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THE BAILY–BOREL COMPACTIFICATION OF A FAMILY OF ORTHOGONAL MODULAR VARIETIES

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Abstract

We study the Baily–Borel compactification of a family of four-dimensional orthogonal modular varieties arising as period spaces of compact hyperkähler manifolds of deformation generalised Kummer type. Our main results concern the classification of boundary components, their incidence relations and combinatorics.

1. Introduction

An *orthogonal modular variety* is a locally symmetric variety given by the quotient of a Hermitian symmetric space of type IV by an arithmetic subgroup of the orthogonal group $\mathrm{O}(L \otimes \mathbb{Q})$ for a lattice L of signature $(2, n)$. The purpose of this paper is to study a family of 4-dimensional orthogonal modular varieties (defined in §1.7) related to moduli and periods of compact hyperkähler manifolds of deformation generalised Kummer type (*deformation generalised Kummer varieties*). Our main results concern the geometry and combinatorics of the Baily–Borel compactification: we describe the isomorphism types of boundary components (Theorem 3.6), their incidence relations (Theorem 3.12 and 3.13) and combinatorics (Corollary 3.7). We believe these are the first such results for orthogonal modular varieties of dimension 4, complementing results in dimension 10 and 19 for moduli spaces of Enriques and K3 surfaces, respectively [23, 22].

1.1. Lattices. A *lattice* L is an even, integral quadratic form on a free abelian group of finite rank. Unless otherwise stated, we will assume that all lattices are non-degenerate. By Sylvester's law of inertia, the quadratic form on $L \otimes \mathbb{R}$ can be diagonalised and the pair consisting of the number of positive and negative terms in the diagonalisation is known as the *signature* of the lattice. We will use $x^2 := (x, x)$ to denote the quadratic form of L evaluated at $x \in L$ and (x, y) to denote the bilinear form of L evaluated at $x, y \in L$ (we will also extend this convention to $L \otimes \mathbb{Q}$ and $L \otimes \mathbb{R}$). Examples of lattices include the rank 1 lattice $\langle d \rangle$ generated by a single element $x \in L$ of length $x^2 = d$; the root lattice A_2; and the hyperbolic plane U, whose Gram matrix is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

on a suitable basis (the *canonical* basis). We use $L_1 \oplus L_2$ to denote the orthogonal direct sum of lattices L_1 and L_2; and nL_1 to denote the orthogonal direct sum of n copies of L_1. We let

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$L(m)$ denote the lattice obtained by multiplying the quadratic form of L by m. If $S \subset L$ is a sublattice, we let $S^\perp \subset L$ denote the orthogonal complement of S in L.

The *dual lattice* $L^\vee$ of L is the free abelian group $\mathrm{Hom}(L, \mathbb{Z}) \subset L \otimes \mathbb{Q}$ with a quadratic form inherited from L. The quotient $D(L) := L^\vee/L$ (known as the *discriminant group* of L) inherits both a $\mathbb{Q}/2\mathbb{Z}$-valued quadratic form q_L (the *discriminant form* of L) and a $\mathbb{Q}/\mathbb{Z}$-valued bilinear form b_L from L [20]. We will often encode b_L by the data $(B, \bigoplus_j C_{i_j})$ where $D(L) \cong \bigoplus_j C_{i_j}$, C_i is the cyclic group of order i and B is the Gram matrix of b_L on a canonical basis of $\bigoplus_j C_{i_j}$.

A (possibly degenerate) sublattice $S \subset L$ is said to be *primitive* if L/S is torsion-free and *totally isotropic* if the restriction of the quadratic form from L to S is identically zero. A non-zero vector $x \in L$ is said to be *primitive* (or *isotropic*) if it defines a primitive (or totally isotropic) sublattice $\langle x \rangle \subset L$. The *divisor* $\mathrm{div}(x)$ of $0 \neq x \in L$ is defined as the positive generator of the ideal (x, L). We note that if $0 \neq x \in L$ is primitive and $x^* := x/\mathrm{div}(x) \in L^\vee$ then x^* mod L is of order $\mathrm{div}(x)$ in $D(L)$.

1.2. The orthogonal group and spinor norm. For a lattice L, we let $\mathrm{O}(L)$ and $\mathrm{O}(L \otimes \mathbb{R})$ denote the orthogonal groups of L and $L \otimes \mathbb{R}$, respectively. As explained in [4], every $g \in \mathrm{O}(L \otimes \mathbb{R})$ can be written as a product

$$g = \sigma_{w_1} \cdots \sigma_{w_m} \tag{1}$$

where

$$\sigma_w : x \mapsto x - \frac{2(x,w)}{(w,w)} w \in \mathrm{O}(L \otimes \mathbb{R})$$

is the reflection defined by $w \in L \otimes \mathbb{R}$. If g is as in (1) then the *spinor norm* $\mathrm{sn}_{\mathbb{R}}(g)$ of g is defined by [15]

$$\mathrm{sn}_{\mathbb{R}}(g) = \left(\frac{-(w_1, w_1)}{2} \right) \cdots \left(\frac{-(w_m, w_m)}{2} \right) \in \mathbb{R}/(\mathbb{R}^*)^2.$$

We let $\mathrm{O}^+(L \otimes \mathbb{R})$ denote the kernel of the spinor norm on $\mathrm{O}(L \otimes \mathbb{R})$ and, for $\Gamma \subset \mathrm{O}(L \otimes \mathbb{R})$, we use Γ^+ to denote the intersection $\Gamma \cap \mathrm{O}^+(L \otimes \mathbb{R})$.

1.3. The stable orthogonal group. There is a natural map

$$\mathrm{O}(L) \to \mathrm{O}(D(L)), \tag{2}$$

where $\mathrm{O}(D(L))$ is the subgroup of $\mathrm{Aut}(D(L))$ preserving q_L. We let $\overline{g}$ denote the image of $g \in \mathrm{O}(L)$ under (2) and use $\widetilde{\mathrm{O}}(L)$ to denote the kernel of (2). For $\Gamma \subset \mathrm{O}(L)$, we use $\widetilde{\Gamma}$ to denote the intersection $\Gamma \cap \widetilde{\mathrm{O}}(L)$. The group $\widetilde{\mathrm{O}}(L)$ (often referred to as the *stable orthogonal group*) has the useful property that $\widetilde{\mathrm{O}}(S) \subset \widetilde{\mathrm{O}}(L)$ for any sublattice $S \subset L$, where $g \in \widetilde{\mathrm{O}}(S) \cap \widetilde{\mathrm{O}}(L)$ acts as the identity on $S^\perp \subset L$ [11, Lemma 7.1].

1.4. The Eichler criterion. We will often need to determine orbits of vectors in lattices. If L is a lattice containing a copy of $2U$ and $v_1, v_2 \in L$ are primitive, then the *Eichler criterion* [8, §10] [10, Proposition 3.3] states that $gv_1 = v_2$ for some $g \in \widetilde{\mathrm{SO}}^+(L)$ if and only if $v_1^2 = v_2^2$ and $v_1^* \equiv v_2^*$ mod L.

1.5. Orthogonal modular varieties. Let L be a lattice of signature $(2, n)$ and let $\Gamma \subset \mathrm{O}^+(L \otimes \mathbb{Q})$ be an arithmetic subgroup. If $\mathcal{D}_L$ is the component of

$$\Omega_L := \{[x] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (x, x) = 0, (x, \overline{x}) > 0\}$$

preserved by $\mathrm{O}^+(L \otimes \mathbb{R})$, then the quotient

$$\mathcal{F}_L(\Gamma) := \mathcal{D}_L/\Gamma$$

is a locally symmetric variety known as an *orthogonal modular variety*. Orthogonal modular varieties are complex analytic spaces (indeed, are even quasi-projective [1]) but are typically non-compact.

1.6. The Baily–Borel compactification. The *Baily–Borel compactification* $\mathcal{F}_L(\Gamma)^*$ of $\mathcal{F}_L(\Gamma)$ is an irreducible normal complex projective variety containing $\mathcal{F}_L(\Gamma)$ as a Zariski-open subset. It is defined by Proj $M_*(\Gamma, \mathbb{1})$ where $M_*(\Gamma, \mathbb{1})$ is the ring of modular forms with trivial character for Γ [1]. In most of the paper we are interested in studying the boundary of $\mathcal{F}_L(\Gamma)^*$, which can be described by Theorem 1.1.

Theorem 1.1 ([11, p. 487])**.** *The Baily–Borel compactification $\mathcal{F}_L(\Gamma)^*$ decomposes as*

$$\mathcal{F}_L(\Gamma)^* = \mathcal{F}_L(\Gamma) \sqcup \bigsqcup_{E \in \mathcal{E}} C_E \sqcup \bigsqcup_{l \in \ell} P_l$$

where ℓ and $\mathcal{E}$ are sets of the finitely many Γ-orbits of primitive totally isotropic sublattices of rank 1 and 2 in L, respectively; and the indices $E \in \mathcal{E}$ and $l \in \ell$ run over a choice of representative for each orbit. Each C_E is a modular curve and each P_l is a point. The point P_l is contained in the closure of C_E if and only if representatives can be chosen such that $l \subset E$.

Furthermore, if $\overline{\mathcal{D}}_L$ is the topological closure of $\mathcal{D}_L$ in the compact dual $\mathcal{D}_L^\vee$, then the boundary curve C_E is isomorphic to $\mathbb{H}^+/G(E)$ where $G(E) := \mathrm{Stab}_\Gamma(E)/\mathrm{Fix}_\Gamma(E)$ and the upper half-plane $\mathbb{H}^+$ is identified with $\mathbb{H}^+ \cong \mathbb{P}(E \otimes \mathbb{C}) \cap \overline{\mathcal{D}}_L$, as explained in [22, §2] (see also [17, §2]).

1.7. A family of orthogonal modular varieties. From now on, we let L_{2d} denote the lattice

$$L_{2d} = 2U \oplus \langle -2d \rangle \oplus \langle -6 \rangle$$

and let $\underline{v}$ and $\underline{w}$ denote generators for the $\langle -2d \rangle$ and $\langle -6 \rangle$ factors of L_{2d}, respectively. We define the group Γ_{2d} by

$$\Gamma_{2d} = \{g \in \mathrm{O}^+(L) \mid g\underline{v}^* \equiv \underline{v}^* \bmod L\}.$$

We will mostly be interested in studying the case of $d = p^2$ for prime $p > 3$, where $\mathcal{F}_{L_{2p^2}}(\Gamma_{2p^2})^*$ has particularly agreeable properties.

1.8. Moduli of deformation generalised Kummer varieties. A more comprehensive account of the moduli theory of compact hyperkähler manifolds can be found in [11], which we follow. A complex manifold Y is said to be a *compact hyperkähler manifold* (or an *irreducible symplectic manifold*) if it is compact, Kähler, simply connected and $H^0(Y, \Omega_X^2)$ is generated by an everywhere non-degenerate holomorphic 2-form [14]. One family of compact hyperkähler manifolds are the generalised Kummer varieties, which are defined as follows: if $A^{[n+1]}$ is the Hilbert scheme parametrising $(n+1)$-points on an abelian surface A, there is a natural projection

$$\pi : A^{[n+1]} \to A$$

and the fibre $X := \pi^{-1}(0)$ is known as a *generalised Kummer variety* [2]. Deformations of X are compact hyperkähler manifolds known as *deformation generalised Kummer varieties*. Many aspects of the geometry of X are encoded in the Beauville–Bogomolov lattice M on $H^2(X, \mathbb{Z})$ [2, 9] which, by the results of Rapagnetta [21], is given by

$$M \cong 3U \oplus \langle -2(n+1) \rangle. \tag{3}$$

One can construct moduli spaces of polarised deformation generalised Kummer varieties. A choice of ample line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ defines a *polarisation* for X: there is an associated vector $h := c_1(\mathcal{L}) \in M$ (given by the first Chern class of $\mathcal{L}$) and a lattice $L := h^\perp \subset M$. $\mathcal{L}$ is said to be *primitive* if h is primitive in M and *split* if $\mathrm{div}(h) = 1$. The *degree* $2d$ of $\mathcal{L}$ is defined by the length $2d := h^2$ and the *polarisation type* of $\mathcal{L}$ is defined as the orbit $\mathrm{O}(M).h$. We assume throughout that all polarisations are primitive. By the work of Viehweg [24], Matsusaka's big theorem [18] and a result of Kollár and Matsusaka [16], there exists a GIT moduli space $\mathcal{M}$ parametrising deformation generalised Kummer varieties of fixed dimension and polarisation type $\mathrm{O}(M).h$. If $\mathrm{O}(M, h)$ is the group

$$\mathrm{O}(M, h) = \{g \in \mathrm{O}(M) \mid gh = h\}$$

then, by [11, Theorem 3.8], there exists a finite-to-one dominant morphism

$$\psi : \mathcal{M}' \to \mathcal{F}_L(\mathrm{O}^+(M, h))$$

for each component $\mathcal{M}'$ of $\mathcal{M}$. In the rest of the paper, we will study the modular varieties $\mathcal{F}_L(\mathrm{O}^+(M, h))$ for split polarisation types, which are classified in Lemma 1.2. (A full classification of polarisation types can be obtained as for irreducible symplectic manifolds of $K3^{[n]}$-type following Proposition 3.6 of [12].)

Lemma 1.2 ([6])**.** *If $h \in M$ corresponds to a split polarisation $\mathcal{L}$ of degree $2d$ then,*

1. *the polarisation type of $\mathcal{L}$ is uniquely determined by the length h^2;*
2. *the lattice $L \cong 2U \oplus \langle -2(n+1) \rangle \oplus \langle -2d \rangle$.*

Proof. Apply the Eichler criterion. □

In Proposition 2.1, we show that when $n = 2$ (corresponding to deformation generalised Kummer varieties of dimension 4) and h corresponds to a split polarisation of degree $2d$ then $\mathcal{F}_L(\mathrm{O}^+(M, h)) \cong \mathcal{F}_{L_{2d}}(\Gamma_{2d})$.

2. Finite geometry and the group Γ

From now on, we assume that $n = 2$ in (3). In this section, we consider the group Γ_{2d}, paying particular attention to the case of $d = p$ for prime $p > 3$, which we study following the approach of §3 of [17]. Where no confusion is likely to arise, we use L to denote L_{2d} and Γ to denote Γ_{2d}. With the exception of Lemmas 2.2 and 2.4, the results of this section are essentially contained in the PhD thesis [6].

Proposition 2.1. *If $h \in M$ corresponds to a split polarisation of degree $2d > 4$ then*

$$\Gamma_{2d} \cong \mathrm{O}^+(M, h).$$

Furthermore, if $d = p^2$ for prime $p > 3$, then $\Gamma_{2d} \subset \Gamma_2$.

Proof. The first part of the proof follows the approach of part (i) of Proposition 3.12 of [12]. As $\mathrm{O}(M, h)$ acts on both $\langle h\rangle$ and $\langle h\rangle^\perp$ but trivially on $\langle h\rangle$, we can immediately identify $\mathrm{O}(M, h)$ with a subgroup of $\mathrm{O}(L_{2d})$.

As in [20, p.111], the series of overlattices

$$\langle h\rangle \oplus h^\perp \subset M \subset M^\vee \subset \langle h\rangle^\vee \oplus (h^\perp)^\vee$$

defines a series of abelian groups

$$M/(\langle h\rangle \oplus \langle h\rangle^\perp) \subset \langle h\rangle^\vee/\langle h\rangle \oplus (\langle h\rangle^\perp)^\vee/(\langle h\rangle^\perp) = D(\langle h\rangle) \oplus D(\langle h\rangle^\perp)$$

and we can regard the isotropic subgroup $H = M/(\langle h\rangle \oplus h^\perp)$ as a subgroup of $D(\langle h\rangle) \oplus D(\langle h\rangle^\perp)$ and define corresponding projections $p_h : H \to D(\langle h\rangle)$ and $p_{h^\perp} : H \to D(\langle h\rangle^\perp)$. Without loss of generality (as h is split) we can assume that $h = e_3 + df_3 \in U \oplus \langle -6\rangle$ where $\{e_i, f_i\}$ is the canonical basis for the i-th copy of U in M. Let $k_1 = e_3 - df_3$, $k_1' = (2d)^{-1}k_1$, $k_2' = (6)^{-1}k_2$ and $k_3' = (2d)^{-1}h$, where k_2 generates the $\langle -6\rangle$ factor of M. Take a basis $\{e_1, f_1, e_2, f_2, k_1', k_2'\}$ for $(h^\perp)^\vee$. By direct calculation, $H = \langle k_3' - k_1', d(k_1' + k_3')\rangle + (\langle h\rangle \oplus h^\perp)$, $p_{h^\perp}(H) = \langle k_1'\rangle$ and $D(h^\perp) = \langle k_1'\rangle \oplus \langle k_2'\rangle$. By Corollary 1.5.2 of [20],

$$\mathrm{O}^+(M, h) \cong \{g \in \mathrm{O}^+(h^\perp) \mid g|_{p_{h^\perp}(H)} = \mathrm{id}\} \cong \Gamma_{2d},$$

and the first part of the claim follows.

For the second part of the claim, we follow the approach of [17, Lemma 3.2]. Let p be an odd prime and embed $L_{2p^2} \subset L_2$ by identifying factors of $2U \oplus \langle -6\rangle$ and mapping

$$L_{2p^2} \ni t + ak_1 \mapsto t + apk \in L_2$$

where $t \in 2U \oplus \langle -6\rangle$, k generates $\langle -2\rangle \subset L_2$ and $a \in \mathbb{Z}$. Define the totally isotropic subgroup $N \subset D(L_{2p^2})$ by $N = L_2/L_{2p^2} \subset D(L_{2p^2})$. If $g \in \Gamma_{2p^2}$ then $g(k_1') = k_1' + L_{2p^2}$. As $N \subset \langle k_1'\rangle + L_{2p^2} \subset D(L_{2p^2})$ and $g(L_{2p^2}) = L_{2p^2}$ then g preserves N and so extends to a unique element of $\mathrm{O}(L_2)$. To verify $g \in \Gamma_2$ one notes that the dual of the $\langle -2\rangle$ factor in L_2 is generated by pk_1' and

$$\begin{aligned} g(pk_1') &\equiv pk_1' \mod L_{2p^2} \\ &\equiv pk_1' \mod L_2, \end{aligned}$$

from which the result follows. □

Lemma 2.2. *Suppose $p > 3$ is prime and let $L = L_{2p^2}$.*

1. *If $g \in \mathrm{O}(L)$ then $g\underline{v}^* \equiv \pm\underline{v}^* \mod L$ and $g\underline{w}^* \equiv \pm\underline{w}^* \mod L$;*
2. $\Gamma_{2p^2} = \widetilde{\mathrm{O}}^+(L) \rtimes \langle \sigma_{\underline{w}}\rangle$.

Proof. We begin by calculating the elements of length $-1/2p^2 \mod 2\mathbb{Z}$ in $D(L)$, much as in [13, Lemma 3.3]. The group $D(L) \cong C_6 \oplus C_{2p^2}$ and

$$q_L(a,b) = -\frac{a^2}{6} - \frac{b^2}{2p^2} \bmod 2\mathbb{Z} \tag{4}$$

for $(a,b) \in D(L)$. Suppose $(a,b) \in D(L)$ is of order $2p^2$ and length $-1/2p^2 \bmod 2\mathbb{Z}$. As the order of (a,b) is coprime to 3 then $a = 0$ or 3. If $a = 0$ then

$$\frac{b^2}{2p^2} \equiv \frac{1}{2p^2} \bmod 2\mathbb{Z} \tag{5}$$

or, equivalently, $(b+1)(b-1) \equiv 0 \bmod 4p^2$. For order reasons, $(b, 2p) = 1$ and so precisely one of $b \pm 1 \equiv 0 \bmod p$ is true. If $b \equiv \pm 1 + xp \bmod p^2$ for $x \in \mathbb{Z}$ then, from (5), $x \equiv 0 \bmod p$. Similarly, as $2b \not\equiv 0 \bmod 4$ then b is odd. Therefore, by the Chinese remainder theorem, $(0,b) = (0,\pm 1)$. The case $a = 3$ cannot occur. From (4), $3p^2 + b^2 \equiv 1 \bmod 4$ and, as p is odd, we obtain the contradiction $b^2 \equiv 2 \bmod 4$. We conclude that $D(L)$ contains two elements of order $2p^2$ and length $-1/2p^2 \bmod 2\mathbb{Z}$, given by $\pm\underline{v}^*$. If $g\underline{w}^* =: (a,b) \in D(L)$ then $(g\underline{w}^*, g\underline{v}^*) \equiv \pm(g\underline{w}^*, \underline{v}^*) \equiv 0 \bmod \mathbb{Z}$. As $((a,b),(0,1)) \equiv b/2p^2 \bmod \mathbb{Z}$ then $b \equiv 0 \bmod 2p^2$ and $a \equiv \pm 1 \bmod 6$, from which the first claim follows. The second claim is immediate from Proposition 2.1. □

We now bound the index $|\Gamma_2 : \Gamma_{2p^2}|$ by using an idea in [17] (attributed to O'Grady). The approach involves considering the quadratic space

$$Q_p := L_2/pL_2,$$

over the finite field $\mathbb{F}_p$, where the quadratic form of Q_p is obtained by reducing the quadratic form of L_2 modulo p. For the rest of the section we will assume that p is an odd prime. We recall (e.g. [7]) that if V is a non-degenerate quadratic space over $\mathbb{F}_p$ then

$$|\mathrm{O}(V)| = \begin{cases} 2p^{m^2}\prod_{i=1}^{m}(p^{2i}-1) & \text{if } \dim V = 2m+1 \\ 2p^{m(m-1)}(p^m-\epsilon)\prod_{i=1}^{m-1}(p^{2i}-1) & \text{if } \dim V = 2m \end{cases} \tag{6}$$

where $m \in \mathbb{N}$ and $\epsilon = 1$ if $(-1)^m \det V \in (\mathbb{F}_p^*)^2$ and $\epsilon = -1$ otherwise.

Lemma 2.3. *For prime $p > 3$ suppose non-zero, non-isotropic $u, v \in Q_p$ define hyperplanes $\Pi_u, \Pi_v \subset Q_p$ given by $\Pi_u \perp u$ and $\Pi_v \perp v$. If $u^2/v^2 \in (\mathbb{F}_p^*)^2$ then Π_u and Π_v are equivalent under* $\mathrm{O}(L_2)$.

Proof. We construct a map $u \mapsto v$ using Eichler transvections, much as in the proof of the Eichler criterion given in [10, Proposition 3.3]. Let $\{e_1, f_1, e_2, f_2, v_1, v_2\}$ be a $\mathbb{Z}$-basis for L_2 where v_1, v_2 are generators for $\langle -6\rangle$ and $\langle -2\rangle$, respectively and $\{e_i, f_i\}$ are canonical bases for the two copies of $U \subset L_2$. We begin by defining some elements of $\mathrm{O}(L_2)$. For isotropic $e \in L_2$ and any $a \in e^\perp \subset L_2$, there exists $t(e,a) \in \mathrm{O}(L_2)$ (an *Eichler transvection*), defined by

$$t(e,a) : w \mapsto w - (a,w)e + (e,w)a - \frac{1}{2}(a,a)(e,w)e \tag{7}$$

for $w \in L_2$ [8, 10]. As $\mathrm{O}(2U) \subset \widetilde{\mathrm{O}}(L_2)$, we can also extend elements of $\mathrm{O}(2U)$. As is well known (e.g. [23]), if $(w,x,y,z) \in 2U$ (with respect to canonical bases of U) then the map

$$U \oplus U \ni (w, x, y, z) \mapsto \begin{pmatrix} w & -y \\ z & x \end{pmatrix} \in M_2(\mathbb{Z}) \tag{8}$$

identifies $2U$ with $M_2(\mathbb{Z})$, where the quadratic form on $M_2(\mathbb{Z})$ is given by $2\det$. Therefore, any $(A, B) \in \mathrm{SL}(2, \mathbb{Z}) \times \mathrm{SL}(2, \mathbb{Z})$ defines an element of $\mathrm{O}(U \oplus U)$ by

$$(A, B) : \begin{pmatrix} w & -y \\ z & x \end{pmatrix} \mapsto A \begin{pmatrix} w & -y \\ z & x \end{pmatrix} B^{-1}. \tag{9}$$

We now use (7) and (9) to show that any $w = (w_1, w_2, w_3, w_4, w_5, w_6) \in L_2/pL_2$ defining a non-degenerate hyperplane $\Pi_w \perp w$ can be put in a standard form. The transvections $t(e_2, v_1)$ and $t(e_2, v_2)$ act on $w = (w_1, w_2, w_3, w_4, w_5, w_6) \in L_2$ by

$$\begin{cases} t(e_2, v_1) : w \mapsto (w_1, w_2, w_3 + 3w_4 + 6w_5, w_4, w_5 + w_4, w_6) \\ t(e_2, v_2) : w \mapsto (w_1, w_2, w_3 + w_4 + 2w_6, w_4, w_5, w_6 + w_4) \end{cases}$$

and so, without loss of generality, we can assume $w_4 \neq 0$ by applying $t(e_2, v_1)$ or $t(e_2, v_2)$, or by permuting $\{w_1, w_2, w_3, w_4\}$ using elements of $\mathrm{O}(2U)$. By rescaling w so that $w_4 = 1$, and by repeated application of $t(e_2, v_1)$ and $t(e_2, v_2)$, w can be transformed to an element of the form $(w'_1, w'_2, w'_3, w'_4, 0, 0)$. By the existence of the Smith normal form for (8) [19], w can be mapped to an element $(w''_1, w''_2, 0, 0, 0, 0)$ using (9). By rescaling as necessary, we can assume w is given by $(1, a, 0, 0, 0, 0)$. We next construct a map between hyperplanes. Without loss of generality, assume that $u = (1, a, 0, 0, 0, 0)$ and $v = (1, b, 0, 0, 0, 0)$. By assumption, $ab^{-1} \in (\mathbb{F}_p^*)^2$ and so there exists $\mu, \lambda \in \mathbb{F}_p$ such that $(\mu u)^2 = (\lambda v)^2$. We define $\hat{u}$ and $\hat{v}$ by $\hat{u} := \mu u = (u_1, u_2, 0, 0, 0, 0)$ and $\hat{v} := \lambda v = (v_1, v_2, 0, 0, 0, 0)$. Without loss of generality, assume that $\hat{u} - \hat{v} = (r, s, 0, 0, 0, 0)$ is non-zero and, by taking representatives for r, s modulo p, let

$$q := \begin{cases} r & \text{if } s = 0 \\ s & \text{if } r = 0 \\ \gcd(r, s) & \text{otherwise.} \end{cases}$$

If $r_1, r_2, s_1, s_2 \in \mathbb{Z}$ are solutions to $r_2 u_1 + r_1 u_2 \equiv q \bmod p$ and $s_2 v_1 + s_1 v_2 \equiv q \bmod p$, define $u', v', w \in e_2^\perp \cap f_2^\perp \subset L_2$ by $u' = (r_1, r_2, 0, 0, 0, 0)$, $v' = (s_1, s_2, 0, 0, 0, 0)$ and $w = (q^{-1}r, q^{-1}s, 0, 0, 0, 0)$. Then, over $\mathbb{F}_p$, $(\hat{u}, u') = q$, $(\hat{v}, v') = q$ and $t(e_2, v')t(f_2, w)t(e_2, u') : \hat{u} \mapsto \hat{v}$, from which the result follows. □

Lemma 2.4. *The group* $\Gamma_2 = \mathrm{O}^+(L)$.

Proof. We first calculate $\mathrm{O}(D(L))$. The group $D(L) \cong C_2 \oplus C_2 \oplus C_3$ and if $(a, b, c) \in D(L)$ then

$$q_L(a, b, c) = -\frac{a^2}{2} - \frac{3b^2}{2} - \frac{2c^2}{3} \bmod 2\mathbb{Z}. \tag{10}$$

The three elements of order 2 in $D(L)$ are of length $q_L(1, 0, 0) \equiv -1/2 \bmod 2\mathbb{Z}$, $q_L(0, 1, 0) \equiv -3/2 \bmod 2\mathbb{Z}$ and $q_L(1, 1, 0) \equiv 0 \bmod 2\mathbb{Z}$. Therefore, $\mathrm{O}(D(L))$ fixes the subgroup $C_2 \oplus C_2 \subset D(L)$ and acts as ± 1 on C_3. Therefore, $\{e, \sigma_{\underline{w}}\}$ represents $\mathrm{O}^+(L)/\widetilde{\mathrm{O}}^+(L)$ where $\sigma_{\underline{w}}$ is the reflection defined by $\underline{w} \in L$ generating the $\langle -6 \rangle$ factor of L. As $\sigma_{\underline{w}} \in \Gamma_2$, the result follows by Proposition 2.1. □

Proposition 2.5. *If $p > 3$ is prime then* $|\Gamma_2 : \Gamma_{2p^2}| \le 2(p^5 + p^2)$.

Proof. We follow the approach of [17, Lemma 3.2]. By definition, if $v \in U \subset L_2$ is of length $v^2 = 2$ then $\mathrm{O}(L_2) = \mathrm{O}^+(L_2) \rtimes \langle \sigma_v \rangle$. The non-degenerate hyperplane $\Pi := L_{2p^2}/pL_2 \subset Q_p$ is stabilised by σ_v and so

$$|\,\mathrm{O}(L_2) : \mathrm{Stab}_{\mathrm{O}(L_2)}(\Pi)| = |\,\mathrm{O}^+(L_2) : \mathrm{Stab}_{\mathrm{O}^+(L_2)}(\Pi)|.$$

By Lemma 2.3, hyperplanes in Q_p have the same orbits under $\mathrm{O}(L_2)$ and $\mathrm{O}(Q_p)$. Therefore

$$(11)\qquad |\,\mathrm{O}^+(L_2) : \mathrm{Stab}_{\mathrm{O}^+(L_2)}(\Pi)| = |\,\mathrm{O}(Q_p) : \mathrm{Stab}_{\mathrm{O}(Q_p)}(\Pi)| = |\,\mathrm{O}(Q_p) : \mathrm{O}(\Pi) \times C_2|,$$

where the last line follows from Witt's theorem. As any element of $\mathrm{Stab}_{\mathrm{O}^+(L_2)}(\Pi)$ extends to $\mathrm{O}^+(L_{2p^2})$ then

$$(12)\qquad \widetilde{\mathrm{O}}^+(L_{2p^2}) \subset \Gamma_{2p^2} \subset \mathrm{Stab}_{\mathrm{O}^+(L_2)}(\Pi) \subset \mathrm{O}^+(L_{2p^2})$$

and so

$$|\,\mathrm{O}^+(L_2) : \Gamma_{2p^2}| = |\,\mathrm{O}^+(L_2) : \mathrm{Stab}_{\mathrm{O}^+(L_2)}(\Pi)||\,\mathrm{Stab}_{\mathrm{O}^+(L_2)}(\Pi) : \Gamma_{2p^2}|.$$

By (12) and Lemma 2.2,

$$|\,\mathrm{Stab}_{\mathrm{O}^+(L_2)}(\Pi) : \Gamma_{2p^2}| \le |\,\mathrm{O}^+(L_{2p^2}) : \widetilde{\mathrm{O}}^+(L_{2p^2})| = 4.$$

By Proposition 2.1, $\Gamma_{2p^2} \subset \Gamma_2$ and by Lemma 2.4, $\mathrm{O}^+(L_2) = \Gamma_2$. Therefore,

$$\begin{aligned}|\Gamma_2 : \Gamma_{2p^2}| &= |\,\mathrm{O}^+(L_2) : \Gamma_{2p^2}| \\ &\le |\,\mathrm{O}^+(L_2) : \mathrm{Stab}_{\mathrm{O}^+(L_2)}(\Pi)||\,\mathrm{Stab}_{\mathrm{O}^+(L_2)}(\Pi) : \Gamma_{2p^2}| \\ &\le 4|\,\mathrm{O}^+(L_2) : \mathrm{Stab}_{\mathrm{O}^+(L_2)}(\Pi)|\end{aligned}$$

then by (11),

$$\le \frac{4|\,\mathrm{O}(Q_p)|}{|\,\mathrm{O}(\Pi) \times C_2|}$$

and by (6),

$$\le \frac{8p^6(p^3+1)(p^4-1)(p^2-1)}{4p^4(p^4-1)(p^2-1)} \le 2(p^5 + p^2),$$

and the result follows. □

3. The Baily–Borel compactification of $\mathcal{F}_{L_{2p^2}}(\Gamma_{2p^2})$

In this section, we study the boundary components of $\mathcal{F}_{L_{2p^2}}(\Gamma_{2p^2})^*$. We begin by counting boundary points in Lemma 3.2 before defining invariants for boundary curves in Proposition 3.3. We use these invariants to classify boundary curves up to isomorphism in Theorem 3.6 and provide bounds for their number in Corollary 3.7. We finish by describing incidence relations in Theorem 3.12 and 3.13. Throughout, we will closely follow the approach of [22], in which each of these questions is addressed for the moduli of K3 surfaces: in particular, §3.1 follows the approach of [22, §4] and §3.2, 3.3, 3.4 follow the approach of [22, §5].

Unless otherwise stated, $L := L_{2p^2}$ and we assume $b_L = ((-1/6) \oplus (-1/2p^2), C_6 \oplus C_{2p^2})$ for prime $p > 3$. Versions of some results in this section were first given in [6].

3.1. Boundary points. We classify boundary points in $\mathcal{F}_L(\Gamma)^*$ using the Eichler criterion. An element $x \in D(L)$ is said to be *isotropic* if $x^2 \equiv 0 \bmod 2\mathbb{Z}$.

Lemma 3.1. *If $D(L) \cong C_6 \oplus C_{2p^2}$ then the isotropic elements of $D(L)$ are given by*

$$\{(0, 2kp), (3, (2k+1)p) \mid k \in \mathbb{Z}\} \subset D(L).$$

Proof. An element $(x, y) \in D(L)$ is isotropic if and only if

$$p^2x^2 + 3y^2 \equiv 0 \bmod 12p^2. \tag{13}$$

As $(3, p) = 1$ then $p|y$ and we define y_1 by $y = py_1$. As $p \equiv \pm 1 \bmod 6$ then $x^2 + 3y_1^2 \equiv 0 \bmod 6$. By considering squares modulo 6, $x \equiv y_1 \bmod 2$ and either $x \equiv 0$ or $3 \bmod 6$. Therefore, as all elements of

$$\{(0, 2kp), (3, (2k+1)p) \mid k \in \mathbb{Z}\} \subset D(L)$$

satisfy (13), the result follows. □

Lemma 3.2. *Let $v \in L$ denote a primitive isotropic vector. Then there are 4 families of points in the boundary of $\mathcal{F}_L(\Gamma)^*$, given by*

1. *p_1 corresponding to $v^* \equiv (0, 0) \bmod L$;*
2. *p_2 corresponding to $v^* \equiv (3, p^2) \bmod L$;*
3. *$p_p(k)$ corresponding to $v^* \equiv (0, 2kp) \bmod L$ for $k = 1, \ldots, p-1$;*
4. *$p_{2p}(k)$ corresponding to $v^* \equiv (3, (2k+1)p) \bmod L$ for $k = 0, \ldots, (p-3)/2$.*

Proof. By Theorem 1.1, points in the boundary of $\mathcal{F}_L(\Gamma)^*$ are in bijection with Γ-orbits of primitive totally isotropic rank 1 sublattices of L. By Lemma 3.1, if $\pm v \in L$ is primitive and isotropic then $\pm v^* \in D(L)$ is given by $(0, 0)$ if v^* is of order 1; $(3, p^2)$ if v^* is of order 2; $(0, 2kp)$ for some $k = 1, \ldots, p-1$ if v^* is of order p; or $(3, (2k+1)p)$ for some $k = 0, \ldots, (p-3)/2$ if v^* is of order $2p$. By Proposition 2.1, $\widetilde{\mathrm{SO}}^+(L) \subset \Gamma$ and so, by Lemma 2.2 and the Eichler criterion, the Γ-orbits of primitive $\pm v \in L$ are uniquely determined by $\pm v^* \bmod L$ as above.

We show that each case can occur. Take a basis $\{v_i\}_{i=1}^6$ where $\{v_1, v_2\}$, $\{v_3, v_4\}$ are canonical bases for U and $v_5 := \underline{w}$, $v_6 := \underline{v}$.

1. By definition of U, $v = (1, 0, 0, 0, 0, 0) \in L$ is primitive, isotropic and $v^* \equiv (0, 0) \bmod L$.
2. If $v \in L$ is of the form $v = (2, 0, 2v_3, 2v_4, 1, 1)$ then $\operatorname{div}(v) = 2$ and v is primitive with $v^* \equiv (3, p^2) \bmod L$. As p is prime then $p^2 \equiv 1 \bmod 8$ and so $v^2 = 8v_3v_4 - 6 - 2p^2 = 0$ admits an integral solution in v_3, v_4.
3. If $v \in L$ is of the form $v = (p, 0, pv_3, pv_4, 0, k) \in L$ with $(k, p) = 1$ then v is primitive and $v^* \equiv (0, 2kp) \bmod L$. As $v^2 = p^2v_3v_4 - 2k^2p^2 = 0$ admits an integral solution in v_3, v_4 for each k, the result follows.
4. If $v \in L$ is of the form $v = (2p, 0, 2pv_3, 2pv_4, p, (2k+1)) \in L$ where $(2k+1, p) = 1$ then, as $(2k+1, p) = 1$ and $(2, p) = 1$, v is primitive and $\operatorname{div}(v) = 2p$. One checks that $v^* \equiv (3, (2k+1)p) \bmod L$. As p and $2k+1$ are odd then $(2k+1)^2p^2 \equiv 1 \bmod 8$

and $v^2 = 8p^2v_3v_4 - 6p^2 - 2(2k+1)^2p^2 = 0$ admits an integral solution in v_3, v_4 for each k. □

3.2. Invariants associated with boundary curves. We show that there exists a normal form for the Gram matrix of L with respect to a primitive totally isotropic sublattice $E \subset L$ of rank 2.

Proposition 3.3. *Let $E \subset L$ be a primitive totally isotropic sublattice of rank 2. Then there exists a $\mathbb{Z}$-basis $\{v_i\}_{i=1}^6$ of L such that $\{v_1, v_2\}$ is a $\mathbb{Z}$-basis for E and $\{v_i\}_{i=1}^4$ is a $\mathbb{Z}$-basis for $E^\perp$. The basis can be chosen so that the Gram matrix*

$$Q = ((v_i, v_j)) = \begin{pmatrix} 0 & 0 & A \\ 0 & B & C \\ {}^{\top}A & {}^{\top}C & D \end{pmatrix} \tag{14}$$

where B is the bilinear form on $E^\perp/E$,

$$A = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}$$

where $a = 1, 2, p$ or $2p$ and $d \in 2\mathbb{Z}$ is taken modulo $2a$. Furthermore,

1. *if $a = 1$ then $C = D = 0$;*
2. *if $a = 2$ then C can be taken modulo 2 and $d = 0$ or 2;*
3. *if $a = p$ then $C = 0$ and $B \cong \langle -2 \rangle \oplus \langle -6 \rangle$ or $B \cong -\left(\begin{smallmatrix} 4 & 2 \\ 2 & 4 \end{smallmatrix}\right)$;*
4. *if $a = 2p$ then $C = 0$ and $B \cong A_2(-1)$.*

Proof. As the lattices E and $E^\perp$ are primitive, there exists a $\mathbb{Z}$-basis of L with Gram matrix of the form (14). As $|\det(Q)| = 12p^2$ then $\det(A)$ is square-free and thus given by 1, 2, p or $2p$. By the existence of the Smith normal form [19], one can apply the change of basis $\operatorname{diag}(P, I, Q)$ for $P, Q \in \mathrm{GL}(2, \mathbb{Z})$ so that A is as in the statement of the lemma. All cases of A are realised: for example, one can take v_1 to be a primitive isotropic vector in U and v_2 to be one of the vectors $(1,0,0,0)$, $(2, 2p^2, p, 1)$, $(p, p, 0, 1)$ or $(2p, 2p, p, 1)$ in $U \oplus \langle -6 \rangle \oplus \langle -2p^2 \rangle$, of divisor 1, 2, p and $2p$, respectively. We now refine the basis further.

1. Suppose $a = 1$. From (14), $\operatorname{div}(v_1) = 1$ and, from the classification of unimodular lattices, $v_1 \in U$. Similarly, $\operatorname{div}(v_2) = 1$ and $v_2 \in U^\perp \subset L$. Therefore, by Proposition 1.15.1 of [20], there exists a sublattice $U \oplus U \oplus L' \subset L$ with v_1 and v_2 each contained in a copy of U. As $|\det(L')| = |\det(L)|$ then $L = 2U \oplus L'$ and we conclude $C = D = 0$.
2. Suppose $a = 2$. To reduce C modulo 2 we apply the change of basis

$$\begin{pmatrix} I & S & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} : Q \mapsto \begin{pmatrix} 0 & 0 & A \\ 0 & B & {}^{\top}AS + C \\ * & * & D \end{pmatrix}$$

for an appropriate choice of S. As L is even and

$$\{{}^{\top}WA + {}^{\top}AW \mid W \in M_2(\mathbb{Z})\} = \left\{ \begin{pmatrix} 2ax & y \\ y & 2z \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{Z} \right\}$$

we can assume that

$$D = 0 \quad \text{or} \quad \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

by applying the change of basis

$$\begin{pmatrix} I & 0 & W \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} : Q \mapsto \begin{pmatrix} 0 & 0 & A \\ 0 & B & C \\ * & * & {}^{\top}WA + {}^{\top}AW + D \end{pmatrix} \tag{15}$$

for an appropriate choice of W.

3. Suppose $a = p$. From (14), $|\det(B)| = 12$ and so, from Table 15.1 in [5],
$$B \cong \langle -2 \rangle \oplus \langle -6 \rangle \quad \text{or} \quad -\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}.$$
To put C in the correct form, we apply the change of basis
$$\begin{pmatrix} I & S & 0 \\ 0 & I & T \\ 0 & 0 & I \end{pmatrix} : Q \mapsto \begin{pmatrix} 0 & 0 & A \\ 0 & B & {}^{\top}SA + BT + C \\ * & * & * \end{pmatrix}. \tag{16}$$
If $C = (c_{ij})$, $S = (s_{ij})$, $T = (t_{ij})$ and $B = \langle -2 \rangle \oplus \langle -6 \rangle$, then
$${}^{\top}SA + BT + C = \begin{pmatrix} as_{21} - 2t_{11} + c_{11} & s_{11} - 2t_{12} + c_{12} \\ as_{22} - 6t_{21} + c_{21} & s_{12} - 6t_{22} + c_{22} \end{pmatrix}.$$
As $(a, 6) = 1$, there exists T such that $-2t_{11} + c_{11} \equiv 0 \bmod a$ and $-6t_{21} + c_{21} \equiv 0 \bmod a$. Therefore, there exists S such that ${}^{\top}SA + BT + C = 0$.
Similarly, if $B = -\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$ then
$${}^{\top}SA + BT + C = \begin{pmatrix} as_{21} - 4t_{11} - 2t_{21} + c_{11} & s_{11} - 4t_{12} - 2t_{22} + c_{12} \\ as_{22} - 2t_{11} - 4t_{21} + c_{21} & s_{12} - 2t_{12} - 4t_{22} + c_{22} \end{pmatrix}$$
and the same conclusion follows. In either case, we put D in the required form by applying an appropriate change of basis (15).
4. If $a = 2p$ then $|\det(B)| = 3$ and, from Table 15.1 in [5], $B \cong A_2(-1)$. One then proceeds as for $a = p$. □

Definition. If $E \subset L$ is a primitive totally isotropic sublattice of rank 2 and a is as in Proposition 3.3, we say that E and the associated boundary curve C_E are of type a.

3.3. Geometry of boundary curves. We now study the groups $G(E) = \operatorname{Stab}_\Gamma(E)/\operatorname{Fix}_\Gamma(E)$ in order to classify the curves C_E up to isomorphism. We assume throughout that $E \subset L$ is a primitive totally isotropic sublattice of rank 2 and type a.

Definition ([22]). If $g \in \operatorname{Stab}_{O(L)}(E)$ then, on the basis of Proposition 3.3,
$$g = \begin{pmatrix} U & V & W \\ 0 & X & Y \\ 0 & 0 & Z \end{pmatrix}. \tag{17}$$
We define the homomorphism $\pi_E : \operatorname{Stab}_{O(L)}(E) \to \mathrm{GL}(2, \mathbb{Z})$ by $\pi_E : g \mapsto U$.

For $n \in \mathbb{N}$, let $\Gamma(n) \subset \mathrm{SL}(2, \mathbb{Z})$ denote the principal congruence subgroup of level n and let

$$\Gamma_0(n) = \left\{ Z \in \mathrm{SL}(2,\mathbb{Z}) \mid Z \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \bmod n \right\}$$

and

$$\Gamma_1(n) = \left\{ Z \in \mathrm{SL}(2,\mathbb{Z}) \mid Z \equiv \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \bmod n \right\}.$$

Lemma 3.4. *If* $g \in \mathrm{Stab}_\Gamma(E)$ *then* $\pi_E(g) \in \mathrm{SL}(2,\mathbb{Z})$ *if* $a = 1$ *and* $\pi_E(g) \in \Gamma_1(a)$ *otherwise.*

Proof. Suppose $g \in \mathrm{Stab}_\Gamma(E)$ is as in (17). Then, by Lemma 5.7.1 of [3], $g \in \mathrm{O}^+(L)$ if and only if $U \in \mathrm{SL}(2,\mathbb{Z})$. As ${}^\top gQg = Q$ then ${}^\top UAZ = A$ and so, if

$$Z = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \quad \text{then} \quad U = \begin{pmatrix} r & -as \\ -a^{-1}t & u \end{pmatrix}.$$

Therefore, $U \in \Gamma_0(a)$ if $a \neq 1$ and $U \in \mathrm{SL}(2,\mathbb{Z})$ otherwise.

Let $\{v_i\}_{i=1}^6$ be the basis defined in Proposition 3.3. By Lemma 2.2, g acts trivially on $C_2 \oplus C_2 \subset D(L)$. If $a = 2$ then, as $\mathrm{div}(v_2) = 2$, $gv_2^* \equiv v_2^* \bmod L$, implying $U \in \Gamma_1(2)$. By definition of Γ, if $a = p$ or $2p$ then g acts trivially on $C_p \subset D(L)$. Therefore, by considering the action of g on v_2^*, we conclude $U \in \Gamma_1(a)$. □

Lemma 3.5. *The image*

$$\pi_E(\mathrm{Stab}_\Gamma(E)) = \begin{cases} \mathrm{SL}(2,\mathbb{Z}) & \text{if } a = 1 \\ \Gamma_1(a) & \text{if } a = 2, p \text{ or } 2p. \end{cases}$$

Proof. We construct a pre-image for π_E. Let

$$Q' = \begin{pmatrix} 0 & A \\ {}^\top A & D \end{pmatrix}$$

be the Gram matrix of $L' := \langle v_1, v_2, v_5, v_6 \rangle \subset L$ where $\{v_i\}_{i=1}^6$ is the $\mathbb{Z}$-basis of L defined in Proposition 3.3. Suppose $U \in \mathrm{SL}(2,\mathbb{Z})$ if $a = 1$ and $U \in \Gamma_1(a)$ otherwise. Assume that $Z \in \mathrm{SL}(2,\mathbb{Z})$ satisfies ${}^\top UAZ = A$. Proceeding much as in [22, p.72], we show that there exist elements of the form

$$g = \begin{pmatrix} U & UW \\ 0 & Z \end{pmatrix} \in \mathrm{O}^+(L')$$

extending to $\mathrm{Stab}_\Gamma(E)$. As

$${}^\top gQ'g = \begin{pmatrix} 0 & {}^\top UAZ \\ {}^\top Z {}^\top AU & {}^\top WA + {}^\top AW + {}^\top ZDZ \end{pmatrix}$$

and ${}^\top UAZ = A$, then W must satisfy

$$(18) \qquad {}^\top WA + {}^\top AW + {}^\top ZDZ = D.$$

If $W = (w_{ij})$ and

$$Z = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$$

then

$$ {}^{\mathsf{T}}WA + {}^{\mathsf{T}}AW + {}^{\mathsf{T}}ZDZ = \begin{pmatrix} 2aw_{21} + dr^2 & aw_{22} + w_{11} + drs \\ aw_{22} + w_{11} + drs & 2w_{12} + ds^2 \end{pmatrix}. \tag{19} $$

Equation (18) is always satisfied for some W:

1. if $a = 1$ or 2 and $D = 0$, set $W := 0$;
2. otherwise, d is even (as L is even) and, by Lemma 3.4, $r^2 \equiv 1 \bmod a$.

Therefore, by (19), there exists W satisfying (18) in both cases. We now show that $g \in \mathrm{O}(L')$ can be extended to Γ by allowing g to act trivially on $(L')^{\perp} \subset L$. We note that as $U \in \mathrm{SL}(2,\mathbb{Z})$ then, by Lemma 5.7.1 of [3], the extension of g automatically belongs to $\mathrm{O}^+(L \otimes \mathbb{R})$.

1. If $a = 1$ or 2 and $D = 0$ then $g \in \widetilde{\mathrm{SO}}^+(L') \subset \widetilde{\mathrm{O}}^+(L) \subset \Gamma$.
2. If $a = 2$ and $d = 2$ then $\mathrm{O}(D(L'))$ is trivial and so $g \in \widetilde{\mathrm{O}}^+(L') \subset \widetilde{\mathrm{O}}^+(L) \subset \Gamma$.
3. If $a = p$ or $2p$ then, by Proposition 3.3, there exists a splitting $L = L' \oplus B$. By construction, g acts trivially on the element $v_2^* \in D(L)$ generating the subgroup $C_p \subset D(L)$. Therefore, by Lemma 2.2, g acts trivially on $C_{p^2} \subset D(L)$ and fixes the subgroup $C_2 \oplus C_2 \subset D(L)$. Therefore, by Proposition 2.1, $g \in \Gamma$. □

Theorem 3.6. *If C_E is the boundary curve of $\mathcal{F}_L(\Gamma)^*$ corresponding to E, then*

$$ C_E \cong \begin{cases} \mathbb{H}^+ / \mathrm{PSL}(2,\mathbb{Z}) & \text{if } a = 1 \\ \mathbb{H}^+ / \Gamma_1(a) & \text{otherwise.} \end{cases} $$

Proof. Immediate from Theorem 1.1 and Lemma 3.5, as $\pi_E(\mathrm{Fix}_\Gamma(E)) \subset \langle \pm I \rangle$. □

3.4. Counting boundary curves. As a corollary to Proposition 3.3, we can bound the number of boundary curves of $\mathcal{F}_L(\Gamma)^*$. We assume $L = L_{2p^2}$ and $\Gamma = \Gamma_{2p^2}$ for prime $p > 3$.

Corollary 3.7. *If $h(D)$ is the class number of discriminant D, then the boundary of $\mathcal{F}_L(\Gamma)^*$ contains at most $4h(-48p^2)$ curves of type 1, $128h(-12p^2)$ curves of type 2, $8p$ curves of type p and $8p$ curves of type $2p$.*

Proof. By Theorem 1.1, it suffices to bound the number of Γ-equivalence classes of primitive totally isotropic sublattices of rank 2 in L. In each case, we first count the number of Gram matrices occurring in Proposition 3.3 for each a, to obtain bounds for equivalence in $\mathrm{O}(L)$. We note that there are at most $h(-48p^2/a^2)$ choices for B for a given a. By Lemma 2.2,

$$ |\mathrm{O}(L) : \Gamma| = |\mathrm{O}(L) : \mathrm{O}^+(L)||\mathrm{O}^+(L) : \Gamma| = 4, $$

from which we obtain a bound for equivalence in Γ. □

3.5. The boundary of $\mathcal{F}_{L_2}(\Gamma_2)^*$. To provide a specific example, we describe the boundary of $\mathcal{F}_{L_2}(\Gamma_2)^*$. Let $L = L_2$ and $\Gamma = \Gamma_2$.

Definition ([3]). If $E \subset L$ is a primitive totally isotropic sublattice, let $H_E := E^{\perp\perp}/E \subset D(L)$ where $E^{\perp\perp} \subset L^\vee$.

Lemma 3.8. *If $E \subset L$ is a primitive totally isotropic sublattice of rank 2, then $E^\perp/E \cong \langle -6 \rangle \oplus \langle -2 \rangle$ or $E^\perp/E \cong A_2(-1)$.*

Proof. The lattice $E^\perp/E$ is negative definite and, by Lemma 4.1 of [3], $D(E^\perp/E) \cong H_E^\perp/H_E$. If $(a,b) \in D(L) \cong C_6 \oplus C_2$ is isotropic then $a^2/6 + b^2/2 = 0 \bmod 2\mathbb{Z}$ and so $(a,b) = (0,0)$ or $(3,1)$. Therefore, $H_E = \langle(0,0)\rangle$ or $\langle(3,1)\rangle$. If $H_E = \langle(0,0)\rangle$ then $D(E^\perp/E)$ has discriminant form $((-1/6) \oplus (-1/2), C_6 \oplus C_2)$. By Table 15.1 of [5], the two negative definite even lattices of determinant 12 are

$$\langle -6\rangle \oplus \langle -2\rangle \quad \text{and} \quad \begin{pmatrix} -4 & -2 \\ -2 & -4 \end{pmatrix},$$

with only $\langle -6\rangle \oplus \langle -2\rangle$ having discriminant form $((-1/6) \oplus (-1/2), C_6 \oplus C_2)$. Therefore, $E^\perp/E \cong \langle -6\rangle \oplus \langle -2\rangle$. If $H_E = \langle(3,1)\rangle$ then $H_E^\perp = \langle(1,1)\rangle$ and $D(E^\perp/E)$ has discriminant form $((-2/3), C_3)$. Therefore, from Table 15.1 of [5], $E^\perp/E \cong A_2(-1)$. □

Lemma 3.9. *Assuming the notation of* Proposition 3.3*, if $E \subset L$ is a primitive totally isotropic sublattice of rank* 2 *then there exists a $\mathbb{Z}$-basis $\{v_i\}_{i=1}^6$ of L such that $\{v_1, v_2\}$ is a $\mathbb{Z}$-basis of E, $\{v_i\}_{i=1}^4$ is a $\mathbb{Z}$-basis of $E^\perp \subset L$ and the Gram matrix of $\{v_i\}_{i=1}^6$ is as in* (14)*. Furthermore, if H_E is trivial then $a = 1$, $B = \langle -6\rangle \oplus \langle -2\rangle$ and $C = D = 0$; otherwise, $a = 2$, $B = A_2(-1)$, $C = 0$ and $d = 2$.*

Proof. As in Proposition 3.3, there exists a basis with Gram matrix

$$Q = \begin{pmatrix} 0 & 0 & A \\ 0 & B & C \\ {}^\top A & {}^\top C & D \end{pmatrix}. \tag{20}$$

By Lemma 3.8, $B \cong \langle -6\rangle \oplus \langle -2\rangle$ if H_E is trivial and $B \cong A_2(-1)$ otherwise. The case of trivial H_E proceeds identically to the case of $a = 1$ in Proposition 3.3. If $H_E = C_2$ then, from (20) and the existence of the Smith normal form, we can assume that

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}.$$

If $S, T \in M_2(\mathbb{Z})$ then

$${}^\top SA + BT + C_1 = \begin{pmatrix} 2s_{21} - 2t_{11} - t_{21} + c_{11} & s_{11} - 2t_{12} - t_{22} + c_{12} \\ 2s_{22} - t_{11} - 2t_{21} + c_{21} & s_{12} - t_{12} - 2t_{22} + c_{22} \end{pmatrix},$$

and so, by applying a change of basis of the form (16) we can assume $C = 0$. Similarly, as

$$\{{}^\top WA + {}^\top AW \mid W \in M_2(\mathbb{Z})\} = \left\{ \begin{pmatrix} 4a & b \\ b & 2c \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{Z} \right\},$$

there exists a change of basis (15) reducing D to $\mathrm{diag}(d, 0)$ where $d = 0$ or 2. As $2U \subset L$, then L is unique in its genus and so uniquely determined by its signature and discriminant form [20, Cor. 1.13.3]. Therefore, by comparing the discriminant forms defined by (20) for $d = 0$ and $d = 2$, only the case $d = 2$ occurs. □

Lemma 3.10. *There are two Γ_2-orbits of primitive totally isotropic sublattices of rank 2 in L.*

Proof. By Lemma 3.9, there are two $\mathrm{O}(L)$-orbits of primitive totally isotropic sublattices of rank 2 in L, which are uniquely determined by the groups H_E. We take representatives E_1

and E_2 for each orbit, where $E_1 = \langle e_1, e_2\rangle$, $E_2 = \langle e_1, l\rangle$ and $l = 2e_2 + 2f_2 + \underline{v} + \underline{w}$. If $x = e_1 + f_1$ and $y = e_1 - f_1$ then $\mathrm{sn}_{\mathbb{R}}(\sigma_x) = -1$, $\mathrm{sn}_{\mathbb{R}}(\sigma_y) = 1$ and one checks that $\sigma_y\sigma_x E_1 = E_1$ and $\sigma_y\sigma_x E_2 = E_2$. As $\{e, \sigma_x\}$ are representatives for $\mathrm{O}(L)/\,\mathrm{O}^+(L)$ and $\sigma_y \in \widetilde{\mathrm{O}}^+(L)$ then there are two $\widetilde{\mathrm{O}}^+(L)$-orbits of primitive totally isotropic sublattices of rank 2 in L. The result then follows from Lemma 2.4. □

Lemma 3.11. *There are two Γ_2-orbits of primitive isotropic vectors in L.*

Proof. As in §3.1, we use the Eichler criterion. The $\widetilde{\mathrm{SO}}^+(L)$-orbits of primitive isotropic $v \in L$ are uniquely determined by $v^* \in D(L)$ and if v_i is isotropic in L then v_i^* is isotropic in $D(L)$. Let $D(L) \cong C_2 \oplus C_2 \oplus C_3$ with q_L as in (10). The only isotropic elements of $D(L)$ are $(0,0,0)$ and $(1,1,0)$. If $v_1 = e_1$ then $v_1^* = (0,0,0)$ and if $v_2 = 2e_2 + 2f_2 + \underline{v} + \underline{w}$ then $v_2^* = (1,1,0)$. By Proposition 2.1, $\widetilde{\mathrm{SO}}^+(L) \subset \Gamma_2$ and, as v_1^* and v_2^* can never be equivalent under Γ_2, the result follows. □

Fig. 1. The boundary of $\mathcal{F}_{L_2}(\Gamma_2)^*$

Theorem 3.12. *The boundary of $\mathcal{F}_{L_2}(\Gamma_2)^*$ consists of curves C_1 and C_2 of type* 1 *and* 2*, respectively and points P_1 and P_2. As illustrated in* Fig. 1*, the only intersections between boundary points and the closures $\overline{C}_1$, $\overline{C}_2$ of C_1, C_2 are $\overline{C}_1 \cap P_1$, $\overline{C}_2 \cap P_1$ and $\overline{C}_2 \cap P_2$.*

Proof. Immediate from Theorem 1.1, Lemma 3.10 and Lemma 3.11. □

3.6. The boundary of $\mathcal{F}_{L_{2p^2}}(\Gamma_{2p^2})^*$. We now describe the boundary of $\mathcal{F}_{L_{2p^2}}(\Gamma_{2p^2})^*$ in general. We let $L = L_{2p^2}$ and $\Gamma = \Gamma_{2p^2}$ for prime $p > 3$. (c.f. [23] and [22] for compactifications of moduli spaces of Enriques and K3 surfaces, respectively.)

Theorem 3.13. *The boundary of $\mathcal{F}_L(\Gamma)^*$ consists of curves C_a of type $a = 1$, 2, p and $2p$, whose isomorphism classes are given by* Theorem 3.6*; and boundary points p_i and $p_i(k)$, as in* Lemma 3.2*. Furthermore, the closure $\overline{C_a}$ of C_a contains p_i or $p_i(k)$ if and only if $i|a$, as illustrated in* Fig. 2*.*

Proof. By Proposition 3.3, the boundary curve C_a corresponds to a primitive totally isotropic sublattice $E = \langle v_1, v_2\rangle \subset L$ for primitive v_1 and v_2 of divisor 1 and a, respectively. Suppose $v \in L$ is a primitive isotropic vector corresponding to a boundary point p_i or $p_i(k)$ intersecting $\overline{C}_a$. We show that $i|a$. By Theorem 1.1, we can assume (without loss of generality) that $v = r_1v_1 + r_2v_2 \in E$ for $r_1, r_2 \in \mathbb{Z}$. By Proposition 3.3, the divisor $\mathrm{div}(v) = (r_1, ar_2)$ and so

$$v^* = \frac{r_1v_1}{(r_1, ar_2)} + \frac{r_2v_2}{(r_1, ar_2)} \equiv \frac{ar_2}{(r_1, ar_2)}v_2^* \bmod L. \tag{21}$$

As $\mathrm{div}(v)$ is equal to the order of v^* in $D(L)$ then, by Lemma 3.2, $\mathrm{div}(v) = i$ and so, by (21), $i|a$.

For the converse, suppose $i|a$. We show that there exist primitive isotropic $v \in E$ corre-

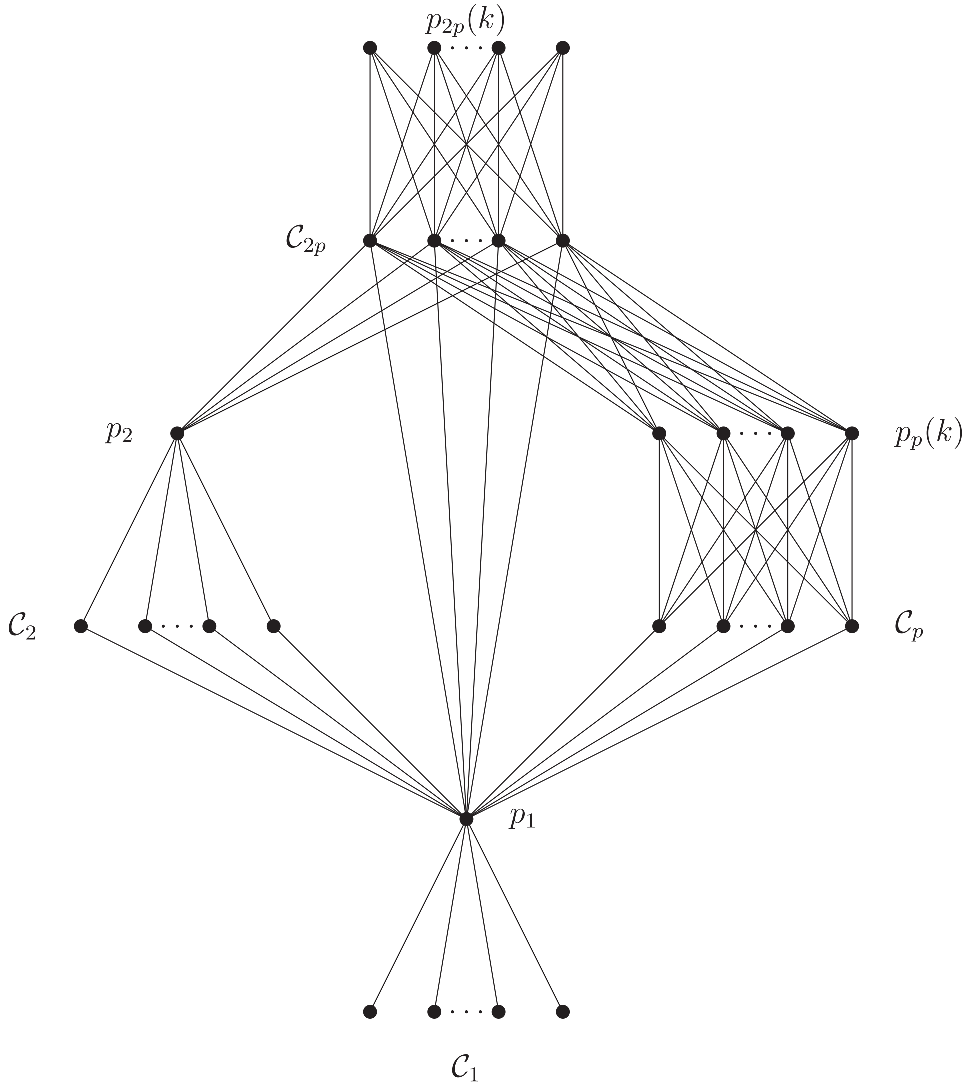

Fig. 2. The boundary of $\mathcal{F}_{L_{2p^2}}(\Gamma_{2p^2})^*$

sponding to each boundary point p_i, $p_i(k)$. Let $w \in L$ be any primitive isotropic vector corresponding to p_i or $p_i(k)$. The primitive isotropic vector v_2 is of divisor a and so v_2^* mod L is given by Lemma 3.2 as in the case of a boundary point p_a or $p_a(k)$. As $i|a$ then, by Lemma 3.2, $w^* \equiv \mu(a/i)v_2^*$ mod L for some μ satisfying $(\mu, i) = 1$. If $v := iv_1 + \mu v_2$ then $\text{div}(v) = (i, \mu a) = i$ and

$$v^* = v_1 + (\mu/i)v_2 \equiv \mu(a/i)v_2^* \bmod L.$$

As $(\mu, i) = 1$ then v is primitive and by Lemma 3.2, v and w define the same boundary point. The result then follows by Theorem 1.1. □

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