ON THE PATHWISE UNIQUENESS FOR A CLASS
OF DEGENERATE ITÔ-STOCHASTIC
DIFFERENTIAL EQUATIONS

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Abstract

We show pathwise uniqueness for a class of degenerate Itô-SDE among all of its weak solutions that spend zero time at the points of degeneracy of the dispersion matrix. Consequently, by the Yamada-Watanabe Theorem and a weak existence result, the pathwise unique solutions can be shown to be strong and to exist. The main tools to show pathwise uniqueness are inequalities associated with maximal functions and a Krylov type estimate derived from elliptic regularity and uniqueness in law.

1. Introduction

An Itô-SDE whose diffusion coefficient is not locally uniformly elliptic is called a degenerate Itô-SDE and in this paper, we aim to show pathwise uniqueness for a class of such SDEs on $\mathbb{R}^d$ with $d \geq 3$. In [13], a Krylov type estimate induced by a parabolic regularity result on $\mathbb{R}^d$ is essentially used to show pathwise uniqueness for a class of non-degenerate Itô-SDEs with singular drift coefficients. However, it is not clear how such an estimate can be derived when the diffusion coefficient is not locally uniformly elliptic. Recently in [7], using elliptic and parabolic regularity results and generalized Dirichlet form theory, the existence and uniqueness in law of a quite large class of degenerate Itô-SDEs with fully discontinuous coefficients was shown (see Theorem 2.2). This leads us to explore a subclass of degenerate Itô-SDE for which a Krylov type estimate can be concluded from resolvent regularity.

Let us introduce our main results. For some $y \in \mathbb{R}^d$, we deal with the following time-homogeneous degenerate Itô-SDE

$$(1) \qquad X_t = y + \int_0^t \sqrt{\frac{1}{\psi}}(X_s) \cdot \sigma(X_s)dW_s + \int_0^t \mathbf{G}(X_s)ds, \quad 0 \leq t < \infty,$$

where $(W_t)_{t \geq 0}$ is a d-dimensional standard Brownian motion and the degeneracy stems from $\sqrt{\frac{1}{\psi}}$. Now consider

(H1): $d \geq 3$, $\mathbf{G} = (g_1, \ldots, g_d) \in L^\infty_{loc}(\mathbb{R}^d, \mathbb{R}^d)$ and $A = (a_{ij})_{1 \leq i,j \leq d}$ is a symmetric matrix of functions with $a_{ji} = a_{ij} \in H^{1,2d+2}_{loc}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ for all $1 \leq i, j \leq d$ such that for every open ball $B \subset \mathbb{R}^d$, there exist constants $\lambda_B, \Lambda_B > 0$ with

$$\lambda_B \|\xi\|^2 \leq \langle A(x)\xi, \xi \rangle \leq \Lambda_B \|\xi\|^2, \quad \text{for all } \xi \in \mathbb{R}^d, \ x \in B.$$

$\sigma = (\sigma_{ij})_{1 \leq i,j \leq d}$ is a matrix of continuous functions with $\sigma\sigma^T = A$. $\psi \in L^q_{loc}(\mathbb{R}^d)$ for some $q > 2d + 2$ with $\psi > 0$ a.e., $\frac{1}{\psi} \in L^\infty_{loc}(\mathbb{R}^d)$, and $\sqrt{\frac{1}{\psi}}(x) \in [0, \infty)$ for any $x \in \mathbb{R}^d$.

If **(H1)** holds and ψ, σ and $\mathbf{G}$ are as in **(H1)**, then we will consider the following conditions.

(H2): *There exist constants $N_0 \in \mathbb{N}$ and $M > 0$ such that for a.e. $x \in \mathbb{R}^d \setminus \overline{B}_{N_0}$*

$$(2) \qquad -\frac{\langle \frac{1}{\psi}(x)A(x)x, x \rangle}{\|x\|^2} + \frac{\text{trace}(\frac{1}{\psi}(x)A(x))}{2} + \langle \mathbf{G}(x), x \rangle \leq M\|x\|^2(\ln \|x\| + 1).$$

(H3): *There exists an open set E in $\mathbb{R}^d$ such that for any open ball U with $\overline{U} \subset E$,*

$$0 < \inf_U \psi \leq \sup_U \psi < \infty.$$

(H4): *For some $\widetilde{q} \in (\frac{d}{2}, \infty)$, $\sqrt{\frac{1}{\psi}} \cdot \sigma_{ij} \in H^{1,2\widetilde{q}}_{loc}(\mathbb{R}^d)$ and $g_i \in H^{1,\widetilde{q}}_{loc}(\mathbb{R}^d)$ for all $1 \leq i, j \leq d$.*

Then our main result is stated as follows.

Theorem 1.1. *Assume* **(H1)–(H4)** *and let E be as in* **(H3)**. *Then for $y \in E$, pathwise uniqueness holds for* (1) *in the following sense. If $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}_y, (\widetilde{\mathcal{F}}_t)_{t \geq 0}, (\widetilde{X}^k_t)_{t \geq 0}, (\widetilde{W}_t)_{t \geq 0})$, $k \in \{1, 2\}$ are weak solutions to* (1) *satisfying both*

$$(3) \qquad \int_0^\infty 1_{\{\sqrt{\frac{1}{\psi}}=0\}}(\widetilde{X}^k_s)ds = 0, \quad \widetilde{\mathbb{P}}_y\text{-a.s},$$

then

$$(4) \qquad \widetilde{\mathbb{P}}_y(\widetilde{X}^1_t = \widetilde{X}^2_t, \ t \geq 0) = 1.$$

Moreover, for $y \in E$ and a d-dimensional Brownian motion $(\widetilde{W}_t)_{t \geq 0}$ on a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, there exists a strong solution $(Y^y_t)_{t \geq 0}$ to (1) *satisfying*

$$\int_0^\infty 1_{\{\sqrt{\frac{1}{\psi}}=0\}}(Y^y_s)ds = 0, \quad \widetilde{\mathbb{P}}\text{-a.s.}$$

The main ingredient for the proof of Theorem 1.1 is a new Krylov type estimate (Corollary 3.2) which is derived from uniqueness in law (Theorem 2.2) and elliptic regularity results for the resolvent (see (9)). Additionally, we adapt a technique used in [13, Proof of Theorem 2.2] and well-known inequalities for the Hardy–Littlewood maximal function (Proposition 2.1). As a concrete application of Theorem 1.1, we obtain the following result.

Corollary 1.2. *Let $d \geq 3$ and $\alpha \in [0, \frac{d}{2d+2})$ be a constant, $\mathbf{G} = (g_1, \ldots, g_d) \in L^\infty_{loc}(\mathbb{R}^d, \mathbb{R}^d)$ satisfying that for some constants $N_0 \in \mathbb{N}$ and $M > 0$,*

$$\langle \mathbf{G}(x), x \rangle \leq M\|x\|^2(\ln\|x\| + 1), \qquad \text{for a.e. } x \in \mathbb{R}^d \setminus \overline{B}_{N_0}$$

and that for some $\varepsilon > 0$ $g_i \in H_{loc}^{1,\frac{d}{2}+\varepsilon}(\mathbb{R}^d)$ for all $1 \leq i \leq d$. Then for $y \in \mathbb{R}^d \setminus \{0\}$, pathwise uniqueness holds for

(5)
$$X_t = y + \int_0^t \|X_s\|^{\frac{\alpha}{2}} \cdot id\, dW_s + \int_0^t \mathbf{G}(X_s)ds, \quad 0 \leq t < \infty$$

in the following sense. If $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}_y, (\widetilde{\mathcal{F}}_t)_{t\geq0}, (\widetilde{X}_t^k)_{t\geq0}, (\widetilde{W}_t)_{t\geq0})$, $k \in \{1, 2\}$ are weak solutions to (5) satisfying both

$$\int_0^\infty 1_{\{0\}}(\widetilde{X}_s^k)ds = 0, \quad \widetilde{\mathbb{P}}_y\text{-a.s,}$$

then (4) follows. Moreover, for $y \in \mathbb{R}^d \setminus \{0\}$ and a d-dimensional Brownian motion $(\widetilde{W}_t)_{t\geq0}$ on a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ there exists a strong solution $(Y_t^y)_{t\geq0}$ to (5) satisfying

$$\int_0^\infty 1_{\{0\}}(Y_s^y)ds = 0, \quad \widetilde{\mathbb{P}}\text{-a.s.}$$

To the author's knowledge, Engelbert and Schmidt first showed in [3] the existence and uniqueness in law of one-dimensional SDEs without drift among the solutions that spend zero time at the zeros of dispersion coefficients. In subsequent studies, they presented in [4] and [5] a sufficient condition for the existence of a pathwise unique and strong solution for one-dimensional SDEs with general drift. As another related literature, we refer to [8] and [2] dealing with similar types of results above. Although here formally similar results are shown for $d \geq 3$, the used methods completely differ from [8], [2], [3], [4], [5] in that we ultimately develop a new Krylov type estimate by using elliptic regularity and uniqueness in law.

In Section 2, we introduce well-known results about the Hardy–Littlewood maximal function and briefly explain the results of [7], where the existence and uniqueness in law of weak solutions to (1) are shown. In Section 3, we prove the already mentioned a new Krylov type estimate for weak solutions to (1) satisfying (3). Finally, in Section 4 we present the proofs of Theorem 1.1 and Corollary 1.2 and some examples where Theorem 1.1 is applied.

2. Preliminaries

For basic notations which are not defined in this paper, we refer to [6, Notations and Conventions]. For $r > 0$ and $x \in \mathbb{R}^d$, denote by $B_r(x)$ the open ball with radius r with center 0 and let $B_r = B_r(0)$. Weak and strong solutions to Itô-SDEs are defined as in [6, Definition 3.50]. For a matrix of functions $B = (b_{ij})_{1\leq i,j\leq d}$, we write $\|B\| = \left(\sum_{i,j=1}^d |b_{ij}|^2\right)^{1/2}$. Denote the $d \times d$ identity matrix by id. $\mathcal{M}$ denotes the Hardy–Littlewood maximal operator defined by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{dx(B_r(x))} \int_{B_r(x)} |f(y)|dy, \qquad f \in L_{loc}^1(\mathbb{R}^d), \ x \in \mathbb{R}^d.$$

The following well-known results are crucially used in the proof of Theorem 1.1.

Proposition 2.1.   (i) *If $f \in L^r(\mathbb{R}^d)$ with $r \in (1, \infty]$, then $\mathcal{M}f \in L^r(\mathbb{R}^d)$ and there exists a constant $c_{d,r} > 0$ which only depends on d and r such that*

$$\|\mathcal{M}f\|_{L^r(\mathbb{R}^d)} \le c_{d,r}\|f\|_{L^r(\mathbb{R}^d)}.$$

(ii) *There exists a constant $\widetilde{c}_d > 0$ which only depends on d such that for any $f \in C_0^1(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$ it holds*

$$|f(x) - f(y)| \le \widetilde{c}_d\|x - y\|(\mathcal{M}\|\nabla f\|(x) + \mathcal{M}\|\nabla f\|(y)).$$

Proof. (i) directly follows from [11, Chapter 1, 1.3, Theorem 1 (c)]. For the proof of (ii), let us choose a measurable set $A \subset \mathbb{R}^d$ in [14, Lemma 3.5] such that $dx(A) = 0$ and that

(6) $$|f(z) - f(w)| \le 2^d\|z - w\|(\mathcal{M}\|\nabla f\|(z) + \mathcal{M}\|\nabla f\|(w)), \quad \forall z, w \in \mathbb{R}^d \setminus A.$$

Since $\|\nabla f\| \in C(\mathbb{R}^d)$, it holds $\|\nabla f\|(x) \le \mathcal{M}\|\nabla f\|(x)$ for all $x \in \mathbb{R}^d$. Thus, by [1, Lemma 3.4] we obtain $\mathcal{M}\|\nabla f\| \in C(\mathbb{R}^d)$, hence the assertion follows from (6) and the continuity of f.

$\square$

Under the assumption **(H1)**, by [7, Theorem 4] there exists $\mu = \rho\psi dx$ with $\rho \in H_{loc}^{1,2d+2}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ and $\rho(x) > 0$ for all $x \in \mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} \langle \mathbf{G} - \frac{1}{2\psi}\nabla A - \frac{1}{2\rho\psi}A\nabla\rho, \nabla f\rangle d\mu = 0, \quad \forall f \in C_0^\infty(\mathbb{R}^d).$$

Moreover, by [7, Theorem 3] there exists a sub-Markovian C_0-semigroup of contractions $(\overline{T}_t)_{t>0}$ on $L^1(\mathbb{R}^d, \mu)$ whose generator extends $(L, C_0^\infty(\mathbb{R}^d))$, where

$$Lf = \frac{1}{2}\text{trace}(\frac{1}{\psi}A\nabla^2 f) + \langle \mathbf{G}, \nabla f\rangle, \quad f \in C_0^\infty(\mathbb{R}^d).$$

Through Riesz–Thorin interpolation, $(\overline{T}_t)_{t>0}$ restricted to $L^1(\mathbb{R}^d, \mu)_b$ can be extended to a sub-Markovian C_0-semigroup of contractions on each $L^r(\mathbb{R}^d, \mu)$, $r \in [1, \infty)$ and to a sub-Markovian semigroup of contractions on $L^\infty(\mathbb{R}^d, \mu)$. We donote all these by $(T_t)_{t>0}$. Denote by $(G_\alpha)_{\alpha>0}$ the sub-Markovian C_0-resolvent of contractions on each $L^r(\mathbb{R}^d, \mu)$, $r \in [1, \infty)$ and the sub-Markovian resolvent of contractions on $L^\infty(\mathbb{R}^d, \mu)$ associated with $(T_t)_{t>0}$, i.e.

$$G_\alpha f := \int_0^\infty e^{-\alpha s}T_s f ds, \qquad f \in \cup_{r\in[1,\infty]}L^r(\mathbb{R}^d, \mu).$$

Then by [7, Theorems 5, 6], there exist $(R_\alpha)_{\alpha>0}$ and $(P_t)_{t>0}$ such that for any $\alpha, t > 0$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$

$$R_\alpha f \in C(\mathbb{R}^d), \quad P.f \in C(\mathbb{R}^d \times (0, \infty)) \quad \text{and} \quad R_\alpha f = G_\alpha f, \quad P_t g = T_t g \quad \mu\text{-a.e.}$$

Moreover, it follows from [7, Theorem 7] that there exists a Hunt process

$$\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, (X_t)_{t\ge0}, (\mathbb{P}_x)_{x\in\mathbb{R}^d})$$

with state space $\mathbb{R}^d$ and life time $\zeta = \inf\{t \ge 0 : X_t = \Delta\}$ satisfying that

$$\mathbb{P}_x\big(\{\omega \in \Omega : X_\cdot(\omega) \in C([0, \infty), \mathbb{R}^d), \ X_t(\omega) = \Delta \text{ for all } t \ge \zeta\}\big) = 1, \quad \forall x \in \mathbb{R}^d,$$

where Δ is a point at infinity and that for all $f \in \mathcal{B}_b(\mathbb{R}^d)$, $t, \alpha > 0$ and $x \in \mathbb{R}^d$,

(7) $$P_t f(x) = \mathbb{E}_x[f(X_t)], \quad R_\alpha f(x) = \mathbb{E}_x\left[\int_0^\infty e^{-\alpha s}f(X_s)ds\right],$$

where $\mathbb{E}_x$ is the expectation with respect to $\mathbb{P}_x$ defined on $(\Omega, \mathcal{F})$. Under the assumptions **(H1)** and **(H2)**, using a similar method to [6, Lemma 3.26], we obtain that $\mathbb{M}$ is non-explosive, i.e. $\mathbb{P}_x(\zeta = \infty) = 1$ for all $x \in \mathbb{R}^d$ (equivalently, $(T_t)_{t>0}$ is conservative, i.e. $T_t 1_{\mathbb{R}^d} = 1$, μ-a.e. for all $t > 0$). As a direct consequence of [7, Theorems 8, 12, Lemma 2 and Remark 2] and the above, we then obtain the following theorem.

Theorem 2.2. *Assume* **(H1)**–**(H2)** *and let* $\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq0}, (X_t)_{t\geq0}, (\mathbb{P}_x)_{x\in\mathbb{R}^d})$ *be the Hunt process defined above. Then there exists an* $(\mathcal{F}_t)_{t\geq0}$-*Brownian motion* $(W_t)_{t\geq0}$ *such that for any* $y \in \mathbb{R}^d$, $(\Omega, \mathcal{F}, \mathbb{P}_y, (\mathcal{F}_t)_{t\geq0}, (X_t)_{t\geq0}, (W_t)_{t\geq0})$ *is a weak solution to* (1) *satisfying*

$$\int_0^\infty 1_{\{\sqrt{\frac{1}{\psi}}=0\}}(X_s)ds = 0, \quad \mathbb{P}_y\text{-a.s.}$$

Furthermore if for some $y \in \mathbb{R}^d$ *there exists a weak solution* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}_y, (\widetilde{\mathcal{F}}_t)_{t\geq0}, (\widetilde{X}_t)_{t\geq0}, (\widetilde{W}_t)_{t\geq0})$ *to* (1) *such that*

$$(8) \qquad \int_0^\infty 1_{\{\sqrt{\frac{1}{\psi}}=0\}}(\widetilde{X}_s)ds = 0, \quad \widetilde{\mathbb{P}}_y\text{-a.s.},$$

then

$$\mathbb{P}_y \circ X^{-1} = \widetilde{\mathbb{P}}_y \circ \widetilde{X}^{-1}, \quad on \ \mathcal{B}(C([0, \infty), \mathbb{R}^d)).$$

REMARK 2.3. In Theorem 2.2, the assumption $d \geq 3$ of **(H1)** can be replaced by $d = 2$. However, we point out that $d \geq 3$ of **(H1)** is needed for all other results of this paper. Moreover, in order to obtain Theorem 2.2, the condition in **(H1)** that σ_{ij} is continuous for all $1 \leq i, j \leq d$ can be replaced by the condition that σ_{ij} is locally bounded and measurable for all $1 \leq i, j \leq d$.

3. Krylov type estimates

Theorem 3.1. *Assume* **(H1)** *and* **(H3)** *and let* E *be as in* **(H3)**. *Let* $\widetilde{q} \in (\frac{d}{2}, \infty)$, $y \in E$, $t > 0$ *and* B *be an open ball in* $\mathbb{R}^d$. *Let* $\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq0}, (X_t)_{t\geq0}, (\mathbb{P}_x)_{x\in\mathbb{R}^d})$ *be the Hunt process of* Section 2. *Then for any* $f \in L^{\widetilde{q}}(\mathbb{R}^d)_0$ *with* $\text{supp}(fdx) \subset B$, *there exists a constant* $C_{y,B,\widetilde{q}} > 0$ *which is independent of* f *and* $t > 0$ *such that*

$$\mathbb{E}_y\left[\int_0^t f(X_s)ds\right] \leq e^t C_{y,B,\widetilde{q}} \|f\|_{L^{\widetilde{q}}(\mathbb{R}^d)}.$$

Proof. Let $(G_\alpha)_{\alpha>0}$ be the resolvent defined on $\cup_{r\in[1,\infty]} L^r(\mathbb{R}^d, \mu)$ as in Section 2. Since E is an open subset of $\mathbb{R}^d$ and $y \in E$, there exist open balls U_1 and U_2 such that $y \in U_1 \subset \overline{U}_1 \subset U_2 \subset \overline{U}_2 \subset E$. First assume $f \in C_0^\infty(\mathbb{R}^d)$ with $\text{supp}(f) \subset B$. Then by the proof of [7, Theorem 5] (cf. [7, Theorem 3 (c)]), $G_1 f \in H^{1,2}(U_2)$ and

$$\int_{U_2} \langle \frac{1}{2}\rho A \nabla G_1 f, \nabla\varphi\rangle dx - \int_{U_2} \langle \rho\psi \mathbf{B}, \nabla G_1 f\rangle\varphi \, dx + \int_{U_2} (\alpha\rho\psi G_1 f) \, \varphi dx$$

$$= \int_{U_2} (\rho\psi f) \, \varphi dx, \qquad \forall\varphi \in C_0^\infty(U_2),$$

where $\mathbf{B} = \mathbf{G} - \frac{1}{2\psi}(\nabla A + \frac{1}{\rho}A\nabla\rho)$. Note that $d \geq 3$ implies $1 - \frac{1}{2d+2} > \frac{2}{d} > \frac{1}{\tilde{q}}$. By [7, Theorems 2, 5], **(H3)** and the $L^1(\mathbb{R}^d, \mu)$-contraction property of $(\alpha G_\alpha)_{\alpha>0}$, there exists a constant $C_1 > 0$ which only depends on the coefficients A, $\mathbf{G}$, ψ and U_1, U_2, $\tilde{q}$, d such that

$$
\begin{aligned}
(9) \qquad \|R_1 f\|_{C(\overline{U}_1)} &\leq C_1(\|G_1 f\|_{L^1(U_2)} + \|\rho\psi f\|_{L^{\tilde{q}}(U_2)}) \\
&\leq C_1((\inf_{U_2}\rho\psi)^{-1}\|G_1 f\|_{L^1(\mathbb{R}^d,\mu)} + (\sup_{U_2}\rho\psi)\|f\|_{L^{\tilde{q}}(\mathbb{R}^d)}) \\
&\leq C_1((\inf_{U_2}\rho\psi)^{-1}\|f\|_{L^1(\mathbb{R}^d,\mu)} + (\sup_{U_2}\rho\psi)\|f\|_{L^{\tilde{q}}(\mathbb{R}^d)}) \\
&\leq C_1((\inf_{U_2}\rho\psi)^{-1}\|\rho\psi\|_{L^{2d+2}(B)}\|f\|_{L^{\frac{2d+2}{2d+1}}(B)} + (\sup_{U_2}\rho\psi)\|f\|_{L^{\tilde{q}}(\mathbb{R}^d)}) \\
&\leq C_1 C_2\|f\|_{L^{\tilde{q}}(\mathbb{R}^d)},
\end{aligned}
$$

where $C_2 := (\inf_{U_2}\rho\psi)^{-1}\|\rho\psi\|_{L^{2d+2}(B)}dx(B)^{\frac{2d+1}{2d+2}-\frac{1}{\tilde{q}}} + \sup_{U_2}\rho\psi$. Using the denseness of $C_0^\infty(B)$ in $L^{\tilde{q}}(B)$, (9) extends to $f \in \mathcal{B}_b(\mathbb{R}^d)_0$ with $\mathrm{supp}(fdx) \subset B$. By (7) and (9), it holds for any $f \in \mathcal{B}_b(\mathbb{R}^d)_0$ with $\mathrm{supp}(fdx) \subset B$

$$
\begin{aligned}
\mathbb{E}_y\left[\int_0^t f(X_s)ds\right] &\leq e^t \mathbb{E}_y\left[\int_0^\infty e^{-s}|f|(X_s)ds\right] \\
&= e^t R_1|f|(y) \leq e^t\|R_1|f|\|_{C(\overline{U}_1)} \leq e^t C_1 C_2\|f\|_{L^{\tilde{q}}(\mathbb{R}^d)}.
\end{aligned}
$$

Finally, using monotone approximation, the assertion follows. $\qquad\square$

As a direct consequence of Theorems 3.1, 2.2, we obtain the following.

Corollary 3.2. *Assume* **(H1)–(H3)** *and let E be as in* **(H3)**. *Let $\tilde{q} \in (\frac{d}{2}, \infty)$, $y \in E$, $t > 0$ and B be an open ball in $\mathbb{R}^d$. Let $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}_y, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, (\widetilde{X}_t)_{t\geq 0}, (\widetilde{W}_t)_{t\geq 0})$ be a weak solution to (1) satisfying (8). Then for any $f \in L^{\tilde{q}}(\mathbb{R}^d)_0$ with $\mathrm{supp}(fdx) \subset B$, there exists a constant $C_{y,B,\tilde{q}} > 0$ which is independent of f and $t > 0$ such that*

$$
\widetilde{\mathbb{E}}_y\left[\int_0^t f(\widetilde{X}_s)ds\right] \leq e^t C_{y,B,\tilde{q}}\|f\|_{L^{\tilde{q}}(\mathbb{R}^d)},
$$

where $\widetilde{\mathbb{E}}_y$ is the expectation with respect to $\widetilde{\mathbb{P}}_y$ defined on $(\widetilde{\Omega}, \widetilde{\mathcal{F}})$.

4. Pathwise uniqueness

Proof of Theorem 1.1. Fix $y \in E$. For $k \in \{1, 2\}$, let $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}_y, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, (\widetilde{X}_t^k)_{t\geq 0}, (\widetilde{W}_t)_{t\geq 0})$ be weak solutions to (1) satisfying (3) and let $\widetilde{Z}_t := \widetilde{X}_t^1 - \widetilde{X}_t^2$, $t \geq 0$. Given $n \in \mathbb{N}$ with $n \geq 2$ and $y \in B_{n-1}$, set $\widetilde{D}_n^k := \inf\{t \geq 0 : \widetilde{X}_t^k \in \mathbb{R}^d \setminus B_{n-1}\}$ and $\widetilde{D}_n := \widetilde{D}_n^1 \wedge \widetilde{D}_n^2$. Let $\widehat{\sigma} = (\widehat{\sigma}_{ij})_{1\leq i,j\leq d}$ be defined by $\widehat{\sigma} = \sqrt{\frac{1}{\psi}} \cdot \sigma$. By Itô's formula, for any $t \geq 0$ it holds $\mathbb{P}_y$-a.s.

$$
\begin{aligned}
\|\widetilde{Z}_{t\wedge D_n}\|^2 &= 2\int_0^{t\wedge D_n} \widetilde{Z}_s\left(\widehat{\sigma}(\widetilde{X}_s^1) - \widehat{\sigma}(\widetilde{X}_s^2)\right)d\widetilde{W}_s + \int_0^{t\wedge D_n} \|\widehat{\sigma}(\widetilde{X}_s^1) - \widehat{\sigma}(\widetilde{X}_s^2)\|^2 ds \\
&\quad + 2\int_0^{t\wedge D_n} \langle\widetilde{Z}_s, \mathbf{G}(\widetilde{X}_s^1) - \mathbf{G}(\widetilde{X}_s^2)\rangle ds.
\end{aligned}
$$

Let $\widehat{\sigma}_{ij}^n \in H^{1,2\tilde{q}}(\mathbb{R}^d)_0$ be an extension of $1_{B_n}\widehat{\sigma}_{ij} \in H^{1,2\tilde{q}}(B_n)$ satisfying $\mathrm{supp}(\widehat{\sigma}_{ij}^n dx) \subset B_{n+1}$ for each $i, j \in \{1, 2 \ldots d\}$. Likewise, let $g_i^n \in H^{1,\tilde{q}}(\mathbb{R}^d)_0$ be an extension of $1_{B_n}g_i$ satisfying

$\text{supp}(g_i^n) \subset B_{n+1}$ for each $i \in \{1, 2 \ldots d\}$ and let $\mathbf{G}^n = (g_1^n, \ldots, g_d^n)$. Given $m \in \mathbb{N}$, let $\eta_m \in C_0^\infty(B_{1/m})$ be a standard mollifier on $\mathbb{R}^d$. Define

$$\widehat{\sigma}_{ij}^{n,m} := \widehat{\sigma}_{ij}^n * \eta_m, \quad g_i^{n,m} := g_i^n * \eta_m, \quad \text{for } i, j \in \{1, 2, \ldots, d\},$$

where $f * g$ is the convolution of f and g. Let $\widehat{\sigma}^{n,m} = (\widehat{\sigma}_{ij}^{n,m})_{1 \le i,j \le d}$ and $\mathbf{G}^{n,m} = (g_1^{n,m}, \ldots, g_d^{n,m})$. Set

$$N_t := M_t + A_t^{(1)} + A_t^{(2)} \quad \text{and} \quad N_t^n := N_{t \wedge \widetilde{D}_n} \quad \text{for } t \ge 0,$$

where

$$M_t := 2 \int_0^t \frac{\widetilde{Z}_s \left(\widehat{\sigma}(\widetilde{X}_s^1) - \widehat{\sigma}(\widetilde{X}_s^2) \right)}{\|\widetilde{Z}_s\|^2} d\widetilde{W}_s, \quad A_t^{(1)} := \int_0^t \frac{\|\widehat{\sigma}(X_s^1) - \widehat{\sigma}(\widetilde{X}_s^2)\|^2}{\|\widetilde{Z}_s\|^2} ds,$$

$$A_t^{(2)} := 2 \int_0^t \frac{\langle \widetilde{Z}_s, \mathbf{G}(\widetilde{X}_s^1) - \mathbf{G}(\widetilde{X}_s^2) \rangle}{\|\widetilde{Z}_s\|^2} ds, \quad t \ge 0,$$

where $\frac{0}{0} := 0$. Then for all $t \ge 0$ it holds

(10)
$$\|\widetilde{Z}_{t \wedge D_n}\|^2 = \int_0^t \|\widetilde{Z}_{s \wedge D_n}\|^2 dN_s^n, \quad \widetilde{\mathbb{P}}_y\text{-a.s.}$$

Let $\widetilde{\mathbb{E}}_y$ be the expectation with respect to $\widetilde{\mathbb{P}}_y$ defined on $(\widetilde{\Omega}, \widetilde{\mathcal{F}})$. By Itô-isometry and Fatou's Lemma, it holds

$$\frac{1}{4} \widetilde{\mathbb{E}}_y \left[|M_{t \wedge \widetilde{D}_n}|^2 \right] \le \widetilde{\mathbb{E}}_y \left[\int_0^{t \wedge \widetilde{D}_n} \frac{\|\widehat{\sigma}^n(\widetilde{X}_s^1) - \widehat{\sigma}^n(\widetilde{X}_s^2)\|^2}{\|\widetilde{Z}_s\|^2} ds \right]$$

$$\le \liminf_{\delta \to 0+} \widetilde{\mathbb{E}}_y \left[\int_0^{t \wedge \widetilde{D}_n} \frac{\|\widehat{\sigma}^n(\widetilde{X}_s^1) - \widehat{\sigma}^n(\widetilde{X}_s^2)\|^2}{\|\widetilde{Z}_s\|^2} 1_{\{\|\widetilde{Z}_s\| > \delta\}} ds \right].$$

Using triangle inequality and Jensen's inequality, we get

$$\widetilde{\mathbb{E}}_y \left[\int_0^{t \wedge \widetilde{D}_n} \frac{\|\widehat{\sigma}^n(\widetilde{X}_s^1) - \widehat{\sigma}^n(\widetilde{X}_s^2)\|^2}{\|\widetilde{Z}_s\|^2} 1_{\{\|\widetilde{Z}_s\| > \delta\}} ds \right]$$

$$\le 3 \left(\limsup_{m \to \infty} \widetilde{\mathbb{E}}_y \left[\int_0^{t \wedge \widetilde{D}_n} \frac{\|\widehat{\sigma}^n(\widetilde{X}_s^1) - \widehat{\sigma}^{n,m}(\widetilde{X}_s^1)\|^2}{\|\widetilde{Z}_s\|^2} 1_{\{\|\widetilde{Z}_s\| > \delta\}} ds \right] \right.$$

$$+ \sup_{m \in \mathbb{N}} \widetilde{\mathbb{E}}_y \left[\int_0^{t \wedge \widetilde{D}_n} \frac{\|\widehat{\sigma}^{n,m}(\widetilde{X}_s^1) - \widehat{\sigma}^{n,m}(\widetilde{X}_s^2)\|^2}{\|\widetilde{Z}_s\|^2} 1_{\{\|\widetilde{Z}_s\| > \delta\}} ds \right]$$

$$\left. + \limsup_{m \to \infty} \widetilde{\mathbb{E}}_y \left[\int_0^{t \wedge \widetilde{D}_n} \frac{\|\widehat{\sigma}^{n,m}(\widetilde{X}_s^2) - \widehat{\sigma}^n(\widetilde{X}_s^2)\|^2}{\|\widetilde{Z}_s\|^2} 1_{\{\|\widetilde{Z}_s\| > \delta\}} ds \right] \right).$$

Using Corollary 3.2,

$$\limsup_{m \to \infty} \widetilde{\mathbb{E}}_y \left[\int_0^{t \wedge \widetilde{D}_n} \frac{\|\widehat{\sigma}^n(\widetilde{X}_s^1) - \widehat{\sigma}^{n,m}(\widetilde{X}_s^1)\|^2}{\|\widetilde{Z}_s\|^2} 1_{\{\|\widetilde{Z}_s\| > \delta\}} ds \right]$$

$$\le \limsup_{m \to \infty} \frac{1}{\delta^2} \widetilde{\mathbb{E}}_y \left[\int_0^t \|\widehat{\sigma}^n(\widetilde{X}_s^1) - \widehat{\sigma}^{n,m}(\widetilde{X}_s^1)\|^2 ds \right]$$

$$\leq \limsup_{m\to\infty} \frac{C_{y,B_{n+1},\widetilde{q}}}{\delta^2} \sum_{i,j=1}^d \|(\widehat{\sigma}_{ij}^n - \widehat{\sigma}_{ij}^{n,m})\|_{L^{2\widetilde{q}}(\mathbb{R}^d)}^2 = 0,$$

where $C_{y,B_{n+1},\widetilde{q}} > 0$ is a constant as in Corollary 3.2. In the same way as above, we also get

$$\limsup_{m\to\infty} \widetilde{\mathbb{E}}_y \left[\int_0^{t\wedge\widetilde{D}_n} \frac{\|\widehat{\sigma}^{n,m}(\widetilde{X}_s^2) - \widehat{\sigma}^n(\widetilde{X}_s^2)\|^2}{\|\widetilde{Z}_s\|^2} 1_{\{\|\widetilde{Z}_s\|>\delta\}} ds \right] = 0.$$

By Proposition 2.1 and Corollary 3.2,

$$(11) \quad \sup_{m\in\mathbb{N}} \widetilde{\mathbb{E}}_y \left[\int_0^{t\wedge\widetilde{D}_n} \frac{\|\widehat{\sigma}^{n,m}(\widetilde{X}_s^1) - \widehat{\sigma}^{n,m}(\widetilde{X}_s^2)\|^2}{\|\widetilde{Z}_s\|^2} 1_{\{\|\widetilde{Z}_s\|>\delta\}} ds \right]$$

$$\leq \sup_{m\in\mathbb{N}} \sum_{i,j=1}^d \widetilde{c}_d \, \widetilde{\mathbb{E}}_y \left[\int_0^t \left(1_{B_n}(\widetilde{X}_s^1)\mathcal{M}\|\nabla\widehat{\sigma}_{ij}^{n,m}\|(\widetilde{X}_s^1) + 1_{B_n}(\widetilde{X}_s^2)\mathcal{M}\|\nabla\widehat{\sigma}_{ij}^{n,m}\|(\widetilde{X}_s^2) \right)^2 ds \right]$$

$$\leq \sup_{m\in\mathbb{N}} \sum_{i,j=1}^d 2e^t \widetilde{c}_d \, C_{y,B_{n+1},\widetilde{q}} \left(\left\| 1_{B_n}\mathcal{M}\|\nabla\widehat{\sigma}_{ij}^{n,m}\| \right\|_{L^{2\widetilde{q}}(\mathbb{R}^d)}^2 + \left\| 1_{B_n}\mathcal{M}\|\nabla\widehat{\sigma}_{ij}^{n,m}\| \right\|_{L^{2\widetilde{q}}(\mathbb{R}^d)}^2 \right)$$

$$\leq 4e^t c_{d,2\widetilde{q}}^2 \widetilde{c}_d \, C_{y,B_{n+1},\widetilde{q}} \sup_{m\in\mathbb{N}} \sum_{i,j=1}^d \|\nabla\widehat{\sigma}_{ij}^{n,m}\|_{L^{2\widetilde{q}}(\mathbb{R}^d)}^2 \leq 4e^t c_{d,2\widetilde{q}}^2 \widetilde{c}_d \, C_{y,B_{n+1},\widetilde{q}} \sum_{i,j=1}^d \|\nabla\widehat{\sigma}_{ij}^n\|_{L^{2\widetilde{q}}(\mathbb{R}^d)}^2,$$

where $c_{d,2\widetilde{q}}, \widetilde{c}_d > 0$ are constants as in Proposition 2.1. Therefore, $\widetilde{\mathbb{E}}_y\left[|M_{t\wedge D_n}|^2\right] < \infty$. Analogously to the above,

$$\widetilde{\mathbb{E}}_y \left[|A_{t\wedge\widetilde{D}_n}^{(1)}| \right] \leq 4e^t c_{d,2\widetilde{q}}^2 \widetilde{c}_d \, C_{y,B_{n+1},\widetilde{q}} \sum_{i,j=1}^d \|\nabla\widehat{\sigma}_{ij}^n\|_{L^{2\widetilde{q}}(\mathbb{R}^d)}^2 < \infty.$$

Using Fatou's Lemma,

$$\frac{1}{2}\widetilde{\mathbb{E}}_y \left[|A_{t\wedge\widetilde{D}_n}^{(2)}| \right] \leq \widetilde{\mathbb{E}}_y \left[\int_0^{t\wedge\widetilde{D}_n} \frac{\|\mathbf{G}^n(\widetilde{X}_s^1) - \mathbf{G}^n(\widetilde{X}_s^2)\|}{\|\widetilde{Z}_s\|} ds \right]$$

$$\leq \liminf_{\delta\to0+} \widetilde{\mathbb{E}}_y \left[\int_0^{t\wedge\widetilde{D}_n} \frac{\|\mathbf{G}^n(\widetilde{X}_s^1) - \mathbf{G}^n(X_s^2)\|}{\|\widetilde{Z}_s\|} 1_{\{\|\widetilde{Z}_s\|>\delta\}} ds \right],$$

and again as above, we have

$$\widetilde{\mathbb{E}}_y \left[\int_0^{t\wedge D_n} \frac{\|\mathbf{G}^n(\widetilde{X}_s^1) - \mathbf{G}^n(\widetilde{X}_s^2)\|}{\|\widetilde{Z}_s\|} 1_{\{\|\widetilde{Z}_s\|>\delta\}} ds \right]$$

$$\leq \left(\limsup_{m\to\infty} \widetilde{\mathbb{E}}_y \left[\int_0^{t\wedge\widetilde{D}_n} \frac{\|\mathbf{G}^n(\widetilde{X}_s^1) - \mathbf{G}^{n,m}(\widetilde{X}_s^1)\|}{\|\widetilde{Z}_s\|} 1_{\{\|\widetilde{Z}_s\|>\delta\}} ds \right] \right.$$

$$+ \sup_{m\in\mathbb{N}} \widetilde{\mathbb{E}}_y \left[\int_0^{t\wedge\widetilde{D}_n} \frac{\|\mathbf{G}^{n,m}(\widetilde{X}_s^1) - \mathbf{G}^{n,m}(\widetilde{X}_s^2)\|}{\|\widetilde{Z}_s\|} 1_{\{\|\widetilde{Z}_s\|>\delta\}} ds \right]$$

$$\left. + \limsup_{m\to\infty} \widetilde{\mathbb{E}}_y \left[\int_0^{t\wedge\widetilde{D}_n} \frac{\|\mathbf{G}^{n,m}(\widetilde{X}_s^2) - \mathbf{G}^n(\widetilde{X}_s^2)\|}{\|\widetilde{Z}_s\|} 1_{\{\|\widetilde{Z}_s\|>\delta\}} ds \right] \right).$$

Using Corollary 3.2,

$$\limsup_{m\to\infty}\widetilde{\mathbb{E}}_y\left[\int_0^{t\wedge\widetilde{D}_n}\frac{\|\mathbf{G}^n(\widetilde{X}_s^1)-\mathbf{G}^{n,m}(\widetilde{X}_s^1)\|}{\|\widetilde{Z}_s\|}1_{\{\|\widetilde{Z}_s\|>\delta\}}ds\right]$$

$$\leq\limsup_{m\to\infty}\frac{1}{\delta^2}\widetilde{\mathbb{E}}_y\left[\int_0^t\|\mathbf{G}^n(\widetilde{X}_s^1)-\mathbf{G}^{n,m}(\widetilde{X}_s^1)\|ds\right]\leq\limsup_{m\to\infty}\frac{e^tC_{y,B_{n+1},\widetilde{q}}}{\delta^2}\|\mathbf{G}^n-\mathbf{G}^{n,m}\|_{L^{\widetilde{q}}(\mathbb{R}^d)}=0,$$

and as before

$$\limsup_{m\to\infty}\widetilde{\mathbb{E}}_y\left[\int_0^{t\wedge\widetilde{D}_n}\frac{\|\mathbf{G}^n(\widetilde{X}_s^2)-\mathbf{G}^{n,m}(\widetilde{X}_s^2)\|}{\|\widetilde{Z}_s\|}1_{\{\|\widetilde{Z}_s\|>\delta\}}ds\right]=0.$$

Using an estimate as in (11), Proposition 2.1 and Corollary 3.2, we obtain

$$\sup_{m\in\mathbb{N}}\widetilde{\mathbb{E}}_y\left[\int_0^{t\wedge D_n}\frac{\|\mathbf{G}^{n,m}(\widetilde{X}_s^1)-\mathbf{G}^{n,m}(\widetilde{X}_s^2)\|}{\|\widetilde{Z}_s\|}1_{\{\|\widetilde{Z}_s\|>\delta\}}ds\right]\leq 2e^t c_{d,2\widetilde{q}}\widetilde{c}_d\,C_{y,B_{n+1},\widetilde{q}}\sum_{i=1}^d\|\nabla g_i\|_{L^{\widetilde{q}}(\mathbb{R}^d)}.$$

Therefore, $\widetilde{\mathbb{E}}_y\left[|A_{t\wedge D_n}^{(2)}|\right]<\infty$. Thus, applying [10, page 378, (2.3) Proposition] to (10), it follows that

$$\|\widetilde{Z}_{t\wedge\widetilde{D}_n}\|^2=\|\widetilde{Z}_0\|^2\exp\left(M_{t\wedge\widetilde{D}_n}+A_{t\wedge\widetilde{D}_n}^{(1)}+A_{t\wedge\widetilde{D}_n}^{(2)}-\frac{1}{2}\langle M\rangle_{t\wedge\widetilde{D}_n}\right)=0,\quad\widetilde{\mathbb{P}}_y\text{-a.s.}$$

Since $\widetilde{D}_n\to\infty$ $\widetilde{\mathbb{P}}_y$-a.s. as $n\to\infty$, letting $n\to\infty$ we obtain $\widetilde{Z}_t=0$, $\widetilde{\mathbb{P}}_y$-a.s, so that pathwise uniqueness is shown. The existence of a strong solution $(Y_t^y)_{t\geq0}$ to (1) follows from the Yamada-Watanabe Theorem ([12, Corollary 1]). □

Proof of Corollary 1.2. Let $A=\sigma=id$, $\sqrt{\frac{1}{\psi}}(x)=\|x\|^{\alpha/2}$, $x\in\mathbb{R}^d$ and $E=\mathbb{R}^d\setminus\{0\}$. Then (H1)–(H4) are satisfied and the SDE (1) becomes the SDE (5). Choose $q\in(2d+2,\frac{d}{\alpha})$ and $\widetilde{q}\in(\frac{d}{2},\frac{d}{2-\alpha}\wedge\frac{d}{2}+\varepsilon)$. Then $\psi\in L_{loc}^q(\mathbb{R}^d)$ $\sqrt{\frac{1}{\psi}}\in H_{loc}^{1,2\widetilde{q}}(\mathbb{R}^d)$ and $g_i\in H_{loc}^{1,\widetilde{q}}(\mathbb{R}^d)$ for all $1\leq i\leq d$. Therefore, the assertion follows from Theorem 1.1. □

REMARK 4.1. In Theorems 1.1, 2.2 and Corollary 3.2, the condition (H2) can be replaced by any ohter condition that implies the non-explosion of $\mathbb{M}$ defined in Section 2 (cf. [7]).

EXAMPLE 4.2. Let $d\geq 3$ and for some $\varepsilon>0$ let $\mathbf{G}=(g_1,\dots,g_d)\in L_{loc}^\infty(\mathbb{R}^d,\mathbb{R}^d)$ with $g_i\in H_{loc}^{1,\frac{d}{2}+\varepsilon}(\mathbb{R}^d)$ for all $1\leq i,j\leq d$. Consider the following Itô-SDE

$$(12)\qquad X_t=y+\int_0^t\sqrt{\frac{1}{\psi}}(X_s)\cdot id\,dW_s+\int_0^t\mathbf{G}(X_s)ds,\qquad 0\leq t<\infty,$$

where y and $\sqrt{\frac{1}{\psi}}$ are specified below in three different cases.

(i) For $\alpha\in[0,\frac{d}{2d+2})$ and $\gamma\in(0,\infty)$, let

$$\sqrt{\frac{1}{\psi}}(x):=\|x\|^{\alpha/2}1_{\{x\neq0\}}(x)+\gamma1_{\{x=0\}}(x),\quad x\in\mathbb{R}^d.$$

Then $\left\{\sqrt{\frac{1}{\psi}}=0\right\}=\emptyset$. Thus by Theorem 1.1 (cf. proof of Corollary 1.2), for $y\in\mathbb{R}^d\setminus\{0\}$ pathwise uniqueness holds for (12) in the usual sense, i.e. if

$$(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}_y, (\widetilde{\mathcal{F}}_t)_{t \geq 0}, (\widetilde{X}_t^k)_{t \geq 0}, (\widetilde{W}_t)_{t \geq 0}), \quad k \in \{1, 2\}$$

are weak solutions to (12), then

$$\widetilde{\mathbb{P}}_y(\widetilde{X}_t^1 = \widetilde{X}_t^2, \ t \geq 0) = 1.$$

Moreover, by Theorem 1.1 and Corollary 3.2 for $y \in \mathbb{R}^d \setminus \{0\}$ and a d-dimensional Brownian motion $(\widetilde{W}_t)_{t \geq 0}$ on a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ there exists a strong solution $(Y_t^y)_{t \geq 0}$ to (12) satisfying

$$\int_0^\infty 1_{\{0\}}(Y_s^y) ds = 0, \quad \widetilde{\mathbb{P}}\text{-a.s.}$$

(ii) Let $\alpha \in [0, \frac{d}{2d+2})$. For $i \in \mathbb{N} \cup \{0\}$, let

$$\phi_i(x) = \|x - 2i\mathbf{e}_1\|^{\frac{\alpha}{2}} 1_{B_1(2i\mathbf{e}_1)}(x), \quad x \in \mathbb{R}^d,$$

where $B_1(2i\mathbf{e}_1)$ denotes an open ball with center $2i\mathbf{e}_1$ and radius 1. Define

$$\sqrt{\frac{1}{\psi}}(x) := 1_{\mathbb{R}^d \setminus \bigcup_{i=0}^\infty B_1(2i\mathbf{e}_1)} + \sum_{i=0}^\infty \phi_i(x), \qquad x \in \mathbb{R}^d.$$

Note that $\left\{\sqrt{\frac{1}{\psi}} = 0\right\} = \{2i\mathbf{e}_1 : i \in \mathbb{N} \cup \{0\}\}$. Thus by Theorem 1.1 (cf. proof of Corollary 1.2), for $y \in \mathbb{R}^d \setminus \{2i\mathbf{e}_1 : i \in \mathbb{N} \cup \{0\}\}$ pathwise uniqueness holds for (12) in the following sense, i.e. if

$$(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}_y, (\widetilde{\mathcal{F}}_t)_{t \geq 0}, (\widetilde{X}_t^k)_{t \geq 0}, (\widetilde{W}_t)_{t \geq 0}), \quad k \in \{1, 2\}$$

are weak solutions to (1) satisfying both

$$\int_0^\infty 1_{\bigcup_{i=0}^\infty \{2i\mathbf{e}_1\}}(\widetilde{X}_s^k) ds = 0, \quad \widetilde{\mathbb{P}}_y\text{-a.s,}$$

then

$$\widetilde{\mathbb{P}}_y(\widetilde{X}_t^1 = \widetilde{X}_t^2, \ t \geq 0) = 1.$$

Moreover, by Theorem 1.1 and Corollary 3.2 for $y \in \mathbb{R}^d \setminus \{2i\mathbf{e}_1 : i \in \mathbb{N} \cup \{0\}\}$ and a d-dimensional Brownian motion $(\widetilde{W}_t)_{t \geq 0}$ on a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ there exists a strong solution $(Y_t^y)_{t \geq 0}$ to (12) satisfying

$$(13) \qquad \int_0^\infty 1_{\{2i\mathbf{e}_1 : i \in \mathbb{N} \cup \{0\}\}}(Y_s^y) ds = 0, \quad \widetilde{\mathbb{P}}\text{-a.s.}$$

(iii) Let $(\gamma_i)_{i \geq 0}$ be a sequence with $\gamma_i \in (0, \infty)$ for all $i \in \mathbb{N} \cup \{0\}$. For $\alpha \in [0, \frac{d}{2d+2})$ and $i \in \mathbb{N} \cup \{0\}$, let ϕ_i be defined as in (ii). Define

$$\sqrt{\frac{1}{\psi}}(x) := 1_{\mathbb{R}^d \setminus \bigcup_{i=0}^\infty B_1(2i\mathbf{e}_1)} + \sum_{i=0}^\infty \gamma_i 1_{\{2i\mathbf{e}_1\}} + 1_{\mathbb{R}^d \setminus \bigcup_{i=0}^\infty \{2i\mathbf{e}_1\}}\left(\sum_{i=0}^\infty \phi_i(x)\right), \qquad x \in \mathbb{R}^d.$$

Then $\left\{\sqrt{\frac{1}{\psi}} = 0\right\} = \emptyset$. Therefore, by Theorem 1.1 (cf. proof of Corollary 1.2), for $y \in \mathbb{R}^d \setminus \{2i\mathbf{e}_1 : i \in \mathbb{N} \cup \{0\}\}$ pathwise uniqueness holds for (12) in the usual sense and by Corollary 3.2 there exists a strong solution $(Y_t^y)_{t \geq 0}$ to (12) satisfying (13).

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