}}{C_{39,p_0,T,A_0,A,B,C_0,C,D,\xi}} \right)^{p_0}. \tag{20}$$

Then, from the estimate (19) in Theorem 2, the Wasserstein distance of the $\mathbb{R}^d$-valued random variables X_t and Y_t can be estimated from the below as

$$d_W(X_t, Y_t) \geq \left| \mathbb{E}[X_t - Y_t] \right| \geq t \left\{ C_{36,k,A_0,B,C_0,D} - C_{39,p_0,T,A_0,A,B,C_0,C,D,\xi} \left(t^{1/2\alpha_1} \vee t^{1/p_0} \right) \right\},$$

which is strictly positive from (20). That is the contradiction. □

4. Study on manifolds

In this section, we shall study the situation on a Riemannian manifold. Let M be a connected, compact and smooth Riemannian manifold of dimension d with the Levi-Civita connection $\nabla = \{\Gamma^i_{jk} ; 1 \leq i, j, k \leq d\}$, and denote by

$$O(M) = \{r = (x, e_x) ; x \in M, e_x = ((e_1)_x, \ldots, (e_d)_x) \text{ is the orthonormal basis in } T_x M\}$$

the bundle of orthonormal frames on M. Let $\pi : O(M) \to M$ be the projection given by $\pi(r) = x$ for $r = (x, e_x) \in O(M)$. For $r = (x, e_x) \in O(M)$, denote by φ a local coordinate in a coordinate neighborhood $V \subset O(M)$ around r, and by ψ a local coordinate in a coordinate neighborhood $\pi(V) \subset M$ around $x = \pi(r)$. Write $\tilde{\pi} = \psi \circ \pi \circ \varphi^{-1}$. Let H_i $(1 \leq i \leq d)$ be the horizontal vector fields over $O(M)$. Then, for $F \in C^\infty(O(M))$, the vector fields H_i $(1 \leq i \leq d)$ can be expressed as

$$(H_i F)(r) = \sum_{j=1}^{d} e_i^j \left(\frac{\partial F^\varphi}{\partial x_j} \right) (\varphi(r)) - \sum_{k,l,p,q=1}^{d} \Gamma^q_{kl}(\pi(r))\, e_i^k\, e_p^l \left(\frac{\partial F^\varphi}{\partial e_p^q} \right) (\varphi(r)) \tag{21}$$

in the local coordinate around $r \in O(M)$, where $F^\varphi = F \circ \varphi^{-1}$ and $\varphi(r) = (x_i, e_i^j ; 1 \leq i, j \leq d)$.

For $r \in O(M)$, let us consider the $O(M)$-valued process $\{\tilde{R}_t ; t \in [0, T]\}$ determined by the Stratonovich stochastic differential equation of the form:

$$d\tilde{R}_t = H(\tilde{R}_t) \circ dW_t, \quad \tilde{R}_0 = r, \tag{22}$$

equivalently,

$$F(\tilde{R}_t) = F(r) + \int_0^t (HF)(\tilde{R}_s) \circ dW_s = F(r) + \int_0^t (\mathcal{L}F)(\tilde{R}_s)\, ds + \int_0^t (HF)(\tilde{R}_s)\, dW_s, \tag{23}$$

for all $F \in C^\infty(O(M))$, where $H = (H_1, \ldots, H_d)$ and

$$(\mathcal{L}F)(r) = \frac{1}{2} \sum_{i=1}^{d} (H_i(H_i F))(r).$$

The operator $\mathcal{L}$ is the infinitesimal generator of the process $\tilde{R}$, which is often called the horizontal Laplacian of Bochner's sense (cf. [9]). Then, the M-valued process $\{\tilde{X}_t ; t \in [0, T]\}$ defined by $\tilde{X}_t := \pi(\tilde{R}_t)$ is the Brownian motion on M with the infinitesimal generator $\Delta_M/2$, where Δ_M is the Laplace-Beltrami operator on M. In fact, for any $f \in C^\infty(M)$, since

$$(\mathcal{L}(f \circ \pi))(r) = \frac{1}{2}(\Delta_M f)(\pi(r)), \tag{24}$$

we have

$$\begin{aligned} f(\tilde{X}_t) &= (f \circ \pi)(r) + \int_0^t (\mathcal{L}(f \circ \pi))(\tilde{R}_s)\, ds + \int_0^t (H(f \circ \pi))(\tilde{R}_s)\, dW_s \\ &= f(x) + \int_0^t \frac{1}{2}(\Delta_M f)(\tilde{X}_s)\, ds + \int_0^t ((\pi_* H) f)(\tilde{X}_s)\, dW_s, \end{aligned} \tag{25}$$

where $((\pi_* A) f)(\pi(r)) = (A(f \circ \pi))(r)$ for a vector field A over $O(M)$, a smooth function f on M, and $r \in O(M)$.

Let $\{\tau_t\,;\, t \in [0, T]\}$ be a subordinator with the Lévy triplet $(\gamma^\tau,\, 0,\, \eta)$, independent of the Brownian motion W, where $\eta(d\theta)$ is the Lévy measure on $(0, +\infty)$ such that the function $\theta \wedge 1$ is integrable with respect to the measure $\eta(d\theta)$, and $\gamma^\tau := \int_0^1 \theta\, \eta(d\theta)$ is the drift. Then, for each $t \in [0, T]$, the Lévy-Itô decomposition theorem (cf. [18]) implies that

$$\tau_t = \gamma^\tau\, t + \int_0^t \int_0^{+\infty} \theta\, d\overline{N}, \tag{26}$$

where $dN(\equiv N(ds, d\theta))$ is the Poisson random measure over $(0, T] \times (0, +\infty)$ with the intensity measure $d\hat{N}(\equiv \hat{N}(ds, d\theta)) = ds\, \eta(d\theta)$, $d\tilde{N}(\equiv \tilde{N}(ds, d\theta)) = dN - d\hat{N}$ is the compensated one, and $d\overline{N}(\equiv \overline{N}(ds, d\theta)) = \mathbb{I}_{(0,1]}(\theta)\, d\tilde{N} + \mathbb{I}_{(1,+\infty)}(\theta)\, dN$. Denote by $p(\theta, z)$ the density of the $\mathbb{R}^d$-valued random variable W_θ for $\theta > 0$. From Theorem 30.1 in [18], the subordinated process $\{W_{\tau_t}\,;\, 0 \le t \le T\}$ is the $\mathbb{R}^d$-valued Lévy process with the Lévy triplet $(\gamma^{W_\tau},\, 0,\, \nu)$, where $\nu(dz)$ is the σ-finite measure on $\mathbb{R}^d_0$ defined by

$$\nu(dz) := \int_0^{+\infty} \mathbb{P}[W_\theta \in dz]\, \eta(d\theta) \left(= \int_0^{+\infty} p(\theta, z)\, \eta(d\theta)\, dz \right), \tag{27}$$

and $\gamma^{W_\tau} := \int_{K_1} z\, \nu(dz)$. Hence, for each $t \in [0, T]$, we see that

$$W_{\tau_t} = t\, \gamma^{W_\tau} + \int_0^t \int_{\mathbb{R}^d_0} z\, d\overline{J} \tag{28}$$

from the Lévy-Itô decomposition theorem (cf. [18]) again, where $dJ(\equiv J(ds, dz))$ is the Poisson random measure over $(0, T] \times \mathbb{R}^d_0$ with the intensity measure $d\hat{J}(\equiv \hat{J}(ds, dz)) = ds\, \nu(dz)$, $d\tilde{J}(\equiv \tilde{J}(ds, dz)) = dJ - d\hat{J}$ and $d\overline{J}(\equiv \overline{J}(ds, dz)) = \mathbb{I}_{K_1}(z)\, d\tilde{J} + \mathbb{I}_{K_1^c}(z)\, dJ$. Then, the $O(M)$-valued subordinated process $\{R_t := \tilde{R}_{\tau_t}\,;\, t \in [0, T]\}$ satisfies that

$$\begin{aligned} F(R_t) &= F(r) + \int_0^{\tau_t} (\mathcal{L}F)(\tilde{R}_s)\, ds + \int_0^{\tau_t} (HF)(\tilde{R}_s)\, dW_s \\ &= F(r) + \int_0^t (\mathcal{L}F)(\tilde{R}_{\tau_{s-}})\, d\tau_s + \int_0^t (HF)(\tilde{R}_{\tau_{s-}})\, dW_{\tau_s} \\ &= F(r) + \int_0^t (\mathcal{A}F)(R_s)\, ds + \int_0^t \int_0^{+\infty} (\mathcal{L}F)(R_{s-})\, \theta\, d\overline{N} + \int_0^t \int_{\mathbb{R}^d_0} (HF)(R_{s-})\, z\, d\overline{J} \end{aligned} \tag{29}$$

for all $F \in C^\infty(O(M))$, where

$$(\mathcal{A}F)(r) := (\mathcal{L}F)(r)\, \gamma^\tau + (HF)(r)\, \gamma^{W_\tau} = \int_0^1 (\mathcal{L}F)(r)\, \theta\, \eta(d\theta) + \int_{K_1} (HF)(r)\, z\, \nu(dz).$$

Define the M-valued process $\{X_t\,;\, t\in[0,T]\}$ by $X_t := \tilde{X}_{\tau_t} = \pi(R_t)$. In particular, we have

$$
\begin{aligned}
(30)\qquad f(X_t) &= (f\circ\pi)(r) + \int_0^t (\mathcal{A}(f\circ\pi))(R_s)\,ds + \int_0^t\int_0^{+\infty} (\mathcal{L}(f\circ\pi))(R_{s-})\,\theta\,d\overline{N}\\
&\quad + \int_0^t\int_{\mathbb{R}_0^d} (H(f\circ\pi))(R_{s-})\,z\,d\overline{J}\\
&= f(x) + \int_0^t ((\pi_*\mathcal{A})f)(X_s)\,ds + \int_0^t\int_0^{+\infty} ((\pi_*\mathcal{L})f)(X_{s-})\,\theta\,d\overline{N}\\
&\quad + \int_0^t\int_{\mathbb{R}_0^d} ((\pi_*H)f)(X_{s-})\,z\,d\overline{J}
\end{aligned}
$$

for $f\in C^\infty(M)$, which implies that

$$
\begin{aligned}
(31)\qquad \mathbb{E}[f(X_t)] &= f(x) + \int_0^t \mathbb{E}\Big[((\pi_*\mathcal{A})f)(X_s)\Big]ds\\
&\quad + \int_0^t\int_1^{+\infty} \mathbb{E}\Big[((\pi_*\mathcal{L})f)(X_s)\,\theta\Big]d\hat{N} + \int_0^t\int_{K_1^c} \mathbb{E}\Big[((\pi_*H)f)(X_s)\,z\Big]d\hat{J}\\
&= f(x) + \int_0^t\int_0^{+\infty} \mathbb{E}\Big[((\pi_*\mathcal{L})f)(X_s)\,\theta\Big]d\hat{N} + \int_0^t\int_{\mathbb{R}_0^d} \mathbb{E}\Big[((\pi_*H)f)(X_s)\,z\Big]d\hat{J}.
\end{aligned}
$$

The second equality can be justified, because

$$
\begin{aligned}
&\int_0^t \mathbb{E}\Big[((\pi_*\mathcal{A})f)(X_s)\Big]ds\\
&= \int_0^t \mathbb{E}\Big[(\mathcal{A}(f\circ\pi))(R_s)\Big]ds\\
&= \int_0^t\int_0^1 \mathbb{E}\Big[(\mathcal{L}(f\circ\pi))(R_s)\,\theta\Big]\,\eta(d\theta)\,ds + \int_0^t\int_{K_1} \mathbb{E}\Big[(H(f\circ\pi))(R_s)\,z\Big]\,\nu(dz)\,ds\\
&= \int_0^t\int_0^1 \mathbb{E}\Big[((\pi_*\mathcal{L})f)(X_s)\,\theta\Big]\,d\hat{N} + \int_0^t\int_{K_1} \mathbb{E}\Big[((\pi_*H)f)(X_s)\,z\Big]\,d\hat{J}.
\end{aligned}
$$

On the other hand, for $r\in O(M)$, let $\{U_t\,;\, t\in[0,T]\}$ be the $O(M)$-valued process determined by the canonical stochastic differential equation with jumps of Marcus type:

$$
\begin{aligned}
(32)\qquad F(U_t) &= F(r) + \int_0^t (\mathcal{B}F)(U_s)\,ds + \int_0^t\int_0^{+\infty} \{F(\mathrm{Exp}(\mathcal{L}\,\theta)(U_{s-})) - F(U_{s-})\}\,d\overline{N}\\
&\quad + \int_0^t\int_{\mathbb{R}_0^d} \{F(\mathrm{Exp}(H\,z)(U_{s-})) - F(U_{s-})\}\,d\overline{J}
\end{aligned}
$$

for all $F\in C^\infty(O(M))$, where

$$
(\mathcal{B}F)(u) := \int_0^1 \{F(\mathrm{Exp}(\mathcal{L}\,\theta)(u)) - F(u)\}\,\eta(d\theta) + \int_{K_1} \{F(\mathrm{Exp}(H\,z)(u)) - F(u)\}\,\nu(dz).
$$

Moreover, for given $\theta\in(0,+\infty)$ and $z\in\mathbb{R}_0^d$, let $\{\Lambda^{\theta,\sigma}(u) := \mathrm{Exp}(\sigma\mathcal{L}\theta)(u)\,;\,\sigma\in[0,1],\ u\in O(M)\}$ and $\{\Xi^{z,\sigma}(u) := \mathrm{Exp}(\sigma\,z\,H)(u)\,;\,\sigma\in[0,1],\ u\in O(M)\}$ be the one parameter groups of diffeomorphisms over $O(M)$, that is, the unique solutions to the ordinary differential equations of the forms:

$$\frac{d}{d\sigma}\Lambda^{\theta,\sigma}(u) = (\theta\mathcal{L})(\Lambda^{\theta,\sigma}(u)), \quad \Lambda^{\theta,0}(u) = u, \tag{33}$$

$$\frac{d}{d\sigma}\Xi^{z,\sigma}(u) = (Hz)(\Xi^{z,\sigma}(u)), \quad \Xi^{z,0}(u) = u. \tag{34}$$

Define the M-valued process $\{Y_t\,;\ t \in [0,T]\}$ by $Y_t = \pi(U_t)$. Since

$$\pi(\mathrm{Exp}(\mathcal{L}\,\theta)(u)) = \mathrm{Exp}(\pi_*\mathcal{L}\,\theta)(\pi(u)), \quad \pi(\mathrm{Exp}(H\,z)(u)) = \mathrm{Exp}(\pi_* H\,z)(\pi(u)),$$

we see that, for $f \in C^\infty(M)$,

$$\begin{aligned} f(Y_t) &= (f\circ\pi)(U_t) \\ &= f(x) + \int_0^t ((\pi_*\mathcal{B})f)(Y_s)\,ds + \int_0^t\int_0^{+\infty} \{f(\mathrm{Exp}(\pi_*\mathcal{L}\,\theta)(Y_{s-})) - f(Y_{s-})\}\,d\overline{N} \\ &\quad + \int_0^t\int_{\mathbb{R}_0^d} \{f(\mathrm{Exp}(\pi_* H\,z)(Y_{s-})) - f(Y_{s-})\}\,d\overline{J}, \end{aligned} \tag{35}$$

which implies that

$$\begin{aligned} \mathbb{E}[f(Y_t)] &= f(\pi(r)) + \int_0^t \mathbb{E}\Big[((\pi_*\mathcal{B})f)(Y_s)\Big]ds \\ &\quad + \int_0^t\int_1^{+\infty} \mathbb{E}\Big[f(\mathrm{Exp}(\pi_*\mathcal{L}\,\theta)(Y_s)) - f(Y_s)\Big]d\hat{N} \\ &\quad + \int_0^t\int_{K_1^c} \mathbb{E}\Big[f(\mathrm{Exp}(\pi_* H\,z)(Y_s)) - f(Y_s)\Big]d\hat{J} \\ &= f(\pi(r)) + \int_0^t\int_0^{+\infty} \mathbb{E}\Big[f(\mathrm{Exp}(\pi_*\mathcal{L}\,\theta)(Y_s)) - f(Y_s)\Big]\,d\hat{N} \\ &\quad + \int_0^t\int_{\mathbb{R}_0^d} \mathbb{E}\Big[f(\mathrm{Exp}(\pi_* H\,z)(Y_s)) - f(Y_s)\Big]\,d\hat{J}. \end{aligned} \tag{36}$$

The second equality can be justified, because

$$\begin{aligned} \int_0^t \mathbb{E}\Big[((\pi_*\mathcal{B})f)(Y_s)\Big]\,ds &= \int_0^t \mathbb{E}\Big[(\mathcal{B}(f\circ\pi))(U_s)\Big]\,ds \\ &= \int_0^t\int_0^1 \mathbb{E}\Big[(f\circ\pi)(\mathrm{Exp}(\mathcal{L}\,\theta)(U_s)) - (f\circ\pi)(U_s)\Big]\,\eta(d\theta)\,ds \\ &\quad + \int_0^t\int_{K_1} \mathbb{E}\Big[(f\circ\pi)(\mathrm{Exp}(H\,z)(U_s)) - (f\circ\pi)(U_s)\Big]\,\nu(dz)\,ds \\ &= \int_0^t\int_0^1 \mathbb{E}\Big[f(\mathrm{Exp}(\pi_*\mathcal{L}\,\theta)(Y_s)) - f(Y_s)\Big]\,d\hat{N} \\ &\quad + \int_0^t\int_{K_1} \mathbb{E}\Big[f(\mathrm{Exp}(\pi_* H\,z)(Y_s)) - f(Y_s)\Big]\,d\hat{J}. \end{aligned}$$

Hence, we have

$$\begin{aligned} &\mathbb{E}[f(Y_t)] - \mathbb{E}[f(X_t)] \\ &= \int_0^t\int_0^{+\infty} \mathbb{E}\Big[((\psi_*\pi_*\mathcal{L})f^\psi)(\psi(Y_s)) - ((\psi_*\pi_*\mathcal{L})f^\psi)(\psi(X_s))\Big]\,\theta\,d\hat{N} \end{aligned} \tag{37}$$

$$
\begin{aligned}
&+ \int_0^t \int_{\mathbb{R}_0^d} \mathbb{E}\Big[((\psi_*\pi_* H) f^\psi)(\psi(Y_s)) - ((\psi_*\pi_* H) f^\psi)(\psi(X_s))\Big]\, z\, d\hat{J} \\
&+ \int_0^t \int_0^{+\infty} \mathbb{E}\Big[f^\psi\Big(\mathrm{Exp}(\psi_*\pi_*\mathcal{L}\,\theta)(\psi(Y_s))\Big) - f^\psi(\psi(Y_s)) \\
&\qquad\qquad - ((\psi_*\pi_*\mathcal{L}) f^\psi)(\psi(Y_s))\,\theta\Big]\, d\hat{N} \\
&+ \int_0^t \int_{\mathbb{R}_0^d} \mathbb{E}\Big[f^\psi\Big(\mathrm{Exp}(\psi_*\pi_* H\, z)(\psi(Y_s))\Big) - f^\psi(\psi(Y_s)) \\
&\qquad\qquad - ((\psi_*\pi_* H) f^\psi)(\psi(Y_s))\, z\Big]\, d\hat{J}.
\end{aligned}
$$

Then, we have

Theorem 3. *Let $C_{40} \in \mathbb{R}^m \otimes \mathbb{R}^m$ be a positive definite symmetric matrix, and $f \in C^\infty(M)$ such that*

$$
\Big((\psi_*\pi_* H)((\psi_*\pi_* H) f)\Big)(\xi) = C_{40}, \quad \xi \in M. \tag{38}
$$

Then, under the condition

$$
\int_{K_1} |z|\, \nu(dz) < \infty,
$$

the law of the subordinated Brownian motion X on M is not the one of the M-valued process Y given by the projection of the $O(M)$-valued process determined by the equation (36).

Remark 2. In order to guarantee the existence of the solution function f to the Poisson equation (38), we need the additional condition on the manifold M. It is sufficient that the manifold M is complete, and has a positive spectrum and a Ricci curvature bounded below by a negative constant. See [4, 14].

Proof of Theorem 3. Recall the stopping time σ given in (7). Denote by $\mathrm{Tr}(C_{40})$ the trace of the matrix C_{40}. Since

$$
((\psi_*\pi_*\mathcal{L}) f^\psi)(\psi(\xi)) = ((\pi_*\mathcal{L}) f)(\xi) = (\mathcal{L}(f\circ\pi))(r) = \frac{(\Delta_M f)(\xi)}{2} = \frac{\mathrm{Tr}(C_{40})}{2}
$$

for $r \in O(M)$ with $\pi(r) = \xi$ from (24) under the condition (38), the Taylor theorem leads us to see that

$$
((\psi_*\pi_*\mathcal{L}) f^\psi)(\psi(Y_s)) - ((\psi_*\pi_*\mathcal{L}) f^\psi)(\psi(X_s)) = \frac{\mathrm{Tr}(C_{40})}{2} - \frac{\mathrm{Tr}(C_{40})}{2} = 0,
$$

and that

$$
\begin{aligned}
& f^\psi\Big(\mathrm{Exp}(\psi_*\pi_*\mathcal{L}\,\theta)(\psi(Y_s))\Big) - f^\psi(\psi(Y_s)) - ((\psi_*\pi_*\mathcal{L}) f^\psi)(\psi(Y_s))\,\theta \\
&= \frac{1}{2}\Big((\psi_*\pi_*\mathcal{L}\,\theta)((\psi_*\pi_*\mathcal{L}\,\theta) f^\psi)\Big)\Big(\mathrm{Exp}(\kappa\,\psi_*\pi_*\mathcal{L}\,\theta)(\psi(Y_s))\Big) \\
&= \frac{1}{2}\Big((\psi_*\pi_*\mathcal{L}\,\theta)((\psi_*\pi_*\mathcal{L}\,\theta) f^\psi)\Big)\Big(\psi(\mathrm{Exp}(\kappa\,\pi_*\mathcal{L}\,\theta)(Y_s))\Big) \\
&= 0
\end{aligned}
$$

from the Taylor theorem, where $0 < \kappa < 1$ is a constant. Thus, we have from (37) that

$$
\begin{aligned}
&\mathbb{E}[f(Y_{(t\wedge\sigma)-})] - \mathbb{E}[f(X_{(t\wedge\sigma)-})]\\
&= \int_0^t\int_0^{+\infty} \mathbb{E}\Big[((\psi_*\pi_*\mathcal{L})f^\psi)(\psi(Y_{(s\wedge\sigma)-})) - ((\psi_*\pi_*\mathcal{L})f^\psi)(\psi(X_{(s\wedge\sigma)-}))\Big]\,\theta\, d\hat{N}\\
&\quad + \int_0^t\int_{\mathbb{R}^d_0} \mathbb{E}\Big[((\psi_*\pi_*H)f^\psi)(\psi(Y_{(s\wedge\sigma)-})) - ((\psi_*\pi_*H)f^\psi)(\psi(X_{(s\wedge\sigma)-}))\Big]\,z\, d\hat{J}\\
&\quad + \int_0^t\int_0^{+\infty} \mathbb{E}\Big[f^\psi\Big(\mathrm{Exp}(\psi_*\pi_*\mathcal{L}\,\theta)(\psi(Y_{(s\wedge\sigma)-}))\Big) - f^\psi(\psi(Y_{(s\wedge\sigma)-}))\\
&\qquad\qquad - ((\psi_*\pi_*\mathcal{L})f^\psi)(\psi(Y_{(s\wedge\sigma)-}))\,\theta\Big]\, d\hat{N}\\
&\quad + \int_0^t\int_{\mathbb{R}^d_0} \mathbb{E}\Big[f^\psi\Big(\mathrm{Exp}(\psi_*\pi_*H\,z)(\psi(Y_{(s\wedge\sigma)-}))\Big) - f^\psi(\psi(Y_{(s\wedge\sigma)-}))\\
&\qquad\qquad - ((\psi_*\pi_*H)f^\psi)(\psi(Y_{(s\wedge\sigma)-}))\,z\Big]\, d\hat{J}\\
&= \int_0^t\int_{K_1} \mathbb{E}\Big[((\psi_*\pi_*H)f^\psi)(\psi(Y_{(s\wedge\sigma)-})) - ((\psi_*\pi_*H)f^\psi)(\psi(X_{(s\wedge\sigma)-}))\Big]\,z\, d\hat{J}\\
&\quad + \frac{1}{2}\int_0^t\int_{K_1} \mathbb{E}\Big[\Big((\psi_*\pi_*H\,z)((\psi_*\pi_*H\,z)f^\psi)\Big)\Big(\psi(\mathrm{Exp}(\kappa\,\pi_*H\,z)(Y_{(s\wedge\sigma)-}))\Big)\Big]\, d\hat{J}\\
&=: I_1 + I_2,
\end{aligned}
$$

where $0 < \kappa < 1$ is a constant.

Since

$$
\mathbb{E}\Big[\Big|\psi(Y_{(s\wedge\sigma)-}) - \psi(X_{(s\wedge\sigma)-})\Big|\Big] \le \sqrt{s}\, C_{41,T,H,x},
$$

similarly to Proposition 1, we can obtain that

$$
\begin{aligned}
I_1 &\ge -\left|\int_0^t\int_{K_1} \mathbb{E}\Big[((\psi_*\pi_*H)f^\psi)(\psi(Y_{(s\wedge\sigma)-})) - ((\psi_*\pi_*H)f^\psi)(\psi(X_{(s\wedge\sigma)-}))\Big]\,z\, d\hat{J}\right|\\
&\ge -\left(\big\|\partial\{(\psi_*\pi_*H)f^\psi\}\big\|_\infty \int_{K_1}|z|\,\nu(dz)\right)\int_0^t \mathbb{E}\big[\big|\psi(Y_{(s\wedge\sigma)-}) - \psi(X_{(s\wedge\sigma)-})\big|\big]\,ds\\
&\ge -t^{3/2}\,\big\|\partial\{(\psi_*\pi_*H)f^\psi\}\big\|_\infty\, C_{37,T,H,x}\int_{K_1}|z|\,\nu(dz),
\end{aligned}
$$

where $\big\|\partial\{(\psi_*\pi_*H)f^\psi\}\big\|_\infty := \sup_{\xi\in\overline{(\psi\circ\pi)(V)}}\Big|(\partial\{(\psi_*\pi_*H)f^\psi\})(\xi)\Big|$. On the other hand, as for I_2, since the matrix C_{40} is positive definite, we see that

$$
I_2 = \frac{1}{2}\int_{K_1} z^*\, C_{40}\, z\,\nu(dz) \ge \frac{t\,C_{42}}{2}\int_{K_1}|z|^2\,\nu(dz),
$$

where $C_{42} > 0$ is the minimum eigenvalue of the matrix C_{40}. Here, we shall choose t as

$$
0 < t < \left\{\frac{C_{42}\displaystyle\int_{K_1}|z|^2\,\nu(dz)}{2\,\big\|\partial\{(\psi_*\pi_*H)f^\psi\big\|_\infty\, C_{41,T,H,x}\displaystyle\int_{K_1}|z|\,\nu(dz)}\right\}^2. \tag{39}
$$

Then, we can get that

$$
\mathbb{E}[f(Y_{(t\wedge\sigma)-})] - \mathbb{E}[f(X_{(t\wedge\sigma)-})] \tag{40}
$$

$$
\begin{aligned}
&\geq -t^{3/2} \left\| \partial\{(\psi_* \pi_* H) f^{\psi}\} \right\|_\infty C_{41,T,H,x} \int_{K_1} |z|\, \nu(dz) + \frac{t\, C_{42}}{2} \int_{K_1} |z|^2\, \nu(dz) \\
&= t \left\| \partial\{(\psi_* \pi_* H) f^{\psi} \right\|_\infty C_{41,T,H,x} \int_{K_1} |z|\, \nu(dz) \\
&\qquad \times \left\{ -\sqrt{t} + \frac{C_{42} \int_{K_1} |z|^2\, \nu(dz)}{2 \left\| \partial\{(\psi_* \pi_* H) f^{\psi} \right\|_\infty C_{41,T,H,x} \int_{K_1} |z|\, \nu(dz)} \right\},
\end{aligned}
$$

which is strictly positive.

Now, let us return the proof on the assertion in Theorem 3. Suppose that the laws of the processes X and Y are equivalent. Then, since those any finite-dimensional distributions are also equivalent, it holds that

$$
d_W(f(Y_{(t\wedge\sigma)-}),\ f(X_{(t\wedge\sigma)-})) = 0
$$

for $0 < t \leq T$, which is the contradiction to the strict positivity on the lower estimate (40). □

REMARK 3. In [5, 16], a similar study to Theorem 3 has been already discussed. Our strategy to attack Theorem 3 can be also regarded as completely different approach to their results.

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Department of Mathematics
Tokyo Woman's Christian University
2–6–1 Zempukuji, Suginami-ku
Tokyo 167–8585
Japan
e-mail: a-takeuchi@lab.twcu.ac.jp