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APPENDIX TO P. MIZERKA'S THEOREM

In a memory of Mitsuru Heiho*

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Abstract

The group $\mathrm{TL}(2,5)$ is a nontrivial double covering of S_5 with the center of order 2, where S_5 is the symmetric group on five letters. In addition, $\mathrm{TL}(2,5)$ has a free irreducible real representation of dimension 8 and contains $\mathrm{SL}(2,5)$ as a normal subgroup up to isomorphisms, where $\mathrm{SL}(2,5)$ is the special linear group of degree 2 over the field consisting of five elements. $\mathrm{TL}(2,5)$ has been denoted by $\mathrm{SL}(2,5).C_2$ as well as $\mathrm{TL}_2(5)$. In this paper, we show that there never exists an effective smooth action of $\mathrm{TL}(2,5)$ on a homology sphere of dimension 21, with exactly one fixed point.

1. Introduction

Let $\mathbb{Z}$ and $\mathbb{N}$ denote the ring of integers and the set of natural numbers, respectively, let S^n denote the standard sphere of dimension n, and let R be a principal ideal domain. For integers m and n, let $[m..n]$ denote the set $\{k \in \mathbb{Z} \mid m \le k \le n\}$. Let $[m..\infty)$ denote the union $\bigcup_n [m..n]$, where n ranges over $\mathbb{N}$. For a compact smooth manifold Σ of dimension n without boundary, we call Σ an *R-homology sphere* if the homology groups $H_i(\Sigma; R)$, $i \in [0..\infty)$, are isomorphic to $H_i(S^n; R)$, respectively. By a *homology sphere* we mean a $\mathbb{Z}$-homology sphere. Let $\mathrm{TL}(2,5)$ denote the group of order 240 with IdGroup $= [240, 89]$ in the GAP SmallGroup library [1]. The center of $\mathrm{TL}(2,5)$ has order 2, therefore we denote the center by Z_2, and $\mathrm{TL}(2,5)$ is a nontrivial double covering of S_5:

$$E \longrightarrow Z_2 \longrightarrow \mathrm{TL}(2,5) \xrightarrow{\pi} S_5 \longrightarrow E, \tag{1.1}$$

where S_5 is the symmetric group on the set $[1..5]$ and E is the trivial group. On the other hand, $\mathrm{TL}(2,5)$ contains $\mathrm{SL}(2,5)$ as a normal subgroup up to isomorphisms, where $\mathrm{SL}(2,5)$ is the special linear group of degree 2: each element of it is 2×2-matrix with determinant 1 over the field consisting of five elements. Therefore we have another exact sequence

$$E \longrightarrow \mathrm{SL}(2,5) \longrightarrow \mathrm{TL}(2,5) \longrightarrow C_2 \longrightarrow E,$$

where C_2 is a group of order 2. The group $\mathrm{TL}(2,5)$ is denoted by $\mathrm{SL}(2,5).C_2$ in [4] as well as $\mathrm{TL}_2(5)$ in [3]. It is remarkable that $\mathrm{SL}(2,5)$ and $\mathrm{TL}(2,5)$ have free irreducible real representations of dimension 4 and 8, respectively. As it is explained in [4, 6, 8], various topologists (e.g. E. Stein, T. Petrie, E. Laitinen–P. Traczyk, M. Morimoto, M. Furuta,

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N. Buchdahl–S. Kwasik–R. Schultz, S. Demichelis, A. Borowiecka, S. Tamura, P. Mizerka, ...) have studied smooth actions with exactly one fixed point on spheres of finite groups G. Recently, P. Mizerka has presented the next result.

Theorem (P. Mizerka [4, Theorem 0.1]). *Let G be* $\mathrm{TL}(2,5)$ *and Σ a homology sphere of dimension n. If $n \in [0..13] \cup \{15, 16, 17\}$, then G cannot act effectively on Σ with exactly one fixed point.*

In the present paper, we give the following two additional results.

First remark that Mizerka's theorem above resembles the theorem [6, Theorem 4.2]: S_5 does not admit a smooth action on S^n with exactly one fixed point if n lies in $[0..5] \cup \{7, 8, 9, 13\}$. Comparing it with Mizerka's theorem, (because $21 = 13 + 8$) we are interested in showing the following fact.

Theorem 1.1. *Let G be* $\mathrm{TL}(2,5)$ *and Σ a homology sphere of dimension* 21. *Then G cannot act effectively on Σ with exactly one fixed point.*

Next recall the theorem [6, Theorem 4.1]: for $G = S_5$ and a homology sphere Σ with a smooth G-action, if $\Sigma^G = \{x_0\}$ then the tangential G-representation $T_{x_0}(\Sigma)$ at x_0 in Σ contains an irreducible G-subrepresentation of dimension 6. It is interesting to show an analogous fact for $G = \mathrm{TL}(2,5)$.

Theorem 1.2. *Let G be* $\mathrm{TL}(2,5)$ *and Σ a homology sphere with an effective smooth G-action. If $\Sigma^G = \{x_0\}$ then the tangential G-representation $T_{x_0}(\Sigma)$ contains an irreducible G-subrepresentation of dimension* 6.

This research was also motivated by the following conjecture.

Conjecture. For the finite groups G and the sets I_G of integers given below, the standard sphere S^n of dimension n has an effective smooth G-action with exactly one fixed point if and only if n belongs to I_G.

(1) $G = S_5$ and $I_G = \{6, 10, 11, 12\} \cup [14..\infty)$.
(2) $G = \mathrm{TL}(2,5)$ and $I_G = \{14, 18, 19, 20\} \cup [22..\infty)$.

By [6, Theorem 4.2], [4, Theorem 0.1] and Theorem 1.1, the 'only if part' of the conjecture is valid. By [2, Theorem B], for an arbitrary Oliver group G and for an arbitrary integer $k \geq 6$, there is an effective smooth G-action on S^n with exactly one fixed point such that

$$n = k\left((|G| - 1) - \sum_p (|G/G^{\{p\}}| - 1)\right),$$

where p ranges over the set of primes dividing $|G|$ and $G^{\{p\}}$ is the smallest normal subgroup of G such that $|G/G^{\{p\}}|$ is a power of p. The author does not know any other established results concerned with the 'if part' of the conjecture.

It should be remarked that the particles necessary to our proof of the two theorems above can be found in P. Mizerka [4].

2. Preliminary

Let G be a finite group. Recall the following elementary fact.

Proposition 2.1. *Let R be a principal ideal domain and let Σ be an R-homology sphere with a smooth G-action and x_0 a fixed point of Σ, i.e. $x_0 \in \Sigma^G$. Let D be a small (closed) disk G-neighborhood of x_0 in Σ. Then $X = \Sigma \setminus \mathrm{Int}(D)$ is R-acyclic.*

The proof of this fact is straightforward.

The family $\mathcal{G}_p^q$ of finite groups is defined in R. Oliver [7], where each of p and q is a prime or 1: $\mathcal{G}_p^q$ is the family of finite groups G possessing a normal series

$$P \trianglelefteq H \trianglelefteq G \text{ such that } |P| \text{ is a power of } p, \quad H/P \text{ is cyclic, and } |G/H| \text{ is a power of } q. \tag{2.1}$$

Let $\mathcal{G}_p = \bigcup_{q \in \mathcal{P}} \mathcal{G}_p^q$ and $\mathcal{G} = \bigcup_{p,q \in \mathcal{P}} \mathcal{G}_p^q$, where $\mathcal{P}$ is the set of all primes. A finite group G is called an *Oliver group* if $G \notin \mathcal{G}$. According to Oliver [7], there exists a smooth G-action on a disk without fixed points if and only if G is an Oliver group. In addition, by [2, Theorem A], there exists a smooth G-action on a sphere with exactly one fixed point if and only if G is an Oliver group.

Proposition 2.2 (cf. [7, Proposition 2])**.** *Let p and q be primes, $G \in \mathcal{G}_p^q$, and Σ a $\mathbb{Z}_p$-homology sphere with a smooth G-action. If $\Sigma^G \neq \emptyset$ then the congruence formula*

$$\chi(\Sigma^G) \equiv \chi(S^m) \mod q$$

holds, where $\chi(-)$ is the Euler characteristic and $m = \dim T_{x_0}(\Sigma)^G$ for an arbitrary $x_0 \in \Sigma^G$.

Proof. Let P and H be as in (2.1), let $x_0 \in \Sigma^G$, and let D and X be as in Proposition 2.1. Since $\Sigma^G = D^G \cup_\partial X^G$, the equality

$$\chi(\Sigma^G) = \chi(X^G) + (-1)^m \tag{2.2}$$

holds for $m = \dim T_{x_0}(\Sigma)^G$. Note that $X^G = ((X^P)^{H/P})^{G/H}$. Since $|P|$ is a power of p, the Smith theory implies that X^P is $\mathbb{Z}_p$-acyclic and hence $\mathbb{Q}$-acyclic. Since H/P is cyclic, the Lefschetz fixed point formula implies the equality $\chi((X^P)^{H/P}) = 1$, see [7, Lemma 1, Proposition 1]. Since $|G/H|$ is a power of q, we readily get $\chi(((X^P)^{H/P})^{G/H}) \equiv \chi((X^P)^{H/P})$ $(= 1) \mod q$, see [7, p.157, ℓ.7]. Therefore we obtain

$$\chi(\Sigma^G) = \chi(X^G) + (-1)^m \equiv 1 + (-1)^m = \chi(S^m) \mod q. \qquad \square$$

Corollary 2.3. *Let G and Σ be as in the above proposition. Then Σ^G is not a singleton, i.e. $|\Sigma^G| \neq 1$.*

Proposition 2.4. *Let G be a finite group, let p be a prime, let P be a subgroup of G such that $|P|$ is a power of p, let H be a subgroup of G containing P, and let Σ be a $\mathbb{Z}_p$-homology sphere with $x_0 \in \Sigma^G$. Set $V = T_{x_0}(\Sigma)$ and suppose $\dim V^P = \dim V^H$. In each of the following cases (1)–(3), it holds that $\Sigma^P = \Sigma^H$ (a $\mathbb{Z}_p$-homology sphere) and hence $|\Sigma^H| \geq 2$.*

(1) $\dim V^P > 0$.

(2) *There is a sequence $P = K_0 \trianglelefteq K_1 \trianglelefteq \cdots \trianglelefteq K_{m-1} \trianglelefteq K_m = H$.*

(3) *$H \in \mathcal{G}_p$.*

Proof. By the Smith theory, Σ^P is a $\mathbb{Z}_p$-homology sphere and $\dim \Sigma^P = \dim V^P$. Let Σ_0^H be the connected component of Σ^H such that $x_0 \in \Sigma_0^H$. We have $T_{x_0}(\Sigma_0^H) = V^H$ as real $N_G(H)$-representations.

Firstly consider the case (1), i.e. $\dim V^P > 0$. In this case, we get $\dim V^H = \dim V^P \geq 1$. Since Σ_0^H and Σ^P are connected closed manifolds of same dimension, the property $\Sigma_0^H \subset \Sigma^H \subset \Sigma^P$ implies $\Sigma_0^H = \Sigma^H = \Sigma^P$.

In the rest of the proof, we can assume $\dim V^P = 0$ without loss of generality. Then we have $|\Sigma^P| = 2$ and hence we write Σ^P as $\{x_0, x_1\}$.

Secondly consider the case (2). In the case $m = 0$, i.e. $P = K_0 = H$, the equality $\Sigma^P = \Sigma^{K_0} = \Sigma^H$ obviously holds. For the case $m \geq 1$, by induction on i, it suffices to show that $\Sigma^P = \Sigma^{K_i}$ under the hypothesis $\Sigma^{K_{i-1}} = \Sigma^P = \{x_0, x_1\}$. Since $K_{i-1} \trianglelefteq K_i$, K_i acts on $\Sigma^{K_{i-1}} = \{x_0, x_1\}$. The point x_0 is a K_i-fixed point and therefore x_1 must be a K_i-fixed point. This shows $\Sigma^{K_i} = \{x_0, x_1\} = \Sigma^P$.

Now consider the case (3). Since $H \in \mathcal{G}_p$, by Corollary 2.3, we get $|\Sigma^H| \neq 1$. Since $x_0 \in \Sigma^H \subset \Sigma^P = \{x_0, x_1\}$, we conclude $\Sigma^H = \Sigma^P$. □

The soul of the next proposition and the corollary to it is due to S. Tamura [8, Proof of Lemma 3.1, Proof in Case 2] and P. Mizerka [4, Lemma 2.3].

Proposition 2.5. *Let G be a finite group, let p be a prime, let P be a subgroup of G such that $|P|$ is a power of p, let H_1 and H_2 be subgroups of G such that $P \subset H_1 \cap H_2$, and let Σ be a $\mathbb{Z}_p$-homology sphere with $x_0 \in \Sigma^G$. Set $V = T_{x_0}(\Sigma)$ and suppose* $\dim V^P = \dim V^{H_1}$. *Further suppose the following two conditions are fulfilled.*

(C1) $\Sigma^{H_1} \neq \{x_0\}$.

(C2) *The elements of $H_1 \cup H_2$ generate G, i.e. $\langle H_1, H_2 \rangle = G$.*

Then $\Sigma^P = \Sigma^{H_1}$ (a $\mathbb{Z}_p$-homology sphere) and $\Sigma^{H_2} = \Sigma^G$.

Proof. We know that Σ^P is a $\mathbb{Z}_p$-homology sphere and $\Sigma^P \supset \Sigma^{H_i}$ for $i = 1, 2$. Once $\Sigma^P = \Sigma^{H_1}$ was proved, we would have

$$\Sigma^G = \Sigma^{\langle H_1, H_2 \rangle} = \Sigma^{H_1} \cap \Sigma^{H_2} = \Sigma^P \cap \Sigma^{H_2} = \Sigma^{H_2}.$$

Firstly consider the case $\dim V^P \geq 1$. Then by Proposition 2.4 (1), we obtain $\Sigma^P = \Sigma^{H_1}$.

Next consider the case $\dim V^P = 0$. Then we have $|\Sigma^P| = 2$. Since $2 \leq |\Sigma^{H_1}| \leq |\Sigma^P| = 2$, we get $\Sigma^P = \Sigma^{H_1}$. □

Corollary 2.6. *In* Proposition 2.5*, further suppose that $H_2 \in \mathcal{G}_p$. Then Σ^G is not a singleton, i.e. $\Sigma^G \neq \{x_0\}$.*

Proof. Since $H_2 \in \mathcal{G}_p$, by Corollary 2.3, we get $(|\Sigma^G| =)\ |\Sigma^{H_2}| \neq 1$. □

Proposition 2.7. *Let X be a connected closed manifold with a smooth G-action. Suppose $X^G = \{x_0\}$ and set $V = T_{x_0}(X)$. Then the real G-representation V does not contain an irreducible summand of dimension* 1.

Proof. The hypothesis $X^G = \{x_0\}$ immediately implies $V^G = \{0\}$.

Recall the fact that $V^L = \{0\}$ for any subgroup L of G with index 2, which has been used in various papers. For the sake of readers' convenience, we give a proof of the fact here.

Assume that L is a subgroup of G such that $|G/L| = 2$ and $V^L \neq \{0\}$. Let X_0^L be the connected component of X^L containing x_0. Then we get $\dim X_0^L = \dim V^L \geq 1$. Note that the group G/L (of order 2) acts on the connected closed manifold X_0^L. Since $(X_0^L)^{G/L} \supset \{x_0\}$, Lemma 2.1 of [5] implies $|X^G| \geq |(X_0^L)^{G/L}| \geq 2$. This contradicts the hypothesis that $X^G = \{x_0\}$. Therefore $V^L = \{0\}$ holds for any subgroup L of G with index 2.

Now we are going to prove that V does not contain an irreducible summand of dimension 1. Assume that U is a 1-dimensional irreducible summand of V. Let $\rho_U : G \to \{1, -1\}$ be the group homomorphism associated with U. Set $K = \mathrm{Ker}(\rho_U)$. Since $V^G = \{0\}$, we have $U^G = \{0\}$. Therefore K is a subgroup of G with index 2. Since $\dim U^K = 1$, we have $\dim V^K \geq \dim U^K = 1$, which contradicts the fact $V^K = \{0\}$. □

Next recall the subgroup lattices of S_5 and $\mathrm{TL}(2, 5)$, see e.g. [4, Figures 1 and 2].

Proposition 2.8. *The symmetric group S_5 contains subgroups*

$$\mathfrak{D}_4 = \langle (2,3),\ (2,3)(4,5) \rangle, \quad \mathfrak{D}_8 = \langle (2,4,3,5),\ (2,3) \rangle, \text{ and}$$
$$\mathfrak{S}_3\mathfrak{C}_2 = \langle (1,2),\ (1,2,3),\ (4,5) \rangle.$$

The subgroup $\mathfrak{D}_4$ is contained in $\mathfrak{D}_8$ and subconjugate to $\mathfrak{S}_3\mathfrak{C}_2$. The subgroups $Q_{8,A}$, Q_{16}, Dic_6 in [4] are conjugate to $\pi^{-1}(\mathfrak{D}_4)$, $\pi^{-1}(\mathfrak{D}_8)$, and $\pi^{-1}(\mathfrak{S}_3\mathfrak{C}_2)$, respectively, where $\pi : \mathrm{TL}(2, 5) \to S_5$ is the projection in (1.1).

The proof of this proposition is straightforward.

3. Proof of Theorem 1.2

Let G be $\mathrm{TL}(2, 5)$ and Σ a homology sphere with an effective smooth G-action. Suppose $\Sigma^G = \{x_0\}$ and set $V = T_{x_0}(\Sigma)$.

As it is described in [4], the irreducible real G-representations of dimension ≥ 2, up to isomorphisms, are

$$U_{4,1},\ U_{4,2},\ U_{5,1},\ U_{5,2},\ U_6,\ W_{8,1},\ W_{8,2},\ W_{8,3},\ W_{12,1}, \text{ and } W_{12,2}.$$

The characters of these real G-representations are given in [4, Table 1.1].

Claim 3.1. *V contains an irreducible summand isomorphic to $W_{8,i}$ or $W_{12,j}$, where i and j can range in $[1..3]$ and $\{1, 2\}$, respectively.*

Proof. This follows from the hypothesis that the G-action on Σ is effective. □

In the rest of this section, assuming that

V does not contain an irreducible summand of dimension 6,

we will argue to find a contradiction.

Claim 3.2. *The following holds.*

1. *V does not contain a summand isomorphic to $U_{4,2}$ nor $U_{5,2}$.*
2. *V contains a summand isomorphic to $U_{5,1}$.*
3. *Any irreducible summand of V is isomorphic to $U_{4,1}$, $U_{5,1}$, $W_{8,i}$, or $W_{12,j}$, where i and j can range in $[1..3]$ and $\{1, 2\}$, respectively.*

Proof. By hypothesis, V does not contain a summand isomorphic to U_6. The first assertion is valid by [4, Theorem 2.5 (1)]. The second assertion is valid by [4, Theorem 2.5 (2)]. The third assertion is clearly valid. □

By Proposition 2.8, we can choose generating subgroups H_1 and H_2 of G, i.e. $\langle H_1, H_2\rangle = G$, such that H_1, H_2, and $H_1 \cap H_2$ are conjugate to Q_{16}, Dic_6, and $Q_{8,A}$, respectively. Set $H = H_1 \cap H_2$.

Claim 3.3. *If k is the multiplicity of $U_{5,1}$ in V up to isomorphisms, then* $\dim V^H = k$, $\dim V^{H_1} = k$, *and* $V^{H_2} = 0$.

In this claim, $k \geq 1$ by Claim 3.2 (2).

Proof. The above claim follows from [4, Table 2.1]. □

Claim 3.4. *The following holds.*

(1) *H and H_1 are 2-groups and $H_2 \in \mathcal{G}_2^2$.*
(2) *$\Sigma^H = \Sigma^{H_1}$ (a connected $\mathbb{Z}_2$-homology sphere).*
(3) *$\Sigma^{H_2} = \Sigma^G$ $(= \{x_0\})$.*

Proof. It is clear that H and H_1 have order 2^3 and 2^4, respectively. Recall that H_2/Z_2 is isomorphic to $S_3 \times C_2$, where Z_2 is the center of G and S_3 is the symmetric group on three letters. Therefore there is a normal series

$$Z_2 \triangleleft T \triangleleft H_2$$

such that $|Z_2| = 2$, T/Z_2 is a cyclic group of order 6, and $|H_2/T| = 2$, which shows $H_2 \in \mathcal{G}_2^2$. This proves the first conclusion.

We have already showed that $\dim V^H = \dim V^{H_1} = k \geq 1$, where $V = T_{x_0}(\Sigma)$. By Proposition 2.5, we obtain the second and third conclusions. □

Claim 3.5. *Σ^{H_2} cannot be a singleton.*

Proof. This follows from Corollary 2.6. □

We have seen that Claim 3.5 contradicts Claim 3.4 (3), and established the proof of Theorem 1.2.

4. Proof of Theorem 1.1

Let G be the group $\mathrm{TL}(2, 5)$ and Σ a homology sphere of dimension 21. In order to prove Theorem 1.1, we assume that Σ has an effective smooth G-action with exactly one fixed point x_0 and will argue to find a contradiction. Let V be the tangential G-representation $T_{x_0}(\Sigma)$.

The decomposition of V to irreducible summands,

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_m,$$

gives the partition of 21, i.e. $21 = d_1 + d_2 + \cdots + d_m$, where $d_i = \dim V_i$.

Claim 4.1. *In the partition $21 = d_1 + d_2 + \cdots + d_m$, any member d_i is not equal to* 6. *Therefore V does not contain a summand isomorphic to U_6.*

Proof. Assume $d_1 = 6$. Then we get the partition $15 = d_2 + \cdots + d_m$.

In the case $V \supset W_{8,i}$ up to isomorphisms for some $i \in [1..3]$, supposing $d_2 = 8$, we get the partition $7 = d_3 + \cdots + d_m$ such that $d_3, \ldots, d_m \in \{4, 5, 6\}$, which is impossible. In the case $V \supset W_{12,j}$ up to isomorphisms for some $j \in \{1, 2\}$, supposing $d_2 = 12$, we get the partition $3 = d_3 + \cdots + d_m$ such that $d_3, \ldots, d_m \in \{4, 5, 6\}$, which is impossible.

The argument shows that any member d_i of the partition $21 = d_1 + d_2 + \cdots + d_m$ is not equal to 6. □

Since Claim 4.1 contradicts Theorem 1.2, G cannot act effectively on Σ with exactly one fixed point.

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