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EXISTENCE OF INFINITELY MANY SOLUTIONS TO SEMILINEAR ELLIPTIC NEUMANN PROBLEMS WITH CONCAVE-CONVEX TYPE NONLINEARITY

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Abstract

In this paper, we consider the semilinear elliptic problem $-\Delta u = a(x)|u|^{p-2}u + \lambda b(x)|u|^{q-2}u$ in a bounded domain Ω with Neumann boundary condition. We show the existence infinitely many solutions by applying critical point theory with a suitable decomposition of the Sobolev space $W^{1,2}(\Omega)$. Also we prove the C^α regularity of the solutions.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be an open bounded domain with smooth boundary. We consider the following semilinear elliptic problem

$$(1.1) \qquad \begin{aligned} -\Delta u &= \lambda a(x)|u|^{p-2}u + \mu b(x)|u|^{q-2}u \text{ in } \Omega, \\ \frac{\partial u}{\partial \eta} &= 0 \text{ on } \partial\Omega, \end{aligned}$$

where $1 < p < 2 < q < 2^* = 2n/(n-2)$, λ, μ are positive real parameters and $a, b : \Omega \to \mathbb{R}$ are functions satisfying the following hypotheses:

(H_1) $a \in L^\infty(\Omega)$ and $\int_\Omega a(x)\,\mathrm{d}x \neq 0$;
(H_1') $a \in L^\infty(\Omega)$ and $\alpha := \inf_{x\in\Omega} a(x) > 0$;
(H_2) $b \in L^\infty(\Omega)$ and $\beta := \inf_{x\in\Omega} b(x) > 0$;
(H_2') $b \in L^\infty(\Omega)$ and $\int_\Omega b(x)\,\mathrm{d}x \neq 0$.

Since the work of Ambrosetti et al. [3], semilinear elliptic problems with concave-convex nonlinearities have been investigated widely. We refer to [1, 2, 9, 12, 10] for more concave-convex problems with Dirichlet boundary conditions. In [3], the authors studied the existence of solutions of the following problem

$$(1.2) \qquad \begin{aligned} -\Delta u &= |u|^{p-2}u + \lambda|u|^{q-2}u \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

and proved that there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ there exist sequences of solutions $\{u_n\}, \{v_n\}$ such that $I(u_n) < 0$ and $I(v_n) > 0$. The authors also studied the existence of positive solutions and proved that there exists $\Lambda > 0$ such that the problem (1.2) has at least two positive solutions for $\lambda < \Lambda$, at least one positive solution for $\lambda = \Lambda$ and no positive solution for $\lambda > \Lambda$. In [8], De Figueiredo et al. extended these previous results to a problem

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with variable coefficients whose prototype is

$$-\Delta u = a(x)|u|^{p-2}u + \lambda b(x)|u|^{q-2}u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{1.3}$$

The function $b(x)$ is assumed to be non-negative and $a(x)$ may change sign. Recently in [15], Quoirin and Umezu considered a problem similar to (1.1) and studied the existence of positive solutions. However the authors did not discuss the existence of infinitely many solutions. We refer to [5, 11, 16] where the existence of infinitely many solutions are studied for some major concave-convex problems with nonlinear boundary conditions. In these papers, the left hand side of the problems involve $-\Delta u + u$, which corresponds to the term $(1/2)\int_\Omega |\nabla u|^2 + u^2$ in the energy functional. The map $u \mapsto (\int_\Omega |\nabla u|^2 + u^2)^(1/2)$, defines a norm in the space $W^{1,2}(\Omega)$. In this paper, we consider (1.1), which does not involve the extra term u. This makes the problem more challenging and the methods in [5, 11, 16] for studying the existence of solutions are not applicable to (1.1). We use some ideas from [10] and consider a suitable decomposition of the space $W^{1,2}(\Omega)$. To the best of our knowledge the existence of infinitely many solutions of (1.1) is not yet addressed. In this paper we address this aspect of (1.1) and prove the following:

Theorem 1.1. *Assume that* $(H_1), (H_2)$ *hold and* $1 < p < 2 < q < 2^*$. *Then there exists* $\Lambda > 0$ *such that for all* $\lambda \in (0, \Lambda)$ *and* $\mu > 0$ *there exists an unbounded sequence of solutions* (u_n) *of* (1.1) *such that* $I(u_n) \to \infty$ *as* $n \to \infty$.

Theorem 1.2. *Assume that* $(H_1'), (H_2')$ *hold and* $1 < p < 2 < q < 2^*$. *Then for all* $\lambda > 0$ *and* $\mu > 0$ *there exists a sequence of solutions* (v_n) *of* (1.1) *such that* $I(v_n) < 0$ *and* $I(v_n) \to 0$ *as* $n \to \infty$.

The paper is organized as follows: In Section 2, we prove some preliminary results required for the proof of the main theorems. In Section 3, we give the proofs of the Theorems 1 and 2. We also prove some regularity results for the solution. More precisely, we prove that the $H^1(\Omega)$ solutions of (1.1) are Holder continuous. This is discussed in Section 4.

2. Preliminaries

Let $E = W^{1,2}(\Omega)$ be the usual Sobolev space with the norm given by

$$\|u\|_{1,2} = \left(\int_\Omega |\nabla u|^2 + u^2\right)^{\frac{1}{2}}.$$

Definition 2.1. We say a function $u \in E$ a weak solution of (1.1) if for every $\varphi \in E$ we have

$$\int_\Omega \nabla u \nabla \varphi \, dx - \lambda \int_\Omega a(x)|u|^{p-2}u\varphi \, dx - \mu \int_\Omega b(x)|u|^{q-2}u\varphi \, dx = 0. \tag{2.4}$$

For $u \in E$ we define the energy functional for problem (1.1) by

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \frac{\lambda}{p}\int_\Omega a(x)|u|^p \, dx - \frac{\mu}{q}\int_\Omega b(x)|u|^q \, dx.$$

Since $1 < p < 2 < q < 2^*$ we have that $I \in C^1(E, \mathbb{R})$ and the critical points of I are the weak solutions of (1.1). Hence the existence of solutions of (1.1) is equivalent to the existence of critical points of I.

We consider the following decomposition of the space E (see [10]) which is crucial in the proof of the main results.

Lemma 2.1. (i) *Let $X = \{u \in W^{1,2}(\Omega) : \int_{\partial\Omega} u \, \mathrm{d}\sigma_x = 0\}$. Then the poincare's inequality holds in X, that is, there exists a constant $C = C(n, \Omega)$ such that*

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}, \text{ for any } u \in X.$$

(ii) *For any $u \in E$ there exists a unique $t_u \in \mathbb{R}$ and $u^\perp \in X$ such that $u = t_u + u^\perp$, that is, $E = \langle 1 \rangle \oplus X$ and for any $u, v \in E(\Omega)$*

$$\langle u, v \rangle = t_u \cdot t_v + \int_\Omega \nabla u \cdot \nabla v$$

defines an inner product in E. Moreover, the corresponding norm is equivalent to the usual norm in E.

Next we prove a technical lemma which we need in the proof of Theorem 1.2.

Lemma 2.2. *Assume that (H_1') holds and $p > 1$. Then there exists $\gamma > 0$ such that for any $u = t_u + u^\perp \in \langle 1 \rangle \oplus X$ with $\|\nabla u\|_{L^2(\Omega)} \leq \gamma |t_u|$, we have*

$$\int_\Omega a(x)|t_u + u^\perp|^p \, \mathrm{d}x \geq \frac{|t_u|^p}{2} \int_\Omega a(x) \, \mathrm{d}x.$$

Proof. Arguing by contradiction, we suppose that there exists a sequence $(u_n) \subset W^{1,2}(\Omega)$ such that $\|\nabla u_n\|_{L^2(\Omega)} \leq \frac{|t_u|}{n}$ and

$$\int_\Omega a(x)|t_{u_n} + u_n^\perp|^q \, \mathrm{d}\sigma_x < \frac{|t_{u_n}|^q}{2} \int_\Omega a(x) \, \mathrm{d}x. \tag{2.5}$$

Now set $v_n := u_n^\perp / t_{u_n}$, then $v_n \in X$ and by our assumption $\|\nabla v_n\|_{L^2(\Omega)} \to 0$ as $n \to \infty$. Since, the Poincare's inequality holds in X, $v_n \to 0$ in X. Also X is continuously embedded in $L^q(\Omega)$, hence $v_n \to 0$ in $L^q(\Omega)$. Consequently, we have $v_n \to 0$ a.e. on Ω. Now dividing both sides of (2.5) by $|t_{u_n}|$ and using Lebesgue theorem we obtain

$$\int_\Omega a(x) \, \mathrm{d}x < 0,$$

which contradicts that $a(x) > 0$ on Ω. □

3. Proofs of the main results

The following abstract results on existence of critical points will be used.

Theorem 3.1. *Let E be an infinite dimensional Banach space and, $I \in C^1(E, R)$ be even and $I(0) = 0$. If $E = V \oplus X$, where V finite dimensional, and I satisfies*

(PS) *any sequence $\{u_n\} \subset E$ for which $I(u_n) \to c$ and $I'(u_n) \to 0$ possesses a convergent subsequence,*

(I_1) *there are constants $r, \alpha > 0$ such that $I|_{\partial B_r \cap X} \geq \alpha$, and*

(I_2) *for each finite dimensional subspace* $\overline{E} \subset E$*, there is an* $R = R(\overline{E})$ *such that* $I \leq 0$ *on* $E \setminus B_{R(\overline{E})}$,

then I possesses an unbounded sequence of critical values.

Theorem 3.2. *Let X be a Banach space and* $I \in C^1(X, \mathbb{R})$ *satisfies*

(I_1) *there exists an admissible representation V of G, a compact lie group, such that* $X = \oplus_{j \in A} X(j)$ *with* $A = \mathbb{N}$ *or* $A = \mathbb{Z}$ *and* $X(j) \cong V$ *for every* $j \in A$*. The space X is then a Banach space with isometric linear G−action. The functional* $I : X \to \mathbb{R}$ *is invariant under this action:* $I(gu) = I(u)$ *for* $g \in G$ *and* $u \in X$,

(I_2) *for every* $k \geq k_0$ *there exists* $R_k > 0$ *such that* $I(u) \geq 0$ *for every* $u \in X_k := \oplus_{j \geq k} X(j)$ *with* $\|u\| = R_k$,

(I_3) $b_k := \inf_{u \in B_k} I(u) \to 0$ *as* $k \to \infty$ *where* $B_k := \{u \in X_k : \|u\| \leq R_k\}$,

(I_4) *for every* $k \geq 1$ *there exists* $r_k \in (0, R_k)$ *and* $d_k < 0$ *such that* $I(u) \leq d_k$ *for every* $u \in X^k := \oplus_{j \leq k} X(j)$ *with* $\|u\| = r_k$*, and*

(I_5) *every sequence* $u_n \in X^n_{-n} := \oplus^n_{j=-n} X(j)$ *with* $I(u_n) < 0$ *bounded and* $\left(I|_{X^n_{-n}}\right)'(u_n) \to 0$ *as* $n \to \infty$ *has a subsequence which converges to a critical point of I.*

Then for each $k \geq k_0$*, I has a critical value* $c_k \in [b_k, d_k]$*, hence* $c_k \to 0$ *as* $k \to \infty$*.*

For the proof of Theorem 3.1 we refer to [14, Theorem 9.12] and for the proof of Theorem 3.2 we refer to [6, Theorem 2].

3.1. Proof of Theorem 1.1. Let $E = \langle 1 \rangle \oplus X$ where X is given by Lemma 2.1. We show that I satisfies hypotheses of Theorem 3.1. Clearly $I(0) = 0$ and I is even.

Let $u \in E$ then there exists t_u and $u^\perp \in X$ such that $u = t_u + u^\perp$. By using the embeddings $L^p \hookrightarrow X, L^q \hookrightarrow X$ and since Poincares inequality holds in X we have

$$\begin{aligned} I(u^\perp) &= \frac{1}{2}\int_\Omega |\nabla u^\perp|^2 \,\mathrm{d}x - \frac{\lambda}{p}\int_\Omega a(x)|u^\perp|^p \,\mathrm{d}x - \frac{\mu}{q}\int_\Omega b(x)|u^\perp|^q \,\mathrm{d}x \\ &\geq \frac{1}{2}\|u^\perp\|^2 - \lambda c_1 \|u^\perp\|^p - \mu c_2 \|u^\perp\|^q \\ &= \frac{1}{2}\rho^2 - \lambda c_1 \rho^p - \mu c_2 \rho^q \\ &= \rho^2 \left(\frac{1}{2} - \lambda c_1 \rho^{p-2} - \mu c_2 \rho^{q-2}\right). \end{aligned}$$

Let $\rho = (1/(8\mu c_2))^{1/(q-2)}$. There exists $\Lambda > 0$ such that for $\lambda \in (0, \Lambda)$ we have $\lambda c_1 \rho^{p-2} \leq 1/8$ and

$$I(u^\perp) \geq \frac{\rho^2}{4} := \alpha,$$

for all $u \in X$ with $\|u\| = \rho$. This shows that I satisfies (I_1).

Let $\overline{E} \subset E$ be any finite dimensional subspace of E and let $u \in \overline{E} \setminus \{0\}$. Let $t > 0$. Since, $\overline{E}$ is finite dimensional and all norms in finite dimensional spaces are equivalent, we have

$$\begin{aligned} I(tu) =& \frac{t^2}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda t^p}{p}\int_\Omega a(x)|u|^p - \frac{\mu t^q}{q}\int_\Omega b(x)|u|^q \\ \leq& \frac{t^2}{2}\|u\|^2 - \lambda c_1 t^p \|u\|^p - \mu c_2 t^q \|u\|^q. \end{aligned}$$

Since $q > 2$ and $b > 0$, we get $I(tu) \to -\infty$ as $t \to \infty$. Hence (I_2) holds.

Next, we show that I satisfies the Palais-smale condition (PS). Let $(u_n) \subset E$ be a sequence and $I(u_n) \to c$, $I'(u_n) \to 0$ as $n \to \infty$. Since the operators $A, B : E \to \mathbb{R}$ defined by

$$A(u) = \int_\Omega a(x)|u|^p \,\mathrm{d}x \text{ and } B(u) = \int_\Omega b(x)|u|^q \,\mathrm{d}x$$

are weakly continuous and their derivatives A', B' are compact it is sufficient to show that (u_n) is bounded in E. Suppose by contradiction, there exists a subsequence denoted by (u_n) with $\|u_n\| \to \infty$ as $n \to \infty$. Set $v_n := u_n/\|u_n\|$. Then $\|v_n\| = 1$ for all n. Hence up to subsequence $v_n \rightharpoonup v$ weakly in E and $v_n \to v$ in $L^p(\Omega)$ and $L^q(\Omega)$. By simple calculations we get

$$I(u_n) - \frac{1}{2}I'(u_n)u_n = \lambda\left(\frac{1}{2} - \frac{1}{p}\right)\int_\Omega a(x)|u_n|^p + \mu\left(\frac{1}{2} - \frac{1}{q}\right)\int_\Omega b(x)|u_n|^q.$$

Thus we can write

$$\int_\Omega b(x)|u|^q \le c + c_1\|u_n\|^p.$$

Also since $I(u_n) \to c$ we have

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2 = \frac{\lambda}{p}\int_\Omega a(x)|u_n|^p + \frac{\mu}{q}\int_\Omega b(x)|u_n|^q + c + o_n(1).$$

Thus $\int_\Omega |\nabla v_n|^2 \le o_n(1) + c_3\|u_n\|^{p-2} \to 0$ as $n \to \infty$. If $v_n = t_{v_n} + v_n^\perp$, then $v_n^\perp \to 0$ in X. Also up to a subsequence $t_{v_n} \to t_v$ in $\mathbb{R}$ and

$$|t_v| = \|t_v\| = \|v\| = \lim_{n\to\infty} \|v_n\| = 1.$$

Hence $v_n \to 1$ strongly in E. Now from $\|u_n\|^{1-q}I'(u_n)t_{v_n} = o_n(1)$ we get

$$\left|\int_\Omega b(x)|v_n|^{q-2}v_n t_{v_n}\right| \le \frac{\lambda}{\mu\|u_n\|^{q-p}}\int_\Omega a(x)|v_n|^{p-1}|t_{v_n}| + o_n(1).$$

Since $q > p$ and $v_n \to 1$ strongly in E the right hand side tends to 0. Hence by Lebesgue theorem we obtain

$$\int_\Omega b(x) = 0,$$

which is a contradiction. Thus (u_n) is bounded in E. That is I satisfies (PS) condition. Hence by Theorem 3.1 problem (1.1) has an unbounded sequence of solutions such that $I(u_n) \to \infty$ as $n \to \infty$.

3.2. Proof of Theorem 1.2. We show that I satisfies the hypotheses of Theorem 3.2. Consider the Neumann eigenvalue problem

$$-\Delta u = \lambda u \text{ in } \Omega,$$
$$\frac{\partial u}{\partial \eta} = 0 \text{ on } \partial\Omega.$$

Then applying the theory of compact self-adjoint operators, one gets that there exists a sequence of eigenvalues $\lambda_j \to \infty$ and the corresponding eigenfunctions $\{e_j\}$ form an orthonormal basis of X (see [4, 7]). Moreover, the eigenvalues has the following characterization

$$\lambda_j = \max_{u\in span\{e_1,\cdots,e_j\}} \frac{\int_\Omega |\nabla u|^2\,\mathrm{d}x}{\int_\Omega u^2\,\mathrm{d}x}. \tag{3.6}$$

Let $E = \langle 1\rangle \oplus \oplus_{j\geq 1} X(j)$, where $X(j) = \operatorname{span}\{e_j\}$. Since I is even, hypothesis (I_1) is satisfied taking $G = \mathbb{Z}/2$ and $V = \mathbb{R}$. For (I_2) we set

$$\delta_k := \sup_{X_k\setminus\{0\}} \frac{\|u\|_{L^p(\Omega)}}{\|u\|}.$$

Then by Rellich's embedding theorem, $\delta_k \to 0$ as $k \to \infty$. Hence for $u \in X_k$ using the embedding $L^p \hookrightarrow X$, we obtain

$$I(u) \geq \frac{1}{2}\int_\Omega |\nabla u^\perp|^2 - \frac{\lambda c_1}{p}(\delta_k)^p\|u\|^p - \frac{\mu c_2}{q}\|u\|^q = \frac{1}{2}\|u\|^2 - \frac{\lambda c_1}{p}(\delta_k)^p\|u\|^p - \frac{\mu c_2}{q}\|u\|^q.$$

Since $q > 2$ choosing $R > 0$ small enough we have $\mu c_2\|u\|^q/q \leq \|u\|^2/4$ if $\|u\| \leq R$. Thus

$$I(u) \geq \frac{1}{4}\|u\|^2 - \frac{\lambda c_1}{p}(\delta_k)^p\|u\|^p.$$

Let $R_k = (4\lambda c_1(\delta_k)^p/p)^{1/(2-p)}$. Then $R_k \to 0$. Hence one can find k_0 such that $R_k \leq R$ for $k \geq k_0$. Now for $u \in X_k$ with $\|u\| = R_k$ we obtain

$$I(u) \geq \frac{1}{4}\|u\|^2 - \frac{\lambda c_1}{p}(\delta_k)^p\|u\|^p = 0.$$

That is (I_2) holds. Since B_k is weakly compact and I is weakly lower semicontinous (I_3) follows from $R_k \to 0$. Next we show (I_4) in two cases.

Case 1: $\|\nabla u^\perp\| \leq \gamma|t_u|$.
Let $\|u\| = r$. Then $\|u\|^2 = t_u^2 + \|\nabla u^\perp\|_2^2 \implies t_u \geq \frac{r}{\sqrt{1+\gamma^2}}$. Now using Lemma 2.2 we get

$$I(u) \leq \frac{1}{2}\|u\|^2 + c_3\|u\|^q - \frac{|t_u|^p}{2}\int_\Omega a(x)\,\mathrm{d}x \leq r^2\left(\frac{1}{2} + c_3r^{q-2} - \frac{1}{2}c_4(1+\gamma^2)^{-p/2}r^{p-2}\right).$$

Since $1 < p < 2$, $r^{p-2} \to \infty$ as $r \to 0$. Thus choosing r_k small enough we get $(1/2) + c_3r^{q-2} - (c_4/2)(1+\gamma^2)^{-p/2}r^{p-2} \leq -1/2$. Hence

$$I(u) \leq -\frac{r_k^2}{2} := d_k < 0.$$

Case 2: $\|\nabla u^\perp\| > \gamma|t_u|$.

In this case for $\|u\| = r$ we have $r^2 = t_u^2 + \|\nabla u^\perp\|_2^2 \implies \|\nabla u^\perp\|_2 > \beta^{-1}r$, where $\beta = \sqrt{1+\gamma^{-2}}$. Now since X^k is finite dimensional, we have $\|u\|_{L^2(\Omega)} \leq c\|u\|_{L^p(\Omega)}$. Hence using (3.6) we obtain

$$\int_\Omega |u|^p \geq c_3\left(\int_\Omega |u|^2\right)^{\frac{p}{2}} \geq c_k\left(\int_\Omega |\nabla u^\perp|^2\right)^{\frac{p}{2}} \geq c_k\beta^{-p}r^p,$$

for some $c_k > 0$ depending on k. Thus we have

$$I(u) \leq \frac{1}{2}\|u\|^2 + c_3\|u\|^q - \frac{\alpha}{p}\int_\Omega |u|^p \leq \frac{r^2}{2} + c_3r^q - c_k\beta^{-p}r^p.$$

Again similar to Case 1 we can find $r_k > 0$ small enough so that

$$I(u) \leq -\frac{r_k^2}{2} = d_k < 0.$$

Hence (I_4) holds. Finally the Palais-smale condition I_5 can be shown as in the proof of part (a).

4. C^α regularity of solutions

In this section we show that the solutions of (1.1) are in $C^\alpha(\Omega)$. We follow the method used in [11, Lemma 4.2] to prove our result. We prove the following.

Proposition 4.1. *Let* $1 < p < 2 < q < 2^*$. *Then any* $H^1(\Omega)$ *solution* u *of* (1.1) *is in* $C^\alpha(\Omega)$ *and satisfies*

$$||u||_{C^\alpha(\Omega)} \leq C\left(||u||_{L^2(\Omega)} + ||u||^{p-1}_{L^\infty(\Omega)} + ||u||^{q-1}_{L^\infty(\Omega)}\right).$$

Proof. Let $u \in H^1(\Omega)$ be any solution of (1.1). We first show that $u \in L^\infty(\Omega)$. We use a bootstrap argument. Let $\nu > 1$ be a parameter to be chosen later. Multiplying both sides of (1.1) by $|u|^{\nu-1}u$, and integrating we obtain

$$\nu \int_\Omega |\nabla u|^2 \, |u|^{\nu-1} = \lambda \int_\Omega a(x) \, |u|^{p+\nu-1} + \mu \int_\Omega b(x) \, |u|^{q+\nu-1}.$$

Since, $a, b \in L^\infty(\Omega)$ we have

$$\nu \int_\Omega |\nabla u|^2 \, |u|^{\nu-1} \leq c\left(||u||^{\nu+p-1}_{\nu+p-1} + ||u||^{\nu+q-1}_{\nu+q-1}\right). \tag{4.7}$$

Using the identity

$$|\nabla(|u|^{\frac{\nu+1}{2}})|^2 = \frac{(\nu+1)^2}{4}|\nabla u|^2 \, |u|^{\nu-1},$$

and putting $w = |u|^{\frac{\nu+1}{2}}$ we obtain

$$\frac{4\nu}{(\nu+1)^2}\int_\Omega |\nabla w|^2 \leq c\left(||u||^{\nu+p-1}_{\nu+p-1} + ||u||^{\nu+q-1}_{\nu+q-1}\right). \tag{4.8}$$

Without loss of generality we can assume

$$q - p = \frac{2}{n-2}. \tag{4.9}$$

Indeed, set q_1 such that $p + 2/(n-2) < q_1 < 2n/(n-2)$ and $q < q_1$. Let $p_1 = q_1 - 2/(n-2)$. Then $p < p_1 < (n-1)/(n-2)$. Then (4.7) and (4.8) are satisfied with p_1, q_1.

The Sobolev embedding shows that

$$||w||^2_{\frac{2n}{n-2}} \leq c||w||^2_{H^1(\Omega)}. \tag{4.10}$$

Combining (4.8) and (4.10) and using $w = |u|^{\frac{\nu+1}{2}}$ we get

$$||u||^{\nu+1}_{\frac{(\nu+1)n}{n-2}} \leq \frac{(\nu+1)^2}{4\nu}c\left(||u||^{\nu+p-1}_{\nu+p-1} + ||u||^{\nu+q-1}_{\nu+q-1}\right) + \int_\Omega |u|^{\nu+1} \leq \nu c\left(||u||^{\nu+p-1}_{\nu+p-1} + ||u||^{\nu+q-1}_{\nu+q-1} + \int_\Omega |u|^{\nu+1}\right).$$

Now using the Holder's inequality

$$\|u\|_{\nu+p-1}^{\nu+p-1} \le |\Omega|^{\frac{q-p}{\nu+q-1}} \|u\|_{\nu+q-1}^{\nu+p-1} \le c\|u\|_{\nu+q-1}^{\nu+p-1},$$
$$\int_\Omega |u|^{\nu+1} \le |\Omega|^{\frac{q-2}{\nu+q-1}} \|u\|_{\nu+q-1}^{\nu+1} \le c\|u\|_{\nu+q-1}^{\nu+1},$$

for some constant $c > 0$. Thus we can write

$$\|u\|_{\frac{(\nu+1)n}{n-2}}^{\nu+1} \le \nu c\left(\|u\|_{\nu+q-1}^{\nu+p-1} + \|u\|_{\nu+q-1}^{\nu+q-1} + \|u\|_{\nu+q-1}^{\nu+1}\right). \tag{4.11}$$

Now define the sequences a_k, b_k by

$$b_1 := 2r,\ b_k := (b_{k-1} - p + 2)r,\ r := \frac{n-1}{n-2}, \tag{4.12}$$

$$a_k := (b_{k-1} - p + 2)\frac{n}{n-2} = \frac{n}{n-1}b_k. \tag{4.13}$$

Then

$$b_k = r^k + (2-p)r\frac{r^k - 1}{r-1}, \forall k \in \mathbb{N}. \tag{4.14}$$

There exist constants $c, C >$ such that

$$cr^k \le b_k \le Cr^k. \tag{4.15}$$

Since $r > 1$, $a_k \to \infty$ and $b_k \to \infty$ as $k \to \infty$. Putting $\nu = b_k - p + 1$ in (4.11) we get

$$\|u\|_{a_{k+1}}^{b_k-p+2} \le cb_k\left(\|u\|_{b_k+\frac{2}{n-2}}^{b_k} + \|u\|_{b_k+\frac{2}{n-2}}^{b_k+\frac{2}{n-2}} + \|u\|_{b_k+\frac{2}{n-2}}^{b_k-p+2}\right), \tag{4.16}$$

where we have used (4.9). Since b_k is increasing it holds that $b_k \ge 2(n-1)/(n-2)$. This implies

$$b_k + \frac{2}{n-2} \le a_k.$$

Let δ_k be such that $1/\delta_k = 1 - ((b_k + 2/(n-2))/a_k)$. Then $1/\delta_k \to 1/n$ as $k \to \infty$. Using this and the Holder inequality implies

$$\|u\|_{b_k+\frac{2}{n-2}}^{b_k+\frac{2}{n-2}} \le |\Omega|^{\frac{1}{\delta_k}} \|u\|_{a_k}^{b_k+\frac{2}{n-2}} \le c\|u\|_{a_k}^{b_k+\frac{2}{n-2}}.$$

By a similar calculation we also have

$$\|u\|_{b_k+\frac{2}{n-2}}^{b_k} \le c\|u\|_{a_k}^{b_k},$$
$$\|u\|_{b_k+\frac{2}{n-2}}^{b_k-p+2} \le c\|u\|_{a_k}^{b_k-p+2}.$$

Substituting this in (4.16) we have

$$\|u\|_{a_{k+1}}^{b_k-p+2} \le cb_k\left(\|u\|_{a_k}^{b_k+\frac{2}{n-2}} + \|u\|_{a_k}^{b_k} + \|u\|_{a_k}^{b_k-p+2}\right). \tag{4.17}$$

Define $\theta_k := \max\{\|u\|_{a_k}, 1\}$. Then (4.17) implies

$$\theta_{k+1}^{b_k-p+2} \le cb_k\theta_k^{b_k+\frac{2}{n-2}}. \tag{4.18}$$

Set

$$\alpha_k = (cb_k)^{\frac{1}{b_k-p+2}},\ \beta_k = \frac{b_k + \frac{2}{n-2}}{b_k - p + 2}.$$

Then (4.18) can be written as

$$\theta_{k+1} \le \alpha_k \theta_k^{\beta_k}. \tag{4.19}$$

Now repeated use of (4.19) produces

$$\theta_k \le \alpha_{k-1} \alpha_{k-2}^{\beta_{k-1}} \alpha_{k-3}^{\beta_{k-1}\beta_{k-2}} \cdots \alpha_1^{\beta_{k-1}\cdots\beta_2} \theta_1^{\beta_{k-1}\cdots\beta_2}. \tag{4.20}$$

We wish to show that $\theta_k \le c$ for all $k \in \mathbb{N}$. For this we first show that

$$0 < \prod_{k=1}^{\infty} \beta_k < \infty. \tag{4.21}$$

By (4.14) there exists a constant $c > 0$ such that $b_k - p + 2 \ge cr^k$ for all $k \in \mathbb{N}$. Since

$$q - p = 2/(n-2), \beta_k = 1 + \frac{q-2}{b_k - p + 2},$$

(4.21) follows from the fact that

$$\sum_{k=1}^{\infty} \frac{q-2}{b_k - p + 2} < \infty.$$

Put $\sigma = \prod_{k=1}^{\infty} \beta_k$. Then $1 < \sigma < \infty$. Since $\beta_k > 1$, it follows that

$$\beta_{k-1} \cdots \beta_i \le \sigma \text{ for } i \le k-1.$$

Since $\alpha_k > 1$, we can write

$$\alpha_{k-1} \alpha_{k-2}^{\beta_{k-1}} \alpha_{k-3}^{\beta_{k-1}\beta_{k-2}} \cdots \alpha_1^{\beta_{k-1}\cdots\beta_2} \le (\alpha_1 \cdots \alpha_{k-1})^{\sigma}.$$

Next, we show that

$$0 < \prod_{k=1}^{\infty} \alpha_k < \infty. \tag{4.22}$$

Using (4.15) and the definition of α_k, we get

$$\log\left(\prod_{k=1}^{m} \alpha_k\right) = \sum_{k=1}^{m} \log \alpha_k \le \sum_{k=1}^{\infty} \log \alpha_k < \infty.$$

Hence (4.22) holds. Thus there exists a constant $c > 0$ such that

$$\theta_k \le c\theta_1^{\sigma}, \forall k \in \mathbb{N}.$$

Taking $k \to \infty$ we obtain

$$||u||_{L^\infty(\Omega)} \le c\theta_1^{\sigma} < \infty.$$

Thus $u \in L^\infty(\Omega)$.

Now using [13, Proposition 3.6] we obtain $u \in C^\alpha(\Omega)$ and u satisfies

$$||u||_{C^\alpha(\Omega)} \le C\left(||u||_{L^2} + ||u||_{L^\infty(\Omega)}^{p-1} + ||u||_{L^\infty(\Omega)}^{q-1}\right).$$

□

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