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A REALIZATION APPROACH TO EXTENDED AFFINE LIE SUPERALGEBRAS

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Abstract

Extended affine Lie superalgebras are super versions of extended affine Lie algebras and, more generally, invariant affine reflection algebras. By employing a method known as "affinization", we construct several classes of extended affine Lie superalgebras of arbitrary nullity.

1. Introduction

Kac-Moody Lie algebras are grouped into three classes; finite dimensional simple Lie algebras, affine Lie algebras and indefinite Kac-Moody Lie algebras [14]. Extended affine Lie algebras are certain generalizations of finite dimensional simple Lie algebras and affine Lie algebras.

This class was introduced in 1990 by two mathematical physicists [12] and was systematically studied in [1], since then it has drawn a considerable amount of attention. In recent years, a super version of extended affine Lie algebras named extended affine Lie superalgebras (EALSA for short) has been introduced and investigated [22].

To each extended affine Lie superalgebra one associates a notion of "nullity", see Definition 2.3. Finite dimensional basic classical Lie superalgebras and affine Lie superalgebras are nullity 0 and nullity 1 extended affine Lie superalgebras, respectively. The class of extended affine Lie (super)algebras are introduced axiomatically, so one asks about the realization and then classification of this class. The method of realizing affine Kac-Moody Lie algebras given in [14] is known in the literature as "twisting" by automorphisms or "affinization". To realize extended affine Lie algebras, people tried to use the idea of affinization by iterating it to higher nullities. The first remarkable achievement in this direction is due to U. Pollman [18] in realization of nullity 2 extended affine Lie algebras, known also as elliptic Lie algebras. This idea was later extended to higher nullities by S. Azam [5]. The "affinization method" for extended affine Lie algebras and, more generally, invariant affine reflection algebras [17] was put in a machinery framework in [3] and [7]. It is now known that centerless cores of almost all extended affine Lie algebras can be realized through the affinization method [2, 4].

For a classification of simple finite-dimensional complex Lie superalgebras, see [13]. In 1985, V. Serganova [19] described the automorphisms of complex finite dimensional simple

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Lie superalgebras. Using the latter results, J. Van de Leur [20, 21] and L. Frappat et al. [10] applied the affinization method in order to realize affine Kac-Moody Lie superalgebras. In [6], the authors provide a solid framework for realizing EALSAs through the affinization method, and using this we consider the realization of low nullity EALSAs.

The paper is arranged as follows. In Section 2, we develop some preliminaries and facts about EALSAs which are needed in the sequel. In particular a method of construction of new EALSAs from old, called affinization, is explained. In Section 3, we consider an automorphism of the ground EALSA and show that its set of fixed points is an EALSA, with respect to a canonical Cartan matrix. In particular the root system of the fixed point subalgebra is an extended affine root supersystem. Also, in this section we consider automorphisms of extended affine root supersystems and investigate conditions under which a natural canonical induced set is again an extended affine root supersystem. In Section 4, we employ the affinization method in order to construct new extended affine Lie superalgebras of arbitrary nullity. In each case we provide a description of the corresponding root system in the form given in [23], which will be crucial in further study of EALSAs. Considering the classification of this class of Lie superalgebras, we should mention that even in the level of root system, the classification of the corresponding root supersystems is not yet completed [23].

2. Preliminaries

Throughout this work, $\mathbb{K}$ is a field of characteristic zero, and $\mathbb{K}$ contains a primitive m-th root of unity for $m \in \mathbb{N}$. Unless otherwise mentioned, all vector spaces are defined over $\mathbb{K}$. For an automorphism σ of period m of a vector space $\mathcal{V}$, we set $\mathcal{V}(j) = \{x \in \mathcal{V} \mid \sigma(x) = \zeta^j x\}$, $j \in \mathbb{Z}$. Since $\mathcal{V}(j) = \mathcal{V}(k)$ if $j = k$ (mod m), $\mathcal{V}(j)$ is also denoted by $\mathcal{V}(\bar{j})$, $\bar{j} \in \mathbb{Z}_m$. We have $\mathcal{V} = \bigoplus_{\bar{j}\in\mathbb{Z}_m} \mathcal{V}(\bar{j})$, and then the projection of $\mathcal{V}$ onto $\mathcal{V}(\bar{j})$ under this direct sum will be denoted by π_j. One sees that

$$\pi_j(x) = \frac{1}{m}\sum_{i=0}^{m-1} \zeta^{-ji}\sigma^i(x).$$

For simplicity of notations, we often denote π_0 by π. For a group A and a subset S, the subgroup generated by S in A will be denoted by $\langle S\rangle$. If $\mathcal{A}$ is a graded algebra, by an automorphism of $\mathcal{A}$, we mean a graded automorphism; an automorphism which preserves homogeneous spaces.

DEFINITION 2.1. A triple $(\mathcal{L}, T, (\cdot,\cdot))$ consisting of a nonzero Lie superalgebra $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$, a non-trivial subalgebra T of $\mathcal{L}_0$ and a non-degenerate invariant even super-symmetric bilinear form $(\cdot,\cdot)$ on $\mathcal{L}$, is called a *super-toral triple* if

(i) $\mathcal{L}$ has a weight space decomposition $\mathcal{L} = \bigoplus_{\alpha\in T^\star} \mathcal{L}^\alpha$ with respect to T via the adjoint representation, namely $\mathcal{L}^\alpha = \{x \in \mathcal{L} \mid [t, x] = \alpha(t)x \text{ for all } t \in T\}$, $\alpha \in T^\star$, where $T^\star$ is the dual of T,

(ii) the restriction of the form $(\cdot,\cdot)$ to T is non-degenerate.

The subalgebra T is called a *toral subalgebra* of $\mathcal{L}$. It can be easily verified that toral subalgebras are abelian. A toral subalgebra T satisfying $T = \mathcal{L}^0 \cap \mathcal{L}_0$ is called a *splitting Cartan subalgebra.* The set of *roots* of $\mathcal{L}$ with respect to T, defined by $R := \{\alpha \in T^\star \mid \mathcal{L}^\alpha \neq \{0\}\}$, is decomposed as $R = R_0 \cup R_1$ in accordance with the even and odd roots. We transfer the form on T to the image of T in $T^\star$ via the injection $\upsilon : t \mapsto (t, \cdot)$. If condition (E1) of the below definition holds, then R is contained in the image of the latter map, so the term (α,β) makes sense for $\alpha,\beta \in R$ (see [16, Remark 1.4]). We denote by t_α, the unique element in T that represents $\alpha \in \upsilon(T)$ through the form $(\cdot,\cdot)$; i.e. $(t_\alpha, t) = \alpha(t)$ for all $t \in T$.

Definition 2.2. A super-toral triple $(\mathcal{L}, T, (\cdot,\cdot))$ with root system $R = R_0 \cup R_1$ is called an *extended affine Lie superalgebra* (EALSA) if

(E1) for $\alpha \in R_i\backslash\{0\}$, $i = 0, 1$, there is a pair $x_{\pm\alpha} \in \mathcal{L}_i^{\pm\alpha}$ with $0 \neq [x_\alpha, x_{-\alpha}] \in T$,

(E2) for $\alpha \in R$ with $(\alpha,\alpha) \neq 0$ and $x \in \mathcal{L}^\alpha$, the map $\mathrm{ad}x : \mathcal{L} \longrightarrow \mathcal{L}$ is locally nilpotent.

An extended affine Lie superalgebra $(\mathcal{L}, T, (\cdot,\cdot))$ is called *division* if (E1) is replaced by the below stronger axiom:

(E1)′ for each $\alpha \in R_i\backslash\{0\}$, $i = 0, 1$, and any $0 \neq x_\alpha \in \mathcal{L}_i^\alpha$, there exists $x_{-\alpha} \in \mathcal{L}_i^{-\alpha}$ such that $0 \neq [x_\alpha, x_{-\alpha}] \in T$.

Finite dimensional basic classical simple Lie superalgebras and affine Lie superalgebras are examples of extended affine Lie superalgebras.

Let A denote the $\mathbb{Z}$-span of R in $T^\star$, for the root system R of an EALSA $(\mathcal{L}, T, (\cdot,\cdot))$. Set

$$A^0 := \{\alpha \in A \mid (\alpha, A) = \{0\}\}, \quad R^0 := R \cap A^0, \quad R^\times := R\backslash R^0,$$
$$R^\times_{re} := \{\alpha \in R^\times \mid (\alpha,\alpha) \neq 0\}, \quad R_{re} := R^\times_{re} \cup \{0\},$$
$$R^\times_{ns} := \{\alpha \in R^\times\backslash R^0 \mid (\alpha,\alpha) = 0\}, \quad R_{ns} := R^\times_{ns} \cup \{0\}.$$

By [22, Corollary 3.9], R satisfies the following conditions:

(S1) $0 \in R$, and $\langle R\rangle = A$,

(S2) $R = -R$,

(S3) for $\alpha \in R^\times_{re}$ and $\beta \in R$, there are non-negative integers d, u with $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = d - u$ such that $(\beta + \mathbb{Z}\alpha) \cap R = \{\beta - d\alpha, \ldots, \beta + u\alpha\}$,

(S4) for $\alpha \in R_{ns}$ and $\beta \in R$ with $(\alpha,\beta) \neq 0$, $\{\beta - \alpha, \beta + \alpha\} \cap R \neq \emptyset$.

Definition 2.3. A triple $(A, (\cdot,\cdot), R)$ consisting of an abelian group A, a non-trivial symmetric map $(\cdot,\cdot) : A \times A \to \mathbb{K}$ which is additive on both components, and a subset R of A satisfying (S1)-(S4), is called an *extended affine root supersystem* (in short, EARSS). For the sake of simplicity, we often call R an extended affine root supersystem. An extended affine root supersystem R is called *tame* if its elements are *non-isolated*, in the sense that for each $\eta \in R^0$, there exists $\alpha \in R^\times$ such that $\alpha + \eta \in R$. It is called *indecomposable* if $R^\times$ cannot be decomposed into a disjoint union of two nonempty subsets which are orthogonal with respect to the form. In this work for the root systems under consideration A is always a free abelian group of finite rank. In this case the rank of A^0 is called the *nullity* of R. The nullity of an EALSA is by definition the nullity of its root system.

Definition 2.4. An extended affine root supersystem $(A, (\cdot,\cdot), R)$ is called *null* if $R^\times = \emptyset$; *irreducible* if R is indecomposable and $R^\times_{re} \neq \emptyset$; *locally finite root supersystem* if $A^0 = \{0\}$; *locally finite root system* if R is a locally finite root supersystem with $R_{ns} = \{0\}$. Also, if a locally finite root (super)system R is finite, then R is called a *finite root (super)system*.

For an extended affine root supersystem $(A, (\cdot,\cdot), R)$, it is known that $\bar{R}$, the image of R

under the canonical epimorphism $^{-}: A \longrightarrow \bar{A} := A/A^0$, endowed with the induced form is a locally finite root supersystem in $\bar{A}$ (see [23, Proposition 1.11]). If R is irreducible, the *type* of R is by definition the type of $\bar{R}$, see [25, Proposition 4.31 and Theorem 4.39]. Since $\bar{A}$ is torsion free, it can be identified as a subgroup of the $\mathbb{K}$-vector space $\mathbb{K}\otimes_{\mathbb{Z}}\bar{A}$ and the from on $\bar{A}$ can be extended naturally to a bilinear form $(\cdot,\cdot)_{\mathbb{K}}$. When R is indecomposable, then $1\otimes\bar{R}$ (resp. $1\otimes\bar{R}_{re}$) is a locally finite root supersystem (resp. locally finite root system) in $\mathbb{K}\otimes\bar{A}$ (resp. $\mathbb{K}\otimes\langle\bar{R}_{re}\rangle$) (see [25, Lemma 3.21 and Lemma 3.5]). It terms out that for a certain subgroup $\dot{A}$ of A^0 with $A=\dot{A}\oplus A^0$, the set $\dot{R} := \{\dot{\alpha}\in\dot{A} \mid \exists\eta\in A^0; \dot{\alpha}+\eta\in R\}$ is a locally finite root supersystem isomorphic to $\bar{R}$.

In what follows we describe irreducible locally finite root supersystems, up to isomorphism.

Suppose V is a vector space with a basis $\{\eta_1,\eta_2,\eta_3\}$ or $\{\epsilon_i,\delta_j \mid i\in I, j\in J\}$ in which I and J are two index sets such that $I\cup J\neq\emptyset$. Define a symmetric bilinear form $(\cdot,\cdot)$ on V by

$$(\eta_1,\eta_1)=1,\quad (\eta_2,\eta_2)=-(1+\lambda),\quad (\eta_3,\eta_3)=\lambda\ (\lambda\in\mathbb{K}\setminus\{0,-1\}),$$
$$(\eta_i,\eta_j)=0\quad (i\neq j),$$

or

$$(\epsilon_i,\epsilon_r)=\delta_{i,r},\quad (\delta_j,\delta_s)=-\delta_{j,s},\quad (\epsilon_i,\delta_j)=0,$$

respectively. Set

$$\dot{A}(I,I) := \pm\{\epsilon_i-\epsilon_r, \delta_i-\delta_r, \epsilon_i-\delta_r-\frac{1}{|I|}\sum_{k\in I}(\epsilon_k-\delta_k) \mid i,r\in I\} \quad (|I|\in\mathbb{Z}_{\geq 2}), \tag{2.1}$$

$$\dot{A}(I,J) := \pm\{\epsilon_i-\epsilon_r, \delta_j-\delta_s, \epsilon_i-\delta_j \mid i,r\in I, j,s\in J\} \quad (|I|\neq|J| \text{ if } I,J \text{ are finite sets}),$$

$$B(I,J) := \pm\{\epsilon_i, \delta_j, \epsilon_i\pm\epsilon_r, \delta_j\pm\delta_s, \epsilon_i\pm\delta_j \mid i,r\in I, j,s\in J, i\neq r\},$$

$$C(I,J) := \pm\{\epsilon_i\pm\epsilon_r, \delta_j\pm\delta_s, \epsilon_i\pm\delta_j \mid i,r\in I, j,s\in J\},$$

$$D(I,J) := \pm\{\epsilon_i\pm\epsilon_r, \delta_j\pm\delta_s, \epsilon_i\pm\delta_j \mid i,r\in I, j,s\in J, i\neq r\},$$

$$BC(I,J) := \pm\{\epsilon_i, \delta_j, \epsilon_i\pm\epsilon_r, \delta_j\pm\delta_s, \epsilon_i\pm\delta_j \mid i,r\in I, j,s\in J\},$$

$$F(4) := \pm\{0, \epsilon, \delta_i\pm\delta_j, \delta_i, \tfrac{1}{2}(\epsilon\pm\delta_1\pm\delta_2\pm\delta_3) \mid 1\leq i\neq j\leq 3\} \quad (\epsilon := \sqrt{3}\epsilon_1,\ I=\{1\}, J=\{1,2,3\}),$$

$$G(3) := \pm\{0, \delta, 2\delta, \epsilon_i-\epsilon_j, 2\epsilon_i-\epsilon_j-\epsilon_t, \delta\pm(\epsilon_i-\epsilon_j) \mid \{i,j,t\}=\{1,2,3\}\} \quad (\delta := \sqrt{2}\delta_1,\ I=\{1,2,3\}, J=\{1\}),$$

$$D(2,1;\lambda) := \{0, \pm 2\eta_i, \pm\eta_1\pm\eta_2\pm\eta_3 \mid 1\leq i\leq 3\}.$$

In the above sets if I or J is an empty set, by convention the terms corresponding to these indices disappear. When the sets I and J in the types above are finite, we replace them with their cardinalities, for example we write $B(|I|,|J|)$ instead of $B(I,J)$. To be compatible with some references which we use, we also denote the type $\dot{A}(|I|,|J|)$ by $A(|I|-1,|J|-1)$.

Either the sets given in (2.1) or an irreducible locally finite root system, provide a complete list of irreducible locally finite root supersystems, up to isomorphism ([25, Theorem

4.37]).

Remark 2.5. EARSS, except types $X \neq A(l,l), C(1,2), C(I,2), BC(1,1)$, are described in [23, Theorem 2.2] in terms of a locally finite root supersystem $\dot{R}$, and certain *symmetric (pointed) reflection subspaces* of A^0. We consider the decomposition $\dot{R}_{re} = \oplus_{i=1}^{n} \dot{R}_{re}^{i}$ of $\dot{R}_{re}$ into irreducible supersystems. We also recall that we denote the short, long and extra long roots of a locally finite root supersystem by $\dot{R}_{sh}$, $\dot{R}_{lg}$ and $\dot{R}_{ex}$, respectively. The description is quite delicate and varies depending on the type. Below, we have summarized the given description without further details on inter relations between the involved symmetric (pointed) reflection subspaces E_1, E_2, L, L_2 (resp. S, L_1) and a subgroup F of A^0:

- If $X \neq A(l,l), BC(I,J), C(I,J), C(1,J)$,

$$R = (S - S) \cup (\dot{R}_{sh} + S) \cup ((\dot{R} \setminus \dot{R}_{sh}) + F).$$

- If $X = BC(1,J), BC(I,J);\ |I|, |J| > 1$,

$$R = (S - S) \cup (\dot{R}_{sh} + S) \cup (\dot{R}_{ex}^{1} + E_1) \cup (\dot{R}_{ex}^{2} + E_2) \cup ((\dot{R}_{lg} \cup \dot{R}_{ns}) + F).$$

- If $X = C(1,J);\ |J| > 2$,

$$R = (S - S) \cup (\dot{R}_{sh}^{1} + S) \cup ((\dot{R}_{sh}^{2} \cup \dot{R}_{ns}) + F) \cup (\dot{R}_{lg}^{2} + L).$$

- If $X = C(I,J);\ |I| \geq 2, |J| > 2$,

$$R = F \cup ((\dot{R}_{sh} \cup \dot{R}_{ns}) + F) \cup ((\dot{R}_{re}^{1})_{lg} + L_1) \cup ((\dot{R}_{re}^{2})_{lg} + L_2).$$

For more details we refer the reader to [23]. As it can be seen from the above description of R, we always have $R^0 = S - S$ or $R^0 = F$.

Remark 2.6. Let R be an EARSS.

(i) It follows from (S3) and (S4) that for any $\alpha \in R^{\times}$ and $\beta \in R$ with $(\alpha, \beta) \neq 0$, we have $\{\alpha + \beta, \alpha - \beta\} \cap R \neq \emptyset$.

(ii) For $k \in \mathbb{K}$ and $\alpha \in R_{re}^{\times}$, $k\alpha \in R$ implies (by (S3)) that $k \in \{0, \pm 1, \pm 2, \pm\frac{1}{2}\}$.

(iii) In the case that R is the root system of an EALSA, $\alpha \in R_{re}^{\times} \cap R_1$ implies $2\alpha \in R_0$. Indeed by condition (E1) of Definition 2.2, there exists $x_{\pm\alpha} \in \mathcal{L}_1^{\pm\alpha}$ for which $0 \neq [x_\alpha, x_{-\alpha}] \in T$, whence

$$[[x_\alpha, x_\alpha], [x_{-\alpha}, x_{-\alpha}]] = -4\alpha(t_\alpha)[x_\alpha, x_{-\alpha}](x_\alpha, x_{-\alpha}) \neq 0.$$

So we have $0 \neq [x_\alpha, x_\alpha] \in \mathcal{L}_0^{2\alpha}$.

Lemma 2.7. *Let R be an EARSS. If $R_{ns}^{\times} \neq \emptyset$ then $R_{re}^{\times} \neq \emptyset$.*

Proof. For $\alpha \in R_{ns}^{\times}$ there exists by definition, $\beta \neq \pm\alpha$ in R such that $(\alpha, \beta) \neq 0$. If $\beta \in R_{re}^{\times}$, then we are done. Otherwise, we get from (S4) that $\alpha + \beta$ or $\alpha - \beta$ is in R. Without loss of generality, we assume that $\alpha + \beta \in R$. Since $(\alpha + \beta, \alpha + \beta) = 2(\alpha, \beta) \neq 0$, $\alpha + \beta \in R_{re}^{\times}$. □

From now on, we fix a 6-tuple $(\mathcal{L}, \sigma, \Lambda, \mathcal{A}, \epsilon, \rho)$ with the following conditions:

- $(\mathcal{L}, T, (\cdot,\cdot))$ is an EALSA with indecomposable root system $R = R_0 \cup R_1$,
- σ is an automorphism of $\mathcal{L}$ satisfying (A1)-(A4) below:

(A1) $\sigma^m = \mathrm{id}_{\mathcal{L}}$,
(A2) $\sigma(T) = T$,
(A3) $(\sigma(x), \sigma(y)) = (x, y)$ for all $x, y \in \mathcal{L}$,
(A4) $C_{\mathcal{L}(0)}(T(0)) \subseteq \mathcal{L}^0$ ($\mathcal{L}^0$ contains the centralizer of $T(0)$ in $\mathcal{L}(0)$).

- Λ is a torsion free abelian group,
- $\mathcal{A} = \oplus_{\lambda\in\Lambda}\mathcal{A}^\lambda$ is a unital associative commutative predivision Λ-graded algebra with $\mathrm{supp}(\mathcal{A}) = \Lambda$, (predivision, means for every $\lambda \in \mathrm{supp}(\mathcal{A})$, there are $a^{\pm\lambda} \in \mathcal{A}^{\pm\lambda}$ such that $a^{+\lambda}a^{-\lambda} = 1$)
- $\epsilon : \mathcal{A} \times \mathcal{A} \to \mathbb{K}$ is a non-degenerate symmetric invariant Λ-graded bilinear form with $\epsilon(1, 1) \neq 0$,
- $\rho : \Lambda \to \mathbb{Z}_m$, $\lambda \mapsto \bar{\lambda}$ is a group epimorphism.

For the rest of this work, we assume that $T(0) \neq \{0\}$. We impose this assumption in order to guarantee that $\pi(R) \neq \{0\}$. We note that if $\mathcal{L}(0) \neq \{0\}$ and $\mathcal{L}^0 = T$, or if m is a prime number then the assumption $T(0) \neq \{0\}$ follows from (A4) (see [6, Lemma 4.2]). We also mention here that in all examples which we provide in this work either m is prime or $\mathcal{L}(0) \neq \{0\}$ and $\mathcal{L}^0 = T$.

In what follows we recall a construction and some results from [6, §6] which will be used in the sequel, we encourage the reader to consult the mentioned reference for details.

Let $\mathcal{V}$ be the $\mathbb{K}-$vector space $\mathbb{K} \otimes_{\mathbb{Z}} \Lambda$ and denote its dual space by $\mathcal{V}^\star$. Since Λ is torsion free, we may consider $\Lambda \subseteq \mathcal{V}$. Then $\widehat{\mathcal{L}} := (\mathcal{L} \otimes \mathcal{A}) \oplus \mathcal{V} \oplus \mathcal{V}^\star$ together with the following multiplication;

$$
\begin{aligned}
&[d, x] = -[x, d] = d(\lambda)x,\ d \in \mathcal{V}^\star, x \in \mathcal{L} \otimes \mathcal{A}^\lambda,\\
&[\mathcal{V}, \widehat{\mathcal{L}}] = [\widehat{\mathcal{L}}, \mathcal{V}] = \{0\},\\
&[\mathcal{V}^\star, \mathcal{V}^\star] = \{0\},\\
&[x \otimes a, y \otimes b] = [x, y] \otimes ab + (x, y)\epsilon(a, b)\lambda,
\end{aligned}
\tag{2.2}
$$

$x\otimes a \in \mathcal{L}\otimes\mathcal{A}^\lambda$, $y\otimes b \in \mathcal{L}\otimes\mathcal{A}$, forms a Lie superalgebra. The form $(\cdot,\cdot)$ on $\mathcal{L}$ can be extended to $\widehat{\mathcal{L}}$, denoted again by $(\cdot,\cdot)$, as follows:

$$
\begin{aligned}
(x \otimes a, y \otimes b) = (x, y)\epsilon(a, b), \quad (x, y \in \mathcal{L}, a, b \in \mathcal{A}),\\
(d, v) := d(v), \quad (d \in \mathcal{V}^\star, v \in \mathcal{V}),\\
(\mathcal{V}, \mathcal{V}) = (\mathcal{V}^\star, \mathcal{V}^\star) = (\mathcal{V}, \mathcal{L} \otimes \mathcal{A}) = (\mathcal{V}^\star, \mathcal{L} \otimes \mathcal{A}) := \{0\}.
\end{aligned}
\tag{2.3}
$$

Then $(\widehat{\mathcal{L}}, \widehat{T}, (\cdot,\cdot))$ is a super-toral triple (see [6, Lemma 6.2]) where $\widehat{T} := (1 \otimes T) \oplus \mathcal{V} \oplus \mathcal{V}^\star$.

The automorphism σ can be extended to $\widehat{\mathcal{L}}$ by acting as identity on $\mathcal{V}\oplus\mathcal{V}^\star$ and $\sigma(x\otimes a) = \zeta^{-\bar{\lambda}}\sigma(x) \otimes a$, for $x \in \mathcal{L}$ and $a \in \mathcal{A}^\lambda$. Our desired algebra under this construction is the fixed point subalgebra $\widetilde{\mathcal{L}} = \sum_{\lambda\in\Lambda} \mathcal{L}(\bar{\lambda}) \otimes A^\lambda \oplus \mathcal{V} \oplus \mathcal{V}^\star$ of $\widehat{\mathcal{L}}$ under σ. The process of obtaining $\widetilde{\mathcal{L}}$ from $\mathcal{L}$ under the above construction is called an *affinization process*. Starting from $\mathcal{L}$, any change in either of the involved data $(\sigma, \mathcal{A}, \epsilon, \rho)$ results in a new, sometimes isomorphic, Lie superalgebra $\widetilde{\mathcal{L}}$. In what follows we discuss certain situations under which $\widetilde{\mathcal{L}}$ is again an EALSA. This procedure can be used to construct EALSAs of higher nullity starting from EALSAs of lower nullities. The goal of this paper is to construct, using the above method, some EALSAs of nullity 2 over field $\mathbb{C}$ of complex numbers.

Remark 2.8. The automorphism $\sigma : T \longrightarrow T$ induces an automorphism of $T^\star$ denoted again by σ, defined by $\sigma(\alpha)(t) = \alpha(\sigma^{-1}(t))$ for $t \in T$ and $\alpha \in T^\star$. Then as we have mentioned before the projection map of $T^\star$ into the subspace of its fixed points is given by $\pi(\alpha) = \frac{1}{m}\sum_{i=0}^{m-1}\sigma^i(\alpha)$. Note that

$$\mathcal{L}^{\pi(\alpha)} := \{x \in \mathcal{L} | [t, x] = \alpha(t)x, \text{ for all } t \in T(0)\} = \sum_{\{\beta\in R|\pi(\beta)=\pi(\alpha)\}} \mathcal{L}^\beta,$$

and

$$\sigma(\mathcal{L}^{\pi(\alpha)}) = \mathcal{L}^{\pi(\alpha)}.$$

Lemma 2.9 ([6, Lemma 3.2])**.** (A4) *holds if and only if* $\pi(\mathcal{L}^\alpha) = \{0\}$ *for all* $\alpha \in R \setminus \{0\}$ *satisfying* $\pi(\alpha) = 0$. *In particular, if m is prime then* (A4) *holds if and only if* $\pi(\alpha) \neq 0$ *for all* $0 \neq \alpha \in R$.

Lemma 2.10 ([26, Lemma 2.4])**.**
(i) *If* $\alpha \in R^\times_{re}, \beta \in R$ *and* $\alpha + \beta \in R$ *then* $[\mathcal{L}^\alpha, \mathcal{L}^\beta] \neq \{0\}$.
(ii) *If* $\alpha, \beta \in R_{ns}$ *and* $\alpha + \beta \in R^\times$ *then* $[\mathcal{L}^\alpha, \mathcal{L}^\beta] \neq \{0\}$.

From [6, Theorem 5.3 and Proposition 6.3] we have the following.

Proposition 2.11. *If* $\mathcal{L}$ *is division, then* $\pi(R)$ *is an indecomposable extended affine root supersystem.*

To recall the main result concerning affinization process, we need the following, "tameness condition":
(A5) $\pi[\mathcal{L}^0_i, \mathcal{L}^0_i] \subseteq T(0)$; for i=0,1.
We note that if T is a splitting Cartan subalgebra of $\mathcal{L}$, then (A5) holds automatically.

Theorem 2.12 ([6, Theorem 8.4])**.** *Suppose* σ *satisfies* (A1)-(A5).
(i) *If* $\mathcal{L}$ *is division, then the triple* $(\widetilde{\mathcal{L}}, \widetilde{T}, (\cdot,\cdot))$ *is an EALSA with root system* $\widetilde{R} = \cup_{\lambda\in\Lambda}(\pi(R(\bar{\lambda})) + \lambda)$ *where* $R(\bar{\lambda}) = \{\alpha \in R \mid \mathcal{L}(\bar{\lambda})^{\pi(\alpha)} \neq \{0\}\}$. *In particular, if* $\mathcal{L}$ *is an EALSA satisfying* $[\mathcal{L}^0_1, \mathcal{L}^0_1] \subseteq T$ *then* $(\widehat{\mathcal{L}}, \widehat{T}, (\cdot,\cdot))$ *is an EALSA with root system* $\widehat{R} = R \oplus \Lambda$.
(ii) *If R is indecomposable then so is* $\widetilde{R}$.

3. Induced root systems

The aim of this section is to study the set $\pi(R)$ for an extended affine root supersystem R under an automorphisms σ of R satisfying certain conditions, see (AR1)-(AR3) below.

Suppose $(\mathcal{L}, T, (\cdot,\cdot))$ is an EALSA with an indecomposable root system R. Let A be the $\mathbb{Z}$-span of R in $T^\star$. Let σ be an automorphism of $\mathcal{L}$ satisfying (A1)-(A4). As we have seen before σ induces an automorphism of $T^\star$ by $\sigma(\alpha)(t) = \alpha(\sigma^{-1}(t))$ for $\alpha \in T^\star$ and $t \in T$. In this case we have
(AR1) $\sigma \in \mathrm{Aut(A)}$, $\sigma^m = 1$,
(AR2) $\sigma(R) = R$,
(AR3) $(\sigma(\alpha), \sigma(\beta)) = (\alpha, \beta)$ for $\alpha, \beta \in A$.
In other words, σ is a finite order automorphism of R. Note that we have two interesting

sets which naturally appear here, namely $\pi(R)$ and

$$R^\sigma := \{\pi(\alpha) \mid \alpha \in R,\ \mathcal{L}^{\pi(\alpha)}(0) \neq \{0\}\}.$$

In general we have $R^\sigma \subseteq \pi(R)$. However, in many examples the equality $R^\sigma = \pi(R)$ occurs, in which cases the type of the corresponding Lie superalgebra $\widetilde{\mathcal{L}}$ can be recognized from $\pi(R)$. As we have seen in Proposition 2.11, if $\mathcal{L}$ is division, then $\pi(R)$ is an indecomposable extended affine root supersystem. However, if we consider σ as an "abstract" finite order automorphism of root system, namely an automorphism of $\langle R\rangle$ satisfying (AR1)-(AR3), the structure of $\pi(R)$ is not fully understood even in the case of extended affine root systems. The structure of $\pi(R)$ under an abstract finite order automorphism of an irreducible extended affine root system is investigated in [9].

Suppose R is an extended affine root supersystem and σ is an automorphism of R satisfying (AR1)-(AR3). For each $\alpha \in R^\times_{re}$, the automorphism $w_\alpha \in \mathrm{Aut}(A)$ given by $w_\alpha(\beta) = \beta - (2(\alpha,\beta)/(\alpha,\alpha))\alpha$, satisfies conditions (AR1)-(AR3). Any such reflection is called an *even* reflection and the subgroup $\mathcal{W}$ of $\mathrm{Aut}(A)$ generated by all even reflections is called the *Weyl group* of R. One knows that for $\alpha \in R^\times_{re}$, the reflection w_α preserves the form on R.

Proposition 3.1. *Let R be an irreducible EARSS of type $X \neq A(l,l)$, $C(1,2)$, $C(I,2)$, $BC(1,1)$ such that $\pi(R)^\times \neq \emptyset$. Then $\pi(R)^0$ is non-isolated.*

Proof. Suppose that $\alpha \in R$ with $\pi(\alpha) \in \pi(R)^0$. It is enough to show that there exists $\beta \in R$ such that $\alpha + \beta \in R$ and $\pi(\beta) \in \pi(R)^\times$. Since R is irreducible, we have from Step 2 of [6, Proposition 6.3] that

$$R^\times = \{\gamma \in R \mid \exists \gamma' \in R \text{ s.t. } (\gamma,\gamma') \neq 0 \text{ and } \pi(\gamma') \in \pi(R)^\times\}.$$

Therefore if $\alpha \in R^\times = R^\times_{re} \cup R^\times_{ns}$, then there exists $\beta \in R^\times$ such that $(\alpha,\beta) \neq 0$ and $\pi(\beta) \in \pi(R)^\times$. Now by root string property or (S4), we have either $\alpha + \beta \in R$ or $\alpha - \beta \in R$, and we are done. Next we assume that $\alpha \in R^0$. By Remark 2.5, R^0 is equal to F or $S - S$, where F is a subgroup of A^0 and S is a pointed reflection subspace of A^0. Thus either $\alpha \in F$ or $\alpha = \delta_1 + \delta_2$ with $\delta_1, \delta_2 \in S$.

It is clear that $\pi(A^0) \subseteq \pi(A)^0$. Since $\pi(R)^\times \neq \emptyset$, $\pi(\dot{R})^\times \neq \emptyset$. Then for some $\beta \in \dot{R}$, $\pi(\beta) \in \pi(\dot{R})^\times$. Now we proceed with the proof by considering the two cases, (I) $\pi(\dot{R}_{re})^\times \neq \emptyset$, or (II) $\pi(\dot{R}_{re})^\times = \emptyset$ and $\pi(\dot{R}_{ns})^\times \neq \emptyset$.

I) Since the group $\langle \dot{R}_{re}\rangle$ is generated by short roots, there exists $\beta \in \dot{R}_{sh}$ with $\pi(\beta) \in \pi(R_{re})^\times$. Note that $\dot{R}_{sh} \subseteq R$. One knows that $\dot{R}_{re} = S_1 \uplus \cdots \uplus S_i$ in which $i \in \{1,2,3\}$ and for $1 \leq t \leq i$, S_t's are locally finite root systems, see [25, Theorem 1.9 and Theorem 1.9]. Let $\beta \in S_{t_0}$ for $1 \leq t_0 \leq i$. Now if $\alpha \in F$, we have

$$\beta + \alpha = \dot{R}_{sh} + F \subseteq R,$$

and if $\alpha = \delta_1 + \delta_2$, then $\widetilde{\beta} := \beta + \delta_1 \in \dot{R}_{sh} + S \subseteq R$ and

$$\widetilde{\beta} + \alpha = \beta + 2\delta_1 + \delta_2 = \dot{R}_{sh} + 2S + S \subseteq \dot{R}_{sh} + S \subseteq R,$$

and we are done.

II) In this case there exists $\beta \in \dot{R}_{ns}$ such that $\pi(\beta) \in \pi(\dot{R}_{ns})^\times$. If R is of real type, i.e. $\mathrm{span}_{\mathbb{Q}}\dot{R}_{re} = \mathbb{Q} \otimes_{\mathbb{Z}} \dot{A}$, then $\pi(R)^\times = \emptyset$ which is a contradiction. If R is of imaginary type (that R is not of real type) then by [23, Theorem 2.2], $S = F = A^0$. In this case, we have:

$$\beta + \alpha \in \dot{R}_{ns} + F \subset R.$$

So, we are done. □

Proposition 3.2. *Let* $(\mathcal{L}, T, (\cdot,\cdot))$ *be an EALSA with root system R, and* σ *be an automorphism of* $\mathcal{L}$ *satisfying* (A1)-(A3). *If further,* $\pi(R)^\times \neq \emptyset$ *and* $C_{\mathcal{L}(0)}(T(0)) = T(0)$*, then* $(\mathcal{L}, T(0), (\cdot,\cdot))$ *is an EALSA with root system* $\pi(R)$*. In particular,* $\pi(R)$ *is an EARSS.*

Proof. Considering Remark 2.8, we have $\mathcal{L} = \bigoplus_{\pi(\alpha)\in\pi(R)} \mathcal{L}^{\pi(\alpha)}$. As $\pi(R)^\times \neq \emptyset$, $T(0) \neq \{0\}$. By (A3), the form $(\cdot,\cdot)$ restricted to $T(0)$ is non-degenerate. Therefore $(\mathcal{L}, T(0), (\cdot,\cdot))$ is a super-toral triple. For $\alpha \in R = R_0 \cup R_1$ and $\bar{j} \in \mathbb{Z}_m$, we have $[\mathcal{L}^{\pi(\alpha)}(\bar{j}), \mathcal{L}^{\pi(-\alpha)}(-\bar{j})] \subseteq \mathcal{L}(0) \cap \mathcal{L}^{\pi(0)} = C_{\mathcal{L}(0)}(T(0)) = T(0)$. Now fix $\pi(\alpha) \in \pi(R)_i \setminus \{0\}$ $(i \in \{0,1\})$, we have $\sigma(\mathcal{L}_i^{\pi(\alpha)}) = \mathcal{L}_i^{\pi(\alpha)}$. So, for some $\bar{j} \in \mathbb{Z}_m$, $\mathcal{L}_i^{\pi(\alpha)}(\bar{j}) \neq \{0\}$. Since the form $(\cdot,\cdot)$ restricted to $\mathcal{L}_i^{\pi(\alpha)}(\bar{j}) \oplus \mathcal{L}_i^{\pi(-\alpha)}(-\bar{j})$ is non-degenerate, we see that there exists $x^\pm \in \mathcal{L}_i^{\pm\pi(\alpha)}(\pm\bar{j})$ with $(x^+, x^-) \neq 0$ such that

$$[x^+, x^-] = (x^+, x^-) t_{\pi(\alpha)}.$$

Thus $0 \neq [x^+, x^-] \in T(0)$ and (E1) holds. An argument similar to [6, Theorem 5.3] shows that (E2) also holds. Thus $(\mathcal{L}, T(0), (\cdot,\cdot))$ is an EALSA, and so by [22, Corollary 3.9], its root system $\pi(R)$ is an EARSS. □

Recall that for $\sigma \in \text{Aut}(T^\star)$ with $\sigma^m = 1$, we have already defined the map $\pi(\alpha) = (1/m)\sum_{i=0}^{m-1} \sigma^i(\alpha)$, $\alpha \in T^\star$. When working with $\sigma \in \text{Aut}(A)$, A an abelian group, instead of the vector space $T^\star$, the map π does not make sense because of the scalar $1/m$. In this case we normalize π by considering $m\pi$ instead of π, using the same notation π.

Corollary 3.3. *Let* $(A, (\cdot,\cdot), R)$ *be a finite root supersystem corresponding to one of the finite dimensional basic classical simple Lie superalgebras, except* $A(1,1)$*. Suppose* $\sigma \in$ *Aut*(A) *satisfies* (AR1)-(AR3). *Further suppose that* $\pi(\alpha) \neq 0$ *for* $\alpha \neq 0$*. Then* $\pi(R)$ *is a finite root supersystem in* $\pi(A)$*.*

Proof. Let $\mathcal{L}$ be a complex finite dimensional basic classical Lie superalgebra with Cartan subalgebra T and root system R. Let $(\cdot,\cdot)$ be the standard non-degenerate bilinear form on $\mathcal{L}$. We transfer the from on T to $T^\star$ in the natural way. We may assume that the form on $T^\star$ restricted to A coincides with the form on $(A, (\cdot,\cdot), R)$. The automorphism σ of R can be lifted to an automorphism σ, of period m, of $\mathcal{L}$ with $\sigma(t_\alpha) = t_{\sigma(\alpha)}$, for every $\alpha \in R$ (see [24, Theorem 4.5 and Lemma 4.6]).

Next, we show that the form on $\mathcal{L}$ is invariant under the extended automorphism σ. Consider the invariant bilinear form $(\cdot,\cdot)' := (\sigma(\cdot), \sigma(\cdot))$. Then $(\cdot,\cdot)$ and $(\cdot,\cdot)'$ are the same up to a multiple nonzero scalar (see [15, Proposition 1.2.4]). Since $\mathcal{L}^0 = T$ is spanned by t_α, $\alpha \in R$, and the form on T is non-degenerate, there exist $\alpha, \beta \in R$ such that $(t_\alpha, t_\beta) \neq 0$. Then, by (AR3),

$$0 \neq (t_\alpha, t_\beta) = (\alpha, \beta) = (\sigma(\alpha), \sigma(\beta)) = (t_{\sigma(\alpha)}, t_{\sigma(\beta)}) = (\sigma(t_\alpha), \sigma(t_\beta)) = (t_\alpha, t_\beta)'.$$

This gives $(\cdot,\cdot) = (\cdot,\cdot)'$. Therefore σ is an automorphism of $\mathcal{L}$ satisfying (A1)-(A3).

Identifying $(T(0))^\star$ with $T^\star(0)$, we conclude that $T(0) \neq \{0\}$. Since by assumption $\pi(R \setminus \{0\}) \subseteq R \setminus \{0\}$, $C_{\mathcal{L}(0)}(T(0)) = T(0)$. Thus using Proposition 3.2, we get that $(\mathcal{L}, T(0), (\cdot,\cdot))$ is an EALSA with root system $\pi(R)$. In particular, $\pi(R)$ is an extended affine root supersystem in $\pi(A)$. Also, by (AR2) and (AR3), for $\alpha, \beta \in R$, we have

$$(\pi(\alpha), \pi(\beta)) = (\pi(\alpha), \sum_{i=0}^{m-1} \sigma^i(\beta)) = \sum_{i=0}^{m-1} (\pi(\alpha), \beta) = m(\pi(\alpha), \beta).$$

So $(\pi(\alpha), \pi(\beta)) = m(\pi(\alpha), \beta)$. This implies that the non-degenerate form $(\cdot,\cdot)$ on A restricted to $\pi(A)$ is non-degenerate. Since $\pi(R)$ is finite, it is a finite root supersystem in $\pi(A)$. □

Definition 3.4. A subset $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ of the root supersystem R is called a *set of simple roots* for R, if any nonzero root $\alpha \in R$ can be written uniquely as a linear combination $\alpha = k_1\alpha_1 + \cdots + k_n\alpha_n$ such that either all k_i are non-negative or all k_i are non-positive. Elements of Π are called *simple* roots.

Remark 3.5. We note that for each root supersystem of basic classical Lie superalgebras, except the type $A(m,m)$, there exists a set of simple roots. For type $A(m,m)$ the set

$$\Pi := \{\alpha_1 = \epsilon_1 - \epsilon_2, \ldots, \alpha_m := \epsilon_m - \epsilon_{m+1}, \alpha_{m+1} = \epsilon_{m+1} - \delta_1, \ldots, \alpha_{2m+1} = \delta_m - \delta_{m+1}\}$$

is usually considered as a set of simple roots. This set is not $\mathbb{Z}$-linearly independent as

$$\alpha_1 + 2\alpha_2 + \cdots + m\alpha_m + (m+1)\alpha_{m+1} + m\alpha_{m+2} + \cdots + 2\alpha_{2m} + \alpha_{2m+1} = 0,$$

(see [11, Table 3.53]). So a proof similar to [9, Corollary 3.2] does not work in Corollary 3.3.

Let $(A, (\cdot,\cdot), R)$ be a tame irreducible EARSS of type $X \neq A(\ell,\ell)$, $C(1,2)$, $C(T,2)$, $BC(1,1)$ satisfying (AR1) and (AR2). Then we have the following lemma.

Lemma 3.6. *Let σ be an automorphism of R satisfying* (AR1)-(AR2). *Then $\sigma(R^0) = R^0$. Also, if $\sigma(R_{ns}) = R_{ns}$ and R_{re} is indecomposable then $(\sigma(x), \sigma(y)) = (x, y)$ for all $x, y \in R_{re}$.*

Proof. By [23, Theorem 2.2], we have $R^0 = S - S$ where S is a pointed reflection subspace of A^0. It follows that if $\alpha \in R^0$, then we have $\mathbb{Z}\alpha \subseteq R^0$. By (AR2), $\mathbb{Z}\sigma(\alpha) \subseteq R$ and so $\mathbb{Z}\overline{\sigma(\alpha)} \subseteq \bar{R}$. But this can happen only if $\overline{\sigma(\alpha)} = 0$, as $\bar{R}$ is a locally finite root supersystem. Thus $\sigma(\alpha) \in R^0$.

Next assume $\sigma(R_{ns}) = R_{ns}$. From this and $\sigma(R^0) = R^0$, we get $\sigma(R_{re}) = R_{re}$. We extend σ to the vector space $\mathbb{K} \otimes_{\mathbb{Z}} \bar{A}$ in the natural manner and identify $\bar{R}$ by $1 \otimes \bar{R}$. As we have seen before, $\bar{R}_{re}$ is a locally finite root system in $\mathbb{K} \otimes \langle \bar{R}_{re} \rangle$. Now a similar argument as in [9, Lemma 2.1], shows that $(\sigma(x), \sigma(y)) = (x, y)$, for all $x, y \in R_{re}$. □

4. Construction of higher nullity EALSAs

In this section we provide new examples of EALSAs of nullity ν ($\nu \in \mathbb{Z}_{>0}$). Roughly speaking, we start with a finite dimensional basic classical simple Lie superalgebra $\mathcal{L}$ and a finite order automorphism σ of $\mathcal{L}$. We then extend this automorphism to

$$\widehat{\mathcal{L}} = (\mathcal{L} \otimes_{\mathbb{C}} \mathcal{A}) \oplus \mathcal{V} \oplus \mathcal{V}^\star,$$

where $\mathcal{A}$ is the algebra of Laurent polynomials in ν variables, and $\mathcal{V}$ is a ν-dimensional vector space consisting of central elements. Its dual space $\mathcal{V}^\star$ is then interpreted as the space of degree derivations. The fixed points $\widetilde{\mathcal{L}}$ of $\widehat{\mathcal{L}}$ under the extended automorphism is our required Lie superalgebra. Almost all of our starting automorphisms are diagram automorphisms of basic classical simple Lie superalgebras, therefore from the type of the diagram automorphism one can guess the type of the resulting EALSA. In our examples we fully describe the resulting root system by giving a description of it in terms of the involved pointed reflection subspaces, see Remark 2.5. Here is an overview of what is done in each example:

Example	Type of $\mathcal{L}$	order of σ	Type of $\widetilde{\mathcal{L}}$
4.1	$G(3)$	2	$G(3)$
4.2	$C(0,n)$	2	$B(0,n)$
4.3	$D(m,n)$	2	$B(m-1,n)$
4.5	$A(2k,2l-1)$	2	$BC(k,l)$
4.5	$A(2k-1,2l-1)$	2	$C(k,l)$
4.5	$A(2k,2l)$	4	$BC(k,l)$
4.6	$A(1,1)$	2	$A(1,1)$

The notation $\mathcal{L}(M,\tau)$ will be used for a contragredient Lie superalgebra associated to a matrix $M=(a_{ij})\in M_{n\times n}(\mathbb{C})$ and a set $\tau\subseteq I:=\{1,\dots,n\}$. For $\mathcal{L}(M,\tau)$ we fix a minimal realization $(\Pi,\Pi^\vee,T)$, where $\Pi=\{\alpha_1,\dots,\alpha_n\}\subseteq T^\star$, and $\Pi^\vee=\{\alpha_1^\vee,\dots,\alpha_n^\vee\}\subseteq T$. Also we fix a set of Chevalley generators $X:=\{e_i,f_i|\ i\in I\}$ associated with this realization. We recall that $\dim(T)=n+\operatorname{corank}(M)$, $\alpha_j(\alpha_i^\vee)=a_{ij}$,

$$(4.4)\qquad [e_i,f_j]=\delta_{i,j}\alpha_i^\vee,\quad [t,t']=0,\quad [t,e_i]=\alpha_i(t)e_i,\quad [t,f_i]=-\alpha_i(t)f_i,$$

$$\deg(t)=0 \text{ and } \deg(e_i)=\deg(f_i)=\begin{cases}0 & \text{if } i\in\tau,\\ 1 & \text{if } i\notin\tau,\end{cases}$$

for $t,t'\in T$ and $i,j\in I$. Note that if (M',τ') is another pair as above for which there exists an invertible diagonal matrix D such that after a suitable renumbering of I we get $(M,\tau)=(DM',\tau')$ then, $\mathcal{L}(M,\tau)\cong\mathcal{L}(M',\tau')$.

If (M,τ) satisfies:

- $\{i\mid a_{ii}=0\}\subseteq\tau$,
- if $a_{ii}\neq 0$, then $2a_{ij}/a_{ii}$ (resp. a_{ij}/a_{ii}) is a non-positive integer for $i\in I\setminus\tau$ (resp. $i\in\tau$) with $i\neq j$,
- $a_{ij}=0$ implies $a_{ji}=0$,

then $\mathcal{L}(M,\tau)$ is called a Kac-Moody Lie superalgebra associated to the generalized Cartan matrix M and the subset τ.

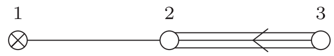
In all examples below the algebra of Laurent polynomials $\mathcal{A}=\mathbb{C}[t_1^{\pm1},\dots,t_\nu^{\pm1}]$ is equipped with the $\mathbb{Z}^\nu$-grading

$$\mathcal{A}=\bigoplus_{(n_1,\dots,n_\nu)\in\mathbb{Z}^\nu}\mathcal{A}^{n_1,\dots,n_\nu},\quad \mathcal{A}^{n_1,\dots,n_\nu}:=\mathbb{C}t_1^{n_1}\dots t_\nu^{n_\nu},$$

and the non-degenerate $\mathbb{Z}^\nu$-graded bilinear form

$$\epsilon(t_1^{n_1}\dots t_\nu^{n_\nu},t_1^{n'_1}\dots t_\nu^{n'_\nu})=\delta_{n_1,-n'_1}\dots\delta_{n_\nu,-n'_\nu}.$$

EXAMPLE 4.1. Let $\mathcal{L} := \mathcal{L}(M,\tau)$ be the simple finite dimensional complex Kac-Moody Lie superalgebra $\mathcal{L} = G(3)$;



where the generalized Cartan matrix M and the set τ are given by:

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 3 \\ 0 & 3 & -6 \end{bmatrix} \quad \text{and} \quad \tau = \{1\}.$$

Let $(\cdot,\cdot)$ be the standard super-symmetric invariant non-degenerate bilinear form on $\mathcal{L}$, see [21, Proposition 4.2]. Consider a 3-dimensional $\mathbb{C}$-vector space $T := \bigoplus_{i=1}^{3} \mathbb{C}h_i$. We consider elements $\epsilon_1, \epsilon_2, \epsilon_3, \delta \in T^\star$ defined by

$\epsilon_1(h_1) = 1$	$\epsilon_1(h_2) = -2$	$\epsilon_1(h_3) = 3$
$\epsilon_2(h_1) = 1$	$\epsilon_2(h_2) = 1$	$\epsilon_2(h_3) = -3$
$\epsilon_3(h_1) = -2$	$\epsilon_3(h_2) = 1$	$\epsilon_3(h_3) = 0$
$\delta(h_1) = 2$	$\delta(h_2) = 0$	$\delta(h_3) = 0$

Let $\Pi := \{\alpha_1 := \delta + \epsilon_3, \alpha_2 := \epsilon_1, \alpha_3 := \epsilon_2 - \epsilon_1\} \subseteq T^\star$ and $\Pi^\vee := \{h_1, h_2, h_3\}$. Then $(\Pi, \Pi^\vee, T)$ is a minimal realization for M. Note that we have $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$. Also note that the form $(\cdot,\cdot)$ restricted to T satisfies $(h_i, h_j) = \alpha_i(h_j)$. As $t_{\alpha_i} = h_i$, $i \in \{1,2,3\}$, we have

$$t_{\epsilon_1} = h_2, t_{\epsilon_2} = t_{\epsilon_1} + t_{\epsilon_2 - \epsilon_1} = h_2 + h_3, t_{\epsilon_3} = -t_{\epsilon_1} - t_{\epsilon_2} = -2h_2 - h_3,$$
$$t_\delta = t_{\alpha_1} - t_{\epsilon_3} = h_1 + 2h_2 + h_3.$$

Transferring this form to $T^\star$ by $(\alpha,\beta) := (t_\alpha, t_\beta)$, we get:

$$(\epsilon_i, \epsilon_j) = -3\delta_{i,j} + 1, \quad (\delta,\delta) = 2, \quad (\epsilon_i, \delta) = 0, \qquad (1 \le i, j \le 3).$$

Let $R = R_0 \cup R_1$ be the root system of $\mathcal{L}$. Then

$$\begin{aligned} R_0 &= \{0, \pm 2\delta, \pm\epsilon_i, \epsilon_i - \epsilon_j \mid 1 \le i \ne j \le 3\} \\ &= \pm\{0, \alpha_2, \alpha_3, \alpha_2 + \alpha_3, 2\alpha_2 + \alpha_3, 3\alpha_2 + \alpha_3, 3\alpha_2 + 2\alpha_3, 2\alpha_1 + 4\alpha_2 + 2\alpha_3\}, \end{aligned}$$

and

$$\begin{aligned} R_1 = \{\pm\delta, \pm\epsilon_i \pm \delta \mid 1 \le i \le 3\} = \pm\{&\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3, \\ &\alpha_1 + 3\alpha_2 + \alpha_3, \alpha_1 + 3\alpha_2 + 2\alpha_3, \alpha_1 + 4\alpha_2 + 2\alpha_3\}. \end{aligned}$$

The group homomorphism $\sigma : \sum_{i=1}^{3} \mathbb{Z}\alpha_i \to \mathbb{C} \setminus \{0\}$, defined by $\sigma(\alpha_1) = \sigma(\alpha_2) = 1$, $\sigma(\alpha_3) = -1$, induces the automorphism $\sigma \in \mathrm{Aut}(\mathcal{L})$ given by $\sigma(x) = \sigma(\alpha)x$ for all x belonging to the root space corresponding to root α. In fact, σ is an automorphism of $\mathcal{L}$ of order 2 which stabilizes T pointwise, forcing $\pi(\alpha) = \alpha$ for every $\alpha \in R$. The automorphism σ satisfies (A3), in fact if we consider the bilinear form $(x,y)' := (\sigma(x), \sigma(y))$ on $\mathcal{L}$, it follows from [15, Proposition 1.2.4], that $(\cdot,\cdot)'$ is a nonzero multiple scalar of $(\cdot,\cdot)$, but σ acts as identity on some plus minus root spaces forcing the scalar to be 1. The above discussion shows that (A1) to (A5) hold.

Next, we consider the Lie superalgebra $\widehat{\mathcal{L}} = (\mathcal{L} \otimes \mathcal{A}) \oplus \mathcal{V} \oplus \mathcal{V}^\star$, see (2.2) and (2.3) for details. We consider the epimorphism $\rho : \mathbb{Z}^\nu \longrightarrow \mathbb{Z}_2$, $(n_1, \ldots, n_\nu) \mapsto \bar{n}_1 + \cdots + \bar{n}_\nu$. By

Theorem 2.12 and Lemma 2.10, the fixed point subalgebra $\widetilde{\mathcal{L}}$ of $\widehat{\mathcal{L}}$, under σ, is an extended affine Lie superalgebra of nullity ν with extended affine root supersystem $\widetilde{R} = \widetilde{R}_0 \cup \widetilde{R}_1$ where

$$\widetilde{R}_0 = \{\sum_{i=1}^{\nu} n_i\delta_i, \sum_{i=1}^{\nu} n_i\delta_i \pm (\epsilon_2 - \epsilon_3), \sum_{i=1}^{\nu} n_i\delta_i \pm 2\delta, \sum_{i=1}^{\nu} n_i\delta_i \pm \epsilon_1, \sum_{i=1}^{\nu} m_i\delta_i \pm (\epsilon_1 - \epsilon_3),$$
$$\sum_{i=1}^{\nu} m_i\delta_i \pm (\epsilon_1 - \epsilon_2), \sum_{i=1}^{\nu} m_i\delta_i \pm \epsilon_2, \sum_{i=1}^{\nu} m_i\delta_i \pm \epsilon_3 | \sum_{i=1}^{\nu} n_i \in 2\mathbb{Z}, \sum_{i=1}^{\nu} m_i \in 2\mathbb{Z}+1\},$$

and

$$\widetilde{R}_1 = \{\sum_{i=1}^{\nu} n_i\delta_i \pm \epsilon_2 \pm \delta, \sum_{i=1}^{\nu} n_i\delta_i \pm \epsilon_3 \pm \delta, \sum_{i=1}^{\nu} m_i\delta_i \pm \epsilon_1 \pm \delta,$$
$$\sum_{i=1}^{\nu} m_i\delta_i \pm \delta | \sum_{i=1}^{\nu} n_i \in 2\mathbb{Z}, \sum_{i=1}^{\nu} m_i \in 2\mathbb{Z}+1\}.$$

In what follows, we provide a description of $\widetilde{R}$ in the form given in Remark 2.5. For this we set $\dot{R} := \dot{R}_{re} \cup \dot{R}_{ns}$, where

$$\dot{R}_{re} := \{0, \pm(\epsilon_2 - \epsilon_3 + 2\delta_1)), \pm 2(\delta + \delta_1), \pm\epsilon_1, \pm(\delta_1 + \epsilon_1 - \epsilon_3), \pm(-\delta_1 + \epsilon_1 - \epsilon_2)$$
$$\pm(\delta_1 + \epsilon_2), \pm(\epsilon_3 - \delta_1), \pm(\delta_1 + \delta)\},$$

and

$$\dot{R}_{ns} := \{0, \pm(\epsilon_2 - \delta), \pm(\epsilon_2 + \delta + 2\delta_1), \pm(\epsilon_3 + \delta), \pm(\delta - \epsilon_3 + 2\delta_1), \pm\epsilon_1 \pm (\delta + \delta_1)\}.$$

One can see that $\dot{R}$ is an irreducible finite root supersystem of type $G(3)$. In fact as $\{\epsilon_1 - \delta, \epsilon_2 - \epsilon_1, \delta\}$ is a set of simple roots for $G(3)$, the assignment

$$\phi : \epsilon_1 - \delta \mapsto \epsilon_1 - \delta - \delta_1, \quad \epsilon_2 - \epsilon_1 \mapsto \epsilon_2 - \epsilon_1 + \delta_1, \quad \delta \mapsto \delta + \delta_1,$$

induces $R \cong \dot{R}$. We see that $\widetilde{R} = (\dot{R}, S, F)$ is an EARSS of type $G(3)$ in which $\dot{R}_{re} = S_1 \uplus S_2$ with

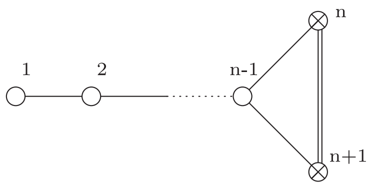
$$S_1 = \phi(\{0, \pm\delta, \pm 2\delta\}), \quad S_2 = \phi(\{0, \pm\epsilon_i, \epsilon_i - \epsilon_j | 1 \leq i \neq j \leq 3\}),$$

and

$$S = F = \{n_1\delta_1 + \cdots n_\nu\delta_\nu | n_1 \cdots + n_\nu \in 2\mathbb{Z}\}.$$

We note that S_1 and S_2 are finite root systems of types BC_1 and G_2, respectively.

Example 4.2. Let $\mathcal{L} := \mathcal{L}(M, \tau)$ be the finite dimensional complex Kac-Moody Lie superalgebra $\mathcal{L} = \mathfrak{osp}(2, 2n)$;



with corresponding generalized Cartan matrix M where

$$M = \begin{bmatrix} 2 & -1 & & & & \\ -1 & \ddots & \ddots & 0 & & \\ & \ddots & 2 & -1 & -1 \\ 0 & & -1 & 0 & 2 \\ & & -1 & 2 & 0 \end{bmatrix}, \quad \tau = \{n, n+1\}.$$

Consider an $(n+1)$-dimensional vector space $T = \sum_{i=1}^{n+1} \mathbb{C}h_i$ with basis $\{h_i \mid 1 \leq i \leq n+1\}$ and set

$$\Pi := \{\alpha_i := \delta_i - \delta_{i+1}, \alpha_n := \delta_n - \epsilon, \alpha_{n+1} := \delta_n + \epsilon \mid 1 \leq i \leq n-1\},$$

$$\Pi^\vee := \{\alpha_i^\vee := h_i - h_{i+1}, \alpha_{n+1}^\vee = h_n + h_{n+1} \mid 1 \leq i \leq n\},$$

where

$$\delta_i(h_j) = \delta_{i,j}, \quad \epsilon(h_i) = 0, \quad \epsilon(h_{n+1}) = -1; \quad 1 \leq i \leq n, 1 \leq j \leq n+1.$$

The triple $(\Pi, \Pi^\vee, T)$ is a minimal realization for M with $\mathcal{L}^0 = T$. Let $\{e_i, f_i | 1 \leq i \leq n+1\}$ be a set of Chevalley generators for this realization such that $e_i \in \mathcal{L}^{\alpha_i}$, $f_i \in \mathcal{L}^{-\alpha_i}$. One knows that the root system of $\mathcal{L}$ with respect to T is

$$R = \underbrace{\{0, \pm\delta_i \pm \delta_j, \pm 2\delta_i \mid 1 \leq i \neq j \leq n\}}_{R_0} \cup \underbrace{\{\pm\epsilon \pm \delta_i | 1 \leq i \leq n\}}_{R_1}.$$

One knows that there exists a super-symmetric invariant non-degenerate bilinear form on $\mathcal{L}$ in which $(\alpha_i^\vee, \alpha_j^\vee) = \alpha_i(\alpha_j^\vee) = M_{ij}$, see [15, Theorem 5.4.1]. Since $t_{\alpha_{n+1}} = h_n + h_{n+1}$ and $t_{\alpha_n} = h_n - h_{n+1}$, we have

$$t_{\delta_n} = \frac{1}{2}(t_{\alpha_n} + t_{\alpha_{n+1}}) = h_n, t_\epsilon = t_{\delta_n + \epsilon} - t_\epsilon = h_{n+1}.$$

Similarly, we can see that $t_{\delta_i} = h_i$, $i \in \{1, \ldots, n-1\}$. Therefore the induced form on $T^\star$ satisfies

$$(\delta_i, \delta_j) = \delta_{i,j}, \quad (\epsilon, \epsilon) = -1 \quad \text{and} \quad (\delta_i, \epsilon) = 0.$$

Now, we consider the period 2 diagram automorphism σ of $\mathcal{L}$ induced by

$$\sigma(e_n) = e_{n+1}, \quad \sigma(f_n) = f_{n+1}, \quad \sigma(e_k) = e_k \quad \text{and} \quad \sigma(f_k) = f_k, \quad (1 \leq k \leq n-1),$$

see [20, Proposition 7.5.5]. An argument similar to the previous example, shows that the automorphism σ preserves T and is invariant on the form. It follows that

$$\pi(R) = \{0, \pm\delta_i \pm \delta_j, \pm 2\delta_i \mid 1 \leq i \neq j \leq n\} \cup \{\pm\delta_i | 1 \leq i \leq n\}.$$

Moreover, it can be seen that for $\alpha \in R$, $\pi(\alpha) = \frac{1}{2}(\alpha + \sigma(\alpha)) = 0$ if and only if $\alpha = 0$, and so by Lemma 2.9, (A4) holds. The above argument shows that conditions (A1) to (A5) are established. In particular, by Proposition 3.2, $(\mathcal{L}, T(0), (\cdot, \cdot))$ is an EALSA with root system $\pi(R)$.

As in Section 2, we extend σ to $\widehat{\mathcal{L}} = (\mathcal{L} \otimes \mathbb{C}[t_1^{\pm 1}, \ldots, t_\nu^{\pm 1}]) \oplus \mathcal{V} \oplus \mathcal{V}^\star$. We consider the epimorphism $\rho : \Lambda := \mathbb{Z}^\nu \longrightarrow \mathbb{Z}_2$ given by $\rho(n_1, \ldots, n_\nu) = \bar{n}_1 + \cdots + \bar{n}_\nu$. By Theorem 2.12, the Lie superalgebra $\widetilde{\mathcal{L}}$ together with the Cartan subalgebra $\widetilde{T}$ and the form $(\cdot, \cdot)$ is an extended affine Lie superalgebra of nullity ν with root system $\widetilde{R} = \{\pi(\alpha) + \lambda | \mathcal{L}^{\pi(\alpha)}(\bar{\lambda}) \neq 0\}$. Using this, we can see that

$$\widetilde{R}_0 = (\bigoplus_{i=1}^{\nu} \mathbb{Z}\gamma_i) \cup \{\sum_{i=1}^{\nu} n_i\gamma_i \pm \delta_i \pm \delta_j, \sum_{i=1}^{\nu} n_i\gamma_i \pm 2\delta_i \mid \sum_{i=1}^{\nu} n_i \in 2\mathbb{Z}, 1 \leq i \neq j \leq n\},$$

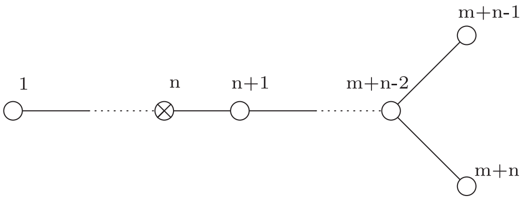
and

$$\widetilde{R}_1 = \{m_1\gamma_1 + \cdots + m_\nu\gamma_\nu \pm \delta_i \mid m_i \in \mathbb{Z},\ 1 \leq i \leq \nu\},$$

(see [21, Table 5]). So, the root system $\widetilde{R} = \widetilde{R}_0 \cup \widetilde{R}_1$ is a root supersystem of type $B(0, n)$. It is worth mentioning here that the root system $\widetilde{R}$ of the Lie superalgebra $\widetilde{\mathcal{L}}$ coincides with an extended affine root system of type BC_n, not necessarily reduced. In fact in the notation of [8, Theorem 1.13], we have $\widetilde{R} = (\dot{R}, S, L, E)$ where

$$\dot{R} = \{\pm\delta_i \pm \delta_j, \pm\delta_i | 1 \leq i, j \leq n\},$$
$$S = \textstyle\bigoplus_{i=1}^{\nu} \mathbb{Z}\gamma_i, \quad E = L = \{\sum_{i=1}^{\nu} n_i\gamma_i \mid \sum_{i=1}^{\nu} n_i \in 2\mathbb{Z}\}.$$

Example 4.3. Let $\mathcal{L} := \mathcal{L}(M, \tau)$ be the finite dimensional complex Kac-Moody Lie superalgebra $\mathcal{L} = D(m, n) = \mathfrak{osp}(2m, 2n)$,



with corresponding generalized Cartan matrix M where

$$M = \begin{bmatrix} 2 & -1 & & & & & & & \\ -1 & \ddots & \ddots & & 0 & & & & \\ & \ddots & 2 & -1 & & & & & \\ & & -1 & 0 & 1 & & & & \\ & & & 1 & -2 & 1 & & & \\ & & 0 & & \ddots & \ddots & 1 & 1 \\ & & & & & 1 & -2 & 0 \\ & & & & & 1 & 0 & -2 \end{bmatrix}, \quad \tau = \{n\}.$$

Consider the $(m + n)$-dimensional vector space $T = \sum_{i=1}^{m+n} \mathbb{C}h_i$ and set

$$\Pi := \{\overbrace{\delta_i - \delta_{i+1}}^{\alpha_i}, \overbrace{\delta_n - \epsilon_1}^{\alpha_n}, \overbrace{\epsilon_j - \epsilon_{j+1}}^{\alpha_{n+j}}, \overbrace{\epsilon_{m-1} + \epsilon_m}^{\alpha_{m+n}} |\ 1 \leq i \leq n - 1, 1 \leq j \leq m - 1\},$$

$$\Pi^\vee := \{\alpha_i^\vee := h_i - h_{i+1}, \alpha_{m+n}^\vee = h_{m+n-1} + h_{m+n} \mid 1 \leq i \leq m + n - 1\},$$

where

$$\delta_i(h_t) = \delta_{i,t}, \quad \epsilon_j(h_t) = -\delta_{j+n,t}; \quad 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq t \leq m + n.$$

The triple $(\Pi, \Pi^\vee, T)$ is a minimal realization for the matrix M with $\mathcal{L}^0 = T$. Let $\{e_i, f_i | 1 \leq i \leq m + n\}$ be a set of Chevalley generators for this realization such that $e_i \in \mathcal{L}^{\alpha_i}$, $f_i \in \mathcal{L}^{-\alpha_i}$.

One knows that the root system of $\mathcal{L}$ with respect to T is

$$R = \underbrace{\{0, \pm\epsilon_i \pm \epsilon_j, \pm\delta_r \pm \delta_t, \pm 2\delta_r\}}_{R_0} \cup \underbrace{\{\pm\epsilon_i \pm \delta_r\}}_{R_1},$$

where $1 \leq i \neq j \leq m, 1 \leq r \neq t \leq n$.

Let $(\cdot,\cdot)$ be the standard super-symmetric invariant non-degenerate bilinear form on $\mathcal{L}$ satisfying $(\alpha_i^\vee, \alpha_j^\vee) = \alpha_i(\alpha_j^\vee) = M_{ij}$. Then we have $t_{\delta_r} = h_r, t_{\epsilon_i} = h_{n+i}$. Therefore the induced form on $T^\star$ satisfies

$$(\delta_r, \delta_t) = \delta_{r,t}, \quad (\epsilon_i, \epsilon_j) = -\delta_{i,j} \quad \text{and} \quad (\delta_r, \epsilon_i) = 0.$$

We consider the automorphism σ on $\mathcal{L}$ induced by

$$\sigma(e_{m+n}) = e_{m+n-1}, \quad \sigma(f_{m+n}) = f_{m+n-1}, \quad \sigma(e_k) = e_k, \quad \sigma(f_k) = f_k, \\ (1 \leq k \leq m+n-2).$$

Now we have

$$\pi(R) = \{0, \pm\epsilon_i \pm \epsilon_j, \pm\epsilon_i, \pm\delta_r \pm \delta_t, \pm 2\delta_r, \pm\epsilon_i \pm \delta_r, \pm\delta_r | \\ 1 \leq i \neq j \leq m-1, 1 \leq r \neq t \leq n\}.$$

Also, by Proposition 3.2, $(\mathcal{L}, T(0), (\cdot,\cdot))$ is an EALSA with root system $\pi(R)$. As in the previous examples, let $\Lambda = \mathbb{Z}^\nu$, $\mathcal{A} = \mathbb{C}[t_1^{\pm 1}, \ldots, t_\nu^{\pm 1}]$ and $\epsilon(\cdot,\cdot)$ be the standard non-degenerate $\mathbb{Z}^\nu$-graded bilinear form on $\mathcal{A}$. Also let $\rho : \Lambda \to \mathbb{Z}_2$ be the epimorphism $\rho(n_1, \ldots, n_\nu) = \bar{n}_1 + \cdots + \bar{n}_\nu$. By Theorem 2.12, $(\widetilde{\mathcal{L}}, \widetilde{T}, (\cdot,\cdot))$ is an extended affine Lie superalgebra of nullity ν. For the sake of notation, we proceed by specifying $\nu = 2$. Then we have

$$\widetilde{R}_0 = \{m_1\gamma_1 + m_2\gamma_2, n_1\gamma_1 + n_2\gamma_2 \pm \epsilon_i \pm \epsilon_j, m_1\gamma_1 + m_2\gamma_2 \pm \epsilon_i, n_1\gamma_1 + n_2\gamma_2 \pm \delta_r \pm \delta_t, \\ n_1\gamma_1 + n_2\gamma_2 \pm 2\delta_t | m_1, m_2 \in \mathbb{Z}, n_1 + n_2 \in 2\mathbb{Z}, 1 \leq i \neq j \leq m-1, 1 \leq r \neq t \leq n\},$$

and odd roots

$$\widetilde{R}_1 = \{n_1\gamma_1 + n_2\gamma_2 \pm \epsilon_i \pm \delta_r, m_1\gamma_1 + m_2\gamma_2 \pm \delta_r | \\ m_1, m_2 \in \mathbb{Z}, n_1 + n_2 \in 2\mathbb{Z}, 1 \leq i \leq m-1, 1 \leq r \leq n\}.$$

So, $\widetilde{R} = \widetilde{R}_0 \cup \widetilde{R}_1$ is an extended affine root supersystem of type $B(m-1, n)$. In the notation of Remark 2.5, we have $\widetilde{R} = (\dot{R}, S, F)$ where

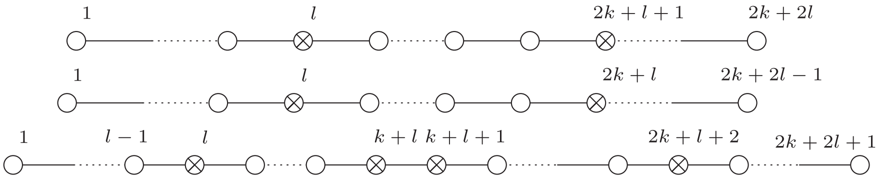
$$\dot{R}_{sh} = \{\pm\epsilon_i \mid 1 \leq i \leq m-1\} \cup \{\pm\delta_r \mid 1 \leq r \leq n\},$$
$$\dot{R}_{ex} = \{\pm 2\delta_r \mid 1 \leq r \leq n\},$$
$$\dot{R}_{lg} = \{\epsilon_i \pm \epsilon_j \mid 1 \leq i \neq j \leq m-1\} \cup \{\delta_r \pm \delta_t \mid 1 \leq r \neq t \leq n\},$$
$$\dot{R}_{ns} = \{\pm\epsilon_i \pm \delta_r \mid 1 \leq i \leq m-1, 1 \leq r \leq n\},$$
$$S = \{m_1\gamma_1 + m_2\gamma_2 | m_1, m_2 \in \mathbb{Z}\},$$
$$F = \{n_1\gamma_1 + n_2\gamma_2 | n_1, n_2 \in \mathbb{Z}, n_1 + n_2 \in 2\mathbb{Z}\}.$$

Remark 4.4. The automorphism σ in Examples 4.2 and 4.3 can be given another description in terms of adjoint action. Namely, we have $\sigma = \mathrm{Ad}(N_\sigma) : \mathcal{L} \to \mathcal{L}$ given by $\mathrm{Ad}(N_\sigma)(z) = N_\sigma z N_\sigma^{-1}$, where

$$N_\sigma = \begin{bmatrix} \begin{bmatrix} & & 0 & & & \\ & I_{m-1} & \vdots & & 0 & \\ 0 & \cdots & 0 & \cdots & 0 & 1 \\ & 0 & \vdots & & I_{m-1} & 0 \\ & & 1 & 0 & \cdots & 0 \end{bmatrix} & 0 \\ 0 & I_{2n} \end{bmatrix},$$

see [20, §7.5.9].

Example 4.5. Let $\mathcal{L}_s := \mathcal{L}(M_s, \tau_s); 1 \leq s \leq 3$, be the finite dimensional complex Kac-Moody Lie superalgebra $A(2k, 2l-1), A(2k-1, 2l-1)$ $((k,l) \neq (1,1))$ and $A(2k, 2l)$, respectively,



in which $\tau_1 = \{l, 2k+l+1\}, \tau_2 = \{l, 2k+l\}$, $\tau_3 = \{l, k+l, k+l+1, 2k+l+2\}$, and

$$(M_1)_{2k+2l}, (M_2)_{2k+2l-1} = \begin{bmatrix} -2 & 1 & & & & & & & & \\ 1 & -2 & 1 & & & & & & & \\ & \ddots & \ddots & \ddots & & & 0 & & & \\ & 1 & 0 & -1 & & & & & & \\ & & -1 & 2 & -1 & & & & & \\ & & & \ddots & \ddots & \ddots & & & & \\ & 0 & & & -1 & 0 & 1 & & & \\ & & & & & 1 & -2 & 1 & & \\ & & & & & & \ddots & \ddots & \ddots & \\ & & & & & & & & 1 & -2 \end{bmatrix},$$

$$(M_3)_{2k+2l+1} = \begin{bmatrix} -2 & 1 & & & & & & & & & \\ 1 & -2 & 1 & & & & & & & & \\ & \ddots & \ddots & \ddots & & & 0 & & & & \\ & 1 & 0 & -1 & & & & & & & \\ & & -1 & 2 & -1 & & & & & & \\ & & & \ddots & \ddots & \ddots & & & & & \\ & & & -1 & 0 & 1 & & & & & \\ & & & & 1 & 0 & -1 & & & & \\ & & 0 & & & -1 & 2 & -1 & & & \\ & & & & & & \ddots & \ddots & \ddots & & \\ & & & & & & -1 & 0 & 1 & & \\ & & & & & & & 1 & -2 & 1 & \\ & & & & & & & \ddots & \ddots & \ddots & \\ & & & & & & & & 1 & -2 \end{bmatrix}.$$

Consider the b_s−dimensional $\mathbb{C}$-vector space $\mathcal{H}_s = \sum_{t=1}^{b_s} \mathbb{C}h_t, 1 \leq s \leq 3$, in which $b_1 = 2k+2l+1, b_2 = 2k+2l$, and $b_3 = 2k+2l+2$. Let

$$\Pi_1 = \{\underbrace{\delta_1 - \delta_2}_{\alpha_1}, \ldots, \underbrace{\delta_{l-1} - \delta_l}_{\alpha_{l-1}}, \underbrace{\delta_l - \epsilon_1}_{\alpha_l}, \underbrace{\epsilon_1 - \epsilon_2}_{\alpha_{l+1}}, \ldots, \underbrace{\epsilon_{2k} - \epsilon_{2k+1}}_{\alpha_{2k+l}}, \underbrace{\epsilon_{2k+1} - \delta_{l+1}}_{\alpha_{2k+l+1}},$$
$$\underbrace{\delta_{l+1} - \delta_{l+2}}_{\alpha_{2k+l+2}}, \ldots, \underbrace{\delta_{2l-1} - \delta_{2l}}_{\alpha_{2k+2l}}\},$$

in which δ_i's and ϵ_i's are defined by, for $1 \leq t \leq 2k+2l+1$;

$$\begin{aligned}\delta_i(h_t) = -\delta_{i,t} \text{ if } 1 \leq i \leq l, \epsilon_i(h_t) &= \delta_{l+i,t} \text{ if } 1 \leq i \leq 2k+1, \delta_i(h_t)\\ &= -\delta_{2k+1+i,t} \text{ if } l+1 \leq i \leq 2l,\end{aligned}$$

$$\Pi_2 = \{\underbrace{\delta_1 - \delta_2}_{\alpha_1}, \ldots, \underbrace{\delta_{l-1} - \delta_l}_{\alpha_{l-1}}, \underbrace{\delta_l - \epsilon_1}_{\alpha_l}, \underbrace{\epsilon_1 - \epsilon_2}_{\alpha_{l+1}}, \ldots, \underbrace{\epsilon_{2k-1} - \epsilon_{2k}}_{\alpha_{2k+l-1}}, \underbrace{\epsilon_{2k} - \delta_{l+1}}_{\alpha_{2k+l}},$$
$$\underbrace{\delta_{l+1} - \delta_{l+2}}_{\alpha_{2k+l+1}}, \ldots, \underbrace{\delta_{2l-1} - \delta_{2l}}_{\alpha_{2k+2l-1}}\},$$

in which, for $1 \leq t \leq 2k+2l$;

$$\begin{array}{ll}\delta_i(h_t) = -\delta_{i,t} & \text{if } 1 \leq i \leq l,\\ \epsilon_i(h_t) = \delta_{l+i,t} & \text{if } 1 \leq i \leq 2k,\\ \delta_i(h_t) = -\delta_{2k+i,t} & \text{if } l+1 \leq i \leq 2l,\end{array}$$

$$\Pi_3 = \{\underbrace{\delta_1 - \delta_2}_{\alpha_1}, \ldots, \underbrace{\delta_{l-1} - \delta_l}_{\alpha_{l-1}}, \underbrace{\delta_l - \epsilon_1}_{\alpha_l}, \underbrace{\epsilon_1 - \epsilon_2}_{\alpha_{l+1}}, \ldots, \underbrace{\epsilon_{k-1} - \epsilon_k}_{\alpha_{k+l-1}}, \underbrace{\epsilon_k - \delta_{l+1}}_{\alpha_{k+l}}, \underbrace{\delta_{l+1} - \epsilon_{k+1}}_{\alpha_{k+l+1}},$$
$$\underbrace{\epsilon_{k+1} - \epsilon_{k+2}}_{\alpha_{k+l+2}}, \ldots, \underbrace{\epsilon_{2k} - \epsilon_{2k+1}}_{\alpha_{2k+l+1}}, \underbrace{\epsilon_{2k+1} - \delta_{l+2}}_{\alpha_{2k+l+2}}, \underbrace{\delta_{l+2} - \delta_{l+3}}_{\alpha_{2k+l+3}}, \ldots, \underbrace{\delta_{2l} - \delta_{2l+1}}_{\alpha_{2k+2l+1}}\},$$

in which, for $1 \leq t \leq 2k+2l+2$;

$$\delta_i(h_t) = -\delta_{i,t} \text{ if } 1 \leq i \leq l, \epsilon_i(h_t) = \delta_{l+i,t} \text{ if } 1 \leq i \leq k, \delta_{l+1}(h_t) = -\delta_{k+l+1,t},$$

$$\epsilon_i(h_t) = \delta_{l+1+i,t} \text{ if } k+1 \leq i \leq 2k+1, \delta_i(h_t) = -\delta_{2k+1+i,t} \text{ if } l+2 \leq i \leq 2l+1.$$

We also set

$$\begin{aligned}\Pi_1^\vee &= \{\alpha_t^\vee = h_t - h_{t+1} | 1 \leq t \leq 2k+2l\},\\ \Pi_2^\vee &= \{\alpha_t^\vee = h_t - h_{t+1} | 1 \leq t \leq 2k+2l-1\},\\ \Pi_3^\vee &= \{\alpha_t^\vee = h_t - h_{t+1} | 1 \leq t \leq 2k+2l+1\}.\end{aligned}$$

Then the triple $(\Pi_s, \Pi_s^\vee, T_s)$ is a minimal realization for $M_s, 1 \leq s \leq 3$, where $T_s := \sum_{t=1}^{b_s} \mathbb{C}(h_t - h_{t+1})$. Also we have $\mathcal{L}_s^0 = T_s$ $(1 \leq s \leq 3)$. In what follows, when there is no ambiguity we often drop the range of indices. Consider a set of Chevalley generators $X = \{e_i, f_i\}$ for this realization such that $e_i \in \mathcal{L}^{\alpha_i}$ and $f_i \in \mathcal{L}^{-\alpha_i}$. One knows that

$$R = R_0 \cup R_1, \text{ where } R_0 = \{0, \epsilon_i - \epsilon_j, \delta_i - \delta_j | i \neq j\} \text{ and } R_1 = \{\pm(\epsilon_i - \delta_j)\},$$

is the root system of $\mathcal{L}_s$. Consider the automorphism σ on $\mathcal{L}_s$ induced by

Case 1: $\mathcal{L}_1 = A(2k, 2l-1)$;

$$\sigma(e_l) = (-1)^l e_{2k+l+1},\ \sigma(e_{2k+l+1}) = (-1)^l e_l,\ \sigma(e_i) = e_{2k+2l+1-i},$$
$$\sigma(f_l) = (-1)^{l+1} f_{2k+l+1}, \sigma(f_{2k+l+1}) = (-1)^{l+1} f_l, \sigma(f_i) = f_{2k+2l+1-i}$$
$$(i \neq l, 2k+l+1).$$

Case 2: $\mathcal{L}_2 = A(2k-1, 2l-1), (k,l) \neq (1,1)$;

$$\sigma(e_l) = (-1)^{k+l+1} e_{2k+l}, \quad \sigma(e_{2k+l}) = (-1)^{k+l+1} e_l, \quad \sigma(e_i) = e_{2k+2l-i},$$
$$\sigma(f_l) = (-1)^{k+l} f_{2k+l}, \quad \sigma(f_{2k+l}) = (-1)^{k+l} f_l, \quad \sigma(f_i) = f_{2k+2l-i}\ (i \neq l, 2k+l).$$

Case 3: $\mathcal{L}_3 = A(2k, 2l)$;

$$\sigma(e_l) = (-1)^l e_{2k+l+2}, \quad \sigma(e_{2k+l+2}) = (-1)^l e_l,$$
$$\sigma(e_{k+l}) = (-1)^{k+1}\mathbf{i}[e_{k+l+1}, e_{k+l+2}], \quad \sigma(f_l) = (-1)^{l+1} f_{2k+l+2},$$
$$\sigma(f_{2k+l+2}) = (-1)^{l+1} f_l, \quad \sigma(f_{k+l}) = -(-1)^{k+1}\mathbf{i}[f_{k+l+1}, \quad f_{k+l+2}],$$
$$\sigma(e_{k+l+1}) = (-1)^k \mathbf{i} f_{k+l+1}, \quad \sigma(e_{k+l+2}) = [e_{k+l}, e_{k+l+1}],$$
$$\sigma(e_i) = e_{2k+2l+2-i}, \quad \sigma(f_{k+l+1}) = (-1)^k \mathbf{i} e_{k+l+1},$$
$$\sigma(f_{k+l+2}) = [f_{k+l}, f_{k+l+1}], \quad \sigma(f_i) = f_{2k+2l+2-i}.$$

Clearly σ preserves T_1, T_2. In addition, the following relations show that σ preserves T_3:

$$\sigma(\alpha^\vee_{k+l+2}) = \sigma[e_{k+l+2}, f_{k+l+2}] = [[e_{k+l}, e_{k+l+1}], [f_{k+l}, f_{k+l+1}]] = -(\alpha^\vee_{k+l} + \alpha^\vee_{k+l+1}),$$

$$\sigma(\alpha^\vee_{k+l}) = \sigma[e_{k+l}, f_{k+l}] = [[e_{k+l+1}, e_{k+l+2}], [f_{k+l+1}, f_{k+l+2}]] = -(\alpha^\vee_{k+l+1} + \alpha^\vee_{k+l+2}),$$

$$\sigma(\alpha^\vee_{k+l+1}) = \sigma[e_{k+l+1}, f_{k+l+1}] = [i f_{k+l+1}, i e_{k+l+1}] = -\alpha^\vee_{k+l+1}.$$

We note that, in the above three cases, σ is of order 2, 2 and 4, respectively. We also note that the automorphism σ can realized as the composition of an adjoint automorphism and the supertranspose, namely suppose that "st" is the supertranspose of the block matrices of desired size; $\mathrm{st}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \begin{bmatrix} A^t & -C^t \\ B^t & D^t \end{bmatrix}$. Set

$$F_n = \begin{bmatrix} & & & & 1 \\ & 0 & & -1 & \\ & & 1 & & \\ & -1 & & 0 & \\ \iddots & & & & \end{bmatrix}_{n\times n}.$$

Then $\sigma = \mathrm{Ad}(N_\sigma) \circ (-\mathrm{st})$, see [20, §7.5.6 - 7.5.8], in which $\mathrm{Ad}(g) : \mathcal{L}_s \to \mathcal{L}_s, \mathrm{Ad}(N_\sigma)(z) = N_\sigma z N_\sigma^{-1}$ for every $z \in \mathcal{L}_s$ with:

Case 1: $N_\sigma = \begin{bmatrix} F_{2k+1} & 0 \\ & \\ 0 & F_{2l} \end{bmatrix} \in \mathfrak{gl}(2k+1, 2l)$,

Case 2: $N_\sigma = \begin{bmatrix} \begin{bmatrix} 0 & F_k \\ F_k^t & 0 \end{bmatrix} & 0 \\ & \\ 0 & F_{2l} \end{bmatrix} \in \mathfrak{gl}(2k, 2l)$,

Case 3: $N_\sigma = \begin{bmatrix} F_{2k+1} & 0 \\ 0 & \begin{bmatrix} F_{2l} & 0 \\ 0 & -\mathbf{i} \end{bmatrix} \end{bmatrix} \in \mathfrak{gl}(2k+1, 2l+1).$

A standard argument shows that σ leaves the form invariant. In fact the form $(x, y)' := (\sigma(x), \sigma(y))$, $x, y \in \mathcal{L}_s$, is a super-symmetric invariant bilinear form on $\mathcal{L}_s$, then as $\mathcal{L}_s$ is simple this form is a scalar multiple of the standard form. But since there exist $x^\pm \in \mathcal{L}_s$ with $(\sigma(x^+), \sigma(x^-)) = (x^+, x^-) \neq 0$, the scalar must be 1, see [15, Proposition 1.2.4]. More precisely, we have $(\alpha_1^\vee, \alpha_1^\vee) = (\alpha_{2k+2l}^\vee, \alpha_{2k+2l}^\vee) = -2 = (\sigma(\alpha_1^\vee), \sigma(\alpha_1^\vee))$ in case 1, $(e_{k+l}, f_{k+l}) = (\sigma(e_{k+l}), \sigma(f_{k+l}))$ in case 2, and $(e_{k+l+1}, f_{k+l+1}) = (\sigma(e_{k+l+1}), \sigma(f_{k+l+1}))$ in case 3.

To describe the root system, we proceed case by case:

Case 1: Taking $\delta'_r := \frac{\delta_r - \delta_{2l+1-r}}{2}$ and $\epsilon'_i := \frac{\epsilon_i - \epsilon_{2k+2-i}}{2}$, we can see that

$$\pi(R) = \{0, \pm\epsilon'_i \pm \epsilon'_j, \pm\epsilon'_i, \pm 2\epsilon'_i, \pm\delta'_r \pm \delta'_t, \pm\delta'_r, \pm 2\delta'_r, \pm\delta'_r \pm \epsilon'_i \mid i \neq j, r \neq t, 1 \leq i, j \leq k, 1 \leq r, t \leq l\}.$$

Case 2: Taking $\delta'_r := \frac{\delta_r - \delta_{2l+1-r}}{2}$ and $\epsilon'_i := \frac{\epsilon_i - \epsilon_{2k+1-i}}{2}$, we obtain

$$\pi(R) = \{0, \pm\epsilon'_i \pm \epsilon'_j, \pm 2\epsilon'_i, \pm\delta'_r \pm \delta'_t, \pm 2\delta'_r, \pm\delta'_r \pm \epsilon'_i \mid i \neq j, r \neq t, 1 \leq i, j \leq k, 1 \leq r, t \leq l\}.$$

Case 3: Taking $\delta'_t = \frac{\delta_r - \delta_{2l+2-r}}{2}$ and $\epsilon'_i = \frac{\epsilon_i - \epsilon_{2k+2-i}}{2}$, we get

$$\pi(R) = \{0, \pm\epsilon'_i \pm \epsilon'_j, \pm\epsilon'_i, \pm 2\epsilon'_i, \pm\delta'_r \pm \delta'_t, \pm\delta'_r, \pm 2\delta'_r, \pm\delta'_r \pm \epsilon'_i \mid i \neq j, r \neq t, 1 \leq i, j \leq k, 1 \leq r, t \leq l\}.$$

The above computations show that in cases 1 and 2, $\pi(\alpha) = 0$ for $\alpha \in R$ if and only if $\alpha = 0$ and in case 3, $\pi(\alpha) \neq 0$ for every $\alpha \in R \setminus \{0, \alpha_{k+l+1}\}$, and $\pi(\mathcal{L}^{\alpha_{k+l+1}}) = \{0\}$, in fact $\pi(e_{k+l+1}) = \frac{1}{4}(e_{k+l+1} + (-1)^k \mathbf{i} f_{k+l+1} - e_{k+l+1} - (-1)^k \mathbf{i} f_{k+l+1}) = 0$. Therefore by Lemma 2.9, (A4) holds. Moreover, by Proposition 3.2, $(\mathcal{L}, T(0), (\cdot, \cdot))$ is an EALSA with root system $\pi(R)$.

Next consider the epimorphism

$$\rho_i : \Lambda \longrightarrow \mathbb{Z}_{m_i}, \quad (n_1, \ldots, n_\nu) \mapsto \bar{n}_1 + \cdots + \bar{n}_\nu \ (\mathrm{mod}\ m_i),$$

where $m_i = 2$, if $i = 1$ or 2 and $m_3 = 4$. Now, based on Theorem 2.12, we can see that $(\widetilde{\mathcal{L}}_s, \widetilde{T}_s, (\cdot, \cdot))$ is an EALSA of nullity ν. As in the previous example, we specify $\nu = 2$. We now describe the root system $\widetilde{R}$ of $\widetilde{\mathcal{L}}_s$, see [20, §7.5.6 - §7.5.8], $1 \leq i \neq j \leq k, 1 \leq r \neq t \leq l$, $(c_1, c_2, c'_1, c'_2, m_1, m_2, n_1, n_2 \in \mathbb{Z})$:

Case 1:

$$\begin{aligned}\widetilde{R}_0 =\{&c_1\gamma_1 + c_2\gamma_2, c_1\gamma_1 + c_2\gamma_2 \pm \epsilon'_i \pm \epsilon'_j, c_1\gamma_1 + c_2\gamma_2 \pm \epsilon'_i, m_1\gamma_1 + m_2\gamma_2 \pm 2\epsilon'_i,\\ &c_1\gamma_1 + c_2\gamma_2 \pm \delta'_r \pm \delta'_t, n_1\gamma_1 + n_2\gamma_2 \pm 2\delta'_r \mid n_1 + n_2 \in 2\mathbb{Z}, m_1 + m_2 \in 2\mathbb{Z} + 1\},\end{aligned}$$

$$\widetilde{R}_1 = \{c_1\gamma_1 + c_2\gamma_2 \pm \delta'_r, c_1\gamma_1 + c_2\gamma_2 \pm \delta'_r \pm \epsilon'_i\},$$

Case 2:

$$\begin{aligned}\widetilde{R}_0 = \{&c_1\gamma_1 + c_2\gamma_2, c_1\gamma_1 + c_2\gamma_2 \pm \epsilon'_i \pm \epsilon'_j, m_1\gamma_1 + m_2\gamma_2 \pm 2\epsilon'_i, c_1\gamma_1 + c_2\gamma_2 \pm \delta'_r \pm \delta'_t,\\ &n_1\gamma_1 + n_2\gamma_2 \pm 2\delta'_r \mid n_1 + n_2 \in 2\mathbb{Z}, m_1 + m_2 \in 2\mathbb{Z} + 1\},\end{aligned}$$

$$\widetilde{R}_1 = \{c_1\gamma_1 + c_2\gamma_2 \pm \delta'_r \pm \epsilon'_i\},$$

Case 3:

$$\widetilde{R}_0 = \{n_1\gamma_1 + n_2\gamma_2, n_1\gamma_1 + n_2\gamma_2 \pm \epsilon'_i \pm \epsilon'_j, n_1\gamma_1 + n_2\gamma_2 \pm \epsilon'_i, n_1\gamma_1 + n_2\gamma_2 \pm \delta'_r \pm \delta'_t,$$
$$m_1\gamma_1 + m_2\gamma_2 \pm \delta'_r, c'_1\gamma_1 + c'_2\gamma_2 \pm 2\epsilon'_i, c_1\gamma_1 + c_2\gamma_2 \pm 2\delta'_r \mid n_1 + n_2 \in 2\mathbb{Z},$$
$$m_1 + m_2 \in 2\mathbb{Z} + 1, c_1 + c_2 \in 4\mathbb{Z}, c'_1 + c'_2 \in 4\mathbb{Z} + 2\},$$

$$\widetilde{R}_1 = \{m_1\gamma_1 + m_2\gamma_2, m_1\gamma_1 + m_2\gamma_2 \pm \epsilon'_i, n_1\gamma_1 + n_2\gamma_2 \pm \delta'_r, n_1\gamma_1 + n_2\gamma_2 \pm \epsilon_i \pm \delta'_r$$
$$n_1 + n_2 \in 2\mathbb{Z}, m_1 + m_2 \in 2\mathbb{Z} + 1\}.$$

Now considering the notations of Remark 2.5, we have $\widetilde{R} = \widetilde{R}_0 \cup \widetilde{R}_1$ is equal to $(\dot{R}, S, F, E_1, E_2)$ or $(\dot{R}, F, L_1, L_2)$, in which $\dot{R}, S, F, E_1, E_2, L_1, L_2$ are described in the following tables

$\dot{R}$	$\dot{R}_{sh}$	$\dot{R}_{lg}$	$\dot{R}^1_{ex}$	$\dot{R}^2_{ex}$	$\dot{R}^\times_{ns}$
Case 1 or 3	$\{\pm\epsilon'_i\} \cup \{\pm\delta'_r\}$	$\{\pm\delta'_r \pm \delta'_t\} \cup \{\pm\epsilon'_i \pm \epsilon'_j\}$	$\{\pm 2\epsilon'_i\}$	$\{\pm 2\delta'_r\}$	$\{\pm\epsilon'_i \pm \delta'_r\}$

$BC(k,l)$	S	F
Case 1	$\mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$	$\mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$
Case 3	$\mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$	$\mathbb{Z}(\gamma_1 + \gamma_2) + \mathbb{Z}(\gamma_1 - \gamma_2)$

$BC(k,l)$	E_1	E_2
Case 1	$\gamma_1 + \mathbb{Z}(\gamma_1 + \gamma_2) + \mathbb{Z}(\gamma_1 - \gamma_2)$	$\mathbb{Z}(\gamma_1 + \gamma_2) + \mathbb{Z}(\gamma_1 - \gamma_2)$
Case 3	$\{c'_1\gamma_1 + c'_2\gamma_2 \mid c'_1 + c'_2 \in 4\mathbb{Z} + 2\}$	$\{c_1\gamma_1 + c_2\gamma_2 \mid c_1 + c_2 \in 4\mathbb{Z}\}$

$\dot{R}$	$\dot{R}_{sh}$	$(\dot{R}^1_{re})_{lg}$	$(\dot{R}^2_{re})_{lg}$	$\dot{R}^\times_{ns}$
Case 2	$\{\pm\delta'_r \pm \delta'_t\} \cup \{\pm\epsilon'_i \pm \epsilon'_j\}$	$\{\pm 2\epsilon'_i\}$	$\{\pm 2\delta'_r\}$	$\{\pm\epsilon'_i \pm \delta'_r\}$

$C(k,l)$	F	L_1	L_2
Case 2	$\mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$	$\gamma_1 + \mathbb{Z}(\gamma_1 + \gamma_2) + \mathbb{Z}(\gamma_1 - \gamma_2)$	$\mathbb{Z}(\gamma_1 + \gamma_2) + \mathbb{Z}(\gamma_1 - \gamma_2)$

Example 4.6. Let $\mathcal{L}$ be the Lie superalgebra of type $A(1,1)$ with the standard Cartan subalgebra $T = \text{span}_{\mathbb{C}}\{e_{11} - e_{22}, e_{33} - e_{44}\}$. We have

$$e_{11} + e_{33} = 1/2(e_{11} - e_{22}) + 1/2(e_{33} - e_{44}) + 1/2(e_{11} + e_{22} + e_{33} + e_{44})$$

and

$$e_{22} + e_{44} = -1/2(e_{11} - e_{22}) - 1/2(e_{33} - e_{44}) + 1/2(e_{11} + e_{22} + e_{33} + e_{44}).$$

Since the following brackets lie in $T \setminus \{0\}$, $\mathcal{L}$ is division;

$$[ae_{23}, e_{32}] = a(e_{22} + e_{33}), [ae_{41}, e_{14}] = a(e_{11} + e_{44}), [ae_{23} + be_{41}, e_{32}] = a(e_{22} + e_{33}),$$

$$[ae_{31}, e_{13}] = a(e_{11} + e_{33}), [ae_{24}, e_{42}] = a(e_{22} + e_{44}), [ae_{24} + be_{31}, e_{42}] = a(e_{22} + e_{44}),$$

$$[ae_{12}, e_{21}] = a(e_{11} - e_{22}), [ae_{34}, e_{43}] = a(e_{33} - e_{44}),$$

for every nonzero $a, b \in \mathbb{C}$. Let σ be the automorphism on $\mathcal{L}$ which acts as identity on $\mathcal{L}_0$ and minus identity on $\mathcal{L}_1$, see [15, Theorem 5.5.22]. Then σ is an automorphism of order 2 with $T(0) = T$ $(T \subseteq \mathcal{L}_0)$. Also, σ preserves the non-degenerate invariant even super-symmetric bilinear form $(x, y) = str(xy)$. On the other hand, we have $\pi(\alpha) = \alpha$ for every $\alpha \in R$. Hence conditions (A1) to (A5) are established. Choose $\mathcal{A}, \rho, \Lambda, \epsilon(\cdot,\cdot)$, as in Example 4.1. Again, we specify $\nu = 2$. By Theorem 2.12, we have $(\widetilde{\mathcal{L}}, \widetilde{T}, (\cdot,\cdot))$ is an EALSA with EARSS $\widetilde{R} = \widetilde{R}_0 \cup \widetilde{R}_1$ of type $A(1,1)$, where

$$\widetilde{R}_0 = \{n_1\gamma_1 + n_2\gamma_2, n_1\gamma_1 + n_2\gamma_2 \pm (\delta_1 - \delta_2), n_1\gamma_1 + n_2\gamma_2 + \pm(\epsilon_1 - \epsilon_2) | n_1 + n_2 \in 2\mathbb{Z}\},$$

$$\widetilde{R}_1 = \{m_1\gamma_1 + m_2\gamma_2 \pm (\epsilon_i - \delta_j) | m_1, m_2 \in 2\mathbb{Z} + 1, 1 \le i, j \le 2\}.$$

It should be noted that in [23, Theorem 2.2], the type $A(l,l)$ is excluded from the list of root systems described in terms of pointed reflection subspaces.

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