Inoue, A.
Osaka J. Math.
60 (2023), 597–611

THE KNOT QUANDLE OF THE TWIST-SPUN TREFOIL IS A CENTRAL EXTENSION OF A SCHLÄFLI QUANDLE

Ayumu INOUE

(Received June 23, 2021, revised May 13, 2022)

Abstract

A quandle is an algebraic system which excels at describing limited symmetries of a space. We introduce the concept of Schläfli quandles which are defined relating to chosen rotational symmetries of regular tessellations. On the other hand, quandles have a good chemistry with knot theory. Associated with a knot we have its knot quandle. We show that the knot quandle of the m-twist-spun trefoil is a central extension of the Schläfli quandle related to the regular tessellation $\{3, m\}$ in the sense of the Schläfli symbol if $m \geq 3$.

1. Introduction

Although all symmetries of a space form a group, its subset which consists of particular symmetries does not in general. For instance, the reflective symmetries of a regular polygon which fix at least one vertex of the polygon do not form a group by themselves. On the other hand, they form a quandle which is an algebraic system. This quandle is called a dihedral quandle. Various kinds of specified symmetries form quandles. For example, the $(2\pi/3)$-rotations of a regular tetrahedron about axes passing through its center and the vertices form the tetrahedral quandle. We also have hexahedral, octahedral, dodecahedral, and icosahedral quandles in the same manner.

Regular polyhedra are identified with regular tessellations of the 2-dimensional spherical space $\mathbb{S}^2$. Two copies of a regular polygon also tessellate $\mathbb{S}^2$ regularly. In this view, each of dihedral and polyhedral quandles is considered as a quandle consisting of the rotations of $\mathbb{S}^2$ about the vertices of a regular tessellation by the same angle which preserve the tessellation setwise. In a similar way, we may have quandles related to regular tessellations of spherical, Euclidean, and hyperbolic spaces. Since those tessellations are characterized by Schläfli symbols, we call them Schläfli quandles[1]. In this paper, we focus on Schläfli quandles related to regular tessellations $\{3, m\}$ $(m \geq 2)$, $\{3, 3, 4\}$, $\{3, 4, 3\}$ and $\{3, 3, 5\}$ in the sense of Schläfli symbols. Since the latter three tessellations are respectively identified with 16-, 24-, and 600-cells, we call correspondent Schläfli quandles 16-, 24-, and 600-cell quandles respectively.

Quandles also have a good chemistry with knot theory. Associated with a knot, we have its knot quandle in a similar way to its knot group. In this paper, we focus on the knot quandle of the m-twist-spun trefoil. Here, the m-twist-spun trefoil is a typical 2-knot, i.e., a 2-sphere embedded in the standard 4-sphere smoothly and locally flatly. We see that the

2020 Mathematics Subject Classification. 57K12, 52B15.

[1]The author has called a Schläfli quandle a mosaic quandle in early version of this paper [6].

16-, 24-, or 600-cell quandle is respectively isomorphic to the knot quandle of the 3-, 4-, or 5-twist-spun trefoil (Theorem 6.1). It is known by Clark et al. [2] with computer calculations that the knot quandle of the 3- or 4-twist-spun trefoil is a central extension of the Schläfli quandle related to $\{3,3\}$ or $\{3,4\}$ respectively, in terms of this paper. We show that this relationship between the knot quandle of the m-twist-spun trefoil and the Schläfli quandle related to $\{3,m\}$ is lasting forever, i.e., the knot quandle of the m-twist-spun trefoil is a central extension of the Schläfli quandle related to $\{3,m\}$ if $m \geq 3$ (Theorem 6.3).

2. Quandle

In this section, we recall some notions about quandles briefly. We refer the reader to [7] for more details.

A *quandle* is a non-empty set X equipped with a binary operation $* : X \times X \to X$ satisfying the following axioms:

(Q1) For each $x \in X$, $x * x = x$

(Q2) For each $x \in X$, the map $*x : X \to X$ ($w \mapsto w * x$) is bijective

(Q3) For each $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$

Let us see a few examples of quandles. Consider a set X which consists of the vertices of a regular n-gon P in $\mathbb{R}^2$ ($n \geq 3$). For each $v \in X$, let l_v be the line passing through the center of P and v. Then for each $v, w \in X$ we have $v * w \in X$ which is the image of v by the reflection through l_w. It is easy to check that $*$ satisfies the axioms of a quandle. We call this quandle the *dihedral quandle* of order n.

Similarly, let X be a set consisting of the vertices of a regular polyhedron P in $\mathbb{R}^3$ and l_v the line passing through the center of P and $v \in X$. Suppose that θ is $\pi/2$ if P is an octahedron, $2\pi/5$ if P is an icosahedron, otherwise $2\pi/3$. Then for each $v, w \in X$ we have $v * w \in X$ which is the image of v by the θ-rotation about l_w (counterclockwise when we see the center from w). It is routine to see that $*$ satisfies the axioms of a quandle. We call this quandle the *tetrahedral*, *hexahedral*, *octahedral*, *dodecahedral*, or *icosahedral quandle* respectively, if P is a tetrahedron, hexahedron, octahedron, dodecahedron, or icosahedron.

The notions of homomorphism, epimorphism, isomorphism and automorphism are appropriately defined for quandles. Suppose that X is a quandle. Axioms (Q2) and (Q3) assert that for each $x \in X$ the map $*x$ is an automorphism of X. Those automorphisms $*x$ ($x \in X$) generate the *inner automorphism group* $\mathrm{Inn}(X)$ of X which is a subgroup of the automorphism group of X. We call an element of $\mathrm{Inn}(X)$ an *inner automorphism* of X. We call X to be *connected* if $\mathrm{Inn}(X)$ acts transitively on X.

3. Schläfli quandle

In this section, we introduce the concept of Schläfli quandles. Then we concretely construct Schläfli quandles related to regular tessellations $\{3,m\}$ ($m \geq 2$), $\{3,3,4\}$, $\{3,4,3\}$ and $\{3,3,5\}$ in the sense of Schläfli symbols.

Consider a regular tessellation T of a spherical, Euclidean, or hyperbolic space $\mathbb{B}$. Let V be the set consisting of vertices of T. For each $v \in V$ choose a rotation r_v of $\mathbb{B}$ which fixes v and preserves T setwise. If we have

$$r_{r_w(v)} = r_w \circ r_v \circ r_w^{-1} \tag{1}$$

for each $v, w \in V$, consider the set $\{(v, r_v) \mid v \in V\}$ and define its binary operation $*$ by $(v, r_v) * (w, r_w) = (r_w(v), r_{r_w(v)})$. It is easy to see that $*$ satisfies the axioms of a quandle. We call this quandle a *Schläfli quandle* related to T.

Let us consider some concrete Schläfli quandles. We first focus on the regular tessellation $\{3, m\}$ ($m \geq 2$). We note that $\{3, m\}$ tessellates $\mathbb{S}^2$ if $2 \leq m \leq 5$, the Euclidean plane if $m = 6$, otherwise the hyperbolic plane (see Figure 1). For each vertex v of $\{3, m\}$, let r_v be the rotation about v by the angle $2\pi/m$. Then the condition (1) is obviously satisfied for each vertices v, w of $\{3, m\}$. We thus have a Schläfli quandle related to $\{3, m\}$. We note that the Schläfli quandles related to $\{3, 2\}$, $\{3, 3\}$, $\{3, 4\}$ and $\{3, 5\}$ are obviously isomorphic to the dihedral quandle of order 3, tetrahedral, octahedral and icosahedral quandles respectively.

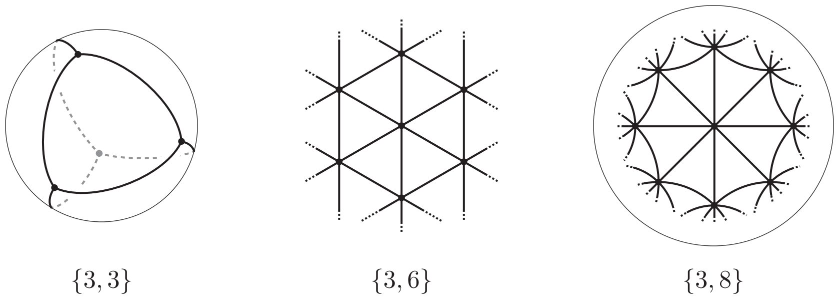

Fig. 1. Regular tessellations $\{3, 3\}$, $\{3, 6\}$ and $\{3, 8\}$, for example

Remark 3.1. Regular tessellations $\{3, m\}$ converge to the Farey tessellation $\{3, \infty\}$ as m goes to infinity. We thus have a Schläfli quandle related to $\{3, \infty\}$ as the "limit" of the Schläfli quandles $\{3, m\}$. Considering a famous relationship between $\{3, \infty\}$ and the mapping class group of a torus, it is routine to see that the Schläfli quandle related to $\{3, \infty\}$ is isomorphic to the Dehn quandle of a torus (see [8], for example, for a Dehn quandle). It is known by Niebrzydowski and Przytycki [8] that the Dehn quandle of a torus (i.e., the Schläfli quandle related to $\{3, \infty\}$) is isomorphic to the knot quandle of the trefoil. In contrast with the fact, we will see that the knot quandle of the m-twist-spun trefoil is a central extension of the Schläfli quandle related to $\{3, m\}$ if $m \geq 3$ (Theorem 6.3). We note that the knot quandle of the 2-twist-spun trefoil is isomorphic to the Schläfli quandle related to $\{3, 2\}$ (Remark 6.2).

We next focus on regular tessellations $\{3, 3, 4\}$, $\{3, 4, 3\}$ and $\{3, 3, 5\}$ of the 3-dimensional spherical space $\mathbb{S}^3$. In the remaining, we identify $\mathbb{S}^3$ with the unit sphere in $\mathbb{R}^4$. We may assume that sets of the vertices of $\{3, 3, 4\}$, $\{3, 4, 3\}$ and $\{3, 3, 5\}$ are respectively

$$\begin{aligned} V_{\{3,3,4\}} &= \{\pm \boldsymbol{e}_1, \pm \boldsymbol{e}_2, \pm \boldsymbol{e}_3, \pm \boldsymbol{e}_4\}, \\ V_{\{3,4,3\}} &= \{\pm \boldsymbol{e}_i \pm \boldsymbol{e}_j \mid 1 \leq i < j \leq 4\}, \\ V_{\{3,3,5\}} &= \{\pm \boldsymbol{e}_1, \pm \boldsymbol{e}_2, \pm \boldsymbol{e}_3, \pm \boldsymbol{e}_4\} \cup \left\{ \frac{1}{2}(\pm \boldsymbol{e}_1 \pm \boldsymbol{e}_2 \pm \boldsymbol{e}_3 \pm \boldsymbol{e}_4) \right\} \\ &\quad \cup \left\{ \frac{1}{2}\left(\pm \phi\, \boldsymbol{e}_{\sigma(1)} \pm \boldsymbol{e}_{\sigma(2)} \pm \phi^{-1} \boldsymbol{e}_{\sigma(3)}\right) \,\middle|\, \sigma \in A_4 \right\}. \end{aligned}$$

Here, $\boldsymbol{e}_i \in \mathbb{R}^4$ denotes the column vector whose j-th entry is δ_{ij}, ϕ the golden ratio

$\left(1+\sqrt{5}\right)/2$, and A_4 the alternating group on $\{1,2,3,4\}$. Associated with $v \in V_S$ ($S \in \{\{3,3,4\}, \{3,4,3\}, \{3,3,5\}\}$), we define the 4×4 matrix R_v as follows:

► $S = \{3,3,4\}$

$$R_{\pm e_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad R_{\pm e_2} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$R_{\pm e_3} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad R_{\pm e_4} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

► $S = \{3,4,3\}$

$$R_{\pm(e_1+e_2)} = R_{\pm(e_1-e_2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$R_{\pm(e_3+e_4)} = R_{\pm(e_3-e_4)} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$R_{\pm(e_1+e_3)} = R_{\pm(e_2+e_4)} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \end{pmatrix},$$

$$R_{\pm(e_1-e_3)} = R_{\pm(e_2-e_4)} = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix},$$

$$R_{\pm(e_1+e_4)} = R_{\pm(e_2-e_3)} = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

$$R_{\pm(e_2+e_3)} = R_{\pm(e_1-e_4)} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}.$$

► $S = \{3, 3, 5\}$

$$R_{\pm e_1} = R_{\pm\frac{1}{2}(\phi e_1+e_2+\phi^{-1}e_3)} = R_{\pm\frac{1}{2}(\phi e_1-e_2-\phi^{-1}e_3)} = R_{\pm\frac{1}{2}(\phi^{-1}e_1+\phi e_2+e_3)} = R_{\pm\frac{1}{2}(\phi^{-1}e_1-\phi e_2-e_3)} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & \phi & -\phi^{-1} \\ 0 & \phi & -\phi^{-1} & 1 \\ 0 & \phi^{-1} & -1 & -\phi \end{pmatrix},$$

$$R_{\pm e_2} = R_{\pm\frac{1}{2}(e_1+\phi e_2+\phi^{-1}e_4)} = R_{\pm\frac{1}{2}(e_1-\phi e_2+\phi^{-1}e_4)} = R_{\pm\frac{1}{2}(\phi e_1+\phi^{-1}e_2+e_4)} = R_{\pm\frac{1}{2}(\phi e_1-\phi^{-1}e_2+e_4)} = \frac{1}{2}\begin{pmatrix} 1 & 0 & \phi^{-1} & \phi \\ 0 & 2 & 0 & 0 \\ -\phi^{-1} & 0 & -\phi & 1 \\ \phi & 0 & -1 & -\phi^{-1} \end{pmatrix},$$

$$R_{\pm e_3} = R_{\pm\frac{1}{2}(e_1+\phi^{-1}e_3-\phi e_4)} = R_{\pm\frac{1}{2}(e_1-\phi^{-1}e_3-\phi e_4)} = R_{\pm\frac{1}{2}(\phi^{-1}e_1+\phi e_3-e_4)} = R_{\pm\frac{1}{2}(\phi^{-1}e_1-\phi e_3-e_4)} = \frac{1}{2}\begin{pmatrix} -\phi^{-1} & 1 & 0 & -\phi \\ -1 & -\phi & 0 & -\phi^{-1} \\ 0 & 0 & 2 & 0 \\ -\phi & \phi^{-1} & 0 & 1 \end{pmatrix},$$

$$R_{\pm e_4} = R_{\pm\frac{1}{2}(e_2-\phi e_3+\phi^{-1}e_4)} = R_{\pm\frac{1}{2}(e_2-\phi e_3-\phi^{-1}e_4)} = R_{\pm\frac{1}{2}(\phi^{-1}e_2-e_3+\phi e_4)} = R_{\pm\frac{1}{2}(\phi^{-1}e_2-e_3-\phi e_4)} = \frac{1}{2}\begin{pmatrix} -\phi & 1 & \phi^{-1} & 0 \\ -1 & -\phi^{-1} & -\phi & 0 \\ -\phi^{-1} & -\phi & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

$$R_{\pm\frac{1}{2}(e_1+e_2-e_3+e_4)} = R_{\pm\frac{1}{2}(e_1-\phi^{-1}e_3+\phi e_4)} = R_{\pm\frac{1}{2}(\phi e_2-\phi^{-1}e_3-e_4)} = R_{\pm\frac{1}{2}(\phi^{-1}e_1+\phi e_2-e_3)} = R_{\pm\frac{1}{2}(\phi^{-1}e_1-e_2+\phi e_4)} = \frac{1}{2}\begin{pmatrix} -\phi^{-1} & 1 & 0 & \phi \\ 0 & 1 & -\phi & -\phi^{-1} \\ -\phi & -1 & -\phi^{-1} & 0 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

$$R_{\pm\frac{1}{2}(e_1-e_2-e_3-e_4)} = R_{\pm\frac{1}{2}(e_1+\phi^{-1}e_2+\phi e_3)} = R_{\pm\frac{1}{2}(\phi e_1-\phi^{-1}e_2-e_4)} = R_{\pm\frac{1}{2}(e_2+\phi e_3+\phi^{-1}e_4)} = R_{\pm\frac{1}{2}(\phi e_1+e_3-\phi^{-1}e_4)} = \frac{1}{2}\begin{pmatrix} 1 & 0 & \phi^{-1} & -\phi \\ -1 & -\phi^{-1} & \phi & 0 \\ 1 & 1 & 1 & 1 \\ -1 & \phi & 0 & -\phi^{-1} \end{pmatrix},$$

$$R_{\pm\frac{1}{2}(e_1-e_2+e_3-e_4)} = R_{\pm\frac{1}{2}(e_1-\phi^{-1}e_2+\phi e_3)} = R_{\pm\frac{1}{2}(\phi^{-1}e_2+e_3+\phi e_4)} = R_{\pm\frac{1}{2}(\phi^{-1}e_1+\phi e_3+e_4)} = R_{\pm\frac{1}{2}(\phi^{-1}e_1-e_2-\phi e_4)} = \frac{1}{2}\begin{pmatrix} -\phi^{-1} & 0 & \phi & -1 \\ -\phi & -\phi^{-1} & 0 & 1 \\ 1 & -1 & 1 & 1 \\ 0 & \phi & \phi^{-1} & 1 \end{pmatrix},$$

$$R_{\pm\frac{1}{2}(e_1-e_2-e_3+e_4)} = R_{\pm\frac{1}{2}(e_1+\phi e_2-\phi^{-1}e_4)} = R_{\pm\frac{1}{2}(\phi e_1-e_3+\phi^{-1}e_4)}$$
$$= R_{\pm\frac{1}{2}(\phi e_1+e_2-\phi^{-1}e_3)} = R_{\pm\frac{1}{2}(\phi e_2+\phi^{-1}e_3-e_4)} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & 1 \\ \phi^{-1} & 1 & 0 & -\phi \\ -\phi & 1 & -\phi^{-1} & 0 \\ 0 & -1 & -\phi & -\phi^{-1} \end{pmatrix},$$

$$R_{\pm\frac{1}{2}(e_1+e_2-e_3-e_4)} = R_{\pm\frac{1}{2}(e_1+\phi^{-1}e_2-\phi e_3)} = R_{\pm\frac{1}{2}(\phi^{-1}e_2+e_3-\phi e_4)}$$
$$= R_{\pm\frac{1}{2}(\phi^{-1}e_1+e_2-\phi e_4)} = R_{\pm\frac{1}{2}(\phi^{-1}e_1-\phi e_3+e_4)} = \frac{1}{2}\begin{pmatrix} -\phi^{-1} & \phi & -1 & 0 \\ 0 & -\phi^{-1} & -1 & -\phi \\ -\phi & 0 & 1 & -\phi^{-1} \\ -1 & -1 & -1 & 1 \end{pmatrix},$$

$$R_{\pm\frac{1}{2}(e_1+e_2+e_3+e_4)} = R_{\pm\frac{1}{2}(e_1-\phi e_2-\phi^{-1}e_4)} = R_{\pm\frac{1}{2}(\phi e_1+e_3+\phi^{-1}e_4)}$$
$$= R_{\pm\frac{1}{2}(\phi e_2+\phi^{-1}e_3+e_4)} = R_{\pm\frac{1}{2}(\phi e_1-e_2+\phi^{-1}e_3)} = \frac{1}{2}\begin{pmatrix} 1 & -\phi^{-1} & \phi & 0 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & -\phi^{-1} & \phi \\ 1 & \phi & 0 & -\phi^{-1} \end{pmatrix},$$

$$R_{\pm\frac{1}{2}(e_1-e_2+e_3+e_4)} = R_{\pm\frac{1}{2}(e_1+\phi^{-1}e_3+\phi e_4)} = R_{\pm\frac{1}{2}(\phi e_2-\phi^{-1}e_3+e_4)}$$
$$= R_{\pm\frac{1}{2}(\phi^{-1}e_1+e_2+\phi e_4)} = R_{\pm\frac{1}{2}(\phi^{-1}e_1-\phi e_2+e_3)} = \frac{1}{2}\begin{pmatrix} -\phi^{-1} & 0 & \phi & 1 \\ -1 & 1 & -1 & 1 \\ 0 & -\phi & -\phi^{-1} & 1 \\ \phi & \phi^{-1} & 0 & 1 \end{pmatrix},$$

$$R_{\pm\frac{1}{2}(e_1+e_2+e_3-e_4)} = R_{\pm\frac{1}{2}(e_1-\phi^{-1}e_2-\phi e_3)} = R_{\pm\frac{1}{2}(\phi e_1+\phi^{-1}e_2-e_4)}$$
$$= R_{\pm\frac{1}{2}(e_2+\phi e_3-\phi^{-1}e_4)} = R_{\pm\frac{1}{2}(\phi e_1-e_3-\phi^{-1}e_4)} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & -\phi^{-1} & 1 & -\phi \\ -\phi^{-1} & \phi & 1 & 0 \\ -\phi & 0 & -1 & -\phi^{-1} \end{pmatrix}.$$

For each $v \in V_S$, let $r_v : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation sending $\boldsymbol{x}$ to $R_v\boldsymbol{x}$. We note that r_v is respectively a $(2\pi/3)$-, $(\pi/2)$-, or $(2\pi/5)$-rotation about a plane in which v and the origin of $\mathbb{R}^4$ lie, if S is $\{3,3,4\}$, $\{3,4,3\}$ or $\{3,3,5\}$. It is routine to check that the condition (1) is satisfied for each $v, w \in V_S$. We thus have Schläfli quandles related to $\{3,3,4\}$, $\{3,4,3\}$ and $\{3,3,5\}$. We call them *16-*, *24-*, and *600-cell quandles* respectively, since convex hulls of $V_{\{3,3,4\}}$, $V_{\{3,4,3\}}$ and $V_{\{3,3,5\}}$ in $\mathbb{R}^4$ are respectively known as 16-, 24-, and 600-cells.

REMARK 3.2. 16- and 24-cell quandles are referred in GAP package Rig [12] as SmallQuandle$(8,1)$ and SmallQuandle$(24,2)$ respectively. We note that they had no geometrical explanations before this.

REMARK 3.3. Reflections of a metric space form a quandle under suitable conditions, too. This quandle is called a Coxeter quandle (see [4] for example).

4. Presentation of a quandle

As well as groups, we have presentations of quandles. Since we will utilize them for our arguments, we briefly recall some notions about presentations of quandles in this section. We refer the reader to [7] for more details.

Let S be a non-empty set, $F(S)$ the free group on S, and $FQ(S)$ the union of the conjugacy classes of $F(S)$ each of which contains an element of S. It is easy to see that the binary operation $*$ on $FQ(S)$ given by

$$(g^{-1}sg) * (h^{-1}th) = (gh^{-1}th)^{-1}s(gh^{-1}th) \quad (s, t \in S,\ g, h \in F(S))$$

satisfies the axioms of a quandle. We call this quandle the *free quandle* on S.

For a given subset R of $FQ(S) \times FQ(S)$, we consider to enlarge R by repeating the following moves:

(a) For each $x \in FQ(S)$, add (x, x) in R
(b) For each $(x, y) \in R$, add (y, x) in R
(c) For each $(x, y), (y, z) \in R$, add (x, z) in R
(d) For each $(x, y) \in R$ and $s \in S$, add $(x * s, y * s)$ and $(x *^{-1} s, y *^{-1} s)$ in R
(e) For each $(x, y) \in R$ and $z \in FQ(S)$, add $(z * x, z * y)$ and $(z *^{-1} x, z *^{-1} y)$ in R

Here, $x *^i y$ denotes the element $(* y)^i(x)$ for each $i \in \mathbb{Z}$. A *consequence* of R is an element of an expanded R by a finite sequence of the above moves. Let $x \sim_R y$ denote that (x, y) is a consequence of R. Then $\sim_R$ is obviously an equivalence relation on $FQ(S)$. The quotient $FQ(S)/\sim_R$ inherits $*$ from $FQ(S)$, and $*$ on $FQ(S)/\sim_R$ still satisfies the axioms of a quandle.

A quandle X is said to have a *presentation* $\langle S \,|\, R\rangle$ if X is isomorphic to the quandle $FQ(S)/\sim_R$. A presentation $\langle S \,|\, R\rangle$ is said to be *finite* if both S and R are finite sets. We refer to an element of S and R as a *generator* and a *relation* of $\langle S \,|\, R\rangle$ respectively. We write a consequence $x \sim_R y$ as $x = y$ and abbreviate a finite presentation $\langle \{s_1, s_2, \ldots, s_n\} \mid \{r_1, r_2, \ldots, r_m\}\rangle$ as $\langle s_1, s_2, \ldots, s_n \mid r_1, r_2, \ldots, r_m\rangle$ in the remaining. Fenn and Rourke [4] essentially showed the following theorem which is similar to the Tietze's theorem for group presentations. We will refer to the moves (T1), (T2) and their inverses as *Tietze moves*.

Theorem 4.1. *Assume that a quandle has two distinct finite presentations. Then the presentations are related to each other by a finite sequence of the following moves or their inverses:*

(T1) *Choose a consequence of the set of relations, and then add it to the set of relations*
(T2) *Choose an element x of the free quandle on the set of generators, and then introduce a new generator s in the set of generators and the new relation $s = x$ in the set of relations*

Suppose that S is a set and X a quandle. A map $f : S \to X$ naturally induces a homomorphism $f_\sharp : FQ(S) \to X$. Further f induces a well-defined homomorphism $f_* : FQ(S)/\sim_R \to X$ if we have $f_\sharp(x) = f_\sharp(y)$ for each relation $x = y$ in R. We note that both $f_\sharp$ and f_* are surjective if the image of f generates X.

5. Knot quandle of the twist-spun trefoil

In this section, we review the twist-spun trefoil and its knot quandle rapidly. We refer the reader [7] for more details.

Consider the oriented knotted arc k, depicted in the left-hand side of Figure 2, which is properly embedded in the upper half space $\mathbb{R}^3_+$. Choose a 3-ball B in $\mathbb{R}^3_+$ so that B wholly contains the knotted part of k (see the left-hand side of Figure 2). We assume that k intersects with ∂B only at the north and south poles of B. Suppose that m is a positive integer. Spin $\mathbb{R}^3_+$ 360 degrees in $\mathbb{R}^4$ along $\partial\mathbb{R}^3_+$, and simultaneously rotate B 360m degrees along the axis of B passing through the north and south poles. Then the locus of k yields an oriented 2-knot after the one point compactification of $\mathbb{R}^4$. We call this 2-knot the *m-twist-spun trefoil.* In the remaining, we let $\tau^m 3_1$ denote the m-twist-spun trefoil.

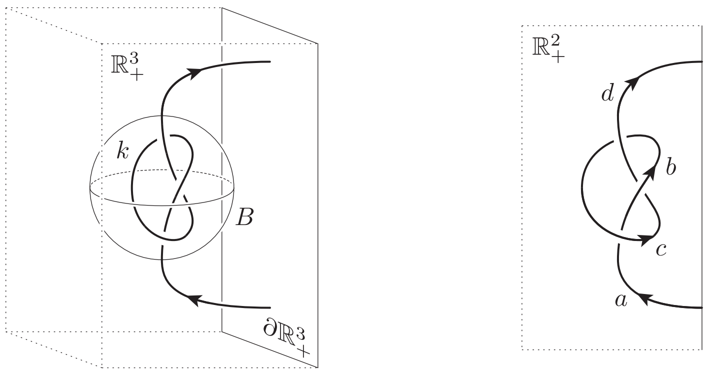

Fig.2. An oriented knotted arc k called the long trefoil and a 3-ball B which wholly contains the knotted part of k (left), and a diagram of k (right)

It is known by Zeeman [13] that $\tau^m 3_1$ is a fibered 2-knot (see [9], for example, for fiberedness of a knot). Therefore, in light of Corollary 3.2 of [5], the knot quandle of $\tau^m 3_1$ is defined as follows, although it is different from the usual way. Let G_m be the fundamental group of a fiber of $\tau^m 3_1$, and φ the monodromy of $\tau^m 3_1$ (which is an automorphism of G_m). Then the *knot quandle of the m-twist-spun trefoil* is defined to be G_m equipped with the binary operation $*$ given by $x * y = \varphi(xy^{-1})y$. In the remaining, we let Q_m denote the knot quandle of $\tau^m 3_1$, although Q_m coincides with G_m as sets. We will count G_m as a group and Q_m as a quandle.

We here study about G_m and φ for subsequent arguments. Since a fiber of $\tau^m 3_1$ has the surgery description depicted in Figure 3, we have the presentation

$$\langle \gamma_1, \gamma_2, \dots, \gamma_m \mid \gamma_1 = \gamma_2\gamma_m,\ \gamma_2 = \gamma_3\gamma_1,\ \dots,\ \gamma_{m-1} = \gamma_m\gamma_{m-2},\ \gamma_m = \gamma_1\gamma_{m-1} \rangle$$

of G_m (see [9], for example, for a surgery description of a 3-manifold). Here, γ_i is the loop depicted in Figure 3. It is easy to see that φ maps γ_i to γ_{i+1} $(1 \leq i \leq m-1)$ and γ_m to γ_1. Let δ be the loop depicted in Figure 3. Since

$$\delta = \gamma_2\gamma_1^{-1}\gamma_2^{-1}\gamma_1 = \cdots = \gamma_m\gamma_{m-1}^{-1}\gamma_m^{-1}\gamma_{m-1} = \gamma_1\gamma_m^{-1}\gamma_1^{-1}\gamma_m,$$

$\varphi(\delta)$ and δ are the same element in G_m. We note that $\delta \neq 1$ if and only if $m \geq 3$. It is routine to check that we have

(2) $$(((\delta^i *^\varepsilon \gamma_1) *^\varepsilon 1) *^\varepsilon 1) *^\varepsilon \gamma_1 = \delta^{i+\varepsilon},$$

(3) $$\gamma *^\varepsilon \delta^i = \gamma *^\varepsilon \delta^j,$$

(4) $$g(\delta^i) = \delta^i g(1)$$

for each $i, j \in \mathbb{Z}$, $\varepsilon \in \{\pm 1\}$, $\gamma \in G_m$, and $g \in \mathrm{Inn}(Q_m)$.

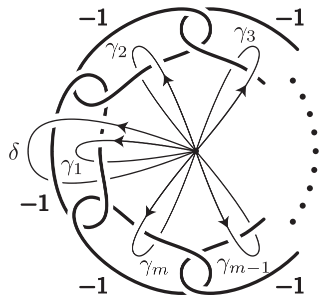

Fig. 3. A surgery description of a fiber of $\tau^m 3_1$ (thick lines) and typical elements of G_m (thin lines)

Since k has the diagram depicted in the right-hand side of Figure 2, it is known by Satoh [11] that Q_m has presentations

$$\begin{aligned} &\langle a, b, c, d \mid a * c = b,\ b * d = c,\ c * b = d,\ b *^m a = b,\ c *^m a = c\rangle \\ &= \langle a, b, c \mid a * c = b,\ (b * c) * b = c,\ b *^m a = b,\ c *^m a = c\rangle \\ &= \langle a, c \mid (a * c) * a = c,\ c *^m a = c\rangle. \end{aligned} \tag{5}$$

Here, equalities of presentations mean being related to each other by Tietze moves. Since Q_m is generated by the set $\{a, c\}$ and c is equal to $(a * c) * a$, Q_m is obviously connected. Studying works [5] and [11], we know that

(6) $$a = 1,\quad b = \gamma_m,\quad c = \gamma_1,\quad d = \delta.$$

6. Relationships between Schläfli quandles and knot quandles

In this section, we study some relationships between Schläfli quandles and knot quandles of twist-spun trefoils. We start with showing the following theorem:

Theorem 6.1. *The* 16-*,* 24-*, or* 600-*cell quandle is respectively isomorphic to the knot quandle of the* 3-*,* 4-*, or* 5-*twist-spun trefoil.*

Proof. Choose adjacent vertices v, w of the regular tessellation $\{3,3,4\}$, $\{3,4,3\}$ or $\{3,3,5\}$ as follows:

$$(v, w) = \begin{cases} (\boldsymbol{e}_1, \boldsymbol{e}_2) & \text{if } \{3,3,4\}, \\ (\boldsymbol{e}_1 + \boldsymbol{e}_2, \boldsymbol{e}_2 + \boldsymbol{e}_4) & \text{if } \{3,4,3\}, \\ \left(\boldsymbol{e}_1, -\dfrac{1}{2}\left(\phi^{-1}\boldsymbol{e}_1 + \phi\,\boldsymbol{e}_3 - \boldsymbol{e}_4\right)\right) & \text{if } \{3,3,5\}. \end{cases}$$

Let X be the 16-, 24-, or 600-cell quandle, and m respectively equal to 3, 4, or 5. Then it is routine to check that the set $\{(v, r_v), (w, r_w)\}$ generates X and we have

$$((v, r_v) * (w, r_w)) * (v, r_v) = (w, r_w), \quad (w, r_w) *^m (v, r_v) = (w, r_w).$$

We thus have the epimorphism $f_* : Q_m \to X$ which sends a and c to (v, r_v) and (w, r_w) respectively. It is known that the cardinality of G_m (i.e., of Q_m) is equal to 8, 24, or 120 respectively (see Section 10.D of [9] for example). Since this number is equal to the cardinality of X, f_* is not only an epimorphism but an isomorphism. □

Remark 6.2. Since $\tau^1 3_1$ is equivalent to the trivial 2-knot [13], Q_1 is the quandle of order 1. Rourke and Sanderson [10] pointed out that Q_2 is isomorphic to the dihedral quandle of order 3 (i.e., the Schläfli quandle related to $\{3, 2\}$). In light of Theorem 4.1 of [5], we know that the cardinality of Q_m is infinite if $m \geq 6$.

We next make discussion on a central extension. Let $\widetilde{X}$ and X be quandles and A a non-trivial abelian group. Suppose that A acts on $\widetilde{X}$ from the left. Then $\widetilde{X}$ is said to be a *central extension* of X if there is an epimorphism $p : \widetilde{X} \to X$ satisfying the following conditions [3]:

(E0) For each $\widetilde{w}, \widetilde{x}, \widetilde{y} \in \widetilde{X}$, $p(\widetilde{x}) = p(\widetilde{y})$ implies $\widetilde{w} * \widetilde{x} = \widetilde{w} * \widetilde{y}$
(E1) For each $\widetilde{x}, \widetilde{y} \in \widetilde{X}$ and $\alpha \in A$, $(\alpha\widetilde{x}) * \widetilde{y} = \alpha(\widetilde{x} * \widetilde{y})$ and $\widetilde{x} * (\alpha\widetilde{y}) = \widetilde{x} * \widetilde{y}$
(E2) For each $x \in X$, A acts on the fiber $p^{-1}(x)$ freely and transitively

A central extension is also called an *abelian extension* (see [1] for example). As well as groups, central extensions of a quandle are closely related to the second cohomology group of the quandle [1, 3].

Recall that tetrahedral and octahedral quandles are respectively the Schläfli quandles related to $\{3, 3\}$ and $\{3, 4\}$. It is known by Clark et al. [2] with computer calculations that SmallQuandle$(8, 1)$ and SmallQuandle$(24, 2)$ are central extensions of tetrahedral and octahedral quandles respectively. Thus, in light of Remark 3.2 and Theorem 6.1, we know that Q_m is a central extension of the Schläfli quandle related to $\{3, m\}$ if m is equal to 3 or 4. This relationship between Q_m and the Schläfli quandle related to $\{3, m\}$ is lasting as follows:

Theorem 6.3. *The knot quandle of the m-twist-spun trefoil is a central extension of the Schläfli quandle related to* $\{3, m\}$ *if* $m \geq 3$.

To show the theorem, we first prepare the following lemma:

Lemma 6.4. *The Schläfli quandle related to* $\{3, m\}$ $(m \geq 2)$ *has the presentation*

$$\langle v, w \mid (v * w) * v = w,\ (w * v) * w = v,\ w *^m v = w \rangle. \tag{7}$$

Proof. Choose adjacent vertices v, w of the regular tessellation $\{3, m\}$. Then it is easy to see that we have

$$((v, r_v) * (w, r_w)) * (v, r_v) = (w, r_w),$$
$$((w, r_w) * (v, r_v)) * (w, r_w) = (v, r_v),$$
$$(w, r_w) *^m (v, r_v) = (w, r_w)$$

on the Schläfli quandle related to $\{3, m\}$. We thus have a homomorphism f from the quandle X having the presentation (7) to the Schläfli quandle related to $\{3, m\}$ sending v and w to (v, r_v) and (w, r_w) respectively. We note that f maps each element of X to a vertex of $\{3, m\}$.

On the other hand, the triple $(v, w, w * v) \in X^3$ forms a triangle as depicted in Figure 4 (a), because we have the second relation and

$$v * (w * v) = (v * w) * v = w.$$

Further, since $*^i v$ is an automorphism of X for each $i \in \mathbb{Z}$, the triple

$$(v *^i v, w *^i v, (w * v) *^i v) = (v, w *^i v, w *^{i+1} v)$$

also forms a triangle. We note that $(v, w*^i v, w*^{i+1} v)$ and $(v, w*^j v, w*^{j+1} v)$ are the same if $i \equiv j \pmod m$, since we have the third relation. Furthermore $w, w * v, \ldots, w *^{m-1} v$ are mutually distinct, because their images by f are so. We thus have m triangles $(v, w *^i v, w *^{i+1} v)$ $(0 \leq i \leq m-1)$ around v as depicted in Figure 4 (b). We here mention that we have

$$v * (w *^i v) = (v *^{i-1} v) * (w *^i v) = ((v * w) * v) *^{i-1} v = w *^{i-1} v,$$
$$(w *^{i+1} v) * (w *^i v) = ((w * v) * w) *^i v = v *^i v = v$$

for each $i \in \mathbb{Z}$.

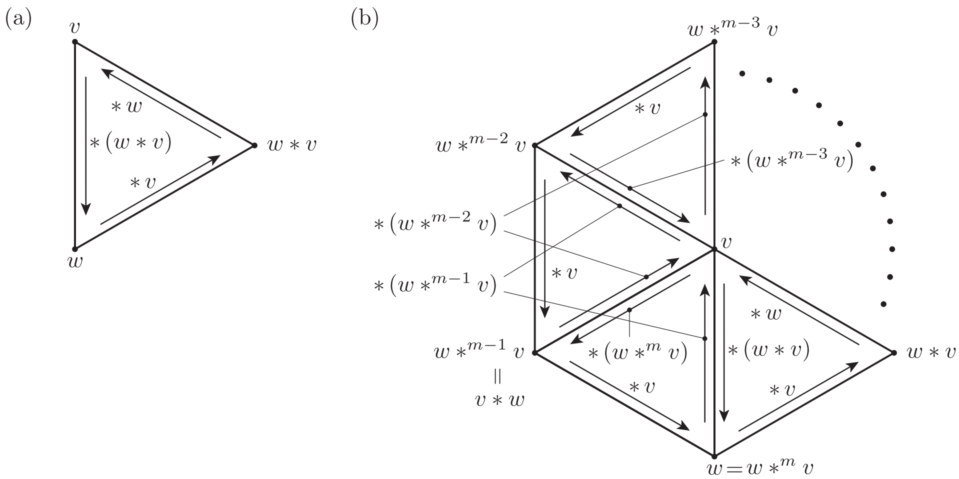

Fig. 4. Some elements of X form triangles

The quandle X is obviously connected, because X is generated by the set $\{v, w\}$ and w is equal to $(v * w) * v$. Thus each element of X is written as $g(v)$ with some $g \in \mathrm{Inn}(X)$. We also have m triangles $(g(v), g(w *^i v), g(w *^{i+1} v))$ $(0 \leq i \leq m-1)$ around $g(v)$. Consider the inner automorphism $g_i = (* g(w *^{i+1} v)) \circ g$ of X. Since we have

$$\begin{aligned}
&(g_i(v), g_i(w *^{i+1} v), g_i(w *^{i+2} v)) \\
&= (g(v * (w *^{i+1} v)), g((w *^{i+1} v) * (w *^{i+1} v)), g((w *^{i+2} v) * (w *^{i+1} v))) \\
&= (g(w *^i v), g(w *^{i+1} v), g(v))
\end{aligned}$$

and

$$\begin{aligned}
&(g_i(v), g_i(w *^{i+2} v), g_i(w *^{i+3} v)) \\
&= (g(v * (w *^{i+1} v)), g((w *^{i+2} v) * (w *^{i+1} v)), g((w *^{i+3} v) * (w *^{i+1} v))) \\
&= (g(w *^i v), g(v), g(((w *^{i+1} v) * (v * (w *^{i+1} v))) * (v * (w *^{i+1} v))))
\end{aligned}$$

$$= (g(w *^i v), g(v), g(((w *^{i+1} v) * (w *^i v)) * (w *^i v)))$$
$$= (g(w *^i v), g(v), g(v * (w *^i v)))$$
$$= (g(w *^i v), g(v), g(w *^{i-1} v))$$

for each i ($0 \le i \le m-1$), the adjacent vertices $g_i(v) = g(w *^i v)$ and $g(v)$ share the adjacent triangles $(g_i(v), g_i(w *^{i+1} v), g_i(w *^{i+2} v)) \sim (g(v), g(w *^i v), g(w *^{i+1} v))$ and $(g_i(v), g_i(w *^{i+2} v), g_i(w *^{i+3} v)) \sim (g(v), g(w *^{i-1} v), g(w *^i v))$. Here, $\sim$ means that the triples are related to each other by a cyclic permutation. Thus this arrangement of triangles can be locally identified with $\{3, m\}$ as depicted in Figure 5. We will see in the next paragraph that this arrangement of triangles totally coincides with $\{3, m\}$ showing the following claim is true for each $g \in \mathrm{Inn}(X)$:

(♣) The m triangles $(g(v), g(w *^i v), g(w *^{i+1} v))$ ($0 \le i \le m-1$) belong to the arrangement $\mathscr{T}$ of triangles to which the triangle $(v, w, w * v)$ belongs.

Since each element of X plays as a vertex of the arrangement of triangles uniquely, this conformableness makes f to be bijective. Therefore X is isomorphic to the Schläfli quandle related to $\{3, m\}$.

We first remark that each inner automorphism g of X may be written as

$$g = (*^{\varepsilon_n} u_n) \circ \cdots \circ (*^{\varepsilon_2} u_2) \circ (*^{\varepsilon_1} u_1)$$

with some $n \ge 0$, $\varepsilon_i \in \{\pm 1\}$, and $u_i \in \{v, w\}$. We already see that (♣) is true if $n = 0$. Assume that (♣) is true for any g with $n \le l$ ($l \ge 0$). Let g be an inner automorphism of X with $n = l$, and

$$g' = g \circ (*^{\varepsilon} u)$$

with some $\varepsilon \in \{\pm 1\}$ and $u \in \{v, w\}$. If $u = v$, since we have

$$g'(v) = g(v *^{\varepsilon} v) = g(v),$$
$$g'(w *^{-\varepsilon} v) = g((w *^{-\varepsilon} v) *^{\varepsilon} v) = g(w),$$
$$g'(w *^{-\varepsilon+1} v) = g((w *^{-\varepsilon+1} v) *^{\varepsilon} v) = g(w * v),$$

we know that the triangle $(g'(v), g'(w *^{-\varepsilon} v), g'(w *^{-\varepsilon+1} v)) = (g(v), g(w), g(w * v))$ belongs to $\mathscr{T}$ by the assumption. Since the triangle is a member of the m triangles $(g'(v), g'(w *^i v), g'(w *^{i+1} v))$ ($0 \le i \le m-1$), the other $m-1$ triangles also belong to $\mathscr{T}$. Otherwise (i.e., $u = w$), since we have

$$g'(w *^{\varepsilon} v) = g((w *^{\varepsilon} v) *^{\varepsilon} w) = g(v),$$
$$g'(w) = g(w *^{\varepsilon} w) = g(w),$$
$$g'(v) = g'(w *^{-\varepsilon} (w *^{\varepsilon} v)) = g'(w) *^{-\varepsilon} g'(w *^{\varepsilon} v) = g(w) *^{-\varepsilon} g(v) = g(w *^{-\varepsilon} v),$$

we know that the triangle $(g'(v), g'(w), g'(w * v)) \sim (g(v), g(w *^{-1} v), g(w))$ or $(g'(v), g'(w *^{-1} v), g'(w)) \sim (g(v), g(w), g(w * v))$ belongs to $\mathscr{T}$ by the assumption, if ε is respectively equal to $+1$ or -1 (remark that both $(w *^{-1} v) *^{-1} w = v$ and $w *^{-1} (w * v) = v$ are obtained from the first relation). Since the triangle is a member of the m triangles $(g'(v), g'(w *^i v), g'(w *^{i+1} v))$ ($0 \le i \le m-1$), the other $m-1$ triangles also belong to $\mathscr{T}$. Thus (♣) is true for any g with $n = l + 1$. □

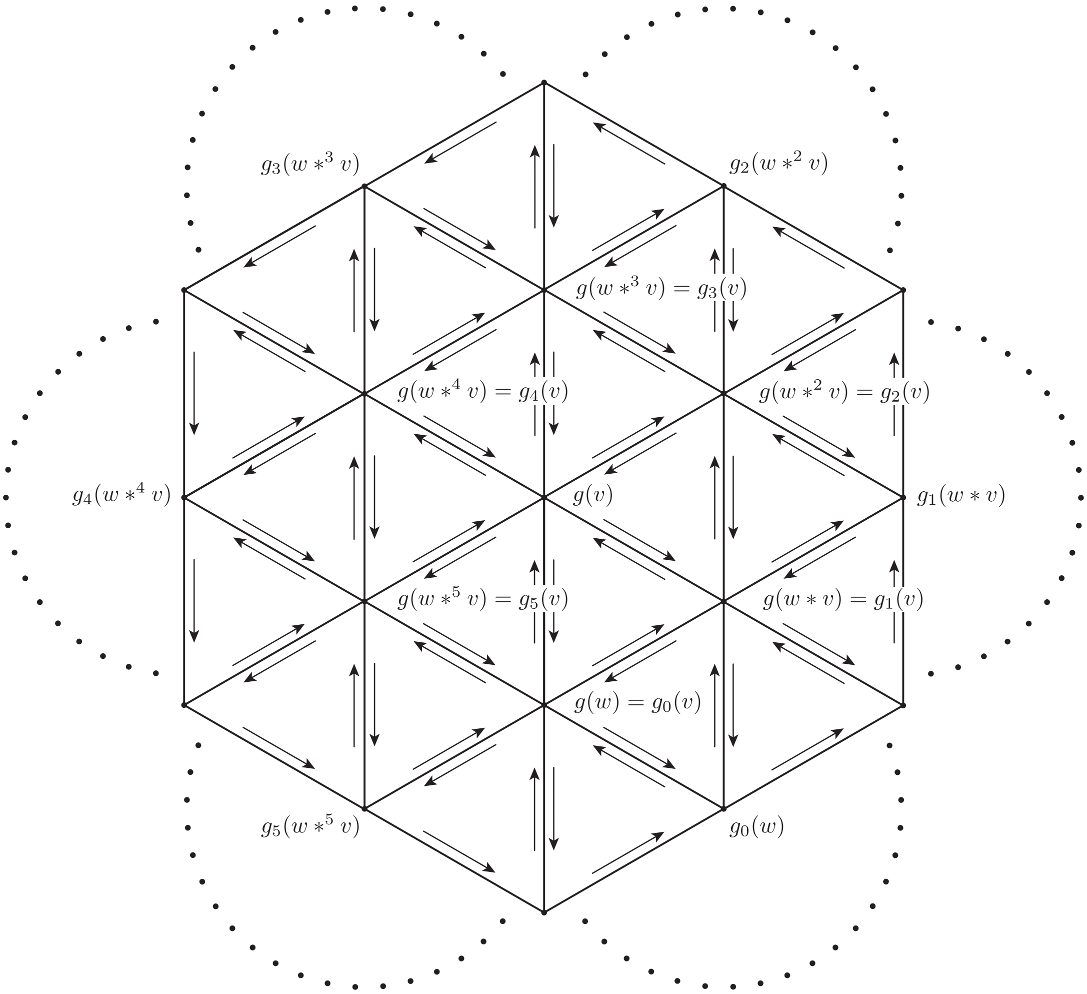

Fig. 5. A part of the arrangement of triangles (in the case $m = 6$)

It is routine to check that the presentation (7) is related to the presentation

$$\langle v, w \mid (v * w) * v = w,\ (((v * w) * v) * v) * w = v,\ w *^m v = w \rangle \tag{8}$$

by Tietze moves.

We next see a property of $\mathrm{Inn}(Q_m)$. Define the inner automorphism $\widehat{g}$ of Q_m by

$$\widehat{g} = (* c) \circ (* a) \circ (* a) \circ (* c).$$

Let g be an inner automorphism of Q_m. Since Q_m is generated by the set $\{a, c\}$, g may be written as

$$g = (*^{\varepsilon_n} e_n) \circ \cdots \circ (*^{\varepsilon_2} e_2) \circ (*^{\varepsilon_1} e_1)$$

with some $n \geq 0$, $\varepsilon_i \in \{\pm 1\}$, and $e_i \in \{a, c\}$. Suppose g' is an inner automorphism of Q_m obtained from g by replacing some $e_i = a$ to $\widehat{g}^{\varepsilon}(a)$ ($\varepsilon \in \{\pm 1\}$). Then, in light of equalities (2), (3) and (6), g and g' are the same inner automorphism (remark that $\widehat{g}^{-1} = (*^{-1} c) \circ (*^{-1} a) \circ (*^{-1} a) \circ (*^{-1} c)$ and $\delta^0 = 1$).

We now prove Theorem 6.3.

Proof of Theorem 6.3. Let X be the Schläfli quandle related to $\{3, m\}$ ($m \geq 2$). Since Q_m and X respectively have the presentations (5) and (8), we have the epimorphism $p : Q_m \to X$ sending a and c to v and w respectively. We note that p maps $\widetilde{g}^i(a)$ to v for each $i \in \mathbb{Z}$.

Let x be an element of X and $\widetilde{x}, \widetilde{x}'$ elements of $p^{-1}(x)$. Since Q_m is connected, $\widetilde{x}$ is written as $g(a)$ with some $g \in \mathrm{Inn}(Q_m)$. Then, in light of the above property of $\mathrm{Inn}(Q_m)$, $\widetilde{x}'$ should be written as $g(\widetilde{g}^i(a))$ with some $i \in \mathbb{Z}$. Further, in light of equalities (2), (4) and (6), we have $g(\widetilde{g}^i(a)) = \delta^i g(a)$. We thus have $\widetilde{x}' = \delta^i \widetilde{x}$.

Let $A = \langle \delta \rangle$ be the abelian subgroup of G_m. Then A surely acts on G_m (i.e., on Q_m) from the left. Obviously p satisfies conditions (E0) and (E2). Further it is routine to see that p also satisfies the condition (E1). Since A is non-trivial if $m \geq 3$, we obtain the claim. □

We conclude the paper with a question. Suppose that A is the above one. Since both of cardinalities of Q_m and the Schläfli quandle related to $\{3, m\}$ are finite if m is equal to 3, 4, or 5, we know that the order of A is respectively 2, 4, or 10. On the other hand, both of cardinalities of Q_m and the Schläfli quandle related to $\{3, m\}$ are infinite if $m \geq 6$. What is the order of A if $m \geq 6$?

Acknowledgements. The author wishes to express his gratitude to the referee for his/her invaluable comments. He is supported by JSPS KAKENHI Grant Numbers JP16K17591 and JP19K03476 partially.

References

[1] J.S. Carter, M. Elhamdadi, M.A. Nikiforou and M. Saito: *Extensions of quandles and cocycle knot invariants*, J. Knot Theory Ramifications **12** (2003), 725–738.

[2] W.E. Clark, M. Saito and L. Vendramin: *Quandle coloring and cocycle invariants of composite knots and abelian extensions*, J. Knot Theory Ramifications **25** (2016), 1650024, 34 pp.

[3] M. Eisermann: *Homological characterization of the unknot*, J. Pure Appl. Algebra **177** (2003), 131–157.

[4] R. Fenn and C. Rourke: *Racks and links in codimension two*, J. Knot Theory Ramifications **1** (1992), 343–406.

[5] A. Inoue: *On the knot quandle of a fibered knot, finiteness and equivalence of knot quandles*, Topology Appl. **265** (2019), 106811, 8 pp.

[6] A. Inoue: *On the knot quandle of the twist-spun trefoil*, arXiv:1808.06276.

[7] S. Kamada: Surface-Knots in 4-Space, An Introduction, Springer Monographs in Mathematics, Springer-Verlag, Singapore, 2017.

[8] M. Niebrzydowski and J.H. Przytycki: *The quandle of the trefoil knot as the Dehn quandle of the torus*, Osaka J. Math. **46** (2009), 645–659.

[9] D. Rolfsen: Knots and links, Mathematics Lecture Series **7**, Publish or Perish, Inc., Houston, TX, 1990. Corrected reprint of the 1976 original.

[10] C. Rourke and B. Sanderson: *A new classification of links and some calculations using it*, arXiv:math/0006062.

[11] S. Satoh: *Surface diagrams of twist-spun 2-knots*, Knots 2000 Korea, Vol. 1 (Yongpyong), J. Knot Theory Ramifications **11** (2002), 413–430.

[12] L. Vendramin: Rig — A GAP package for racks and quandles, available at http://mate.dm.uba.ar/~lvendram/rig/.

[13] E.C. Zeeman: *Twisting spun knots*, Trans. Amer. Math. Soc. **115** (1965), 471–495.

Department of Mathematics, Tsuda University
2–1–1 Tsuda-machi, Kodaira-shi
Tokyo 187–8577
Japan
e-mail: ayminoue@tsuda.ac.jp