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AFFINE KILLING REEB VECTOR FIELD FOR A REAL HYPERSURFACE IN THE COMPLEX QUADRIC

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Abstract

In this paper, first we introduce a general notion of affine Killing vector fields on the complex quadric Q^m, which is weaker than usual Killing vector field. Next, we give a complete classification of Hopf real hypersurfaces M with affine Killing Reeb vector field in the complex quadric Q^m, $m \geq 3$.

1. Introduction

It is well known that the importance of Killing vector fields on Riemannian manifolds and Mathematical Physics is highly evaluated (see [2], [22], [28], and [34]). Recall that a vector field V on a Riemannian manifold $(\bar{M}, g)$ is said to be *Killing* if the Lie derivative of the metric tensor g along the direction of V is invariant, that is,

$$(\mathcal{L}_V g)(X, Y) = 0, \tag{1.1}$$

or equivalently,

$$g(\bar{\nabla}_X V, Y) = -g(\bar{\nabla}_Y V, X), \quad X, Y \in \mathfrak{X}(\bar{M}), \tag{1.2}$$

where $\bar{\nabla}$ denotes the Riemannian connection on $(\bar{M}, g)$ and $\mathfrak{X}(\bar{M})$ the set of differentiable vector fields on $\bar{M}$. In terms of local components, (1.1) can be expressed as $\mathcal{L}_V g_{ij} = 0$.

When we consider a real hypersurface (M, g) in a Hermitian symmetric space $(\bar{M}, g)$, there exists a Reeb vector field ξ on M defined by $\xi = -JN$, where J denotes the Kähler structure on $\bar{M}$. Then, the Reeb vector field ξ is Killing (or *M has isometric Reeb flow*) if and only if the Lie derivative of the induced metric tensor g along the Reeb direction vanishes, $\mathcal{L}_\xi g = 0$. By using the Lie algebraic methods given in [1] and [3], Berndt and Suh [7] gave a complete classification of real hypersurfaces with isometric Reeb flow in Hermitian symmetric spaces.

On a Riemannian manifold $(\bar{M}, g)$ we say that a vector field V is *affine Killing* if it satisfies

$$\mathcal{L}_V \Gamma^i_{jk} = \bar{\nabla}_j \bar{\nabla}_k V^i + R^i_{\ kmj} V^m = 0$$

where Γ^i_{jk} and $\bar{\nabla}$ denote Christoffel symbols and the Riemannian connection defined on $(\bar{M}, g)$. From such a view point let us define an affine Killing vector field on a Riemannian

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manifold $(\bar{M}, g)$ as follows:

Definition 1.1. Let $(\bar{M}, g)$ be a Riemannian manifold with Riemannian connection $\bar{\nabla}$. A vector field V is said to be an *affine Killing vector field*, if it satisfies

$$(\mathcal{L}_V\bar{\nabla})(X, Y) = 0, \tag{1.3}$$

or equivalently,

$$[V, \bar{\nabla}_X Y] - \bar{\nabla}_{[V,X]}Y - \bar{\nabla}_X[V, Y] = 0 \tag{1.4}$$

for all differentiable vector fields X and $Y \in \mathfrak{X}(\bar{M})$. In particular, the Reeb vector field ξ is said to be *affine Killing*, if $(\mathcal{L}_\xi\bar{\nabla})(X, Y) = 0$ (see also [11] and [13]).

In fact, the Riemannian connection $\bar{\nabla}$ of $\bar{M}$ is the unique linear connection being torsion-free defined by $\bar{\nabla}(X, Y) = \bar{\nabla}_X Y$ for any X, $Y \in \mathfrak{X}(\bar{M})$. From this and the properties of Lie derivative, $\mathcal{L}_X Y = [X, Y]$ and $[X, Y] = \bar{\nabla}_X Y - \bar{\nabla}_Y X$, we get

$$\begin{aligned}(\mathcal{L}_V\bar{\nabla})(X, Y) &= \mathcal{L}_V(\bar{\nabla}(X, Y)) - \bar{\nabla}(\mathcal{L}_V X, Y) - \bar{\nabla}(X, \mathcal{L}_V Y)\\ &= [V, \bar{\nabla}_X Y] - \bar{\nabla}_{[V,X]}Y - \bar{\nabla}_X[V, Y]\\ &= \bar{\nabla}_V\bar{\nabla}_X Y - \bar{\nabla}_{\bar{\nabla}_X Y}V - \bar{\nabla}_{[V,X]}Y - \bar{\nabla}_X\bar{\nabla}_V Y + \bar{\nabla}_X\bar{\nabla}_Y V\\ &= \bar{R}(V, X)Y - \bar{\nabla}_{\bar{\nabla}_X Y}V + \bar{\nabla}_X\bar{\nabla}_Y V,\end{aligned} \tag{1.5}$$

where the curvature tensor $\bar{R}$ of M is a (1,3) tensor field defined by

$$\bar{R}(X, Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z \tag{1.6}$$

for any X, Y, $Z \in \mathfrak{X}(\bar{M})$.

In [35], Yano proved the existence of Killing vector fields on a compact orientable Riemannian manifold. Moreover, from the integrability condition of Killing vector fields we see that *every Killing vector field on a Riemannian manifold $(\bar{M}, g)$ is affine Killing, but, the converse is not necessarily true* (see [11], [34], and [35]). Thus, in this paper we are interested in the study of real hypersurfaces with affine Killing Reeb vector field in the complex quadric Q^m.

The complex quadric $Q^m = SO_{m+2}/SO_mSO_2$ which is a complex hypersurface in the complex projective space $\mathbb{C}P^{m+1}$ can be regarded as a kind of real Grassmann manifold of compact type with rank 2 (see [4], [6], [12], and [16]). Accordingly, Q^m admits two important geometric structures, so-called a real structure A and a complex structure J which anti-commute with each other, that is, $AJ = -JA$. By using the method of Lie algebra in [17], the triple (Q^m, J, g) is a Hermitian symmetric space of compact type with rank 2 and its maximal sectional curvature is equal to 4 (see also [12], [27], and [29]).

As a typical characterization, Berndt–Suh [5] considered a notion of isometric Reeb flow for real hypersurfaces in Q^m and gave a classification theorem as follows:

Theorem A ([5]). *Let M be a connected orientable real hypersurface in the complex quadric Q^m, $m \geq 3$. Then, the Reeb flow on M is isometric if and only if m is even, say $m = 2k$, and M is locally congruent to an open part of a tube around some totally geodesic $\mathbb{C}P^k$ in Q^{2k}.*

It can be easily checked that the isometric Reeb flow is equivalent to the fact that the shape operator S of M commutes with the structure tensor ϕ, that is, $S\phi = \phi S$. From this, a real hypersurface M with isometric Reeb flow in Q^m is Hopf. The notion of *Hopf* means that the Reeb vector field ξ of M is principal for the shape operator S of M, that is, $S\xi = g(S\xi, \xi)\xi = \alpha\xi$. If the Reeb function $\alpha = g(S\xi, \xi)$ identically vanishes on M, we say that M has vanishing geodesic Reeb flow.

A nonzero tangent vector $W \in T_{[z]}Q^m$ is called *singular* if it is tangent to more than one maximal flat in Q^m. The complex quadric Q^m is a Hermitian symmetric space of rank 2. So, there exist two types of singular tangent vectors in Q^m: Let $V(A) = \{Z \in T_{[z]}Q^m \,|\, AZ = Z\}$ and $JV(A) = \{Z \in T_{[z]}Q^m \,|\, AZ = -Z\}$ be the $(+1)$-eigenspace and (-1)-eigenspace for the involution A on the tangent space $T_{[z]}Q^m$ of Q^m at any point $[z] \in Q^m$.

- If there exists a conjugation $A \in \mathfrak{A} = \{A_{\lambda\bar{z}} \,|\, \lambda \in S^1 \subset \mathbb{C}\}$ such that $W \in V(A)$, then W is singular, and it is called $\mathfrak{A}$-*principal*.
- If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $Z_1, Z_2 \in V(A)$ such that $W/\|W\| = (Z_1 + JZ_2)/\sqrt{2}$, then W is singular, and it is called $\mathfrak{A}$-*isotropic*.

Related to the singularity of tangent vector fields of Q^m, Lee and Suh [20] gave a classification of Hopf real hypersurfaces in the complex quadric Q^m as follows:

Theorem B ([20]). *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$. Then the unit normal vector field N of M is $\mathfrak{A}$-principal if and only if M is locally congruent to an open part of a tube around the m-dimensional sphere S^m which is totally real and totally geodesic in Q^m.*

Remark 1.2. Usually, if the Reeb vector field ξ is Killing, that is, $\mathcal{L}_\xi g_{ij} = 0$, then it is affine Killing. So, the notion of affine Killing is more general and weaker than usual notion of Killing. In fact, we note that

$$\mathcal{L}_\xi \Gamma^i_{jk} = \frac{1}{2} g^{im} \Big\{ \bar{\nabla}_j (\mathcal{L}_\xi\, g_{km}) + \bar{\nabla}_k (\mathcal{L}_\xi\, g_{jm}) - \bar{\nabla}_m (\mathcal{L}_\xi\, g_{jk}) \Big\} = 0.$$

Therefore, the Reeb vector field ξ is affine Killing (see [13]). The detailed proof also will be given in Lemma 4.1 in section 4.

Motivated by Theorems A and B, and above Remark 1.2, we give a characterization of Hopf real hypersurfaces with affine Killing Reeb vector field in the complex quadric Q^m as follows:

Theorem 1.3. *Let M be a Hopf real hypersurface with affine Killing Reeb vector field ξ in the complex quadric Q^m, $m \geq 3$. Then, the unit normal vector field N of M is singular, that is, either $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic.*

Then, we can obtain a characterization for real hypersurfaces in Theorem A as follows:

Theorem 1.4. *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$. The Reeb vector field ξ is affine Killing if and only if M is locally congruent to an open part of a tube around some totally geodesic $\mathbb{C}P^k$ in Q^m, where $m = 2k$.*

Finally, as another generalization of Killing vector fields on a Riemannian manifold $(\bar{M}, g)$, we introduce the notion of conformal Killing vector field on $(\bar{M}, g)$ as follows: A vector field V is said to be *conformal Killing*, if it satisfies

$$(\mathcal{L}_V g)(X, Y) = 2\delta g(X, Y) \tag{1.7}$$

for a differentiable function δ, or equivalently,

$$g(\bar{\nabla}_X V, Y) + g(\bar{\nabla}_Y V, X) = 2\delta g(X, Y) \tag{1.8}$$

for all differentiable vector fields X, $Y \in \mathfrak{X}(\bar{M})$. In this case, we say that the Reeb vector field ξ is *conformal Killing* if it satisfies (1.7).

From (1.7), a Killing vector field becomes a conformal Killing for a vanishing function, $\delta = 0$. As a converse problem, by Theorem 1.4, we obtain as a corollary that the Reeb vector field ξ should be Killing if ξ is conformal Killing as follows:

Corollary 1.5. *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$. The Reeb vector field ξ is conformal Killing if and only if M is locally congruent to an open part of a tube around some totally geodesic $\mathbb{C}P^k$ in Q^m, where $m = 2k$.*

2. The complex quadric

As mentioned in section 1, the complex quadric Q^m is the complex hypersurface in $\mathbb{C}P^{m+1}$ defined by the equation $z_1^2 + \cdots + z_{m+2}^2 = 0$, where $z_1, \ldots, z_{m+2}$ are homogeneous coordinates on $\mathbb{C}P^{m+1}$. We equip Q^m with the Riemannian metric which is induced from the Fubini Study metric on $\mathbb{C}P^{m+1}$ with constant holomorphic sectional curvature 4. The Kähler structure on $\mathbb{C}P^{m+1}$ induces canonically a Kähler structure (J, g) on the complex quadric. The complex quadric Q^1 is isometric to a sphere S^2 with constant curvature, and Q^2 is isometric to the Riemannian product of two 2-spheres with constant curvature. For this reason we will assume $m \geq 3$ from now on (see [5]).

For a nonzero vector $z \in \mathbb{C}^{m+2}$ we denote by $[z]$ the complex span of z, that is,

$$[z] = \mathbb{C}z = \{\lambda z \,|\, \lambda \in S^1 \subset \mathbb{C}\}.$$

Note that, by definition, $[z]$ is a point in $\mathbb{C}P^{m+1}$. For each $[z] \in Q^m \subset \mathbb{C}P^{m+1}$ we identify $T_{[z]}\mathbb{C}P^{m+1}$ with the orthogonal complement $\mathbb{C}^{m+2} \ominus \mathbb{C}z$ of $\mathbb{C}z$ in $\mathbb{C}^{m+2}$. Then, the tangent space $T_{[z]}Q^m$ can be identified canonically with the orthogonal complement $\mathbb{C}^{m+2} \ominus (\mathbb{C}z \oplus \mathbb{C}\rho)$ of $\mathbb{C}z \oplus \mathbb{C}\rho$ in $\mathbb{C}^{m+2}$, where $\rho \in \nu_{[z]}Q^m$ is a normal vector of Q^m in $\mathbb{C}P^{m+1}$ at the point $[z]$ (see [17]).

For a unit normal vector ρ of Q^m at a point $[z] \in Q^m$ we denote by $A = A_\rho$ the shape operator of Q^m in $\mathbb{C}P^{m+1}$ with respect to ρ. The shape operator A is an involution on the tangent space $T_{[z]}Q^m$ and

$$T_{[z]}Q^m = V(A_\rho) \oplus JV(A_\rho),$$

where $V(A_\rho)$ is the $(+1)$-eigenspace and $JV(A_\rho)$ is the (-1)-eigenspace of A_ρ. Geometrically this means that the shape operator A_ρ defines a real structure on the tangent space $T_{[z]}Q^m$. Since the real codimension of Q^m in $\mathbb{C}P^{m+1}$ is 2, this induces an S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^m)$ consisting of real structures, that is, $\mathfrak{A} = \{A_{\lambda\rho} \,|\, \lambda \in S^1 \subset \mathbb{C}\}$. Then, the subbundle $\mathfrak{A}$ is parallel, which means that there exists a certain 1-form q defined on TQ^m such that

$$(\bar{\nabla}_U A)W = q(U)JAW$$

for any vector fields U and W on Q^m (see [27] and [29]). Moreover, for every unit tangent vector $W \in T_{[z]}Q^m$ there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $Z_1, Z_2 \in V(A)$ such that

$$W = \cos(t)Z_1 + \sin(t)JZ_2 \tag{2.1}$$

for some $t \in [0, \pi/4]$ (see Proposition 3 in [27]). The singular tangent vectors of Q^m correspond to the values $t = 0$ and $t = \pi/4$.

The Gauss equation for Q^m in $\mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor $\bar{R}$ of Q^m can be described in terms of the complex structure J and the complex conjugations $A \in \mathfrak{A}$:

$$\begin{aligned} \bar{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY \\ & - 2g(JX,Y)JZ + g(AY,Z)AX \\ & - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY. \end{aligned} \tag{2.2}$$

For more background to this section we refer to [5], [8], [14], [17], [19], [26], [27], [31], [32], and [33].

3. Real hypersurfaces in Q^m

Let M be a real hypersurface in Q^m and denote by N a unit normal vector field of M. Then, we obtain the induced almost contact metric structure (ϕ, ξ, η, g) on M. From this, for any vector field X tangent to M, we may put $JX = \phi X + \eta(X)N$ and $JN = -\xi$, where ϕX is the tangential component of JX. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker \eta$ is the maximal complex subbundle of TM. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$.

Denote by ∇ and S the induced Riemannian connection and the shape operator on M, respectively. Then the Gauss and Weingarten formulas are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + g(SX,Y)N, \quad \bar{\nabla}_X Y = -SX,$$

where $\bar{\nabla}$ is the connection on Q^m. Also, we have

$$(\nabla_X \phi)Y = \eta(Y)SX - g(SX,Y)\xi, \quad \nabla_X \xi = \phi SX. \tag{3.1}$$

Moreover, since the complex quadric Q^m has also a real structure A, we decompose AX into its tangential and normal components for a fixed $A \in \mathfrak{A}_{[z]}$ and $X \in T_{[z]}M$:

$$AX = BX + g(AX,N)N$$

where BX denotes the tangential component of AX.

On the other hand, since the normal vector field N belongs to $T_{[z]}Q^m$, $[z] \in M$, from (2.1), we can choose $A \in \mathfrak{A}_{[z]}$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2 \tag{3.2}$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \leq t \leq \frac{\pi}{4}$. If $t = 0$, then $N = Z_1 \in V(A)$, therefore we see that the unit normal vector field N of M becomes an $\mathfrak{A}$-principal vector field. On the other hand, if $t = \frac{\pi}{4}$, then $N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$. That is, N is $\mathfrak{A}$-isotropic.

It is known that the Reeb vector field ξ is defined by $\xi = -JN$. Then, it follows that

$$\begin{cases} \xi = \sin(t)Z_2 - \cos(t)JZ_1, \\ AN = \cos(t)Z_1 - \sin(t)JZ_2, \\ A\xi = \sin(t)Z_2 + \cos(t)JZ_1. \end{cases} \tag{3.3}$$

From this, we have $g(A\xi, N) = 0$ and $g(A\xi, \xi) = -g(AN, N) = -\cos(2t)$ on M. In particular, from the former equation we know that the unit vector field $A\xi$ is tangent to M. So, we get

$$AN = AJ\xi = -JA\xi = -\phi A\xi - \eta(A\xi)N, \tag{3.4}$$

where we have used the property of $JA = -AJ$. Hereafter, for our convenience sake, we denote the smooth function $g(A\xi, \xi)$ by κ, that is,

$$\kappa = g(A\xi, \xi) = -\cos(2t).$$

Moreover, by using the Gauss and Weingarten formulas, the left side of (2.2) becomes

$$\bar{R}(X, Y)Z = R(X, Y)Z - g(SY, Z)SX + g(SX, Z)SY + \{g((\nabla_X S)Y, Z) - g((\nabla_Y S)X, Z)\}N,$$

where R and S denote the Riemannian curvature tensor and the shape operator of M in Q^m, respectively. From this formula, taking tangential and normal components of (2.2), we obtain respectively

$$\begin{aligned} R(X, Y)Z = {} & g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & - 2g(\phi X, Y)\phi Z + g(BY, Z)BX - g(BX, Z)BY \\ & + g(\phi BY, Z)\phi BX + g(\phi BY, Z)g(X, \phi A\xi)\xi \\ & + g(Y, \phi A\xi)\eta(Z)\phi BX - g(\phi BX, Z)\phi BY \\ & - g(\phi BX, Z)g(Y, \phi A\xi)\xi - g(X, \phi A\xi)\eta(Z)\phi BY \\ & + g(SY, Z)SX - g(SX, Z)SY \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} (\nabla_X S)Y - (\nabla_Y S)X = {} & \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\ & - g(\phi A\xi, X)BY + g(\phi A\xi, Y)BX \\ & + g(A\xi, X)\phi BY + g(A\xi, X)g(\phi A\xi, Y)\xi \\ & - g(A\xi, Y)\phi BX - g(A\xi, Y)g(\phi A\xi, X)\xi, \end{aligned} \tag{3.6}$$

which are called the equations of Gauss and Codazzi.

Now let us assume that M is a Hopf real hypersurface in the complex quadric Q^m. That is, the shape operator S of M in Q^m satisfies $S\xi = \alpha\xi$ where the Reeb function α of M is given by $\alpha = g(S\xi, \xi)$. By Codazzi equation (3.6) and (3.4), we obtain:

Lemma 3.1 ([5])**.** *Let M be a Hopf hypersurface in Q^m, $m \geq 3$. Then, we obtain*

$$X\alpha = (\xi\alpha)\eta(X) - 2\kappa g(\phi A\xi, X) \tag{3.7}$$

and

$$(3.8)\qquad 2S\phi SX - \alpha\phi SX - \alpha S\phi X - 2\phi X - 2g(\phi A\xi, X)A\xi + 2g(A\xi, X)\phi A\xi + 2\kappa g(\phi A\xi, X)\xi - 2\kappa\eta(X)\phi A\xi = 0$$

for any tangent vector fields X and Y on M.

Remark 3.2 ([18]). By (3.7) we know that if M has vanishing geodesic Reeb flow, then the unit normal vector field N of M in Q^m becomes singular. Moreover, by the definition, the unit vector field N is $\mathfrak{A}$-isotropic if the smooth function $\kappa = g(A\xi, \xi)$ identically vanishes on M in Q^m.

From (3.8) we want to give some information about principal curvatures for Hopf real hypersurfaces in Q^m with $\mathfrak{A}$-principal normal vector field as follows:

Lemma 3.3 ([30]). *Let M be a Hopf hypersurface in Q^m such that the normal vector field N is $\mathfrak{A}$-principal everywhere. Then, the Reeb function α is constant. Moreover, if $X \in \mathcal{C}$ is a principal curvature vector of M with principal curvature λ, then $2\lambda \neq \alpha$ and its corresponding vector ϕX is a principal curvature vector of M with principal curvature $\frac{\alpha\lambda+2}{2\lambda-\alpha}$.*

In addition, when the unit normal vector field N of M is $\mathfrak{A}$-principal, we obtain that $A\xi = -\xi$ and $AN = N$ from (3.2) and (3.3). By using these formulas we get

Lemma 3.4 ([21]). *Let M be a real hypersurface with $\mathfrak{A}$-principal normal vector field N in the complex quadric Q^m, $m \geq 3$. Then we obtain:*

(a) *$AX = BX$,*
(b) *$A\phi X = -\phi AX$,*
(c) *$A\phi SX = -\phi SX$ and $q(X) = 2g(SX, \xi)$,*
(d) *$ASX = SX - 2g(SX, \xi)\xi$ and $SAX = SX - 2\eta(X)S\xi$*

for all $X \in TM$.

If a real hypersurface M in Q^m has an $\mathfrak{A}$-isotropic unit normal vector field N, it yields

$$g(A\xi, N) = 0 \text{ and } g(AN, N) = g(A\xi, \xi) = 0.$$

From this and (3.4), we see that the two unit vector fields $A\xi$ and AN belong to the distribution $\mathcal{C} = [\xi]^\perp$, which is the orthogonal complement of the Reeb vector field ξ. Thus, it follows that $AN = -\phi A\xi$, so $A\xi \perp AN$. This implies that the tangent vector bundle TM of M can be decomposed as

$$TM = [\xi] \oplus \text{span}\{A\xi, AN\} \oplus \mathcal{Q},$$

where $[\xi] = \text{span}\{\xi\}$ and $\mathcal{C} \ominus \mathcal{Q} = \mathcal{Q}^\perp = \text{span}\{A\xi, AN\}$. Using such a decomposition and (3.8), we get the following:

Lemma 3.5 ([21]). *Let M be a Hopf hypersurface in Q^m, $m \geq 3$, such that the normal vector field N is $\mathfrak{A}$-isotropic everywhere. Then the following statements hold.*

(a) *The Reeb function α is constant.*
(b) *The unit tangent vector fields $A\xi$ and $AN = -\phi A\xi$ are principal for the shape operator and their principal curvature is zero, that is,*

$$SA\xi = SAN = S\phi A\xi = 0.$$

(c) *If $X \in \mathcal{Q}$ is a principal curvature vector of M with principal curvature λ, then $2\lambda \neq \alpha$ and its corresponding vector ϕX is a principal curvature vector of M with principal curvature $\frac{\alpha\lambda+2}{2\lambda-\alpha}$.*

4. Hopf real hypersurfaces with affine Killing Reeb vector field

In this section, first, we will give an important result concerned with Killing and affine Killing vector fields on Riemannian manifolds $(\bar{M}, g)$. In addition, by using this property, we will give a some basic formula about Hopf real hypersurfaces with affine Killing Reeb vector field in the complex quadric Q^m, $m \geq 3$.

We will prove the following.

Lemma 4.1. *Let $(\bar{M}, g)$ be a Riemannian manifold with Riemannian connection $\bar{\nabla}$ and V be a differentiable vector field on $\bar{M}$. If V is a Killing vector field, then it is also affine Killing.*

Proof. The curvature tensor $\bar{R}$ satisfies

$$g(\bar{R}(X,Y)U,W) = g(\bar{R}(U,W)X,Y) \quad \text{(Interchange Symmetry)}$$

and

$$g(\bar{R}(X,Y)U,W) = -g(\bar{R}(X,Y)W,U) \quad \text{(Skew Symmetry)}$$

for any $X, Y, U, W \in \mathfrak{X}(\bar{M})$. By using these properties and (1.6), we obtain

$$\begin{aligned} g(\bar{R}(X,Y)V,W) &+ g(\bar{R}(Y,V)X,W) - g(\bar{R}(V,X)Y,W) \\ &= g(\bar{R}(X,Y)V,W) - g(\bar{R}(X,W)V,Y) - g(\bar{R}(Y,W)V,X) \\ &= g(\bar{\nabla}_X\bar{\nabla}_Y V,W) - g(\bar{\nabla}_Y\bar{\nabla}_X V,W) - g(\bar{\nabla}_{[X,Y]}V,W) \\ &\quad - g(\bar{\nabla}_X\bar{\nabla}_W V,Y) + g(\bar{\nabla}_W\bar{\nabla}_X V,Y) + g(\bar{\nabla}_{[X,W]}V,Y) \\ &\quad - g(\bar{\nabla}_Y\bar{\nabla}_W V,X) + g(\bar{\nabla}_W\bar{\nabla}_Y V,X) + g(\bar{\nabla}_{[Y,W]}V,X) \end{aligned} \tag{4.1}$$

for any vector fields X, Y, V and $W \in \mathfrak{X}(\bar{M})$. By the first Bianchi identity,

$$\bar{R}(X,Y)V + \bar{R}(Y,V)X + \bar{R}(V,X)Y = 0,$$

the left side of (4.1) becomes

$$g(\bar{R}(X,Y)V,W) + g(\bar{R}(Y,V)X,W) - g(\bar{R}(V,X)Y,W) = -2g(\bar{R}(V,X)Y,W). \tag{4.2}$$

On the other hand, from the assumption of V being Killing, (1.2) yields

$$g(\bar{\nabla}_X V,W) = -g(\bar{\nabla}_W V,X) \tag{4.3}$$

for all vector fields $X, W \in \mathfrak{X}(\bar{M})$. Since $[X,Y] = \bar{\nabla}_X Y - \bar{\nabla}_Y X \in \mathfrak{X}(\bar{M})$, (4.3) gives us

$$g(\bar{\nabla}_{[X,Y]}V,W) = -g(\bar{\nabla}_W V,[X,Y]) = -g(\bar{\nabla}_W V,\bar{\nabla}_X Y) + g(\bar{\nabla}_W V,\bar{\nabla}_Y X). \tag{4.4}$$

Moreover, taking the covariant derivative of (4.3) along the Y-direction gives

$$g(\bar{\nabla}_Y\bar{\nabla}_X V,W) = -g(\bar{\nabla}_Y\bar{\nabla}_W V,X) - g(\bar{\nabla}_W V,\bar{\nabla}_Y X) - g(\bar{\nabla}_X V,\bar{\nabla}_Y W). \tag{4.5}$$

Then, from (4.4) and (4.5), we obtain

$$g(\bar{\nabla}_X\bar{\nabla}_Y V, W) - g(\bar{\nabla}_Y\bar{\nabla}_X V, W) - g(\bar{\nabla}_{[X,Y]}V, W) \\ = g(\bar{\nabla}_X\bar{\nabla}_Y V, W) + g(\bar{\nabla}_Y\bar{\nabla}_W V, X) + g(\bar{\nabla}_X V, \bar{\nabla}_Y W) + g(\bar{\nabla}_W V, \bar{\nabla}_X Y). \tag{4.6}$$

Similarly, we get

$$g(\bar{\nabla}_X\bar{\nabla}_W V, Y) - g(\bar{\nabla}_W\bar{\nabla}_X V, Y) - g(\bar{\nabla}_{[X,W]}V, Y) \\ = g(\bar{\nabla}_X\bar{\nabla}_W V, Y) + g(\bar{\nabla}_W\bar{\nabla}_Y V, X) + g(\bar{\nabla}_X V, \bar{\nabla}_W Y) + g(\bar{\nabla}_Y V, \bar{\nabla}_X W) \tag{4.7}$$

and

$$g(\bar{\nabla}_Y\bar{\nabla}_W V, X) - g(\bar{\nabla}_W\bar{\nabla}_Y V, X) - g(\bar{\nabla}_{[Y,W]}V, X) \\ = g(\bar{\nabla}_Y\bar{\nabla}_W V, X) + g(\bar{\nabla}_W\bar{\nabla}_X V, Y) + g(\bar{\nabla}_Y V, \bar{\nabla}_W X) + g(\bar{\nabla}_X V, \bar{\nabla}_Y W). \tag{4.8}$$

Substituting (4.6), (4.7), and (4.8) into the right side of (4.1), it follows:

$$\begin{aligned} &g(\bar{R}(X,Y)V, W) + g(\bar{R}(Y,V)X, W) - g(\bar{R}(V,X)Y, W) \\ &= g(\bar{\nabla}_X\bar{\nabla}_Y V, W) + g(\bar{\nabla}_Y\bar{\nabla}_W V, X) + g(\bar{\nabla}_X V, \bar{\nabla}_Y W) + g(\bar{\nabla}_W V, \bar{\nabla}_X Y) \\ &\quad - g(\bar{\nabla}_X\bar{\nabla}_W V, Y) - g(\bar{\nabla}_W\bar{\nabla}_Y V, X) - g(\bar{\nabla}_X V, \bar{\nabla}_W Y) - g(\bar{\nabla}_Y V, \bar{\nabla}_X W) \\ &\quad - g(\bar{\nabla}_Y\bar{\nabla}_W V, X) - g(\bar{\nabla}_W\bar{\nabla}_X V, Y) - g(\bar{\nabla}_Y V, \bar{\nabla}_W X) - g(\bar{\nabla}_X V, \bar{\nabla}_Y W) \\ &= g(\bar{\nabla}_X\bar{\nabla}_Y V, W) + g(\bar{\nabla}_X V, \bar{\nabla}_Y W) + g(\bar{\nabla}_W V, \bar{\nabla}_X Y) \\ &\quad - g(\bar{\nabla}_X\bar{\nabla}_W V, Y) - g(\bar{\nabla}_W\bar{\nabla}_Y V, X) - g(\bar{\nabla}_X V, \bar{\nabla}_W Y) - g(\bar{\nabla}_Y V, \bar{\nabla}_X W) \\ &\quad - g(\bar{\nabla}_W\bar{\nabla}_X V, Y) - g(\bar{\nabla}_Y V, \bar{\nabla}_W X) - g(\bar{\nabla}_X V, \bar{\nabla}_Y W). \end{aligned} \tag{4.9}$$

Applying the formula (4.5) to the two terms $g(\bar{\nabla}_X\bar{\nabla}_W V, Y)$ and $g(\bar{\nabla}_W\bar{\nabla}_X V, Y)$ in the right side of (4.9), then it follows that

$$g(\bar{R}(X,Y)V, W) + g(\bar{R}(Y,V)X, W) - g(\bar{R}(V,X)Y, W) \\ = 2g(\bar{\nabla}_X\bar{\nabla}_Y V, W) + 2g(\bar{\nabla}_W V, \bar{\nabla}_X Y). \tag{4.10}$$

Then, by using (4.2) and (4.3), the formula (4.10) can be rewritten as follows:

$$\begin{aligned} -2g(\bar{R}(V,X)Y, W) &= g(\bar{R}(X,Y)V, W) + g(\bar{R}(Y,V)X, W) - g(\bar{R}(V,X)Y, W) \\ &= 2g(\bar{\nabla}_X\bar{\nabla}_Y V, W) + 2g(\bar{\nabla}_W V, \bar{\nabla}_X Y) \\ &= 2g(\bar{\nabla}_X\bar{\nabla}_Y V, W) - 2g(\bar{\nabla}_{\bar{\nabla}_X Y} V, W), \end{aligned}$$

that is,

$$g(\bar{R}(V,X)Y, W) = -g(\bar{\nabla}_X\bar{\nabla}_Y V, W) + g(\bar{\nabla}_{\bar{\nabla}_X Y} V, W)$$

for any vector fields X, Y and $W \in \mathfrak{X}(\bar{M})$. From this, we get

$$\bar{R}(V,X)Y = -\bar{\nabla}_X\bar{\nabla}_Y V + \bar{\nabla}_{\bar{\nabla}_X Y} V.$$

By (1.5), it implies that for any differentiable vector fields $X, Y \in \mathfrak{X}(\bar{M})$

$$(\mathcal{L}_V\bar{\nabla})(X,Y) = 0,$$

which means that the Killing vector field V is affine Killing. □

Bearing this result in mind, let us consider the case of the Reeb vector field ξ of M being affine Killing. Here, M denotes a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$. That is, the Reeb vector field ξ of M satisfies $\mathcal{L}_\xi \nabla = 0$, where ∇ is the Levi-Civita connection of M. Then, (1.5), together with $\nabla_X \xi = \phi SX$, yields

$$\begin{aligned} 0 = (\mathcal{L}_\xi \nabla)(X, Y) &= R(\xi, X)Y + \nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi \\ &= R(\xi, X)Y + \nabla_X(\phi SY) - \phi S(\nabla_X Y) \\ &= R(\xi, X)Y + (\nabla_X \phi)SY + \phi(\nabla_X S)Y, \end{aligned} \tag{4.11}$$

where we used $\nabla_X(\phi SY) - \phi S(\nabla_X Y) = (\nabla_X \phi)SY + \phi(\nabla_X S)Y$ for any vector fields X and $Y \in TM$. Moreover, from (3.1) we get

$$(\nabla_X \phi)SY = \eta(SY)SX - g(SX, SY)\xi.$$

By this formula and (3.5), the equation (4.11) becomes

$$\begin{aligned} 0 &= R(\xi, X)Y + \eta(SY)SX - g(SX, SY)\xi + \phi(\nabla_X S)Y \\ &= g(X, Y)\xi - \eta(Y)X + g(BX, Y)A\xi - g(A\xi, Y)BX + g(\phi BX, Y)\phi A\xi \\ &\quad + g(X, \phi A\xi)\eta(Y)\phi A\xi - g(\phi A\xi, Y)\phi BX - g(\phi A\xi, Y)g(X, \phi A\xi)\xi \\ &\quad + g(SX, Y)S\xi - g(SX, SY)\xi + \phi(\nabla_X S)Y. \end{aligned} \tag{4.12}$$

Moreover, from our assumption of M being Hopf, the Reeb vector field ξ satisfies $S\xi = \alpha\xi$. Then, (4.12) can be written as

$$\begin{aligned} & g(X, Y)\xi - \eta(Y)X + g(BX, Y)A\xi - g(A\xi, Y)BX + g(\phi BX, Y)\phi A\xi \\ &\quad + g(X, \phi A\xi)\eta(Y)\phi A\xi - g(\phi A\xi, Y)\phi BX - g(\phi A\xi, Y)g(X, \phi A\xi)\xi \\ &\quad + \alpha g(SX, Y)\xi - g(SX, SY)\xi + \phi(\nabla_X S)Y = 0 \end{aligned} \tag{4.13}$$

for any tangent vector fields X and Y on M. Consequently, (4.13) is equivalent to the assumption of the Reeb vector field ξ being affine Killing.

5. Proof of Theorem 1.3

In this section, we want to show that the unit normal vector field N is singular for Hopf real hypersurfaces M in Q^m with affine Killing Reeb vector field. Then, we can use (4.13). So, if we put $Y = \xi$ in (4.13), it follows that

$$\begin{aligned} 0 &= \eta(X)\xi - X + g(A\xi, X)A\xi - \kappa BX + g(\phi A\xi, X)\phi A\xi + \phi(\nabla_X S)\xi \\ &= \eta(X)\xi - X + g(A\xi, X)A\xi - \kappa BX + g(\phi A\xi, X)\phi A\xi \\ &\quad - \alpha SX + \alpha^2 \eta(X)\xi - \phi S\phi SX, \end{aligned}$$

where we have used $(\nabla_X S)\xi = (X\alpha)\xi + \alpha\phi SX - S\phi SX$ and $\phi^2 SX = -SX + g(SX, \xi)\xi$. Applying the structure tensor ϕ of M to this equation becomes

$$-\phi X + g(A\xi, X)\phi A\xi - \kappa\phi BX - g(\phi A\xi, X)A\xi + \kappa g(\phi A\xi, X)\xi - \alpha\phi SX + S\phi SX = 0,$$

that is,

$$S\phi SX - \phi X + g(A\xi, X)\phi A\xi - g(\phi A\xi, X)A\xi + \kappa g(\phi A\xi, X)\xi = \alpha\phi SX + \kappa\phi BX. \tag{5.1}$$

Moreover, from (3.8) we obtain

$$(5.2)\qquad 2S\phi SX - 2\phi X + 2g(A\xi, X)\phi A\xi - 2g(\phi A\xi, X)A\xi + 2\kappa g(\phi A\xi, X)\xi = \alpha\phi SX + \alpha S\phi X + 2\kappa\eta(X)\phi A\xi.$$

Hence, both equations (5.1) and (5.2) give

$$(5.3)\qquad \alpha(\phi S - S\phi)X = 2\kappa\{\eta(X)\phi A\xi - \phi BX\}$$

for any tangent vector field X on M.

As mentioned in Remark 3.2, we know that *if either the Reeb function $\alpha = g(S\xi, \xi)$ or the smooth function $\kappa = g(A\xi, \xi)$ identically vanishes on M, then the unit normal vector field N becomes singular*. Thus, in the remaining part of this section we only consider the case of both α and κ being non-vanishing.

Proposition 5.1. *Let M be a Hopf real hypersurface with non-vanishing geodesic Reeb flow in the complex quadric Q^m, $m \geq 3$. If the Reeb vector field ξ of M is affine Killing and $\kappa \neq 0$, then the unit normal vector field N of M is $\mathfrak{A}$-principal.*

Proof. Putting $X = A\xi$ in (5.3) and using $BA\xi = A^2\xi - g(A^2\xi, N)N = \xi$, together with $\alpha \neq 0$, we get

$$(5.4)\qquad S\phi A\xi = \phi SA\xi + \sigma\phi A\xi,$$

where $\sigma = -\frac{2\kappa^2}{\alpha}$. Taking $X = \phi A\xi$ in (3.8) and using (5.4), together with $g(\phi A\xi, \phi A\xi) = 1 - \kappa^2$, we obtain

$$(5.5)\qquad S^2A\xi = -2\kappa^3\xi + (\alpha - \sigma)SA\xi.$$

On the other hand, taking the inner product of (4.13) with ξ and using $B\xi = A\xi$, we obtain

$$(5.6)\qquad g(X, Y) - \eta(X)\eta(Y) + \kappa g(BX, Y) - g(A\xi, X)g(A\xi, Y) - g(\phi A\xi, X)g(\phi A\xi, Y) + \alpha g(SX, Y) - g(SX, SY) = 0$$

for any tangent vector fields X and Y on M. Taking $X = A\xi$ in (5.6) we obtain

$$\alpha g(SA\xi, Y) - g(S^2A\xi, Y) = 0 \quad \forall Y \in TM,$$

which implies that

$$(5.7)\qquad S^2A\xi = \alpha SA\xi.$$

From this and (5.5), we see that

$$(5.8)\qquad SA\xi = \alpha\kappa\xi.$$

Moreover, from (5.8) equation (5.4) becomes

$$(5.9)\qquad S\phi A\xi = \sigma\phi A\xi.$$

Putting $X = \phi A\xi$ in (5.6), and using (5.9), we get

$$(5.10)\qquad \begin{aligned} 0 &= \phi A\xi + \kappa B\phi A\xi - (1 - \kappa^2)\phi A\xi + \alpha\sigma\phi A\xi - \sigma^2\phi A\xi \\ &= (2\kappa^2 + \alpha\sigma - \sigma^2)\phi A\xi \\ &= -\sigma^2\phi A\xi, \end{aligned}$$

where we used

$$\begin{aligned} B\phi A\xi &= A\phi A\xi - g(A\phi A\xi, N)N \\ &= A(-AN - \kappa N) - g(AN, -AN - \kappa N)N \\ &= -N - \kappa AN + g(AN, AN)N + \kappa g(AN, N)N \\ &= -\kappa AN + \kappa g(AN, N)N \\ &= \kappa\phi A\xi + \kappa^2 N + \kappa g(AN, N)N \\ &= \kappa\phi A\xi, \end{aligned}$$

together with $AN = -\phi A\xi - \kappa N$, $A^2X = X$ and $\kappa = g(A\xi, \xi) = -g(AN, N)$.

Since the smooth functions α and κ are non-vanishing on M, we get

$$\sigma = -\frac{2\kappa^2}{\alpha} \neq 0.$$

Thus, (5.10) tells that the vector field $\phi A\xi$ vanishes on M, that is, $\phi A\xi = 0$. Since $g(\phi A\xi, \phi A\xi) = 1 - \kappa^2$, it means $\kappa = \pm 1$. Meanwhile, from (3.3) we see that the smooth function $\kappa = g(A\xi, \xi)$ is given by $\kappa(t) = -\cos(2t)$ where $t \in [0, \frac{\pi}{4})$. By such two facts related to κ, we consequently have $t = 0$. This means that the unit normal vector field N satisfies $N = Z_1 \in V(A)$. Therefore, we claim that the unit normal vector field N is $\mathfrak{A}$-principal. □

Summing up Remark 3.2 and Proposition 5.1 we give a complete proof of our Theorem 1.3.

6. Proof of Theorem 1.4

In this section, unless otherwise stated, let M be a Hopf real hypersurface with affine Killing Reeb vector field in the complex quadric Q^m, $m \geq 3$.

First, by virtue of our Theorem 1.3, we consider the case where M has an $\mathfrak{A}$-principal normal vector field N in Q^m. Then, Theorem B tells that *M with $\mathfrak{A}$-principal normal vector field N is locally congruent to an open part of a tube around the m-dimensional sphere S^m*. Hereafter, we call such a model space (resp. a model space in Theorem A) *a real hypersurface of type* (B) (resp. *a real hypersurface of type* (A)) and denote such a model space by $(\mathcal{T}_B)$ (resp. $(\mathcal{T}_A)$).

As a converse of this statement, naturally, the following question arises.

> *Has the real hypersurface $(\mathcal{T}_B)$ of type (B) in Q^m an affine Killing Reeb vector field ξ?*

To solve this problem, we introduce the following proposition given in [30].

Proposition A. *Let $(\mathcal{T}_B)$ be the tube of radius $0 < r < \frac{\pi}{2\sqrt{2}}$ around the m-dimensional sphere S^m in Q^m. Then the following statements hold:*

(i) *$(\mathcal{T}_B)$ is a Hopf hypersurface.*
(ii) *The normal bundle of $(\mathcal{T}_B)$ consists of $\mathfrak{A}$-principal vector fields.*
(iii) *$(\mathcal{T}_B)$ has three distinct constant principal curvatures. The principal curvatures and corresponding principal curvature spaces of $(\mathcal{T}_B)$ are as follows:*

principal curvature	eigenspace	multiplicity
$\alpha = -\sqrt{2}\cot(\sqrt{2}r)$	$T_\alpha = \mathbb{R}JN$	1
$\lambda = \sqrt{2}\tan(\sqrt{2}r)$	$T_\lambda = V(A) \cap \mathcal{C} = \{X \in \mathcal{C} \mid AX = X\}$	$m-1$
$\mu = 0$	$T_\mu = JV(A) \cap \mathcal{C} = \{X \in \mathcal{C} \mid AX = -X\}$	$m-1$

(iv) $S\phi + \phi S = 2\tau\phi$, $\tau = -\frac{1}{\alpha} \neq 0$ *(contact hypersurface).*

By (i) and (ii) in Proposition A, we know that $(\mathcal{T}_B)$ is a Hopf real hypersurface with $\mathfrak{A}$-principal normal vector field N in Q^m, $m \geq 3$.

Assume that the Reeb vector field ξ of $(\mathcal{T}_B)$ satisfies the affine Killing property (4.11). It implies (4.13) for all tangent vector fields X and Y on $T(\mathcal{T}_B) = T_\alpha \oplus T_\lambda \oplus T_\mu$. Then, by Lemma 3.4 and (4.13) we obtain for $Y = \xi$

$$2\eta(X)\xi - X + AX - \alpha SX + \alpha^2\eta(X)\xi - \phi S\phi SX = 0, \tag{6.1}$$

where we have used $A\xi = -\xi$ and $\phi(\nabla_X S)\xi = \alpha^2\eta(X)\xi - \alpha SX - \phi S\phi SX$. Then, by Proposition A, the left side of (6.1) becomes

$$2\eta(X)\xi - X + AX - \alpha SX + \alpha^2\eta(X)\xi - \phi S\phi SX = \begin{cases} 0 & \text{if } X \in T_\alpha \\ 2X & \text{if } X \in T_\lambda \\ -2X & \text{if } X \in T_\mu. \end{cases} \tag{6.2}$$

Then, comparing (6.1) with (6.2), we get a contradiction. Summing up all the facts, we get the following

Lemma 6.1. *There does not exist any Hopf real hypersurface with affine Killing Reeb vector field in the complex quadric Q^m, $m \geq 3$, whose unit normal vector field N is $\mathfrak{A}$-principal.*

From this lemma, together with Theorem 1.3, it follows that

Proposition 6.2. *Let M be a Hopf real hypersurface with affine Killing Reeb vector field in the complex quadric Q^m, $m \geq 3$. Then, the unit normal vector field N in Q^m is $\mathfrak{A}$-isotropic.*

By virtue of Proposition 6.2, the unit normal vector field N of M is $\mathfrak{A}$-isotropic. Thus, N is expressed as

$$N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$$

for some orthonormal vector fields $Z_1, Z_2 \in V(A)$, where $V(A)$ denotes the $(+1)$-eigenspace of the complex conjugation $A \in \mathfrak{A}$. Since $AJ = -JA$ and $J^2X = -X$, it follows that

$$AN = \frac{1}{\sqrt{2}}(Z_1 - JZ_2),\ JN = \frac{1}{\sqrt{2}}(JZ_1 - Z_2),\ \text{ and }\ AJN = -\frac{1}{\sqrt{2}}(JZ_1 + Z_2).$$

Then it gives that

$$\kappa = g(\xi, A\xi) = g(JN, AJN) = 0 = -g(AN, N)$$

and

$$g(\xi, AN) = -g(JN, AN) = 0,$$

which means that the vector fields $AN = -\phi A\xi$ and $A\xi$ are tangent to M. From $\kappa = 0$ and

(5.3), it follows that

$$\alpha(\phi S - S\phi)X = 0 \tag{6.3}$$

for any tangent vector field X on M. If M has non-vanishing geodesic Reeb flow, that is, $\alpha \neq 0$, (6.3) gives that the shape operator S commutes with the structure tensor ϕ on M, that is, $S\phi = \phi S$. So, *the Reeb vector field ξ becomes Killing* if the Reeb function α is non-vanishing on M.

In the following proposition let us consider the case that M has vanishing geodesic Reeb flow, that is, $\alpha = 0$. In fact, from Lemma 3.5 we obtain:

Proposition 6.3. *Let M be a Hopf real hypersurface with affine Killing Reeb vector field in the complex quadric Q^m, $m \geq 3$. If M has vanishing geodesic Reeb flow, the Reeb vector field ξ is Killing. That is, the shape operator S of M commutes with the structure tensor ϕ on M.*

Proof. Since M is a Hopf real hypersurface with affine Killing Reeb vector field ξ in Q^m, we see that the unit normal vector field N is $\mathfrak{A}$-isotropic. That is, it implies $\kappa = g(A\xi, \xi) = 0$. By using this fact and our assumption $\alpha = 0$, if we take the inner product of (4.13) with ξ, it follows that

$$g(X, Y) - \eta(X)\eta(Y) - g(A\xi, X)g(A\xi, Y) - g(\phi A\xi, X)g(\phi A\xi, Y) - g(SX, SY) = 0$$

for any tangent vector fields X and Y on M. Then, we get

$$X - \eta(X)\xi - g(A\xi, X)A\xi - g(\phi A\xi, X)\phi A\xi - S^2X = 0 \tag{6.4}$$

for any $X \in TM$.

On the other hand, by the fact (b) in Lemma 3.5, the tangent bundle TM of M is given by

$$TM = [\xi] \oplus \mathcal{C} = [\xi] \oplus \mathcal{Q}^{\perp} \oplus \mathcal{Q},$$

where $\mathcal{C} = [\xi]^{\perp}$ and $\mathcal{Q}^{\perp} = \text{span}\{A\xi, AN\} \subset \mathcal{C}$. So, (6.4) provides

$$S^2X = X - \eta(X)\xi - g(A\xi, X)A\xi - g(\phi A\xi, X)\phi A\xi = \begin{cases} 0 & \text{if } X \in [\xi] \oplus \mathcal{Q}^{\perp} \\ X & \text{if } X \in \mathcal{Q}\,. \end{cases} \tag{6.5}$$

Now, choose some unit tangent vector field X_0 of $\mathcal{Q}$ such that $SX_0 = \lambda X_0$. Substituting $X = X_0$ in (6.5) yields $\lambda^2 = 1$, that is, $\lambda = \pm 1$. Moreover, from Lemma 3.5 we see that the corresponding unit vector field ϕX_0 is also a principal curvature vector field of M with principal curvature $\mu := \frac{1}{\lambda}$. That is, we obtain either $S\phi X_0 = \phi X_0$ if $SX_0 = X_0$ or $S\phi X_0 = -\phi X_0$ if $SX_0 = -X_0$. From this, we know that the eigenspace $T_\lambda = \{X \in \mathcal{Q} \subset \mathcal{C} \,|\, SX = \lambda X\}$ is ϕ-invariant, that is, $\phi T_\lambda = T_\lambda$.

On the other hand, it is well-known that the shape operator S is diagonalizable, that is, S has a basis of eigenvectors. From such a view point, we take

$$\mathfrak{B} = \left\{ \begin{array}{l} \xi, A\xi, AN, e_1, e_2 = \phi e_1, \ldots, e_{p-1}, e_p = \phi e_{p-1}, \\ \quad e_{p+1}, e_{p+2} = \phi e_{p+1}, \ldots, e_{2m-5}, e_{2m-4} := \phi e_{2m-5} \end{array} \right\}$$

as the basis of M satisfying

$$S\xi = SA\xi = SAN = 0$$

and

$$Se_k = \begin{cases} e_k & \text{for } k = 1, 2, \dots, p, \\ -e_k & \text{for } k = p+1, \dots, 2m-4. \end{cases}$$

Then, any tangent vector field $X \in \mathcal{Q}$ is expressed by the basis $\mathfrak{B}$ as

$$X = g(X,\xi)\xi + g(X,A\xi)A\xi + g(X,AN)AN + \sum_{k=1}^{2m-4} g(X,e_k)e_k = \sum_{k=1}^{2m-4} g(X,e_k)e_k. \tag{6.6}$$

Applying the operator ϕS of TM to (6.6) yields

$$\begin{aligned} \phi SX &= \sum_{k=1}^{2m-4} g(X,e_k)\phi Se_k = \sum_{k=1}^{p} g(X,e_k)\phi Se_k + \sum_{k=p+1}^{2m-4} g(X,e_k)\phi Se_k \\ &= \sum_{k=1}^{p} g(X,e_k)\phi e_k - \sum_{k=p+1}^{2m-4} g(X,e_k)\phi e_k \\ &= g(X,e_1)\phi e_1 + g(X,e_2)\phi e_2 + \cdots + g(X,e_{p-1})\phi e_{p-1} \\ &\quad + g(X,e_p)\phi e_p - g(X,e_{p+1})\phi e_{p+1} - g(X,e_{p+2})\phi e_{p+2} \\ &\quad - \cdots - g(X,e_{2m-5})\phi e_{2m-5} - g(X,e_{2m-4})\phi e_{em-4} \\ &= g(X,e_1)e_2 + g(X,e_2)\phi^2 e_1 + \cdots + g(X,e_{p-1})e_p \\ &\quad + g(X,e_p)\phi^2 e_{p-1} - g(X,e_{p+1})e_{p+2} - g(X,e_{p+2})\phi^2 e_{p+1} \\ &\quad - \cdots - \cdots - g(X,e_{2m-5})e_{2m-4} - g(X,e_{2m-4})\phi^2 e_{em-5} \\ &= g(X,e_1)e_2 - g(X,e_2)e_1 + \cdots + g(X,e_{p-1})e_p \\ &\quad - g(X,e_p)e_{p-1} - g(X,e_{p+1})e_{p+2} + g(X,e_{p+2})e_{p+1} \\ &\quad - \cdots - g(X,e_{2m-5})e_{2m-4} + g(X,e_{2m-4})e_{em-5} \\ &= g(X,e_1)e_2 + \cdots + g(X,e_{p-1})e_p + g(X,e_{p+2})e_{p+1} + \cdots \\ &\quad + g(X,e_{2m-4})e_{2m-5} - g(X,e_2)e_1 - \cdots - g(X,e_p)e_{p-1} \\ &\quad - g(X,e_{p+1})e_{p+2} - \cdots - g(X,e_{2m-5})e_{2m-4}, \end{aligned} \tag{6.7}$$

where we have used $\phi^2 e_k = -e_k + \eta(e_k)\xi = -e_k$ for any $k = 1, 2, \dots, 2m-4$.

On the other hand, by using linear combination with our basis and the symmetric property of S, the tangent vector field $S\phi X$ is given as follows:

$$\begin{aligned} S\phi X &= g(S\phi X,\xi)\xi + g(S\phi X,A\xi)A\xi + g(S\phi X,AN)AN \\ &\quad + g(S\phi X,e_1)e_1 + g(S\phi X,\phi e_1)\phi e_1 + \cdots + g(S\phi X,e_{p-1})e_{p-1} \\ &\quad + g(S\phi X,\phi e_{p-1})\phi e_{p-1} + g(S\phi X,e_{p+1})e_{p+1} + g(S\phi X,\phi e_{p+1})\phi e_{p+1} \\ &\quad + \cdots + g(S\phi X,e_{2m-5})e_{2m-5} + g(S\phi X,\phi e_{2m-5})\phi e_{2m-5} \\ &= g(\phi X,Se_1)e_1 + g(\phi X,S\phi e_1)\phi e_1 + \cdots + g(\phi X,Se_{p-1})e_{p-1} \\ &\quad + g(\phi X,S\phi e_{p-1})\phi e_{p-1} + g(\phi X,Se_{p+1})e_{p+1} + g(\phi X,S\phi e_{p+1})\phi e_{p+1} \\ &\quad + \cdots + g(\phi X,Se_{2m-5})e_{2m-5} + g(\phi X,S\phi e_{2m-5})\phi e_{2m-5} \\ &= g(\phi X,e_1)e_1 + g(\phi X,\phi e_1)\phi e_1 + \cdots + g(\phi X,e_{p-1})e_{p-1} \\ &\quad + g(\phi X,\phi e_{p-1})\phi e_{p-1} - g(\phi X,e_{p+1})e_{p+1} - g(\phi X,\phi e_{p+1})\phi e_{p+1} \end{aligned} \tag{6.8}$$

$$
\begin{aligned}
&- \cdots - g(\phi X, e_{2m-5})e_{2m-5} - g(\phi X, \phi e_{2m-5})\phi e_{2m-5} \\
&= -g(X, \phi e_1)e_1 - g(X, \phi^2 e_1)\phi e_1 + \cdots - g(X, \phi e_{p-1})e_{p-1} \\
&\quad - g(X, \phi^2 e_{p-1})\phi e_{p-1} + g(X, \phi e_{p+1})e_{p+1} + g(X, \phi^2 e_{p+1})\phi e_{p+1} \\
&\quad + \cdots + g(X, \phi e_{2m-5})e_{2m-5} + g(X, \phi^2 e_{2m-5})\phi e_{2m-5} \\
&= -g(X, \phi e_1)e_1 + g(X, e_1)\phi e_1 + \cdots - g(X, \phi e_{p-1})e_{p-1} + g(X, e_{p-1})\phi e_{p-1} \\
&\quad + g(X, \phi e_{p+1})e_{p+1} - g(X, e_{p+1})\phi e_{p+1} \\
&\quad + \cdots + g(X, \phi e_{2m-5})e_{2m-5} - g(X, e_{2m-5})\phi e_{2m-5} \\
&= -g(X, e_2)e_1 + g(X, e_1)e_2 + \cdots - g(X, e_p)e_{p-1} + g(X, e_{p-1})e_p \\
&\quad + g(X, e_{p+2})e_{p+1} - g(X, e_{p+1})e_{p+2} \\
&\quad + \cdots + g(X, e_{2m-4})e_{2m-5} - g(X, e_{2m-5})e_{2m-4} \\
&= g(X, e_1)e_2 + \cdots + g(X, e_{p-1})e_p + g(X, e_{p+2})e_{p+1} + \cdots + g(X, e_{2m-4})e_{2m-5} \\
&\quad - g(X, e_2)e_1 - \cdots - g(X, e_p)e_{p-1} - g(X, e_{p+1})e_{p+2} - \cdots - g(X, e_{2m-5})e_{2m-4}
\end{aligned}
$$

for any tangent vector field $X \in \mathcal{Q}$. From (6.7) and (6.8) we see that the shape operator S of M commutes with the structure tensor ϕ on M, that is, $S\phi = \phi S$ on $\mathcal{Q}$.

Bearing in mind $S\xi = SA\xi = SAN = 0$, together with $\phi^2 A\xi = -A\xi$, we naturally obtain that the commuting property $S\phi = \phi S$ holds on $[\xi] \oplus \mathcal{Q}^{\perp}$.

From these facts, we conclude that the shape operator S commutes with the structure tensor ϕ on M, when the Reeb function α identically vanishes on M. It completes the proof of our proposition. □

Consequently, Proposition 6.3 and (6.3) assure that *the affine Killing Reeb vector field ξ of M must be Killing*. Moreover, by virtue of this fact and Lemma 4.1 we give a complete proof of Theorem 1.4.

7. Proof of Corollary 1.5

As a generalized notion of Killing vector field of Riemannian manifolds $(\bar{M}, g)$, we introduced conformal vector fields in the introduction. In fact, in (1.7)

$$(\mathcal{L}_V g)(X, Y) = 2\delta g(X, Y)$$

for any vector fields X and Y and the smooth function $\delta = 0$ on $\bar{M}$ implies that any conformal Killing vector field V satisfies the Killing property (1.1). From this, it assures:

Fact A. *Any Killing vector field V of $\bar{M}$ becomes conformal Killing with $\delta = 0$.*

Motivated by this assertion, let us consider the converse problem of Fact A. That is, we prove that if the Reeb vector field ξ of a Hopf real hypersurface in Q^m is conformal, it satisfies the Killing property as follows:

Lemma 7.1. *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$, with conformal Killing Reeb vector field ξ. Then, the Reeb vector field ξ becomes Killing. That is, the shape operator S of M commutes with the structure tensor ϕ on M.*

Proof. From the assumption of the Reeb vector field being conformal Killing, (1.8) gives

$$\phi SX - S\phi X = 2\delta X \tag{7.1}$$

for any tangent vector field X on M. Substituting $X = \xi$ in this equation and using M being Hopf implies the smooth function δ identically vanishes on M. From this, the conformal Killing property (7.1) becomes for any tangent vector field X on M

$$\phi SX - S\phi X = 0,$$

which means that the vector field ξ is Killing. □

From Lemma 7.1, together with Theorem A, we give a complete proof of Corollary 1.5.

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