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DYNAMICAL NUMBER OF BASE-POINTS OF NON BASE-WANDERING JONQUIÈRES TWISTS

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Abstract

We give some properties of the dynamical number of base-points of birational self-maps of the complex projective plane.

In particular we give a formula to determine the dynamical number of base-points of non base-wandering Jonquières twists.

1. Introduction

The *plane Cremona group* $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ is the group of birational maps of the complex projective plane $\mathbb{P}^2_\mathbb{C}$. It is isomorphic to the group of $\mathbb{C}$-algebra automorphisms of $\mathbb{C}(X,Y)$, the function field of $\mathbb{P}^2_\mathbb{C}$. Using a system of homogeneous coordinates $(x:y:z)$ a birational map $f\in\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ can be written as

$$(x:y:z)\dashrightarrow(P_0(x,y,z):P_1(x,y,z):P_2(x,y,z)),$$

where P_0, P_1 and P_2 are homogeneous polynomials of the same degree without common factor. This degree does not depend on the system of homogeneous coordinates. We call it the *degree* of f and denote it by $\deg(f)$. Geometrically it is the degree of the pull-back by f of a general projective line. Birational maps of degree 1 are homographies and form the group $\mathrm{Aut}(\mathbb{P}^2_\mathbb{C})=\mathrm{PGL}(3,\mathbb{C})$ of automorphisms of the projective plane.

$\diamond$ **Four types of elements**.

The elements $f\in\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ can be classified into exactly one of the four following types according to the growth of the sequence $(\deg(f^k))_{k\in\mathbb{N}}$ (*see* [8, 4]):

(1) The sequence $(\deg(f^k))_{k\in\mathbb{N}}$ is bounded, f is either of finite order or conjugate to an automorphism of $\mathbb{P}^2_\mathbb{C}$; we say that f is an *elliptic element*.

(2) The sequence $(\deg(f^k))_{k\in\mathbb{N}}$ grows linearly, f preserves a unique pencil of rational curves and f is not conjugate to an automorphism of any rational projective surface; we call f a *Jonquières twist*.

(3) The sequence $(\deg(f^k))_{k\in\mathbb{N}}$ grows quadratically, f is conjugate to an automorphism of a rational projective surface preserving a unique elliptic fibration; we call f a *Halphen twist*.

(4) The sequence $(\deg(f^k))_{k\in\mathbb{N}}$ grows exponentially and we say that f is *hyperbolic*.

$\diamond$ **The Jonquières group**.

Let us fix an affine chart of $\mathbb{P}^2_\mathbb{C}$ with coordinates (x,y). The *Jonquières group* J is

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the subgroup of the Cremona group of all maps of the form

$$(x,y) \dashrightarrow \left(\frac{A(y)x+B(y)}{C(y)x+D(y)}, \frac{ay+b}{cy+d}\right), \tag{1.1}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2,\mathbb{C})$ and $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{PGL}(2,\mathbb{C}(y))$. The group J is the group of all birational maps of $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$ permuting the fibers of the projection onto the second factor; it is isomorphic to the semi-direct product $\mathrm{PGL}(2,\mathbb{C}(y))\rtimes\mathrm{PGL}(2,\mathbb{C})$.

We can check with (1.1) that if f belongs to J, then $(\deg(f^k))_{k\in\mathbb{N}}$ grows at most linearly; elements of J are either elliptic or Jonquières twists. Let us denote by $\mathcal{J}$ the set of Jonquières twist:

$$\mathcal{J} = \{f \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}}) \mid f \text{ Jonquières twist }\}.$$

A Jonquières twist is called a *base-wandering Jonquières twist* if its action on the basis of the rational fibration has infinite order. Let us denote by J_0 the normal subgroup of J that preserves fiberwise the rational fibration, that is the subgroup of those maps of the form

$$(x,y) \dashrightarrow \left(\frac{A(y)x+B(y)}{C(y)x+D(y)}, y\right).$$

The group J_0 is isomorphic to $\mathrm{PGL}(2,\mathbb{C}(y))$. The group J_0 has three maximal (for the inclusion) uncountable abelian subgroups

$$\mathrm{J}_a = \{(x+a(y),y) \mid a \in \mathbb{C}(y)\}, \qquad \mathrm{J}_m = \{(b(y)x,y) \mid b \in \mathbb{C}(y)^*\},$$

and

$$\mathrm{J}_F = \left\{(x,y), \left(\frac{c(y)x+F(y)}{x+c(y)}, y\right) \mid c \in \mathbb{C}[y]\right\},$$

where F denotes an element of $\mathbb{C}[y]$ that is not a square ([7]).

Let us associate to $f = \left(\frac{A(y)x+B(y)}{C(y)x+D(y)}, y\right) \in \mathrm{J}_0$ the matrix $M_f = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. The *Baum Bott index* of f is $\mathrm{BB}(f) = \frac{(\mathrm{Tr}(M_f))^2}{\det(M_f)}$ (by analogy with the Baum Bott index of a foliation) which is well defined in PGL and is invariant by conjugation. This invariant BB indicates the degree growth:

Proposition 1.1 ([5]). *Let f be a Jonquières twist that preserves fiberwise the rational fibration. The rational function* $\mathrm{BB}(f)$ *is constant if and only if f is an elliptic element.*

A direct consequence is the following:

Corollary 1.2. *Let f be a non-base wandering Jonquières twist*; *the rational function* $\mathrm{BB}(f)$ *is constant if and only if f is an elliptic element.*

Every $f \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ admits a resolution

$$\begin{array}{ccc} & S & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathbb{P}^2_{\mathbb{C}} & \overset{}{\underset{f}{\dashrightarrow}} & \mathbb{P}^2_{\mathbb{C}} \end{array}$$

where π_1, π_2 are sequences of blow-ups. The resolution is *minimal* if and only if no (-1)-curve of S are contracted by both π_1 and π_2. Assume that the resolution is minimal; the *base-points* of f are the points blown-up by π_1, which can be points of S or infinitely near points. If f belongs to J, then f has one base-point p_0 of multiplicity $d-1$ and $2d-2$ base-points $p_1, p_2, \ldots, p_{2d-2}$ of multiplicity 1. Similarly the map f^{-1} has one base-point q_0 of multiplicity $d-1$ and $2d-2$ base-points q_1, q_2, $\ldots$, q_{2d-2} of multiplicity 1. Let us denote by $f_\sharp$ the action of f on the Picard-Manin space of $\mathbb{P}^2_{\mathbb{C}}$, by $\mathbf{e}_m \in \mathrm{NS}(S)$ the Néron-Severi class of the total transform of m under π_j (for $1 \leq j \leq 2$), and by ℓ the class of a line in $\mathbb{P}^2_{\mathbb{C}}$. The action of f on ℓ and the classes $(\mathbf{e}_{p_j})_{0\leq j\leq 2d-2}$ is given by:

$$\begin{cases} f_\sharp(\ell) = d\ell - (d-1)\mathbf{e}_{q_0} - \displaystyle\sum_{i=1}^{2d-2} \mathbf{e}_{q_i}, \\ f_\sharp(\mathbf{e}_{p_0}) = (d-1)\ell - (d-2)\mathbf{e}_{q_0} - \displaystyle\sum_{i=1}^{2d-2} \mathbf{e}_{q_i}, \\ f_\sharp(\mathbf{e}_{p_i}) = \ell - \mathbf{e}_{q_0} - \mathbf{e}_{q_i} \qquad \forall\, 1 \leq i \leq 2d-2. \end{cases}$$

◇ **Dynamical degree**.

Given a birational self-map $f\colon S \dashrightarrow S$ of a complex projective surface, its dynamical degree $\lambda(f)$ is a positive real number that measures the complexity of the dynamics of f. Indeed $\log(\lambda(f))$ provides an upper bound for the topological entropy of f and is equal to it under natural assumptions (*see* [3, 9]). The dynamical degree is invariant under conjugacy; as shown in [2] precise knowledge on $\lambda(f)$ provides useful information on the conjugacy class of f. By definition a *Pisot number* is an algebraic integer $\lambda \in]1, +\infty[$ whose other Galois conjugates lie in the open unit disk; Pisot numbers include integers $d \geq 2$ as well as reciprocal quadratic integers $\lambda > 1$. A *Salem number* is an algebraic integer $\lambda \in]1, +\infty[$ whose other Galois conjugates are in the closed unit disk, with at least one on the boundary. Diller and Favre proved the following statement:

Theorem 1.3 ([8])**.** *Let f be a birational self-map of a complex projective surface.*

If $\lambda(f)$ is different from 1, then $\lambda(f)$ is a Pisot number or a Salem number.

One of the goal of [2] is the study of the structure of the set of all dynamical degrees $\lambda(f)$ where f runs over the group of birational maps $\mathrm{Bir}(S)$ and S over the collection of all projective surfaces. In particular they get:

Theorem 1.4 ([2])**.** *Let Λ be the set of all dynamical degrees of birational maps of complex projective surfaces. Then*

- ◇ *Λ is a well ordered subset of $\mathbb{R}_+$;*
- ◇ *if λ is an element of Λ, there is a real number $\varepsilon > 0$ such that $]\lambda, \lambda+\varepsilon]$ does*

not intersect Λ;

- ⋄ *there is a non-empty interval* $]\lambda_G, \lambda_G + \varepsilon]$, *with* $\varepsilon > 0$, *on the right of the golden mean that contains infinitely many Pisot and Salem numbers, but does not contain any dynamical degree.*

⋄ **Dynamical number of base-points** ([4]).

If S is a projective smooth surface, every $f \in \mathrm{Bir}(S)$ admits a resolution

$$\begin{array}{ccc} & Z & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ S & \underset{f}{\dashrightarrow} & S \end{array}$$

where π_1, π_2 are sequences of blow-ups. The resolution is *minimal* if and only if no (-1)-curve of Z are contracted by both π_1 and π_2. Assume that the resolution is minimal; the *base-points* of f are the points blown-up by π_1, which can be points of S or infinitely near points. We denote by $\mathfrak{b}(f)$ the number of such points, which is also equal to the difference of the ranks of the Picard group $\mathrm{Pic}(Z)$ of Z and the Picard group $\mathrm{Pic}(S)$ of S, and thus equal to $\mathfrak{b}(f^{-1})$.

Let us define the *dynamical number of base-points* of f by

$$\mu(f) = \lim_{k\to+\infty} \frac{\mathfrak{b}(f^k)}{k}.$$

Since $\mathfrak{b}(f \circ \varphi) \leq \mathfrak{b}(f) + \mathfrak{b}(\varphi)$ for any $f, \varphi \in \mathrm{Bir}(S)$ we see that $\mu(f)$ is a non-negative real number. Moreover, $\mathfrak{b}(f^{-1})$ and $\mathfrak{b}(f)$ being equal we get $\mu(f^k) = |k\mu(f)|$ for any $k \in \mathbb{Z}$. Furthermore, the dynamical number of base-points is an invariant of conjugation: if $\psi \colon S \dashrightarrow Z$ is a birational map between smooth projective surfaces and if f belongs to $\mathrm{Bir}(S)$, then $\mu(f) = \mu(\psi \circ f \circ \psi^{-1})$. In particular if f is conjugate to an automorphism of a smooth projective surface, then $\mu(f) = 0$. The converse holds, *i.e.* $f \in \mathrm{Bir}(S)$ is conjugate to an automorphism of a smooth projective surface if and only if $\mu(f) = 0$ ([4, Proposition 3.5]). This follows from the geometric interpretation of μ we will recall now. If $f \in \mathrm{Bir}(S)$ is a birational map, a (possibly infinitely near) base-point p of f is a *persistent base-point* of f if there exists an integer N such that p is a base-point of f^k for any $k \geq N$ but is not a base-point of f^{-k} for any $k \geq N$. We put an equivalence relation on the set of points that belongs to S or are infinitely near: take a minimal resolution of f

$$\begin{array}{ccc} & Z & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ S & \underset{f}{\dashrightarrow} & S \end{array}$$

where π_1, π_2 are sequences of blow-ups; the point p is *equivalent* to q if there exists an integer k such that $(\pi_2 \circ \pi_1^{-1})^k(p) = q$. Denote by ν the number of equivalence classes of persistent base-points of f; then the set

$$\{\mathfrak{b}(f^k) - \nu k \,|\, k \geq 0\} \subset \mathbb{Z}$$

is bounded. In particular, $\mu(f)$ is an integer, equal to ν (*see* [4, Proposition 3.4]).

This gives a bound for $\mu(f)$; indeed, if $f \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is a map whose base-points have multiplicities $m_1 \geq m_2 \geq \ldots \geq m_r$ then (*see for instance* [1, §2.5] and [1, Corollary 2.6.7])

$$\begin{cases} \displaystyle\sum_{i=1}^{r} m_i = 3(\deg(f) - 1), \\ \displaystyle\sum_{i=1}^{r} m_i^2 = \deg(f)^2 - 1, \\ m_1 + m_2 + m_3 \geq \deg(f) + 1, \end{cases}$$

in particular, $r \leq 2\deg(f) - 1$ so $\nu \leq 2\deg(f) - 1$ and $\mu(f) \leq 2\deg(f) - 1$.

If $f \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is a Jonquières twist, then there exists an integer $a \in \mathbb{N}$ such that

$$\lim_{k\to+\infty} \frac{\deg(f^k)}{k} = a^2 \frac{\mu(f)}{2};$$

moreover, a is the degree of the curves of the unique pencil of rational curves invariant by f (*see* [4, Proposition 4.5]). In particular, $a = 1$ if and only if f preserves a pencil of lines. On the one hand $\{\mu(f) \,|\, f \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})\} \subseteq \mathbb{N}$ and on the other hand if f belongs to $\mathcal{J}$, then $\mu(f) > 0$; as a result

$$\{\mu(f) \,|\, f \in \mathcal{J}\} \subseteq \mathbb{N} \setminus \{0\}.$$

Let us recall that if $f_{\alpha,\beta} = \left(\frac{\alpha x+y}{x+1}, \beta y\right)$ then $\mu(f_{\alpha,\beta}) = 1$. Indeed, by induction one can prove that $f^{2n}_{\alpha,\beta} = \left(\frac{P_n(x,y)}{Q_n(x,y)}, \beta^{2n} y\right)$ with

$$P_n(x,y) = \sum_{0\leq i+j\leq n+1} a_{ij} x^i y^j, \qquad Q_n(x,y) = \sum_{0\leq i+j\leq n} b_{ij} x^i y^j,$$

and $a_{ij} \geq 0$, $b_{ij} \geq 0$ for any $n \geq 0$, so that $\deg f^{2n}_{\alpha,\beta} = n+1$ for any $n \geq 0$; we conclude using the fact that $\mu(f) = 2 \lim_{k\to+\infty} \frac{\deg(f^k)}{k}$. Furthermore, $\mu(f^k_{\alpha,\beta}) = |k\mu(f_{\alpha,\beta})| = |k|$ for any $k \in \mathbb{Z}$. Hence

$$\{\mu(f) \,|\, f \in \mathcal{J}\} = \mathbb{N} \setminus \{0\},$$

and

$$\{\mu(f) \,|\, f \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})\} = \mathbb{N}.$$

As we have seen if f belongs to $\mathcal{J}$, then $\mu(f) = 2 \lim_{k\to+\infty} \frac{\deg(f^k)}{k}$. Can we express $\mu(f)$ in a simpliest way ? We will see that if f is a non base-wandering Jonquières twist, the answer is yes.

◇ **Results**.

The dynamical number of base-points of birational self maps of the complex projective plane satisfies the following properties:

Theorem A. 1. *If f is a birational self-map from $\mathbb{P}^2_{\mathbb{C}}$ into itself, then its dynamical number of base-points is bounded*: *if $f \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$, then $\mu(f) \leq 2\deg(f) - 1$.*

2. *We can precise the set of all dynamical numbers of base-points of birational*

maps of $\mathbb{P}^2_{\mathbb{C}}$ *(resp. of Jonquières maps of* $\mathbb{P}^2_{\mathbb{C}}$*) :*

$$\{\mu(f)\,|\,f\in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})\}=\mathbb{N}\qquad \text{and}\qquad \{\mu(f)\,|\,f\in\mathcal{J}\}=\mathbb{N}\smallsetminus\{0\}.$$

3. *There exist sequences* $(f_n)_n$ *of birational self-maps of* $\mathbb{P}^2_{\mathbb{C}}$ *such that*
   - ⋄ $\mu(f_n)>0$ *for any* $n\in\mathbb{N}$;
   - ⋄ $\mu(\lim_{n\to+\infty} f_n)=0$.
4. *There exist sequences* $(f_n)_n$ *of birational self-maps of* $\mathbb{P}^2_{\mathbb{C}}$ *such that*
   - ⋄ $\mu(f_n)=0$ *for any* $n\in\mathbb{N}$;
   - ⋄ $\mu(\lim_{n\to+\infty} f_n)>0$.

Let us now give a formula to determine the dynamical number of base-points of Jonquières twists that preserves fiberwise the fibration.

Theorem B. *Let* $f=\left(\frac{A(y)x+B(y)}{C(y)x+D(y)},y\right)$ *be a Jonquières twist that preserves fiberwise the fibration, and let* M_f *be its associated matrix. Denote by* $\mathrm{Tr}(M_f)$ *the trace* M_f*, by* χ_f *the characteristic polynomial of* M_f*, and by* Δ_f *the discriminant of* χ_f*. Then exactly one of the following holds*:

1. *If* χ_f *has two distinct roots in* $\mathbb{C}[y]$*, then* f *is conjugate to* $g=\left(\frac{\mathrm{Tr}(M_f)+\delta_f}{\mathrm{Tr}(M_f)-\delta_f}\,x,y\right)$*, where* $\delta_f^2=\Delta_f$*, and*
$$\mu(f)=\mu(g)=2(\deg(g)-1).$$
2. *If* χ_f *has no root in* $\mathbb{C}[y]$*, set*
$$\Omega_f=\gcd\left(\frac{\mathrm{Tr}(M_f)}{2},\left(\frac{\mathrm{Tr}(M_f)}{2}\right)^2-\det(M_f)\right),$$
and let us define P_f *and* S_f *as*
$$\frac{\mathrm{Tr}(M_f)}{2}=P_f\Omega_f,\qquad\qquad \left(\frac{\mathrm{Tr}(M_f)}{2}\right)^2-\det(M_f)=S_f\Omega_f.$$
   - 2.a. *If* $\gcd(\Omega_f,S_f)=1$*, then*
     - ⋄ *if* $\deg(S_f)\leq\deg(\Omega_f)+2\deg(P_f)$*, then* $\mu(f)=\deg(\Omega_f)+2\deg(P_f)$;
     - ⋄ *otherwise* $\mu(f)=\deg(S_f)$.
   - 2.b. *If* $S_f=\Omega_f^pT_f$ *with* $p\geq 1$ *and* $\gcd(T_f,\Omega_f)=1$*, then*
     - ⋄ *if* $\deg(S_f)\leq\deg(\Omega_f)+2\deg(P_f)$*, then* $\mu(f)=2\deg(P_f)$;
     - ⋄ *otherwise* $\mu(f)=\deg(S_f)-\deg(\Omega_f)$.
   - 2.c. *If* $\Omega_f=S_f^pT_f$ *with* $p\geq1$ *and* $\gcd(T_f,S_f)=1$*, then* $\mu(f)=2\deg(P_f)+\deg(\Omega_f)-\deg(S_f)$.

As a consequence we are able to determine the dynamical number of base-points of non base-wandering Jonquières twists:

Corollary C. *Let* $f=(f_1,f_2)$ *be a non base-wandering Jonquières twist. If* ℓ *is the order of* f_2*, then* $\mu(f)=\frac{\mu(f^\ell)}{\ell}$ *where* $\mu(f^\ell)$ *is given by* Theorem B.

Combining the inequalities obtained in Theorem A and Theorem B we get the following statement (we use the notations introduced in Theorem B):

Corollary D. *Let f be a Jonquières twist that preserves fiberwise the fibration. Assume that χ_f has two distinct roots in $\mathbb{C}[y]$.*

Then there exists a conjugate g of f such that g belongs to J_m and $\deg(g) \leq \deg(f)$. For instance $g = \left(\frac{\mathrm{Tr}(M_f)+\delta_f}{\mathrm{Tr}(M_f)-\delta_f}\, x, y\right)$ suits.

2. Dynamical number of base-points of Jonquières twists

In this section we will prove Theorem B.

Let f be an element of J_0; write f as $\left(\frac{A(y)x+B(y)}{C(y)x+D(y)}, y\right)$ with $A, B, C, D \in \mathbb{C}[y]$. The characteristic polynomial of $M_f = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is $\chi_f(X) = X^2 - \mathrm{Tr}(M_f)X + \det(M_f)$. There are three possibilities:

(1) χ_f has one root of multiplicity 2 in $\mathbb{C}[y]$;
(2) χ_f has two distinct roots in $\mathbb{C}[y]$;
(3) χ_f has no root in $\mathbb{C}[y]$.

Let us consider these three possibilities.

(1) If χ_f has one root of multiplicity 2 in $\mathbb{C}[y]$, then f is conjugate to the elliptic birational map $(x + a(y), y)$ of J_a. In particular f does not belong to $\mathcal{J}$.

(2) Assume that χ_f has two distinct roots. The discriminant of χ_f is

$$\Delta_f = (\mathrm{Tr}(M_f))^2 - 4\det(M_f) = \delta_f^2,$$

and the roots of χ_f are

$$\frac{\mathrm{Tr}(M_f) + \delta_f}{2} \qquad \text{and} \qquad \frac{\mathrm{Tr}(M_f) - \delta_f}{2}.$$

Furthermore, M_f is conjugate to $\begin{pmatrix} \frac{\mathrm{Tr}(M_f)+\delta_f}{2} & 0 \\ 0 & \frac{\mathrm{Tr}(M_f)-\delta_f}{2} \end{pmatrix}$, *i.e.* f is conjugate to $g = (a(y)x, y) \in \mathrm{J}_m$ with $a(y) = \frac{\mathrm{Tr}(M_f)+\delta_f}{\mathrm{Tr}(M_f)-\delta_f}$. Let us first express $\mu(g)$ thanks to $\deg(g)$. Remark that $g^k = (a(y)^k x, y)$. Write $a(y)^j$ as $\frac{P_j(y)}{Q_j(y)}$ where $P_j, Q_j \in \mathbb{C}[y]$, $\gcd(P_j, Q_j) = 1$, then $\deg(g^j) = \max(\deg(P_j), \deg(Q_j)) + 1$. But $\deg(P_j) = j\deg(P)$ and $\deg(Q_j) = j\deg(Q)$ so

$$\deg(g^k) = \max(k\deg(P_f), k\deg(Q_1)) + 1 = k\underbrace{\max(\deg(P_f), \deg(Q_1))}_{\deg(g)-1} + 1.$$

As a consequence $\deg(g^k) = k\deg(g) - k + 1$. According to $\mu(g) = 2\lim_{k\to+\infty} \frac{\deg(g^k)}{k}$, we get

$$\mu(g) = 2\lim_{k\to+\infty}\left(\deg(g) - 1 + \frac{1}{k}\right) = 2(\deg(g) - 1).$$

Let us now express $\mu(f)$ thanks to f. Since f and g are conjugate $\mu(f) = \mu(g)$, hence $\mu(f) = 2(\deg(g) - 1)$. But $g = \left(\frac{\mathrm{Tr}(M_f)+\delta_f}{\mathrm{Tr}(M_f)-\delta_f} x, y\right)$; in particular

$$\deg(g) \leq 1 + \max\left(\deg(\mathrm{Tr}(M_f) + \delta_f), \deg(\mathrm{Tr}(M_f) - \delta_f)\right),$$

and $\mu(f)\leq 2\max\big(\deg(\mathrm{Tr}(M_f)+\delta_f),\deg(\mathrm{Tr}(M_f)-\delta_f)\big)$.

(3) Suppose that χ_f has no root in $\mathbb{C}[y]$. This means that $(\mathrm{Tr}(M_f))^2-4\det(M_f)$ is not a square in $\mathbb{C}[y]$ (hence $BC\neq 0$). Note that

$$\begin{pmatrix} C & \frac{D-A}{2}\\ 0 & 1\end{pmatrix}\begin{pmatrix} A & B\\ C & D\end{pmatrix}\begin{pmatrix} C & \frac{D-A}{2}\\ 0 & 1\end{pmatrix}^{-1}=\begin{pmatrix} \frac{\mathrm{Tr}(M_f)}{2} & \left(\frac{\mathrm{Tr}(M_f)}{2}\right)^2-\det(M_f)\\ 1 & \frac{\mathrm{Tr}(M_f)}{2}\end{pmatrix}.$$

In other words f is conjugate to

$$g=\left(\frac{\frac{\mathrm{Tr}(M_f)}{2}x+\left(\frac{\mathrm{Tr}(M_f)}{2}\right)^2-\det(M_f)}{x+\frac{\mathrm{Tr}(M_f)}{2}},y\right)\in \mathrm{J}_{\frac{\mathrm{Tr}(M_f)}{2}}.$$

Set $P(y)=\frac{\mathrm{Tr}(M_f)}{2}\in\mathbb{C}[y]$ and $F(y)=\left(\frac{\mathrm{Tr}(M_f)}{2}\right)^2-\det(M_f)\in\mathbb{C}[y]$, *i.e.* f is conjugate to $g=\left(\frac{P(y)x+F(y)}{x+P(y)},y\right)$ with P, $F\in\mathbb{C}[y]$. Denote by d_P (resp. d_F) the degree of P (resp. F). Remark that $\deg(g)=\max(d_P+1,d_F,2)$.

Let us now express $\deg(g^k)$. Consider $M_g=\begin{pmatrix} P & F\\ 1 & P\end{pmatrix}$ and set

$$Q=\begin{pmatrix}\sqrt{F} & -\sqrt{F}\\ 1 & 1\end{pmatrix}\qquad\text{and}\qquad D=\begin{pmatrix} P+\sqrt{F} & 0\\ 0 & P-\sqrt{F}\end{pmatrix}.$$

Then $M_g^k=QD^kQ^{-1}$, hence

$$M_g^k=\begin{pmatrix}\sqrt{F}\frac{(P+\sqrt{F})^k+(P-\sqrt{F})^k}{(P+\sqrt{F})^k-(P-\sqrt{F})^k} & F\\ 1 & \sqrt{F}\frac{(P+\sqrt{F})^k+(P-\sqrt{F})^k}{(P+\sqrt{F})^k-(P-\sqrt{F})^k}\end{pmatrix}.$$

Let us set

$$\Upsilon_k=\sqrt{F}\frac{(P+\sqrt{F})^k+(P-\sqrt{F})^k}{(P+\sqrt{F})^k-(P-\sqrt{F})^k},$$

and let us denote by D_k (resp. N_k) the denominator (resp. numerator) of Υ_k.

Lemma 2.1. *Let* $\Omega_f=\gcd(P,F)$ *and write* P (*resp.* F) *as* Ω_fP_f (*resp.* Ω_fS_f). *Assume* $\gcd(S_f,\Omega_f)=1$. *Then*

- *if* $d_{S_f}\leq d_{\Omega_f}+2d_{P_f}$, *then* $\mu(g)=d_{\Omega_f}+2d_{P_f}$;
- *otherwise* $\mu(g)=d_{S_f}$.

Proof. (a) Assume k even, write k as 2ℓ. A straightforward computation yields to

$$\Upsilon_{2\ell}=\frac{\displaystyle\sum_{j=0}^{\ell}\binom{2\ell}{2j}\Omega_f^{\ell-j}P_f^{2(\ell-j)}S_f^j}{\displaystyle P_f\sum_{j=0}^{\ell-1}\binom{2\ell}{2j+1}\Omega_f^{\ell-1-j}P_f^{2(\ell-1-j)}S_f^j}.$$

Recall that $\gcd(\Omega_f,S_f)=1$ by assumption and $\gcd(\Omega_f,P_f)=1$ by construction. On the one hand

$$\deg(N_{2\ell})=\begin{cases}\ell(d_{\Omega_f}+2d_{P_f}) & \text{if } d_{S_f}\leq d_{\Omega_f}+2d_{P_f},\\ \ell d_{S_f} & \text{otherwise.}\end{cases}$$

On the other hand

$$\deg(D_{2\ell}) = \begin{cases} d_{P_f} + (\ell-1)(d_{\Omega_f} + 2d_{P_f}) & \text{if } d_{S_f} \leq d_{\Omega_f} + 2d_{P_f}, \\ d_{P_f} + (\ell-1)\, d_{S_f} & \text{otherwise.} \end{cases}$$

Finally

$$\deg(g^{2\ell}) = \begin{cases} \max\Big(\ell(d_{\Omega_f}+2d_{P_f})+1, d_{S_f}+\ell d_{\Omega_f}+(2\ell-1)d_{P_f}, (\ell-1)\, d_{\Omega_f}+(2\ell-1)d_{P_f}+2\Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } d_{S_f} \leq d_{\Omega_f}+2d_{P_f}, \\ \max\Big(\ell d_{S_f}+1, d_{\Omega_f}+d_{P_f}+\ell d_{S_f}, d_{P_f}+(\ell-1)\, d_{S_f}+2\Big) \quad \text{otherwise.} \end{cases}$$

(b) Suppose k odd, write k as $2\ell+1$. A straightforward computation yields to

$$\Upsilon_{2\ell+1} = P_f \Omega_f \; \frac{\displaystyle\sum_{j=0}^{\ell} \binom{2\ell+1}{2j} \Omega_f^{\ell-j} P_f^{2(\ell-j)} S_f^j}{\displaystyle\sum_{j=0}^{\ell} \binom{2\ell+1}{2j+1} \Omega_f^{\ell-j} P_f^{2(\ell-j)} S_f^j}.$$

Let us recall that $\gcd(\Omega_f, S_f) = 1$ by assumption and $\gcd(\Omega_f, P_f) = 1$ by construction.

On the one hand

$$\deg(N_{2\ell+1}) = \begin{cases} \ell(d_{\Omega_f} + 2d_{P_f}) + d_{\Omega_f} + d_{P_f} & \text{if } d_{S_f} \leq d_{\Omega_f} + 2d_{P_f}, \\ \ell d_{S_f} + d_{P_f} + d_{\Omega_f} & \text{otherwise.} \end{cases}$$

On the other hand

$$\deg(D_{2\ell+1}) = \begin{cases} \ell(d_{\Omega_f} + 2d_{P_f}) & \text{if } d_{S_f} \leq d_{\Omega_f} + 2d_{P_f}, \\ \ell d_{S_f} & \text{otherwise.} \end{cases}$$

Finally

$$\deg(g^{2\ell+1}) = \begin{cases} \max\Big((\ell+1)d_{\Omega_f}+(2\ell+1)d_{P_f}+1, (\ell+1)d_{\Omega_f}+2\ell d_{P_f}+d_{S_f}, \ell(d_{\Omega_f}+2d_{P_f})+2\Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } d_{S_f} \leq d_{\Omega_f}+2d_{P_f}, \\ \max\Big(\ell d_{S_f}+d_{P_f}+d_{\Omega_f}+1, (\ell+1)d_{S_f}+d_{\Omega_f}, \ell d_{S_f}+2\Big) \quad \text{otherwise.} \end{cases}$$

We conclude with the equality $\mu(g) = 2 \lim_{k\to+\infty} \dfrac{\deg(g^k)}{k}$. □

Lemma 2.2. *Let* $\Omega_f = \gcd(P, F)$ *and write* P *(resp.* F*) as* $\Omega_f P_f$ *(resp.* $\Omega_f S_f$*). Suppose that* $S_f = \Omega_f^p T_f$ *with* $p \geq 1$ *and* $\gcd(T_f, \Omega_f) = 1$. *Then*

- ⋄ *if* $d_{S_f} \leq d_{\Omega_f} + 2d_{P_f}$, *then* $\mu(g) = 2d_{P_f}$;
- ⋄ *otherwise* $\mu(g) = d_{S_f} - d_{\Omega_f}$.

Proof. (a) Assume k even, write k as 2ℓ. We get

$$\Upsilon_{2\ell} = \frac{\displaystyle\sum_{j=0}^{\ell} \binom{2\ell}{2j} \Omega_f^{\ell-j} P_f^{2(\ell-j)} S_f^j}{\displaystyle P_f \sum_{j=0}^{\ell-1} \binom{2\ell}{2j+1} \Omega_f^{\ell-1-j} P_f^{2(\ell-1-j)} S_f^j} = \frac{\displaystyle \Omega_f \sum_{j=0}^{\ell} \binom{2\ell}{2j} \Omega_f^{(p-1)j} P_f^{2(\ell-j)} T_f^j}{\displaystyle P_f \sum_{j=0}^{\ell-1} \binom{2\ell}{2j+1} \Omega_f^{(p-1)j} P_f^{2(\ell-1-j)} T_f^j}.$$

Recall that $\gcd(\Omega_f, T_f) = 1$ and that $d_{S_f} = pd_{\Omega_f} + d_{T_f}$, *i.e.* $d_{T_f} = d_{S_f} - pd_{\Omega_f}$. On the one hand

$$\deg(N_{2\ell}) = \begin{cases} 2\ell d_{P_f} + d_{\Omega_f} & \text{if } d_{S_f} \leq d_{\Omega_f} + 2d_{P_f}, \\ \ell d_{S_f} + (1-\ell)d_{\Omega_f} & \text{otherwise.} \end{cases}$$

On the other hand

$$\deg(D_{2\ell}) = \begin{cases} (2\ell-1)d_{P_f} & \text{if } d_{S_f} \leq d_{\Omega_f} + 2d_{P_f}, \\ (\ell-1)(d_{S_f} - d_{\Omega_f}) + d_{P_f} & \text{otherwise.} \end{cases}$$

Finally

$$\deg(g^{2\ell}) = \begin{cases} \max\Big(2\ell d_{P_f} + d_{\Omega_f} + 1, (2\ell-1)d_{P_f} + d_{\Omega_f} + d_{S_f}, (2\ell-1)d_{P_f} + 1\Big) & \text{if } d_{S_f} \leq d_{\Omega_f} + 2d_{P_f}, \\ \max\Big(\ell d_{S_f} - (\ell-1)d_{\Omega_f} + 1, \ell d_{S_f} + (2-\ell)d_{\Omega_f} + d_{P_f}, (\ell-1)(d_{S_f} - d_{\Omega_f}) + d_{P_f} + 1\Big) & \\ & \text{otherwise.} \end{cases}$$

(b) Suppose k odd, write k as $2\ell+1$. We get

$$\Upsilon_{2\ell+1} = \frac{P_f\Omega_f \displaystyle\sum_{j=0}^{\ell} \binom{2\ell+1}{2j} \Omega_f^{\ell-j} P_f^{2(\ell-j)} S_f^j}{\displaystyle\sum_{j=0}^{\ell} \binom{2\ell+1}{2j+1} \Omega_f^{\ell-j} P_f^{2(\ell-j)} S_f^j} = \frac{P_f\Omega_f \displaystyle\sum_{j=0}^{\ell} \binom{2\ell+1}{2j} \Omega_f^{(p-1)j} P_f^{2(\ell-j)} T_f^j}{\displaystyle\sum_{j=0}^{\ell} \binom{2\ell+1}{2j+1} \Omega_f^{(p-1)j} P_f^{2(\ell-j)} T_f^j}.$$

On the one hand

$$\deg(N_{2\ell+1}) = \begin{cases} (2\ell+1)d_{P_f} + d_{\Omega_f} & \text{if } d_{S_f} \leq d_{\Omega_f} + 2d_{P_f}, \\ \ell d_{S_f} - (\ell-1)d_{\Omega_f} + d_{P_f} & \text{otherwise.} \end{cases}$$

On the other hand

$$\deg(D_{2\ell+1}) = \begin{cases} 2\ell d_{P_f} & \text{if } d_{S_f} \leq d_{\Omega_f} + 2d_{P_f}, \\ \ell d_{S_f} - \ell d_{\Omega_f} & \text{otherwise.} \end{cases}$$

Finally

$$\deg(g^{2\ell+1}) = \begin{cases} \max\Big((2\ell+1)d_{P_f} + d_{\Omega_f} + 1, 2\ell d_{P_f} + d_{\Omega_f} + d_{S_f}, 2\ell d_{P_f} + 1\Big) & \text{if } d_{S_f} \leq d_{\Omega_f} + 2d_{P_f}, \\ \max\Big(\ell d_{S_f} - (\ell-1)d_{\Omega_f} + d_{P_f} + 1, (\ell+1)d_{S_f} - (\ell-1)d_{\Omega_f}, \ell d_{S_f} - \ell d_{\Omega_f} + 1\Big) & \\ & \text{otherwise.} \end{cases}$$

We conclude with the equality $\mu(g) = 2 \lim_{k \to +\infty} \dfrac{\deg(g^k)}{k}$. □

Lemma 2.3. *Let* $\Omega_f = \gcd(P, F)$ *and write* P *(resp.* F*) as* $\Omega_f P_f$ *(resp.* $\Omega_f S_f$*). Suppose that* $\Omega_f = S_f^p T_f$ *with* $p \geq 1$ *and* $\gcd(T_f, S_f) = 1$. *Then*

$$\mu(g) = 2d_{P_f} + d_{\Omega_f} - d_{S_f}.$$

Proof. (a) Assume k even, write k as 2ℓ. We obtain

$$\Upsilon_{2\ell} = \frac{\displaystyle\sum_{j=0}^{\ell}\binom{2\ell}{2j}\Omega_f^{\ell-j}P_f^{2(\ell-j)}S_f^j}{\displaystyle P_f\sum_{j=0}^{\ell-1}\binom{2\ell}{2j+1}\Omega_f^{\ell-1-j}P_f^{2(\ell-1-j)}S_f^j} = \frac{\displaystyle S_f\sum_{j=0}^{\ell}\binom{2\ell}{2j}S_f^{j(p-1)}P_f^{2j}T_f^j}{\displaystyle P_f\sum_{j=0}^{\ell-1}\binom{2\ell}{2j+1}S_f^{j(p-1)}P_f^{2j}T_f^j}.$$

Recall that $\gcd(S_f, T_f) = 1$; one has

$$\deg(N_{2\ell}) = (p\ell - \ell + 1)d_{S_f} + 2\ell d_{P_f} + \ell d_{T_f},$$

and

$$\deg(D_{2\ell}) = (\ell - 1)(p - 1)d_{S_f} + (2\ell - 1)d_{P_f} + (\ell - 1)d_{T_f}.$$

Finally

$$\begin{aligned}\deg(g^{2\ell}) = \max\Big((p\ell - \ell + 1)d_{S_f} + 2\ell d_{P_f} + \ell d_{T_f} + 1,\\ (\ell(p-1) + 2)d_{S_f} + (2\ell - 1)d_{P_f} + \ell d_{T_f},\\ (\ell - 1)(p - 1)d_{S_f} + (2\ell - 1)d_{P_f} + (\ell - 1)d_{T_f} + 2\Big).\end{aligned}$$

(b) Suppose k odd, write k as $2\ell + 1$. We get

$$\Upsilon_{2\ell+1} = \frac{\displaystyle\sum_{j=0}^{\ell}\binom{2\ell+1}{2j}P^{2\ell+1-2j}F^j}{\displaystyle\sum_{j=0}^{\ell}\binom{2\ell+1}{2j+1}P^{2\ell-2j}F^j} = \frac{\displaystyle S_f^pT_fP_f\sum_{j=0}^{\ell}\binom{2\ell+1}{2(\ell-j)}S_f^{j(p-1)}T_f^jP_f^{2j}}{\displaystyle\sum_{j=0}^{\ell}\binom{2\ell+1}{2j}S_f^{j(p-1)}T_f^jP_f^{2j}}.$$

On the one hand

$$\deg(N_{2\ell+1}) = (p + \ell(p - 1))d_{S_f} + (\ell + 1)d_{T_f} + (2\ell + 1)d_{P_f},$$

and on the other hand

$$\deg(D_{2\ell+1}) = 2\ell d_{P_f} + \ell d_{T_f} + \ell(p - 1)d_{S_f}.$$

Finally

$$\begin{aligned}\deg(g^{2\ell+1}) = \max\Big((p + \ell(p - 1))d_{S_f} + (\ell + 1)d_{T_f} + (2\ell + 1)d_{P_f} + 1,\\ (p + 1 + \ell(p - 1))d_{S_f} + 2\ell d_{P_f} + (\ell + 1)d_{T_f},\\ 2\ell d_{P_f} + \ell d_{T_f} + \ell(p - 1)d_{S_f} + 2\Big).\end{aligned}$$

We conclude with the equality $\mu(g) = 2\lim\limits_{k\to+\infty}\dfrac{\deg(g^k)}{k}$. □

3. Examples

In this section we will give examples that illustrate Theorem B; more precisely §3.1 (resp. §3.2) illustrates Theorem B.1. (resp. Theorem B.2.)

3.1. Examples that illustrate Theorem B.1.

3.1.1. First example. Consider the birational map of J given in the affine chart $x = 1$ by $f = (y, (1-y)yz)$. The matrix associated to f is

$$M_f = \begin{pmatrix} (1-y)y & 0 \\ 0 & 1 \end{pmatrix},$$

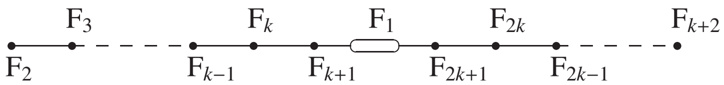
and the Baum Bott index $\mathrm{BB}(f)$ of f is $\frac{\big((1-y)y+1\big)^2}{(1-y)y}$; in particular f belongs to $\mathcal{J}$ (Proposition 1.1). The characteristic polynomial of M_f is

$$\chi_f(X) = (X - (1-y)y)(X-1).$$

According to Theorem B.1. one has $\mu(f) = 4 \leq 2\max\big(\deg(2), \deg(2(1-y)y\big) = 4$.

We can see it another way: [5] asserts that $\deg(f^k) = k\deg(f) - k + 1 = 3k - k + 1 = 2k+1$. Consequently $\mu(f) = 2\lim\limits_{k\to+\infty}\dfrac{\deg(f^k)}{k} = 2\lim\limits_{k\to+\infty}\dfrac{2k+1}{k} = 4$.

A third way to see this is to look at the configuration of the exceptional divisors. For any $k \geq 1$ one has $f^k = (x^{2k+1} : x^{2k}y : (x-y)^k y^k z)$. The configuration of the exceptional divisors of f^k is



where

- ⋄ two curves are related by an edge if their intersection is positive;
- ⋄ the self-intersections correspond to the shape of the vertices;
- ⋄ the point means self-intersection -1, the rectangle means self-intersection $-2k$.

In particular the number of base-points of f^k is $2k + 2k + 1 = 4k + 1$ and

$$\mu(f) = \lim_{k\to+\infty}\frac{\#\mathfrak{b}(f^k)}{k} = 4.$$

3.1.2. Second example. Consider the birational map of J given in the affine chart $z = 1$ by $f = (x, xy + x(x-1))$. The matrix associated to f is

$$M_f = \begin{pmatrix} x & x(x-1) \\ 0 & 1 \end{pmatrix};$$

according to Proposition 1.1 the map f is a Jonquières twist (indeed $\mathrm{BB}(f) = \frac{(1+x)^2}{x} \in \mathbb{C}(x) \smallsetminus \mathbb{C}$). The characteristic polynomial of M_f is $\chi_f(X) = (X-x)(X-1)$. And f is conjugate to $g = (x, xy)$. According to Theorem B.1 one has

$$\mu(f) = \mu(g) = 2(\deg(g) - 1) = 2 \leq 2\max\big(\deg(2), \deg(2(1-y)y\big) = 2.$$

We can see it another way: for any $k \geq 1$ one has $f^k = (x, x^k y + x^{k+1} - x)$ and thus

$\deg(f^k) = k+1$. As a result $\mu(f) = 2\lim_{k\to+\infty} \dfrac{\deg(f^k)}{k} = 2\times 1 = 2$.

3.2. Examples that illustrate Theorem B.2.

3.2.1. First example. Consider the map of J given in the affine chart $y = 1$ by

$$f = \left(x, \frac{x(1-xz)}{z}\right).$$

The matrix associated to f is

$$M_f = \begin{pmatrix} -x^2 & x \\ 1 & 0 \end{pmatrix},$$

the Baum Bott index $\mathrm{BB}(f)$ of f is $-x^3$ and f belongs to $\mathcal{J}$ (Proposition 1.1).

Theorem B.2.a asserts that $\mu(f) = 3$. We can see it another way: a computation gives $\deg(f^{2k}) = 3k+1$ and $\deg(f^{2k+1}) = 3(k+1)$ for any $k \geq 0$. Since $\mu(f) = 2\lim_{k\to+\infty} \dfrac{\deg(f^k)}{k}$ one gets $\mu(f) = 3$.

3.2.2. Second example. Consider the map f of J associated to the matrix

$$M_f = \begin{pmatrix} y & 2y^8 \\ y & 1 \end{pmatrix}.$$

The Baum Bott index $\mathrm{BB}(f)$ of f is $\frac{(y+1)^2}{y(1-2y^8)}$ and f belongs to $\mathcal{J}$ (Proposition 1.1). Theorem B.2.a asserts that $\mu(f) = 9$. We can see it another way: a computation gives $\deg(f^{2k}) = 9k+1$ and $\deg(f^{2k+1}) = 9k+8$ for any $k \geq 0$. Since $2\lim_{k\to+\infty} \dfrac{\deg(f^k)}{k} = \mu(f)$ one gets $\mu(f) = 9$.

3.2.3. Third example. Let us consider the Jonquières map of $\mathbb{P}^2_{\mathbb{C}}$ given in the affine chart $z = 1$ by

$$f = \left(\frac{y(y+2)x + y^5}{x + y(y+2)}, y\right).$$

The matrix associated to f is

$$M_f = \begin{pmatrix} y(y+2) & y^5 \\ 1 & y(y+2) \end{pmatrix},$$

and the Baum Bott index $\mathrm{BB}(f)$ of f is $\frac{4(y+2)^2}{(y+2)^2-y^5}$. In particular f is a Jonquières twist (Proposition 1.1).

According to Theorem B.2.b one has $\mu(f) = 3$. An other way to see that is to compute $\deg f^k$ for any k: for any $\ell \geq 1$ one has

$$\deg(f^{2\ell}) = 3(\ell+1), \qquad \deg(f^{2\ell+1}) = 3\ell + 5.$$

Then we find again $\mu(f) = 2\lim_{k\to+\infty} \dfrac{\deg(f^k)}{k} = 3$.

3.2.4. Fourth example. Consider the map f of J associated to the matrix

$$M_f = \begin{pmatrix} y(y+2)^8 & y^5 \\ 1 & y(y+2)^8 \end{pmatrix}.$$

The Baum Bott index $\mathrm{BB}(f)$ of f is $\frac{4(y+2)^{16}}{(y+2)^{16}-y^3}$ and f belongs to $\mathcal{J}$ (Proposition 1.1). According to Theorem B.2.b one has $\mu(f) = 16$. An other way to see that is to compute $\deg f^k$ for any k: for any $k \geq 1$ one has $\deg f^k = 8k+2$. Then we find again $\mu(f) = 2\lim_{k\to+\infty} \dfrac{\deg(f^k)}{k} = 2 \times 8 = 16$.

3.2.5. Fifth example. Let us consider the Jonquières map of $\mathbb{P}^2_{\mathbb{C}}$ given in the affine chart $z = 1$ by

$$f = \left(\frac{y(y+1)(y+2)x + y^2}{(y+2)x + y(y+1)(y+2)}, y\right).$$

The matrix associated to f is

$$M_f = \begin{pmatrix} y(y+1)(y+2) & y^2 \\ y+2 & y(y+1)(y+2) \end{pmatrix},$$

and the Baum-Bott index $\mathrm{BB}(f)$ of f is $\frac{4(y+1)^2(y+2)}{(y+1)^2(y+2)-1}$; in particular f is a Jonquières twist (Proposition 1.1).

Theorem B.2.c asserts that $\mu(f) = 3$. An other way to see that is to compute $\deg f^k$ for any k: for any $k \geq 1$

$$\deg(f^{2k}) = 3k+2, \qquad \deg(f^{2k+1}) = 3k+4,$$

so $2\lim_{k\to+\infty} \dfrac{\deg(f^k)}{k} = 3$ and we find again $\mu(f) = 3$.

3.3. Families.

3.3.1. First family. Let us consider the family $(f_t)_t$ of elements of J given by $f_t = \left(x+t, y\frac{x}{x+1}\right)$. A straightforward computation yields to

$$f_t^n = \left(x+nt, y\frac{x}{x+1}\frac{x+t}{x+t+1}\cdots\frac{x+(n-1)t}{x+(n-1)t+1}\right).$$

The birational map f_t belongs to $\mathcal{J}$ if some multiple of t is equal to 1, and to $\mathrm{J} \smallsetminus \mathcal{J}$ otherwise. Furthermore,

- ⋄ if no multiple of t is equal to 1, then $\mu(f_t) = 2$ $\left(\text{because } \lim_{k\to+\infty} \dfrac{\deg f_t^k}{k} = 1\right)$;
- ⋄ otherwise $\mu(f_t) = 0$.

3.3.2. Second family, illustration of Theorem A.3. Let us recall a result of [6]: let f be any element of $\mathrm{PGL}_3(\mathbb{C})$, or any elliptic element of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ of infinite order; then f is a limit of pairwise conjugate loxodromic elements (resp. Jonquières twists) in the Cremona group. Hence there exist families $(f_n)_n$ of birational self-maps of the complex projective plane such that

- ⋄ $\mu(f_n) > 0$ for any $n \in \mathbb{N}$;

◇ $\mu(\lim_{n\to+\infty} f_n) = 0$.

3.3.3. Third family, illustration of Theorem A.4. Let us recall a construction given in [6]. Consider a pencil of cubic curves with nine distinct base points p_i in $\mathbb{P}^2_{\mathbb{C}}$. Given a point m in $\mathbb{P}^2_{\mathbb{C}}$, draw the line $(p_1 m)$ and denote by m' the third intersection point of this line with the cubic of our pencil that contains m: the map $m \mapsto \sigma_1(m) = m'$ is a birational involution. Replacing p_1 by p_2, we get a second involution and, for a very general pencil, $\sigma_1 \circ \sigma_2$ is a Halphen twist that preserves our cubic pencil. At the opposite range, consider the degenerate cubic pencil, the members of which are the union of a line through the origin and the circle $C = \{x^2 + y^2 = z^2\}$. Choose $p_1 = (1 : 0 : 1)$ and $p_2 = (0 : 1 : 1)$ as our distinguished base points. Then, $\sigma_1 \circ \sigma_2$ is a Jonquières twist preserving the pencil of lines through the origin; if the plane is parameterized by $(s, t) \mapsto (st, t)$, this Jonquières twist is conjugate to $(s, t) \mapsto \left(s, \frac{(s-1)t+1}{(s^2+1)t+s-1}\right)$. Now, if we consider a family of general cubic pencils converging towards this degenerate pencil, we obtain a sequence of Halphen twists converging to a Jonquières twist. So there exists a sequence $(f_n)_n$ of birational self-maps of $\mathbb{P}^2_{\mathbb{C}}$ whose limit is also a birational self-map of $\mathbb{P}^2_{\mathbb{C}}$ and such that

◇ $\mu(f_n) = 0$ for any $n \in \mathbb{N}$;

◇ $\mu(\lim_{n\to+\infty} f_n) > 0$.

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