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ON EQUIVARIANT INDEX OF A GENERALIZED BOTT MANIFOLD

Yuki SUGIYAMA

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Abstract

In this paper, we consider the equivariant index of a generalized Bott manifold. We show the multiplicity function of the equivariant index is given by the density function of a generalized twisted cube. In addition, we give a Demazure-type character formula of this representation.

1. Introduction

A *Bott tower* of height n is a sequence:

$$M_n \xrightarrow{\pi_n} M_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} M_1 \xrightarrow{\pi_1} M_0 = \{\text{a point}\}$$

of complex manifolds $M_j = \mathbb{P}(\underline{\mathbb{C}} \oplus E_j)$, where $\underline{\mathbb{C}}$ is the trivial line bundle over M_{j-1}, E_j is a holomorphic line bundle over M_{j-1}, $\mathbb{P}(\cdot)$ denotes the projectivization, and $\pi_j : M_j \to M_{j-1}$ is the projection of the $\mathbb{C}P^1$-bundle. We call M_j a *j-stage Bott manifold*. The notion of a Bott tower was introduced by Grossberg and Karshon ([6]).

A *generalized Bott tower* is a generalization of a Bott tower. A generalized Bott tower of height m is a sequence:

$$B_m \xrightarrow{\pi_m} B_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\},$$

of complex manifolds $B_j = \mathbb{P}(\underline{\mathbb{C}} \oplus E_j^{(1)} \oplus \cdots \oplus E_j^{(n_j)})$, where $\underline{\mathbb{C}}$ is the trivial line bundle over B_{j-1}, $E_j^{(k)}$ is a holomorphic line bundle over B_{j-1} for $k = 1, \ldots, n_j$. We call B_j a *j-stage generalized Bott manifold*. A generalized Bott tower has been studied from various points of view (see, e.g., [2, 3, 8]). Generalized Bott manifolds are a certain class of toric manifolds, so it is interesting to investigate the specific properties of generalized Bott towers.

In [6], Grossberg and Karshon showed the multiplicity function of the *equivariant index* (see §2.4) for a holomorphic line bundle over a Bott manifold is given by the density function of a *twisted cube*, which is determined by the structure of the Bott manifold and the line bundle over it. From this, they derived a Demazure-type character formula.

The purpose of this paper is to generalize the results in [6] to generalized Bott manifolds. We generalize the twisted cube, and we call it the *generalized twisted cube*. It is a special case of twisted polytope introduced by Karshon and Tolman [9] for the presymplectic toric manifold, and it is a special case of multi-polytope introduced by Hattori and Masuda [7] for the torus manifold. We show the multiplicity function of the equivariant index for a holomorphic line bundle over the generalized Bott manifold is given by the density function

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of the associated generalized twisted cube. From this, we derive a Demazure-type character formula. In order to state the main results, we give some notation. Let $\mathbf{L}$ be a holomorphic line bundle over a generalized Bott manifold B_m, which is constructed from integers $\{\ell_i\}$ and $\{c_{i,j}^{(k)}\}$ (see §2.1). Let $N = \sum_{j=1}^{m} n_j$, and let $T^N = S^1 \times \cdots \times S^1$. We consider the action of T^N on B_m as follows:

$$(\mathbf{t}_1, \dots, \mathbf{t}_m) \cdot [\mathbf{z}_1, \dots, \mathbf{z}_m] = [\mathbf{t}_1\mathbf{z}_1, \dots, \mathbf{t}_m\mathbf{z}_m],$$

where $\mathbf{t}_i = (t_{i,1}, \dots, t_{i,n_i}), \mathbf{z}_i = (z_{i,0}, \dots, z_{i,n_i}), \mathbf{t}_i\mathbf{z}_i = (z_{i,0}, t_{i,1}z_{i,1}, \dots, t_{i,n_i}z_{i,n_i})$ for $i = 1, \dots, m$. Also we consider the action of $T = T^N \times S^1$ on $\mathbf{L}$ as follows:

$$(\mathbf{t}_1, \dots, \mathbf{t}_m, t_{m+1}) \cdot [\mathbf{z}_1, \dots, \mathbf{z}_m, v] = [\mathbf{t}_1\mathbf{z}_1, \dots, \mathbf{t}_m\mathbf{z}_m, t_{m+1}v]. \tag{1.1}$$

We define the generalized twisted cube as follows. It is defined to be the set of $x = (x_{1,1}, \dots, x_{m,n_m}) \in \mathbb{R}^N$ which satisfies

$$A_i(x) \le \sum_{k=1}^{n_i} x_{i,k} \le 0, \;\; x_{i,k} \le 0 \;\; (1 \le k \le n_i)$$
$$\text{or } 0 < \sum_{k=1}^{n_i} x_{i,k} < A_i(x), \;\; x_{i,k} > 0 \;\; (1 \le k \le n_i),$$

for $1 \le i \le m$, where

$$A_i(x) = \begin{cases} -\ell_m & (i = m) \\ -(\ell_i + \sum_{j=i+1}^{m} \sum_{k=1}^{n_j} c_{i,j}^{(k)} x_{j,k}) & (1 \le i \le m-1). \end{cases}$$

We denote the generalized twisted cube by C. We also define $\operatorname{sgn}(x_{i,k}) = 1$ for $x_{i,k} > 0$ and $\operatorname{sgn}(x_{i,k}) = -1$ for $x_{i,k} \le 0$. The *density function* of the generalized twisted cube is defined to be $\rho(x) = (-1)^N \prod_{1 \le i \le m, 1 \le k \le n_i} \operatorname{sgn}(x_{i,k})$ when $x \in C$ and 0 elsewhere.

Let $\mathfrak{t}$ be the Lie algebra of T and let $\mathfrak{t}^*$ be its dual space. Let $\ell^* \subset i\mathfrak{t}^*$ be the integral weight lattice and let $\text{mult} : \ell^* \to \mathbb{Z}$ be the multiplicity function of the equivariant index. The first main result of this paper is the following:

Theorem 1.1. *Fix integers $\{c_{i,j}^{(k)}\}$ and $\{\ell_j\}$. Let $\mathbf{L} \to B_m$ be the corresponding line bundle over a generalized Bott manifold. Let $\rho : \mathbb{R}^N \to \{-1, 0, 1\}$ be the density function of the generalized twisted cube C which is determined by these integers. Consider the torus action of $T = T^N \times S^1$ as in* (1.1). *Then the multiplicity function for $\ell^* \cong \mathbb{Z}^N \times \mathbb{Z}$ is given by*

$$\text{mult}(x, k) = \begin{cases} \rho(x) & (k = 1) \\ 0 & (k \neq 1). \end{cases}$$

Karshon and Tolman found a toric manifold for which the multiplicities of the equivariant index are $0, -1$, or -2 ([9, Example 6.7]). A generalized Bott manifold is different from this case by Theorem 1.1.

Next, we give our character formula. Let $\{e_{1,1}, \dots, e_{m,n_m}, e_{m+1}\}$ be the standard basis in $\mathbb{R}^{N+1}$, $x_i = (x_{i,1}, \dots, x_{i,n_i})$, and $e_i = (e_{i,1}, \dots, e_{i,n_i})$. Let $\Delta_{n,r}^{-} = \left\{z = (z_1, \dots, z_n) \in \mathbb{Z}_{\le 0}^n \;\middle|\; z_1 + \cdots + z_n = -r\right\}$, and let $\Delta_{n,r}^{+} = \left\{z = (z_1, \dots, z_n) \in \mathbb{Z}_{>0}^n \;\middle|\; z_1 + \cdots + z_n = r-1\right\}$. Let $\langle x_i, e_i \rangle = x_{i,1}e_{i,1} + \cdots + x_{i,n_i}e_{i,n_i}$. For every integral weight $\mu \in \ell^*$ we have a homomorphism $\lambda^\mu : T \to S^1$. We denote the integral combinations of these λ^μ's by $\mathbb{Z}[T]$. Then the operators

$D_i : \mathbb{Z}[T] \to \mathbb{Z}[T]$ are defined using $c_{i,j}^{(k)}$ and ℓ_j in the following way:

$$D_i(\lambda^\mu) = \begin{cases} \sum_{0 \le r \le k_i} \sum_{x_i \in \Delta^-_{n_i,r}} \lambda^{\mu + \langle x_i, e_i \rangle} & \text{if } k_i \ge 0 \\ 0 & \text{if } -n_i \le k_i \le -1 \\ \sum_{n_i+1 \le r \le -k_i} \sum_{x_i \in \Delta^+_{n_i,r}} (-1)^{n_i} \lambda^{\mu + \langle x_i, e_i \rangle} & \text{if } k_i \le -n_i - 1, \end{cases}$$

where the functions k_i are defined as follows: if $\mu = e_{m+1} + \sum_{j=i+1}^m \sum_{k=1}^{n_j} x_{j,k} e_{j,k}$, then $k_i(\mu) = \ell_i + \sum_{j=i+1}^m \sum_{k=1}^{n_j} c_{i,j}^{(k)} x_{j,k}$. From Theorem 1.1, we obtain the following theorem:

Theorem 1.2. *Consider the action of the torus T on $\mathbf{L} \to B_m$ as in* (1.1). *Denote the $(N+1)$-th component of the standard basis in $\mathbb{R}^{N+1}$ by e_{m+1}. Then the character is given by the following element of $\mathbb{Z}[T]$:*

$$\chi = D_1 \cdots D_m(\lambda^{e_{m+1}}).$$

This is a Demazure-type character formula. On the other hand, the character is also given by the localization formula with respect to the action of T ([7, Corollary 7.4]). We compare our formula with the localization formula (see Remark 3.8).

This paper is organized as follows. In Section 2, we recall the generalized Bott towers and the equivariant index, and we give the definition of generalized twisted cubes. In Section 3, we prove the main theorems.

2. Preliminaries

In this section, we set up the tools to prove the main theorems.

2.1. Generalized Bott manifolds.

Definition 2.1 ([2]). A *generalized Bott tower* of height m is a sequence:

$$B_m \xrightarrow{\pi_m} B_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\},$$

of manifolds $B_j = \mathbb{P}(\underline{\mathbb{C}} \oplus E_j^{(1)} \oplus \cdots \oplus E_j^{(n_j)})$, where $\underline{\mathbb{C}}$ is the trivial line bundle over B_{j-1}, $E_j^{(k)}$ is a holomorphic line bundle over B_{j-1} for $k = 1, \dots n_j$, and $\mathbb{P}(\cdot)$ denotes the projectivization. We call B_j a j-stage generalized Bott manifold.

The construction of the generalized Bott tower is as follows. A 1-step generalized Bott tower can be written as $B_1 = \mathbb{C}P^{n_1} = (\mathbb{C}^{n_1+1})^\times / \mathbb{C}^\times$, where $\mathbb{C}^\times$ acts diagonally. We construct a line bundle over B_1 by $E_2^{(k)} = (\mathbb{C}^{n_1+1})^\times \times_{\mathbb{C}^\times} \mathbb{C}$ for $k = 1, \dots, n_2$, where $\mathbb{C}^\times$ acts on $\mathbb{C}$ by $a : v \mapsto a^{-c_k} v$ for some integer c_k. In $E_2^{(k)}$ we have $[z_{1,0}, \dots, z_{1,n_1}, v] = [z_{1,0}a, \dots, z_{1,n_1}a, a^{c_k} v]$ for all $a \in \mathbb{C}^\times$. A 2-step generalized Bott tower $B_2 = \mathbb{P}(\underline{\mathbb{C}} \oplus E_2^{(1)} \oplus \cdots \oplus E_2^{(n_2)})$ can be written as $B_2 = ((\mathbb{C}^{n_1+1})^\times \times (\mathbb{C}^{n_2+1})^\times)/G$, where the right action of $G = (\mathbb{C}^\times)^2$ is given by

$$(\mathbf{z}_1, \mathbf{z}_2) \cdot (a_1, a_2) = (z_{1,0}a_1, z_{1,1}a_1, \dots, z_{1,n_1}a_1, z_{2,0}a_2, a_1^{c_1} z_{2,1}a_2, \dots, a_1^{c_{n_2}} z_{2,n_2}a_2),$$

where $\mathbf{z}_j = (z_{j,0}, z_{j,1}, \dots, z_{j,n_j})$ for $j = 1, 2$.

We can construct higher generalized Bott tower in a similar way. In this way we get an m-step generalized Bott manifold $B_m = \mathbb{P}(\underline{\mathbb{C}} \oplus E_m^{(1)} \oplus \cdots \oplus E_m^{(n_m)})$ from any collection of integers $\{c_{i,j}^{(k)}\}$:

$$B_m = ((\mathbb{C}^{n_1+1})^\times \times \cdots \times (\mathbb{C}^{n_m+1})^\times)/G,$$

where the right action of $G = (\mathbb{C}^\times)^m$ is given by

$$(\mathbf{z}_1, \ldots, \mathbf{z}_m) \cdot \mathbf{a} = (\mathbf{z}_1', \mathbf{z}_2', \ldots, \mathbf{z}_m'),$$

where $\mathbf{z}_i = (z_{i,0}, \ldots, z_{i,n_i})$ for $i = 1, \ldots, m$, $\mathbf{a} = (a_1, \ldots, a_m) \in (\mathbb{C}^\times)^m$, $\mathbf{z}_1' = (z_{1,0}a_1, z_{1,1}a_1, \ldots, z_{1,n_1}a_1)$ and $\mathbf{z}_j' = (z_{j,0}a_j, a_1^{c_{1,j}^{(1)}} \cdots a_{j-1}^{c_{j-1,j}^{(1)}} z_{j,1}a_j, \ldots, a_1^{c_{1,j}^{(n_j)}} \cdots a_{j-1}^{c_{j-1,j}^{(n_j)}} z_{j,n_j}a_j)$ for $j = 2, \ldots, m$. We can construct a line bundle over B_m from the integers $(\ell_1, \ldots, \ell_m)$ by

$$\mathbf{L} = ((\mathbb{C}^{n_1+1})^\times \times \cdots \times (\mathbb{C}^{n_m+1})^\times) \times_G \mathbb{C},$$

where $G = (\mathbb{C}^\times)^m$ acts by

$$((\mathbf{z}_1, \ldots, \mathbf{z}_m), v) \cdot \mathbf{a} = (\mathbf{z}_1', \mathbf{z}_2', \ldots, \mathbf{z}_m', a_1^{\ell_1} \cdots a_m^{\ell_m} v). \tag{2.1}$$

2.2. Torus action on generalized Bott towers. Let $N = \sum_{j=1}^m n_j$ and let $T^N = S^1 \times \cdots \times S^1$. We consider the action of T^N on B_m as follows:

$$(\mathbf{t}_1, \ldots, \mathbf{t}_m) \cdot [\mathbf{z}_1, \ldots, \mathbf{z}_m] = [\mathbf{t}_1 \cdot \mathbf{z}_1, \ldots, \mathbf{t}_m \cdot \mathbf{z}_m],$$

where $\mathbf{t}_i = (t_{i,1}, \ldots, t_{i,n_i})$ and $\mathbf{t}_i \cdot \mathbf{z}_i = (z_{i,0}, t_{i,1}z_{i,1}, \ldots, t_{i,n_i}z_{i,n_i})$ for $i = 1, \ldots, m$. Also we consider the action of $T = T^N \times S^1$ on $\mathbf{L}$ as follows:

$$(\mathbf{t}_1, \ldots, \mathbf{t}_m, t_{m+1}) \cdot [\mathbf{z}_1, \ldots, \mathbf{z}_m, v] = [\mathbf{t}_1 \cdot \mathbf{z}_1, \ldots, \mathbf{t}_m \cdot \mathbf{z}_m, t_{m+1}v]. \tag{2.2}$$

2.3. Generalized twisted cubes.

DEFINITION 2.2. A *generalized twisted cube* C is defined to be the set of $x = (x_{1,1}, \ldots, x_{m,n_m}) \in \mathbb{R}^N$ which satisfies

$$\begin{aligned} &A_i(x) \le \sum_{k=1}^{n_i} x_{i,k} \le 0, \ \ x_{i,k} \le 0 \ \ (1 \le k \le n_i) \\ &\text{or } 0 < \sum_{k=1}^{n_i} x_{i,k} < A_i(x), \ \ x_{i,k} > 0 \ \ (1 \le k \le n_i), \end{aligned} \tag{2.3}$$

for all $1 \le i \le m$, where

$$A_i(x) = \begin{cases} -\ell_m & (i = m) \\ -(\ell_i + \sum_{j=i+1}^m \sum_{k=1}^{n_j} c_{i,j}^{(k)} x_{j,k}) & (1 \le i \le m-1). \end{cases}$$

REMARK 2.3. (i) The generalized twisted cube is a special case of multi-polytope defined in [7]. In particular, it is a special case of twisted polytope defined in [9].

(ii) When $n_i = 1$ for all $1 \le i \le m$, the generalized twisted cube is the twisted cube given in [6, (2.21)].

DEFINITION 2.4. We define $\mathrm{sgn}(x_{i,k}) = 1$ for $x_{i,k} > 0$ and $\mathrm{sgn}(x_{i,k}) = -1$ for $x_{i,k} \le 0$. The *density function* of the generalized twisted cube is then defined to be $\rho(x) = (-1)^N \prod_{1 \le i \le m, 1 \le k \le n_i} \mathrm{sgn}(x_{i,k})$ when $x \in C$ and 0 elsewhere.

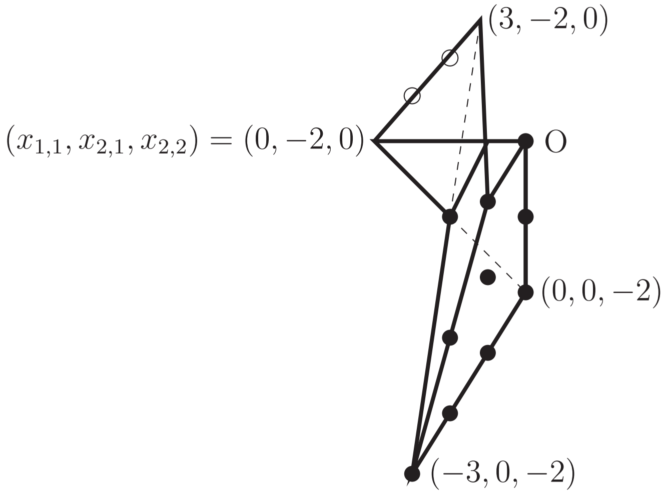

Fig. 1.

EXAMPLE 2.5. Suppose that $m = 2, n_1 = 1, n_2 = 2, \ell_1 = 1$, and $\ell_2 = 2$. We set $c^{(1)}_{1,2} = 2$ and $c^{(2)}_{1,2} = -1$. Then the generalized twisted cube is the set of $x = (x_{1,1}, x_{2,1}, x_{2,2})$ which satisfies

- $-2 \leq x_{2,1} + x_{2,2} \leq 0,\ \ x_{2,1}, x_{2,2} \leq 0,$
- $-1 - 2x_{2,1} + x_{2,2} \leq x_{1,1} \leq 0$ or $0 < x_{1,1} < -1 - 2x_{2,1} + x_{2,2}$.

In Figure 1, the black dots represent the lattice points of the sign $+1$ and the white dots represent the sign -1.

EXAMPLE 2.6. Suppose that $m = 2, n_1 = 2, n_2 = 1, \ell_1 = 2$, and $\ell_2 = -6$. We set $c^{(1)}_{1,2} = -1$. Then the generalized twisted cube is the set of $x = (x_{1,1}, x_{1,2}, x_{2,1})$ which satisfies

- $0 < x_{2,1} < 6,$
- $-2 + x_{2,1} \leq x_{1,1} + x_{1,2} \leq 0,\ \ x_{1,1}, x_{1,2} \leq 0$ or $0 < x_{1,1} + x_{1,2} < -2 + x_{2,1},\ \ x_{1,1}, x_{1,2} > 0.$

In Figure 2, the white dots represent the sign -1.

EXAMPLE 2.7. Suppose that $m = 2, n_1 = n_2 = 2, \ell_1 = 1$, and $\ell_2 = 2$. We set $c^{(1)}_{1,2} = 2$ and $c^{(2)}_{1,2} = -1$. Then the generalized twisted cube is the set of $x = (x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2})$ which satisfies

- $-2 \leq x_{2,1} + x_{2,2} \leq 0,\ \ x_{2,1}, x_{2,2} \leq 0,$
- $-1 - 2x_{2,1} + x_{2,2} \leq x_{1,1} + x_{1,2} \leq 0,\ \ x_{1,1}, x_{1,2} \leq 0$
  or $0 < x_{1,1} + x_{1,2} < -1 - 2x_{2,1} + x_{2,2},\ \ x_{1,1}, x_{1,2} > 0.$

The lattice points in the generalized twisted cube represent the sign -1.

2.4. Equivariant index. Let $\mathbf{L}$ be a holomorphic line bundle over a generalized Bott manifold B_m with the action of the torus T as in (2.2). Let $\mathcal{O}_{\mathbf{L}}$ be the sheaf of holomorphic sections. The *equivariant index* of a generalized Bott manifold is the formal sum of representation of T:

$$\mathrm{index}(B_m, \mathcal{O}_{\mathbf{L}}) = \sum (-1)^i H^i(B_m, \mathcal{O}_{\mathbf{L}}).$$

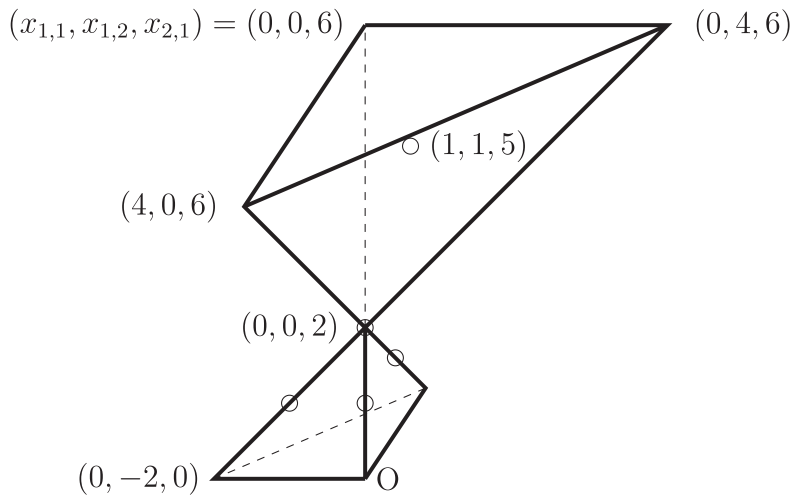

Fig. 2.

The *character* of the equivariant index is the function $\chi : T \to \mathbb{C}$ which is given by $\chi = \sum (-1)^i \chi^i$ where $\chi^i(a) = \text{trace}\{a : H^i(B_m, \mathcal{O}_{\mathbf{L}}) \to H^i(B_m, \mathcal{O}_{\mathbf{L}})\}$ for $a \in T$. Let $\mathfrak{t}$ be the Lie algebra of T and let $\mathfrak{t}^*$ be its dual space. Every μ in the integral weight lattice $\ell^* \subset i\mathfrak{t}^*$ defines a homomorphism $\lambda^\mu : T \to S^1$. We can write $\chi = \sum_{\mu \in \ell^*} m_\mu \lambda^\mu$. The coefficients are given by a function mult $: \ell^* \to \mathbb{Z}$, sending $\mu \mapsto m_\mu$, called the *multiplicity function* for the equivariant index.

3. Main theorems

3.1. Multiplicity function of the equivariant index. We will show that the multiplicity function of the equivariant index of a generalized Bott manifold is given by the density function of a generalized twisted cube C. In particular, all the weights occur with a multiplicity $-1, 0$, or 1.

Theorem 3.1. *Fix integers $\{c_{i,j}^{(k)}\}$ and $\{\ell_j\}$. Let $\mathbf{L} \to B_m$ be the corresponding line bundle over a generalized Bott manifold. Let $\rho : \mathbb{R}^N \to \{-1, 0, 1\}$ be the density function of the generalized twisted cube C which is determined by these integers as in* (2.3). *Consider the torus action of $T = T^N \times S^1$ as in* (2.2). *Then the multiplicity function for $\ell^* \cong \mathbb{Z}^N \times \mathbb{Z}$ is given by*

$$\text{mult}(x, k) = \begin{cases} \rho(x) & (k = 1) \\ 0 & (k \neq 1). \end{cases}$$

Proof. We compute $H^*(B_m, \mathcal{O}_{\mathbf{L}})$. Take the covering $\tilde{\mathcal{U}} = \{U_{r_1} \times \cdots \times U_{r_m}\}$ of $(\mathbb{C}^{n_1+1})^\times \times \cdots \times (\mathbb{C}^{n_m+1})^\times$ for $r_1, \ldots, r_m \in \{0, 1, \ldots, n_\ell\}$ $(\ell = 1, \ldots, m)$, where $U_{r_j} = \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{r_j} \times \mathbb{C}^\times \times \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{n_\ell - r_j}$. This descends to the covering $\mathcal{U}$ of B_m; every intersection of sets in $\mathcal{U}$ is isomorphic to a product of $\mathbb{C}$'s and $\mathbb{C}^\times$'s. The coverings $\tilde{\mathcal{U}}$ and $\mathcal{U}$ are the Leray coverings ([5]).

Let $\mathcal{O}$ be the sheaf of holomorphic functions, and let $G = (\mathbb{C}^\times)^m$. Since holomorphic sections of $\mathcal{O}_{\mathbf{L}}$ are given by holomorphic sections of $\mathcal{O}$ which are G-invariant with respect to the action (2.1) ([9]), $H^*(\mathcal{U}, \mathcal{O}_{\mathbf{L}})$ is isomorphic to the G-invariant part of $H^*(\tilde{\mathcal{U}}, \mathcal{O})$. By the Leray theorem, $H^*(B_m, \mathcal{O}_{\mathbf{L}})$ is isomorphic to the G-invariant part of $H^*((\mathbb{C}^{n_1+1})^\times \times \cdots \times (\mathbb{C}^{n_m+1})^\times, \mathcal{O})$.

In order to compute $H^*((\mathbb{C}^{n_1+1})^\times \times \cdots \times (\mathbb{C}^{n_m+1})^\times, \mathcal{O})$, we compute $H^*((\mathbb{C}^{n+1})^\times, \mathcal{O})$. Let $\mathcal{U}' = \{U_0, U_1, \dots, U_n\}$ be the covering of $(\mathbb{C}^{n+1})^\times$, let $j_0, j_1, \dots, j_k \in \{0, 1, \dots, n\}$ for $k = 0, 1, \dots, n$ and let $U_{j_0 j_1 \cdots j_k} = U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_k}$. Let $I = (i_0, i_1, \dots, i_n) \in \mathbb{Z}^{n+1}$. The holomorphic functions on $U_{j_0 j_1 \cdots j_k}$ are given by

$$\Gamma_{\mathrm{hol}}(U_{j_0 j_1 \cdots j_k}) = \left\{ \sum_{I \in \mathbb{Z}^{n+1}, i_\ell \geq 0 (\ell \neq j_0, j_1, \dots, j_k)} a_I z_0^{i_0} z_1^{i_1} \cdots z_n^{i_n} \right\}.$$

Consider the Čech cochain complex

$$0 \to \check{C}^0(\mathcal{U}', \mathcal{O}) \xrightarrow{\delta^0} \check{C}^1(\mathcal{U}', \mathcal{O}) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{n-1}} \check{C}^n(\mathcal{U}', \mathcal{O}) \xrightarrow{\delta^n} 0,$$

where $\check{C}^i(\mathcal{U}', \mathcal{O}) = \oplus \Gamma_{\mathrm{hol}}(U_{j_0 j_1 \cdots j_i})$ $(i = 0, \dots, n)$. The map $\delta^p : \check{C}^p(\mathcal{U}', \mathcal{O}) \to \check{C}^{p+1}(\mathcal{U}', \mathcal{O})$ is given by $\{f_{j_0 j_1 \cdots j_p}\} \mapsto \{g_{j_0 j_1 \cdots j_{p+1}}\}, g_{j_0 j_1 \cdots j_{p+1}} = \sum (-1)^k f_{j_0 j_1 \cdots \hat{j}_k \cdots j_{p+1}}$. Recall that $H^0((\mathbb{C}^{n+1})^\times, \mathcal{O}) = \operatorname{Ker} \delta^0$, and $H^n((\mathbb{C}^{n+1})^\times, \mathcal{O}) = \operatorname{Coker} \delta^{n-1}$. The torus $T^{n+1} = (S^1)^{n+1}$ acts on the holomorphic functions by $((t_0, \dots, t_n) \cdot f)(z_0, \dots, z_n) = f(t_0^{-1} z_0, \dots, t_n^{-1} z_n)$. This action descends to the cohomology. The corresponding weight spaces for the weight $I \in \mathbb{Z}^{n+1}$ are

$$H^0((\mathbb{C}^{n+1})^\times, \mathcal{O})_I = \begin{cases} \operatorname{span}(z_0^{-i_0} \cdots z_n^{-i_n}) & (I \in \mathbb{Z}^{n+1}_{\leq 0}) \\ 0 & \text{otherwise} \end{cases}$$

$$H^n((\mathbb{C}^{n+1})^\times, \mathcal{O})_I = \begin{cases} \operatorname{span}(z_0^{-i_0} \cdots z_n^{-i_n}) & (I \in \mathbb{Z}^{n+1}_{> 0}) \\ 0 & \text{otherwise.} \end{cases}$$

We now prove $H^q((\mathbb{C}^{n+1})^\times, \mathcal{O}) = 0$ for $1 \leq q \leq n-1$. Let Δ be the fan of $(\mathbb{C}^{n+1})^\times$, and let $|\Delta| = \cup_{\sigma \in \Delta} \sigma$ be the support of Δ. Let

$$Z(I) := \{v \in |\Delta|\, ;\, \langle I, v \rangle \leq \varphi(v)\},$$

where φ is the support function. From [4],

$$H^q((\mathbb{C}^{n+1})^\times, \mathcal{O})_I = H^q(|\Delta|, |\Delta| \setminus Z(I)\, ; \mathbb{C}).$$

Since $\mathcal{O}$ is the sheaf of holomorphic function, $\varphi(v) = 0$ for all $v \in |\Delta|$. In the case that $i_j \leq 0$ for all j, since $|\Delta|$ is contractible,

$$H^q((\mathbb{C}^{n+1})^\times, \mathcal{O})_I = 0 \ (q \geq 1).$$

In the case that $i_j > 0$ for all j, $Z(I) = \{0\}$. Since $|\Delta| \setminus \{0\}$ is homotopic to S^{n-1},

$$H^q((\mathbb{C}^{n+1})^\times, \mathcal{O})_I = 0 \ (q \neq n).$$

In other case, since $|\Delta| \setminus Z(I)$ is path-connected and contractible,

$$H^q((\mathbb{C}^{n+1})^\times, \mathcal{O})_I = 0$$

for all q.

We now compute $H^*((\mathbb{C}^{n_1+1})^\times \times \cdots \times (\mathbb{C}^{n_m+1})^\times, \mathcal{O})$. Consider the natural action of $T^{N+m} = (S^1)^{N+m}$ on the holomorphic function. The weights are multi-indices $I' \in \mathbb{Z}^{N+m}$; we write $I' = (\mathbf{i}'_1, \ldots, \mathbf{i}'_m)$, where $\mathbf{i}'_j = (i_{j,0}, i_{j,1}, \ldots, i_{j,n_j})$ for $j = 1, \ldots, m$. From the cohomology of $(\mathbb{C}^{n+1})^\times$ that we have computed and from the Künneth formula ([1]), it follows that

$$H^q((\mathbb{C}^{n_1+1})^\times \times \cdots \times (\mathbb{C}^{n_m+1})^\times, \mathcal{O})_{I'} = \begin{cases} \mathrm{span}(z_{1,0}^{-i_{1,0}} z_{1,1}^{-i_{1,1}} \cdots z_{m,n_m}^{-i_{m,n_m}}) \\ 0. \end{cases}$$

The former occurs if for all ℓ we have $\mathrm{sgn}(i_{\ell,0}) = \mathrm{sgn}(i_{\ell,1}) = \cdots = \mathrm{sgn}(i_{\ell,n_\ell}) =: \varepsilon_\ell$, here $q = \sum_{\{\ell \mid \varepsilon_\ell = 1, 1 \le \ell \le m\}} n_\ell$, and $q = 0$ when $\varepsilon_\ell = -1$ for all ℓ. In particular, $(-1)^q = (-1)^N \prod_{1 \le \ell \le m, 1 \le p \le n_\ell} \mathrm{sgn}(i_{\ell,p})$.

The action (2.1) induces an action on functions given by

$$\begin{aligned} &(a_k f)(z_{1,0}, \ldots, z_{m,n_m}) \\ &= a_k^{\ell_k} f(z_{1,0}, \ldots, z_{k-1,n_{k-1}}, z_{k,0} a_k^{-1}, z_{k,1} a_k^{-1}, \ldots, z_{k,n_k} a_k^{-1}, \ldots, z_{\ell,0}, a_k^{-c_{k,\ell}^{(1)}} z_{\ell,1}, \ldots, a_k^{-c_{k,\ell}^{(n_\ell)}} z_{\ell,n_\ell}, \ldots). \end{aligned}$$

The monomial $z_{1,0}^{-i_{1,0}} z_{1,1}^{-i_{1,1}} \cdots z_{m,n_m}^{-i_{m,n_m}}$ is then a weight vector with a weight whose k-th coordinate is $\ell_k + i_{k,0} + \cdots + i_{k,n_k} + \sum_{\ell=k+1}^m \sum_{p=1}^{n_\ell} c_{k,\ell}^{(p)} i_{\ell,p}$. Thus the G-invariant part of $H^*((\mathbb{C}^{n_1+1})^\times \times \cdots \times (\mathbb{C}^{n_m+1})^\times, \mathcal{O})$ consists of those monomials $z_{1,0}^{-i_{1,0}} z_{1,1}^{-i_{1,1}} \cdots z_{m,n_m}^{-i_{m,n_m}}$ for which

$$\begin{aligned} \ell_1 + i_{1,0} + \cdots + i_{1,n_1} + \sum_{\ell=2}^m \sum_{p=1}^{n_\ell} c_{1,\ell}^{(p)} i_{\ell,p} &= 0 \\ \ell_2 + i_{2,0} + \cdots + i_{2,n_2} + \sum_{\ell=3}^m \sum_{p=1}^{n_\ell} c_{2,\ell}^{(p)} i_{\ell,p} &= 0 \\ &\vdots \\ \ell_m + i_{m,0} + \cdots + i_{m,n_m} &= 0. \end{aligned} \tag{3.1}$$

The action (2.2) induces a T action on the functions given by

$$\begin{aligned} &((t_{1,1}, \ldots, t_{m,n_m}, t_{m+1}) \cdot f)(z_{1,0}, \ldots, z_{m,n_m}) \\ &= t_{m+1} f(z_{1,0}, t_{1,1}^{-1} z_{1,1}, \ldots, z_{m,0}, t_{m,1}^{-1} z_{m,1}, \ldots, t_{m,n_m}^{-1} z_{m,n_m}). \end{aligned}$$

The weight of the monomial $z_{1,0}^{-i_{1,0}} z_{1,1}^{-i_{1,1}} \cdots z_{m,n_m}^{-i_{m,n_m}}$ with respect to this T action is $(\mathbf{i}_1, \mathbf{i}_2, \ldots, \mathbf{i}_m, 1)$, where $\mathbf{i}_j = (i_{j,1}, \ldots, i_{j,n_j})$ for $j = 1, \ldots, m$. Thus the index of $(B_m, \mathcal{O}_\mathbf{L})$ is given by the set of $x = (x_{1,1}, \ldots, x_{m,n_m}, 1) = (i_{1,1}, \ldots, i_{m,n_m}, 1)$ for which there exist $(i_{1,0}, \ldots, i_{m,0})$ such that (3.1) is satisfied and such that $\mathrm{sgn}(i_{\ell,0}) = \mathrm{sgn}(i_{\ell,1}) = \cdots = \mathrm{sgn}(i_{\ell,n_\ell})$ for all ℓ. This is exactly the set (2.3). Therefore the multiplicity of the equivariant index is $(-1)^N \prod_{1 \le \ell \le m, 1 \le p \le n_\ell} \mathrm{sgn}(i_{\ell,p}) = (-1)^N \prod_{1 \le \ell \le m, 1 \le p \le n_\ell} \mathrm{sgn}(x_{\ell,p}) = \rho(x)$. □

3.2. Character formula for the equivariant index. In the following the theorem we give a formula for the character $\chi : T \to \mathbb{C}$ of the equivariant index of a generalized Bott manifold. For every integral weight $\mu \in \ell^*$ we have a homomorphism $\lambda^\mu : T \to S^1$. We denote the integral combinations of these λ^μ's by $\mathbb{Z}[T]$. Then $\chi \in \mathbb{Z}[T]$ is given by

$\chi = \sum_{\mu \in \ell^*} m_\mu \lambda^\mu$ where $m_\mu = \mathrm{mult}(\mu)$.

DEFINITION 3.2. Let $\{e_{1,1}, \dots, e_{m,n_m}, e_{m+1}\}$ be the standard basis in $\mathbb{R}^{N+1}$, $x_i = (x_{i,1}, \dots, x_{i,n_i})$ and $e_i = (e_{i,1}, \dots, e_{i,n_i})$. Let $\Delta^-_{n,r} = \left\{ z = (z_1, \dots, z_n) \in \mathbb{Z}^n_{\le 0} \;\middle|\; z_1 + \dots + z_n = -r \right\}$, and let $\Delta^+_{n,r} = \left\{ z = (z_1, \dots, z_n) \in \mathbb{Z}^n_{>0} \;\middle|\; z_1 + \dots + z_n = r - 1 \right\}$. Let $\langle x_i, e_i \rangle = x_{i,1}e_{i,1} + \dots + x_{i,n_i}e_{i,n_i}$. Then the operators $D_i : \mathbb{Z}[T] \to \mathbb{Z}[T]$ are defined using $c^{(k)}_{i,j}$ and ℓ_j in the following way:

$$D_i(\lambda^\mu) = \begin{cases} \displaystyle\sum_{0 \le r \le k_i} \sum_{x_i \in \Delta^-_{n_i,r}} \lambda^{\mu + \langle x_i, e_i \rangle} & \text{if } k_i \ge 0 \\ 0 & \text{if } -n_i \le k_i \le -1 \\ \displaystyle\sum_{n_i+1 \le r \le -k_i} \sum_{x_i \in \Delta^+_{n_i,r}} (-1)^{n_i} \lambda^{\mu + \langle x_i, e_i \rangle} & \text{if } k_i \le -n_i - 1, \end{cases}$$

where the functions k_i are defined as follows: if $\mu = e_{m+1} + \sum_{j=i+1}^m \sum_{k=1}^{n_j} x_{j,k} e_{j,k}$, then $k_i(\mu) = \ell_i + \sum_{j=i+1}^m \sum_{k=1}^{n_j} c^{(k)}_{i,j} x_{j,k}$.

From Theorem 3.1, we immediately obtain the following theorem.

Theorem 3.3. *Consider the action of the torus T on $\mathbf{L} \to B_m$ as in* (2.2). *Denote the* $(N+1)$*-th component of the standard basis in $\mathbb{R}^{N+1}$ by e_{m+1}. Then the character is given by the following element of $\mathbb{Z}[T]$:*

$$\chi = D_1 \cdots D_m(\lambda^{e_{m+1}}).$$

REMARK 3.4. When $n_i = 1$ for all i, the operator D_i is given by

$$D_i(\lambda^\mu) = \begin{cases} \lambda^\mu + \lambda^{\mu - e_{i,1}} + \dots + \lambda^{\mu - k_i e_{i,1}} & \textit{if } k_i \ge 0 \\ 0 & \textit{if } k_i = -1 \\ -\lambda^{\mu + e_{i,1}} - \lambda^{\mu + 2e_{i,1}} - \dots - \lambda^{\mu - (k_i+1)e_{i,1}} & \textit{if } k_i \le -2. \end{cases}$$

We can check that this operator agrees with the one in [6, Proposition 2.32].

EXAMPLE 3.5. Suppose that $m = 2, n_1 = 1$, and $n_2 = 2$. We set $\ell_1 = 1, \ell_2 = 2, c^{(1)}_{1,2} = 2$, and $c^{(2)}_{1,2} = -1$ as in Example 2.5. Then the corresponding character χ is given by

$$\begin{aligned} \chi &= D_1 D_2(\lambda^{e_3}) \\ &= D_1(\lambda^{e_3} + \lambda^{e_3 - e_{2,1}} + \lambda^{e_3 - e_{2,2}} + \lambda^{e_3 - 2e_{2,1}} + \lambda^{e_3 - e_{2,1} - e_{2,2}} + \lambda^{e_3 - 2e_{2,2}}) \\ &= \lambda^{e_3} + \lambda^{e_3 - e_{1,1}} + \lambda^{e_3 - e_{2,2}} + \lambda^{e_3 - e_{2,2} - e_{1,1}} + \lambda^{e_3 - e_{2,2} - 2e_{1,1}} - \lambda^{e_3 - 2e_{2,1} + e_{1,1}} - \lambda^{e_3 - 2e_{2,1} + 2e_{1,1}} \\ &\quad + \lambda^{e_3 - e_{2,1} - e_{2,2}} + \lambda^{e_3 - 2e_{2,2}} + \lambda^{e_3 - 2e_{2,2} - e_{1,1}} + \lambda^{e_3 - 2e_{2,2} - 2e_{1,1}} + \lambda^{e_3 - 2e_{2,2} - 3e_{1,1}}. \end{aligned}$$

EXAMPLE 3.6. Suppose that $m = 2, n_1 = 2$, and $n_2 = 1$. We set $\ell_1 = 2, \ell_2 = -6$, and $c^{(1)}_{1,2} = -1$ as in Example 2.6. Then the corresponding character χ is given by

$$\begin{aligned} \chi &= D_1 D_2(\lambda^{e_3}) \\ &= D_1(-\lambda^{e_3 + e_{2,1}} - \lambda^{e_3 + 2e_{2,1}} - \lambda^{e_3 + 3e_{2,1}} - \lambda^{e_3 + 4e_{2,1}} - \lambda^{e_3 + 5e_{2,1}}) \\ &= -\lambda^{e_3 + e_{2,1}} - \lambda^{e_3 + e_{2,1} - e_{1,1}} - \lambda^{e_3 + e_{2,1} - e_{1,2}} - \lambda^{e_3 + 2e_{2,1}} - \lambda^{e_3 + 5e_{2,1} + e_{1,1} + e_{1,2}}. \end{aligned}$$

EXAMPLE 3.7. Suppose that $m = 2, n_1 = 2$, and $n_2 = 2$. We set $\ell_1 = 1, \ell_2 = 2, c_{1,2}^{(1)} = 2$, and $c_{1,2}^{(2)} = -1$ as in Example 2.7. Then the corresponding character χ is given by

$$\begin{aligned}
\chi &= D_1 D_2(\lambda^{e_3}) \\
&= D_1(\lambda^{e_3} + \lambda^{e_3-e_{2,1}} + \lambda^{e_3-e_{2,2}} + \lambda^{e_3-2e_{2,1}} + \lambda^{e_3-e_{2,1}-e_{2,2}} + \lambda^{e_3-2e_{2,2}}) \\
&= \lambda^{e_3} + \lambda^{e_3-e_{1,1}} + \lambda^{e_3-e_{1,2}} + \lambda^{e_3-e_{2,2}} + \lambda^{e_3-e_{2,2}-e_{1,1}} + \lambda^{e_3-e_{2,2}-e_{1,2}} + \lambda^{e_3-e_{2,2}-2e_{1,1}} \\
&\quad + \lambda^{e_3-e_{2,2}-e_{1,1}-e_{1,2}} + \lambda^{e_3-e_{2,2}-2e_{1,2}} + \lambda^{e_3-2e_{2,1}+e_{1,1}+e_{1,2}} + \lambda^{e_3-e_{2,1}-e_{2,2}} + \lambda^{e_3-2e_{2,2}} \\
&\quad + \lambda^{e_3-2e_{2,2}-e_{1,1}} + \lambda^{e_3-2e_{2,2}-e_{1,2}} + \lambda^{e_3-2e_{2,2}-2e_{1,1}} + \lambda^{e_3-2e_{2,2}-e_{1,1}-e_{1,2}} + \lambda^{e_3-2e_{2,2}-2e_{1,2}} \\
&\quad + \lambda^{e_3-2e_{2,2}-3e_{1,1}} + \lambda^{e_3-2e_{2,2}-2e_{1,1}-e_{1,2}} + \lambda^{e_3-2e_{2,2}-e_{1,1}-2e_{1,2}} + \lambda^{e_3-2e_{2,2}-3e_{1,2}}.
\end{aligned}$$

REMARK 3.8. We gave the formula for the character using the Demazure-type operators. On the other hand, the character is also given by the localization formula ([7, Corollary 7.4]). For example, when we set the parameters as in Example 3.5, the character is computed using the localization formula as follows:

$$\begin{aligned}
\chi = \lambda^{e_3}\Bigg(&\frac{1}{(1-\lambda^{-e_{1,1}})(1-\lambda^{-e_{2,1}})(1-\lambda^{-e_{2,2}})} + \frac{\lambda^{-2e_{2,2}}}{(1-\lambda^{-e_{1,1}})(1-\lambda^{-e_{2,1}+e_{2,2}})(1-\lambda^{e_{2,2}})} \\
&+ \frac{\lambda^{-2e_{2,1}}}{(1-\lambda^{-e_{1,1}})(1-\lambda^{e_{2,1}-e_{2,2}})(1-\lambda^{e_{2,1}})} + \frac{\lambda^{-e_{1,1}}}{(1-\lambda^{e_{1,1}})(1-\lambda^{2e_{1,1}-e_{2,1}})(1-\lambda^{-e_{1,1}-e_{2,2}})} \\
&+ \frac{\lambda^{-3e_{1,1}-2e_{2,2}}}{(1-\lambda^{e_{1,1}})(1-\lambda^{3e_{1,1}-e_{2,1}+e_{2,2}})(1-\lambda^{e_{1,1}+e_{2,2}})} \\
&+ \frac{\lambda^{3e_{1,1}-2e_{2,1}}}{(1-\lambda^{e_{1,1}})(1-\lambda^{-3e_{1,1}+e_{2,1}-e_{2,2}})(1-\lambda^{-2e_{1,1}+e_{2,1}})}\Bigg).
\end{aligned}$$

We can check that this result agrees with the result in Example 3.5.

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Department of Mathematics, Graduate School of Science and Engineering
Chuo University
Kasuga, Bunkyo-Ku, Tokyo, 112–8551
Japan
e-mail: y-sugi@gug.math.chuo-u.ac.jp