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ON THE JACOBIAN OF A FAMILY OF HYPERELLIPTIC CURVES

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Abstract

In this paper, we study the algebraic rank and the analytic rank of the Jacobian of hyperelliptic curves $y^2 = x^5 + m^2$ for integers m. Namely, we first provide a condition on m that gives a bound of the size of Selmer group and then we provide a condition on m that makes L-functions non-vanishing. As a consequence, we construct a Jacobian that satisfies the rank part of the Birch–Swinnerton-Dyer conjecture.

1. Introduction

For each integer A, we define a hyperelliptic curve $C_A : y^2 = x^5 + A$ and its Jacobian J_A. In [6, 7] Stoll studied the arithmetic of C_A and in [9] Stoll and Yang studied the L-values of C_A. In this paper, we focus on the case of $A = m^2$ where m is a square-free integer. More precisely, we study the algebraic rank and the analytic rank of J_{m^2}. We note that every hyperelliptic curve in our family does not satisfy the conditions [6, (1.3)], so this curve is not covered in [6].

To get an algebraic rank, a standard method is to give a bound of the Selmer groups of the Jacobians. Using the result of Schaefer [5] and the calculation of the root numbers [7], we obtain the following.

Theorem 1.1. *There are infinitely many integers m where $J = J_{m^2}$ satisfies*

$$J(\mathbb{Q}) \cong \mathbb{Z}/5\mathbb{Z}.$$

On the other hand, there are infinitely many m such that

$$J(\mathbb{Q}) \cong \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}$$

under the parity conjecture.

We recall that the parity conjecture claims that the algebraic rank and the analytic rank are equal modulo 2.

For simplicity, we mainly consider the case where m is a prime. However, our proof of this theorem can be applied to general J_{m^2} for square-free m such that all of the prime divisors p of m satisfy $p \not\equiv 1 \pmod 5$, and there is at most one $p \equiv 4 \pmod 5$ among them. In this case, the primes of K above m satisfy a certain kind of orthogonality (i.e. there exist generators $\pi_{\mathfrak{p}}, \pi_{\mathfrak{p}'}$ such that $\pi_{\mathfrak{p}}$ is trivial in $K_{\mathfrak{p}'}^{\times}/K_{\mathfrak{p}'}^{\times 5}$ and vice versa). This property makes the descent computation much easier as we will see in Proposition 3.3. For the case where m

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is not a prime, see Remark 3.2 and Example 3.6. As an example, we consider $m = 101$ a prime equivalent to 1 modulo 5 in Proposition 3.5.

On the analytic side, there are results on the special L-value of the hyperelliptic curves C_A like [9, 2]. Such curves have complex multiplication, so there is a Hecke character η_A satisfying

$$L(s, C_A) = L(s, J_A) = L(s, \eta_A).$$

Based on the work [10, 11, 12] on the non-vanishings of L-functions of Hecke characters and [6, 7] on hyperelliptic curves C_A, Stoll and Yang showed that

$$L(1, J_1) \neq 0$$

in [9]. In this paper, we extend this result for the curve C_A with certain conditions on A, in Proposition 4.3 which gives an expression of $L(1, \eta_A)$. As a consequence, we obtain

Theorem 1.2. *Let J_A be a Jacobian of C_A whose root number is $+1$. If A is a square integer such that every prime divisor is a prime equivalent to 1 modulo 5, and $(A^4 - 1)$ is divided by 25, then $L(1, J_A) \neq 0$.*

Note that the rational primes $p \equiv 1 \pmod 5$ are exactly the ones split completely in K. In formula (8), one can see from (7) that the factors involving primes v of F split in K are non-zero. To see whether the factors involving primes of F inert in K vanish or not, one need to evaluate integral (5), which seems to be complicated. However, when it comes to the descent on C_{m^2}, the situation seems complementary. More precisely, if m only has prime factors which are not totally split, then the descent is manageable. However, if m has prime factors which split completely in K, then the descent become more complicated to deal with. This explains why we cannot obtain an infinite family of Jacobians of the form J_{m^2} satisfying the rank part of the Birch–Swinnerton-Dyer conjecture. Instead of this, we give an illustration for the case $p \equiv 1 \pmod 5$:

Corollary 1.3. *A Jacobian J_{101^2} satisfies the rank part of the Birch–Swinnerton-Dyer conjecture.*

We note that Corollary 1.3 may be deduced from 2-descent available in Magma and the numerical computation of L-values since the rank of J_{101^2} is zero, but we want to emphasize that the analogous result for other primes $p \equiv 1 \pmod 5$ may be deduced from our $(1 - \zeta_5)$-descent with less computational complexity.

In Section 2, we list some facts on local fields and recall the computation of the root number of J_{m^2}. Based on these results, we describe descent for Jacobians in Section 3 and give a proof of Theorem 1.1. After computing the special L-value in Section 4, we will show Theorem 1.2 and Corollary 1.3.

2. Preliminaries

2.1. Local field computation. We list some notations which will be used in Sections 2 and 3. We fix a fifth root of unity ζ_5 in $\overline{\mathbb{Q}}$. Let $K = \mathbb{Q}(\zeta_5)$ and $F = \mathbb{Q}(\sqrt{5})$. We recall that a rational prime p is inert, splits into two primes, splits completely in $K/\mathbb{Q}$ if and only if $p \equiv 2$ or 3, $p \equiv 4$, $p \equiv 1$ modulo 5, respectively. In each case, we denote primes of K above a rational prime p by p, w, v and its generator by p, π_w, π_v, respectively. The unique prime above 5 is denoted by v_5, but we also admit the notations K_5 and π_5 for K_{v_5} and π_{v_5}. We use a symbol $\mathfrak{p}$ to indicate a prime ideal of K and π to a prime element. For the integer ring of a local field with a maximal ideal $\mathfrak{p}$,

$$U^{(i)} := 1 + \mathfrak{p}^i.$$

Also we use the notation ζ_n for a primitive n-th root of unity in K or any local fields, if it exists.

In this section, we compute the images of prime elements π in $K_{\mathfrak{p}}^{\times}/K_{\mathfrak{p}}^{\times 5}$. We first compute the group $K_{\mathfrak{p}}^{\times}/K_{\mathfrak{p}}^{\times 5}$. When $\mathfrak{p} = v_5$, we fix a generator π_5 by $(1-\zeta_5)$. Since

$$K_5^{\times} \cong \pi_5^{\mathbb{Z}} \times \mu_4 \times U^{(1)} \quad \text{and} \quad U^{(2)} \cong \mathbb{Z}_5^4,$$

we have

$$K_5^{\times}/K_5^{\times 5} \cong \langle \pi_5, 1+\pi_5, 1+\pi_5^2, 1+\pi_5^3, 1+\pi_5^4, 1+\pi_5^5 \rangle \tag{1}$$

and every element in $U^{(6)}$ is a fifth-power. We rename the generating elements by $\langle \alpha, \beta, \gamma, \delta, \epsilon, \eta \rangle$. For all other primes $\mathfrak{p} \neq v_5$, 5 is invertible in the ring of integers $\mathcal{O}_{K,\mathfrak{p}}$. So we have

$$K_{\mathfrak{p}}^{\times}/K_{\mathfrak{p}}^{\times 5} \cong \langle \pi_{\mathfrak{p}}, \zeta_{5^n} \rangle \tag{2}$$

where ζ_{5^n} generates the 5-part of the root of unities of $K_{\mathfrak{p}}^{\times}$. We also rename the generating elements by $\langle \alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}} \rangle$ and drop the subscript whenever the meaning is clear from the context. We note that every element in $U^{(2)}$ is a fifth-power in this case.

We need π_5-expansions of some elements in K_5. By expanding $\pi_5^4 = (1-\zeta_5)^4$, we have

$$5 = 4\pi_5^4 + 3\pi_5^5 + 3\pi_5^6 + 4\pi_5^7 + \pi_5^8 + 3\pi_5^9 + O(\pi_5^{11}).$$

We choose $\sqrt{5}$ and ζ_4 in K_5 such that

$$\sqrt{5} \equiv 2\pi_5^2 \pmod{\pi_5^3} \quad \text{and} \quad \zeta_4 \equiv 2 \pmod{\pi_5}$$

respectively. Then, one may verify that

$$\begin{aligned} \sqrt{5} &= 2\pi_5^2 + 2\pi_5^3 + \pi_5^4 + O(\pi_5^7), \\ \zeta_4 &= 2 + 4\pi_5^4 + 3\pi_5^5 + O(\pi_5^6), \\ \zeta_4^3 &= 3 + 2\pi_5^4 + 4\pi_5^5 + O(\pi_5^6), \\ -\left(\frac{1+\sqrt{5}}{2}\right) &= 2 + 4\pi_5^2 + 4\pi_5^3 + \pi_5^5 + O(\pi_5^6), \end{aligned}$$

where the last one is a fundamental unit of F, which we will denote by u_F. We note that $\{1, u_F\}$ is an integral basis of $\mathcal{O}_F$, so we can choose a generator $\pi_w = a + b\sqrt{5}$ for $a, b \in \frac{1}{2}\mathbb{Z}$, or $\pi_w = a + bu_F$ for $a, b \in \mathbb{Z}$.

Now we can describe the images of the prime elements of K which is not above a rational prime $p \equiv 1 \pmod 5$ in $K_5^{\times}/K_5^{\times 5}$.

Lemma 2.1. (1) *Let n be a rational integer not divided by 5. Then, the image of n in $K_5^{\times}/K_5^{\times 5}$ is*

$$\begin{array}{rl} 1 & \textit{if } n \equiv 1,7,18,24 \pmod{25} \\ \epsilon\eta^2 & \textit{if } n \equiv 3,4,21,22 \pmod{25} \\ \epsilon^2\eta^4 & \textit{if } n \equiv 9,12,13,16 \pmod{25} \\ \epsilon^3\eta & \textit{if } n \equiv 2,11,14,23 \pmod{25} \\ \epsilon^4\eta^3 & \textit{if } n \equiv 6,8,17,19 \pmod{25} \end{array}$$

(2) *For a prime w above a rational prime $p \equiv 4 \pmod 5$ and its generator $\pi_w = a + b\sqrt{5}$ with $a, b \in \frac{1}{2}\mathbb{Z}$, the image of π_w in $K_5^\times/K_5^{\times 5}$ is given by the following table.*

$a \pmod 5$	$p \equiv 4$	$p \equiv 9$	$p \equiv 14$	$p \equiv 19$	$p \equiv 24$
2	$\gamma^b\delta^b\epsilon^{b+3}\eta$	$\gamma^b\delta^b\epsilon^{b+1}\eta^2$	$\gamma^b\delta^b\epsilon^{b+4}\eta^3$	$\gamma^b\delta^b\epsilon^{b+2}\eta^4$	$\gamma^b\delta^b\epsilon^b$
4	$\gamma^{3b}\delta^{3b}\epsilon^{3b+3}\eta$	$\gamma^{3b}\delta^{3b}\epsilon^{3b+1}\eta^2$	$\gamma^{3b}\delta^{3b}\epsilon^{3b+4}\eta^3$	$\gamma^{3b}\delta^{3b}\epsilon^{3b+2}\eta^4$	$\gamma^{3b}\delta^{3b}\epsilon^{3b}$
3	$\gamma^{4b}\delta^{4b}\epsilon^{4b+3}\eta$	$\gamma^{4b}\delta^{4b}\epsilon^{4b+1}\eta^2$	$\gamma^{4b}\delta^{4b}\epsilon^{4b+4}\eta^3$	$\gamma^{4b}\delta^{4b}\epsilon^{4b+4}\eta^4$	$\gamma^{4b}\delta^{4b}\epsilon^{4b}$
1	$\gamma^{2b}\delta^{2b}\epsilon^{2b+3}\eta$	$\gamma^{2b}\delta^{2b}\epsilon^{2b+1}\eta^2$	$\gamma^{2b}\delta^{2b}\epsilon^{2b+4}\eta^3$	$\gamma^{2b}\delta^{2b}\epsilon^{2b+1}\eta^4$	$\gamma^{2b}\delta^{2b}\epsilon^{2b}$

Here $p \equiv a$ means p is equivalent to a modulo 25.

Proof. For a generator $\sigma : \zeta_5 \mapsto \zeta_5^2$ of $\mathrm{Gal}(K_5/\mathbb{Q}_5)$, we have

$$\begin{aligned} &\sigma(1+\pi_5, 1+\pi_5^2, 1+\pi_5^3, 1+\pi_5^4, 1+\pi_5^5) \\ &\equiv (1+2\pi_5+4\pi_5^2, 1+4\pi_5^2+\pi_5^3+\pi_5^4, 1+3\pi_5^3+3\pi_5^4+\pi_5^5, 1+\pi_5^4+3\pi_5^5, 1+2\pi_5^5), \end{aligned}$$

modulo $K_5^{\times 5}$, which implies

$$\sigma(\beta,\gamma,\delta,\epsilon,\eta) \equiv (\beta^2\gamma^3\delta^4\epsilon\eta, \gamma^4\delta\eta, \delta^3\epsilon^3\eta, \epsilon\eta^3, \eta^2) \pmod{K_5^{\times 5}}.$$

For a prime $\mathfrak{p}$ not above 5, any generator $\pi_\mathfrak{p}$ of $\mathfrak{p}$ is not divided by π_5 so we can write

$$\pi_\mathfrak{p} \equiv \zeta_4^i\beta^b\gamma^c\delta^d\epsilon^e\eta^f \pmod{\pi_5^6}.$$

A (multiplicative) $\mathbb{F}_5$-vector space $\langle\beta,\gamma,\delta,\epsilon,\eta\rangle$ is decomposed by eigenvectors $\left\{\epsilon\eta^2, \gamma\delta\epsilon, \eta, \beta\gamma\epsilon, \delta\epsilon^4\eta^3\right\}$ of σ such that

$$\sigma(\epsilon\eta^2, \gamma\delta\epsilon, \eta, \beta\gamma\epsilon, \delta\epsilon^4\eta^3) \equiv (\epsilon\eta^2, (\gamma\delta\epsilon)^4, \eta^2, (\beta\gamma\epsilon)^2, (\delta\epsilon^4\eta^3)^3) \pmod{K_5^{\times 5}}.$$

(1) Since $\sigma(n) = n$ for all $n \in \mathbb{Z}$, the class of n in $K_5^\times/K_5^{\times 5}$ is a power of $\epsilon\eta^2$, which is the unique eigenvector with eigenvalue $+1$. Note that

$$\epsilon\eta^2(1+\pi_5^6)^2(1+\pi_5^7) \equiv 1+\pi_5^4+2\pi_5^5+2\pi_5^6+\pi_5^7 \equiv 21 \pmod{\pi_5^8}, \quad \text{and} \quad \zeta_4 \equiv 7 \pmod{\pi_5^8}.$$

So for $i = 0, 1, 2, 3$,

$$\begin{array}{rcl} \zeta_4^i\epsilon\eta^2(1+\pi_5^6)^2(1+\pi_5^7) & \equiv & 21, 22, 3, 4 \pmod{25} \\ \zeta_4^i\epsilon^2\eta^4(1+\pi_5^6)^4(1+\pi_5^7)^2 & \equiv & 16, 12, 9, 13 \pmod{25} \\ \zeta_4^i\epsilon^3\eta(1+\pi_5^6)^6(1+\pi_5^7)^3 & \equiv & 11, 2, 14, 23 \pmod{25} \\ \zeta_4^i\epsilon^4\eta^3(1+\pi_5^6)^8(1+\pi_5^7)^4 & \equiv & 6, 17, 19, 8 \pmod{25} \\ \zeta_4^i & \equiv & 1, 7, 24, 18 \pmod{25} \end{array}$$

where $(1+\pi_5^6)^2(1+\pi_5^7)$ is a 5^{th}-power in $K_5^\times$.

(2) Since $p \equiv 4 \pmod 5$, p splits into two primes. For a generator π_w, $\sigma\pi_w \neq \pi_w$ but $\sigma^2\pi_w = \pi_w$. Hence the image of π_w in $K_5^\times/K_5^{\times 5}$ is a product of a nontrivial power of the eigenvector $\gamma\delta\epsilon$ with eigenvalue -1 and a power of the eigenvector $\epsilon\eta^2$ with eigenvalue $+1$,

say

$$\pi_w = (\gamma\delta\epsilon)^c(\epsilon\eta^2)^e \pmod{K_5^{\times 5}}.$$

Also, $\pi_w \cdot \sigma\pi_w \equiv (\epsilon\eta^2)^{2e} \pmod{K_5^{\times 5}}$ and $\pi_w \cdot \sigma\pi_w \equiv p \pmod{K_5^{\times 5}}$ imply that the exponent e is $0, 1, 2, 3, 4$ when $p \equiv 24, 9, 19, 4, 14 \pmod{25}$ respectively. We also have

$$\begin{aligned}-u_F &\equiv 2 + 4\pi_5^2 + 4\pi_5^3 + \pi_5^5 \equiv \zeta_4(1 + 2\pi_5^2 + 2\pi_5^3 + 3\pi_5^4 + 4\pi_5^5) \pmod{\pi_5^6}\\ &\equiv \zeta_4\gamma^2\delta^2\epsilon^2 \pmod{\pi_5^6}.\end{aligned}$$

Since u_F is a fundamental unit of $\mathbb{Q}(\sqrt{5})$, we note that another choice of a generator of the form $a' + b'\sqrt{5}$ for $a', b' \in \frac{1}{2}\mathbb{Z}$ should be a product of power of $-1, u_F$, and $a + b\sqrt{5}$. Let $\pi_w = a + b\sqrt{5}$ be a generator for w with $a, b \in \frac{1}{2}\mathbb{Z}$ and let $a \equiv 2^k \pmod 5$ with $1 \le k \le 4$. Since

$$\frac{-1-\sqrt{5}}{2}(a + b\sqrt{5}) = -\frac{a+5b}{2} - \left(\frac{a+b}{2}\right)\sqrt{5}$$

and $(-a - 5b)/2 \equiv 2a \pmod 5$, we can find another generator

$$\pi_w' = a' + b'\sqrt{5} = \left(-\frac{1+\sqrt{5}}{2}\right)^{5-k}\pi_w$$

of w, where $a' \equiv 2 \pmod 5$. We also note that every generator of w is equivalent to one of π_w' up to $K^{\times 5}$.

Now assume $a \equiv 2 \pmod 5$. Then

$$\begin{aligned}\zeta_4^3 \cdot (a + b\sqrt{5}) &= (3 + 2\pi_5^4 + 4\pi_5^5 + O(\pi_5^6))(a + b(2\pi_5^2 + 2\pi_5^3 + \pi_5^4 + O(\pi_5^6)))\\ &= 1 + b\pi_5^2 + O(\pi_5^3)\end{aligned}$$

implies that $\pi_w = (\gamma\delta\epsilon)^b(\epsilon\eta^2)^e$ in $K^\times/K^{\times 5}$. This induces the first row of the table. The other rows are determined by the relation between π_w' and π_w and the value of $-(1+\sqrt{5})/2$ in $K_5^\times/K_5^{\times 5}$. □

In the next section, we will need the images of $\{\zeta_5, 1 \pm \zeta_5, 2\}$ in $K_\mathfrak{p}^\times/K_\mathfrak{p}^{\times 5}$ also. We begin with $\mathfrak{p} = 2$. Recall that $K_2^\times/K_2^{\times 5} \cong \langle 2, \zeta_5\rangle = \langle \alpha, \beta\rangle$ in (2).

Lemma 2.2. (1) *The image of* $(\zeta_5, 1+\zeta_5, 1-\zeta_5, 2)$ *in* $K_2^\times/K_2^{\times 5}$ *is* $(\beta, \beta^3, \beta^3, \alpha)$.

(2) *The images of odd integers and prime elements* $\pi_w = a + bu_F$ *for* $a, b \in \mathbb{Z}$ *in* $K_2^\times/K_2^{\times 5}$ *are trivial.*

Proof. (1) To describe 2-expansions of elements of K_2, we fix an isomorphism

$$\mathbb{F}_{16} \cong \mathbb{F}_2[t]/(t^4 + t + 1).$$

We choose an embedding of K in K_2 which sends $\zeta_5 \in K$ to $t^3 \in \mathbb{F}_{16}$. Since

$$(t^3+1)(t^2+t+1) = t^3 + t, \qquad (t^2+t+1)^3 = 1, \qquad t^9 = t^3 + t,$$

we know that $(1 + \zeta_5)\zeta_3 = \zeta_5^3$ in K_2. Since ζ_3 is trivial in $K_2^\times/K_2^{\times 5}$, the image of $(1 + \zeta_5)$ in $K_2^\times/K_2^{\times 5}$ is β^3. Also, the 2-expansion of the image of $(1 - \zeta_5)$ in K_2 is

$$1 - \zeta_5 = 1 + t^3(1 + 2 + O(2^2)) = (1 + t^3)(1 + (1 + t^3)^{-1}t^3 2 + O(2^2)).$$

Hence the image of $(1-\zeta_5)$ in $K_2^\times/K_2^{\times 5}$ is β^3 also.

(2) Since $U^{(1)}$ vanishes in $K_2^\times/K_2^{\times 5}$, every odd integer maps to the trivial element in $K_2^\times/K_2^{\times 5}$. In K_2, one has

$$\sqrt{5} = 1 + (t^2+t)2 + O(2^2) \quad \text{and} \quad u_F = (t^2+t+1) + O(2).$$

Therefore, the image of $a+bu_F$ in $\mathbb{F}_{16}^\times$ is contained in $\left\{t^2+t+1, t^2+t, 1\right\}$ which is the group generated by ζ_3. □

Lemma 2.3. *Let $p \neq 2$ be a rational prime inert in $K/\mathbb{Q}$ and let π_w be a prime element defined by $a+b\sqrt{5}$ for $a,b \in \frac{1}{2}\mathbb{Z}$.*

(1) *For $\mathfrak{p} = (p)$ or (π_w), the image of $\{\zeta_5, 1+\zeta_5, 1-\zeta_5\}$ in $K_\mathfrak{p}^\times/K_\mathfrak{p}^{\times 5}$ is in $\langle\beta_\mathfrak{p}\rangle$.*

(2) *For $\mathfrak{p} = (p)$, the images of rational primes relatively prime to $\mathfrak{p}$ and prime elements $\pi_{w'} = a' + b'\sqrt{5}$ for $a', b' \in \frac{1}{2}\mathbb{Z}$ are trivial in $K_\mathfrak{p}^\times/K_\mathfrak{p}^{\times 5}$.*

(3) *For $\mathfrak{p} = (\pi_w)$, the images of rational primes relatively prime to $\mathfrak{p}$ and a prime element $\pi_{\overline{w}} := a - b\sqrt{5}$ are trivial in $K_\mathfrak{p}^\times/K_\mathfrak{p}^{\times 5}$.*

Proof. (1) We recall that $K_p^\times \cong p^\mathbb{Z} \times \mu_{p^4-1} \times U^{(1)}$ and $K_w^\times \cong \pi_w^\mathbb{Z} \times \mu_{p^2-1} \times U^{(1)}$, i.e. $K_\mathfrak{p}^\times/K_\mathfrak{p}^{\times 5} = \langle\alpha_\mathfrak{p}, \beta_\mathfrak{p}\rangle$ for $\mathfrak{p} = (p)$ or (w) in (2). Especially, the $U^{(1)}$-part vanishes in $K_\mathfrak{p}^\times/K_\mathfrak{p}^{\times 5}$. Since $\zeta_5, 1\pm\zeta_5$ are not divided by $\mathfrak{p}$, their images are in $\langle\beta_\mathfrak{p}\rangle$.

(2) Every rational integer relatively prime to p and $\pi_{w'}$ maps to $\mathbb{F}_{p^2}^\times$ modulo p. Since the fifth-power map on $\mathbb{F}_{p^2}^\times$ is bijective, every element maps to $\mathbb{F}_{p^2}^\times$ vanish in $K_p^\times/K_p^{\times 5}$.

(3) Similarly, every integer and $\pi_{\overline{w}}$ maps to $\mathbb{F}_{p_w}^\times$ where p_w is the rational prime divided by π_w. □

2.2. The root numbers. We recall the result of [7] on the root numbers of $y^2 = x^l + A$, where l is an odd prime.

Theorem 2.4 ([7, Theorem 3.2])**.** *The root number $w(A)$ of the curve $y^2 = x^l + A$ over $\mathbb{Q}$ where A is a $2l$-th power free integer not divisible by l, is given by*

$$w(A) = \begin{cases} \left(\frac{2Av_A}{l}\right) & \text{if } l \mid q_l(A), \\ -\left(\frac{2q_l(A)v_A}{l}\right) & \text{if } l \nmid q_l(A), \end{cases}$$

where $q_l(A) = (A^{l-1}-1)/l$ and $v_A = 2^{f_2(A)} \prod_{p|A, p\neq 2} p$ where f_2 is given by

$$f_2(A) = \begin{cases} 0 & \text{if } e = 2l-2 \text{ and } B \equiv 1 \pmod 4, \\ 1 & \text{if } e < 2l-2 \text{ and is even and } B \equiv 1 \pmod 4, \\ 2 & \text{if } e \text{ is even and } B \equiv -1 \pmod 4, \\ 3 & \text{if } e \text{ is odd} \end{cases}$$

for $A = 2^e B$ with B odd.

In this paper, we only need the following special case.

Corollary 2.5. *For an odd square-free integer m, the root number $w(m^2)$ of the hyperelliptic curve $y^2 = x^5 + m^2$ over $\mathbb{Q}$ is given by*

$$w(m^2) = \begin{cases} +1 & \text{if } m \equiv 1,2,4,6,12,13,19,21,23,24 \pmod{25}, \\ -1 & \text{if } m \equiv 3,7,8,9,11,14,16,17,18,22 \pmod{25}. \end{cases}$$

3. Descent for Jacobian of hyperelliptic curves

We recall the general facts on the descent for Jacobian of hyperelliptic curves of odd prime degree. The main reference is [5].

Let p be an odd prime, let K be a number field containing ζ_p, and let C be a curve defined by an equation $y^p = f(x)$. Let J be the Jacobian of C and consider an endomorphism ϕ of J. The ϕ-Selmer group of J/K is defined by

$$\mathrm{Sel}_{\phi}(J/K) := \ker\left(H^1(K, J[\phi]) \to \prod_{\mathfrak{p}} H^1(K_{\mathfrak{p}}, J)\right)$$

where $\mathfrak{p}$ is taken over all primes of K. Following the Schaefer's idea, instead of using the first cohomology group we will use more concrete object which we will describe as follows. Assume that $J[\phi]$ has a prime power exponent q. We define

$$L := K[T]/(f(T)), \qquad H := \ker\left(\mathrm{Norm} : L^{\times}/L^{\times q} \to K^{\times}/K^{\times q}\right).$$

Let $\lambda : J \to \widehat{J}$ be the canonical polarization of J and let $\widehat{\phi}$ be the dual isogeny of ϕ. Let $\Psi := \lambda^{-1}(\widehat{J}[\widehat{\phi}]) \subset J[q]$ and choose a G_K-invariant set of divisor classes that generate Ψ. We also define $\mathrm{Div}^0_{\perp}(C)$ as a set of degree zero divisors of C with support not intersecting with the generating set of Ψ. For each element of $J(K)$, we may choose its representative in $\mathrm{Div}^0_{\perp}(C)$. There is a map

$$F : \mathrm{Div}^0_{\perp}(C) \to L^{\times}$$

which induces $F : J(K)/\phi J(K) \to L^{\times}/L^{\times q}$ by [5, Lemma 2.1, Theorem 2.3].

Now we consider our cases $p = 5$, $K = \mathbb{Q}(\zeta_5)$, $C_{m^2} : y^2 = x^5 + m^2$ and $\phi = (1 - \zeta_5)$ where $\zeta_5(x_0, y_0) := (\zeta_5 x_0, y_0)$. We note that the class number of K is one and there is a fundamental unit $(1 + \zeta_5)$. Let J_{m^2} be the Jacobian of C_{m^2}. The polynomial $f(T) = T^2 - m^2$ is reducible so we have $L \cong K \oplus K$, and the norm map is given by $(k_1, k_2) \to k_1 k_2$. After identifying H with $K^{\times}$, we have

$$H^1(K, J_{m^2}[\phi]; S) \cong K(S, 5)$$

where $K(S, 5)$ is a subset of $K^{\times}/K^{\times 5}$ consisting of elements trivial outside S, by [5, Proposition 3.4]. Since the set of bad primes S consists of the primes above $10m$, we note that $K(S, 5)$ is generated by

$$\zeta_5, 1 + \zeta_5, 2, 1 - \zeta_5$$

and prime elements dividing m. We also have $\lambda^{-1}(\widehat{J_{m^2}}[\widehat{\phi}]) = J_{m^2}[\phi]$ and $(0, m) - \infty$ generates $J_{m^2}[\phi]$ by [5, Propositions 3.1, 3.2]. Furthermore, we have

$$\mathrm{Sel}_{\phi}(J/K) \cong \bigcap_{\mathfrak{p} \in S} (i_{\mathfrak{p}}^{-1} \circ F_{\mathfrak{p}})\left(J_{m^2}(K_{\mathfrak{p}})/\phi J_{m^2}(K_{\mathfrak{p}})\right), \tag{3}$$

where $i_{\mathfrak{p}}$ is a natural map $L^{\times} \to L^{\times}_{\mathfrak{p}}$. For the concrete computation, we remind that

$$\dim_{\mathbb{F}_p}(J_{m^2}(K_{\mathfrak{p}})/\phi J_{m^2}(K_{\mathfrak{p}})) = \begin{cases} 3 & \text{if } \mathfrak{p} \mid 5, \\ 1 & \text{otherwise,} \end{cases} \tag{4}$$

by [5, Corollary 3.6]. This result guides us when we stop finding the independent points of $J_{m^2}(K_\mathfrak{p})/\phi J_{m^2}(K_\mathfrak{p})$. Also, for $D = Q_1 + \cdots + Q_r - r\infty$ where Q_i are K-conjugates with $x(Q_i) \neq 0$,

$$F_\mathfrak{p}([D]) \equiv \prod_{i=1}^{r}(y(Q_i) - T) \pmod{L_\mathfrak{p}^{\times 5}}$$

and for $D = (0, \pm m) - \infty = Q - \infty$,

$$F_\mathfrak{p}([D]) \equiv (-y(Q) - T)^{-1} + (y(Q) - T) \pmod{L_\mathfrak{p}^{\times 5}}$$

by [5, Proposition 3.3]. As Schaefer did in [5, Propositions 3.9, 3.12], we denote $F_\mathfrak{p}$ by the composition of the original $F_\mathfrak{p}$ and the isomorphism $L \cong K \oplus K$. For example, the image of $F_\mathfrak{p}$ of $D = (0, m) - \infty$ is $(-2m, (-2m)^{-1})$ and written by

	$y+m$	$y-m$
$[(0, m) - \infty]$	$-2m$	$(-2m)^{-1}$

We remark that

$$\mathrm{rank}(J_{m^2}(\mathbb{Q})) = \dim_{\mathbb{F}_5}(J_{m^2}(K)/\phi J_{m^2}(K)) - \dim_{\mathbb{F}_5} J_{m^2}(K)[\phi],$$

by [5, Corollary 3.7, Proposition 3.8].

One of the main goals of the paper is computing the Selmer group of Jacobian of C_{m^2}.

Proposition 3.1. *Let m be an odd integer and let J_{m^2} be a Jacobian of C_{m^2}. Under the identifications of $K_\mathfrak{p}^\times/K_\mathfrak{p}^{\times 5}$ as in* (1) *and* (2)*, we have*

$$F_5(J_{m^2}(K_5)/\phi J_{m^2}(K_5)) = \langle \delta, \epsilon, \eta \rangle \qquad \text{if } m \equiv \pm 1, \pm 7 \pmod{25}.$$

If the prime $\mathfrak{p}$ does not divide 5 *or totally split primes, and* $\mathrm{ord}_\mathfrak{p}(m) \not\equiv 0 \pmod 5$*, then we have*

$$F_\mathfrak{p}(J_{m^2}(K_\mathfrak{p})/\phi J_{m^2}(K_\mathfrak{p})) = \langle \alpha_\mathfrak{p} \rangle.$$

Proof. In the proof, we denote J by J_{m^2}. The F_5-case is a generalization of [5, Proposition 3.12]. We recall that

$$K_5^\times/K_5^{\times 5} \cong \langle \pi_5, 1+\pi_5, 1+\pi_5^2, 1+\pi_5^3, 1+\pi_5^4, 1+\pi_5^5 \rangle := \langle \alpha, \beta, \gamma, \delta, \epsilon, \eta \rangle$$

and every element of $K_5^\times$ which is one modulo π_5^6 is a fifth power. When $m^2 \pm 1 \equiv 0 \pmod{25}$, either $y^2 - m^2 \equiv 1 \pmod{\pi_5^6}$ or $m^2 - y^2 \equiv 1 \pmod{\pi_5^6}$ has solutions π_5^i for $i = 3, 4, 5$. Hence, in each case, there is an x_i such that $[(x_i, \pi_5^i) - \infty]$ for $i = 3, 4, 5$ is the point of $J(K_5)/\phi J(K_5)$. The value of $F_5((x_i, \pi_5^i) - \infty)$ is determined by the image of $\pi_5^i + m$ in $K_5^\times/K_5^{\times 5}$. For $m \equiv \pm 1, \pm 7 \pmod{25}$, the images of $\pi_5^i + m$ in $U^{(2)}$ are

$$(1+\pi_5^i), \qquad (1-\pi_5^i), \qquad \zeta_4^3(7+\pi_5^i), \qquad \zeta_4^3(7-\pi_5^i)$$

respectively. Computing the π_5-expansion, we get

	$y+1$	$y-1$	$y+7$	$y-7$
$[(x_3, \pi_5^3) - \infty]$	δ	δ^{-1}	δ^3	δ^2
$[(x_4, \pi_5^4) - \infty]$	ϵ	ϵ^{-1}	ϵ^3	ϵ^2
$[(x_5, \pi_5^5) - \infty]$	η	η^{-1}	η^3	η^2

Together with (4) we have

$$F_5(J(K_5)/\phi J(K_5)) = \langle \delta, \epsilon, \eta \rangle.$$

Again by (4) for $\mathfrak{p} \nmid 5$, we have $\dim_{\mathbb{F}_5}(J(K_\mathfrak{p})/\phi J(K_\mathfrak{p})) = 1$. By Lemma 2.2, arbitrary odd integer m maps to 1 in $K_2^\times/K_2^{\times 5} \cong \langle 2, \zeta_5 \rangle = \langle \alpha_2, \beta_2 \rangle$. Hence,

$$\begin{array}{ccc} & y+m & y-m \\ [(0,m)-\infty] & 2 & 2^{-1} \end{array}$$

and $F_2(J(K_2)/\phi J(K_2))$ is $\langle \alpha_2 \rangle$. Similarly for $\mathfrak{p}$ which does not divide 10 or the totally splitting primes, the image of 2 in $K_\mathfrak{p}^\times/K_\mathfrak{p}^{\times 5}$ is trivial by Lemma 2.3. So

$$\begin{array}{ccc} & y+m & y-m \\ [(0,m)-\infty] & m & m^{-1} \end{array}$$

shows that $F_\mathfrak{p}(J(K_\mathfrak{p})/\phi J(K_\mathfrak{p})) = \langle \alpha_\mathfrak{p} \rangle$, when $\mathrm{ord}_\mathfrak{p}(m) \not\equiv 0 \pmod 5$. □

Remark 3.2. We note that Proposition 3.1 is enough to prove the main theorem, but the same strategy gives $F_5(J_{m^2}(K_5)/\phi J_{m^2}(K_5))$ when one knows the generators of $J_{m^2}(K_5)/\phi J_{m^2}(K_5)$. For example,

$$(-\pi_5, 2 + 3\pi_5^4 + 2\pi_5^5), \qquad (1, \pi_5^2 + \pi_5^3 + 3\pi_5^4), \qquad (2, 1)$$

are solutions of $y^2 \equiv x^5 + m^2 \pmod{\pi_5^6}$ when $m \equiv \pm 12 \pmod{25}$. Therefore,

$$\begin{aligned} &(\zeta_4^2(2 + 3\pi_5^4 + 2\pi_5^5 + 12), \zeta_4^3(\pi_5^2 + \pi_5^3 + 3\pi_5^4 + 12), \zeta_4(1 + 12)) \\ &\qquad \equiv (1 + 4\pi_5^5, 1 + 3\pi_5^2 + 3\pi_5^3 + \pi_5^4 + 4\pi_5^5, 1 + 2\pi_5^4 + 4\pi_5^5) \pmod{\pi_5^6} \\ &\qquad \equiv (\eta^4, \gamma^3\delta^3\epsilon, \epsilon^2\eta^4) \quad \text{in } K_5^\times/K_5^{\times 5}. \end{aligned}$$

Hence,

$$F_5(J_{m^2}(K_5)/\phi J_{m^2}(K_5)) = \langle \gamma\delta, \epsilon, \eta \rangle$$

when $m \equiv \pm 12 \pmod{25}$. Similarly we can compute $F_5(J_{m^2}(K_5)/\phi J_{m^2}(K_5))$ for other cases. Also, Lemmas 2.2 and 2.3 describe an image of prime element not lying above $p \equiv 1 \pmod 5$. Therefore, we can calculate the Selmer group of J_{m^2} when m is square-free and

(a) if p divides m then $p \not\equiv 1 \pmod 5$,
(b) there is at most one prime divisor p of m such that $p \equiv 4 \pmod 5$,

even though we do not fully describe the result. We will give an example in the end of this section.

Proposition 3.3. *Let m be an odd square-free integer satisfying the above two conditions* (a), (b) *and let $\mathfrak{p} \nmid 5$ be a prime of K dividing m. Then, $(i_\mathfrak{p}^{-1} \circ F_\mathfrak{p})(J_{m^2}(K_\mathfrak{p})/\phi J_{m^2}(K_\mathfrak{p}))$ contains* 2 *and prime generators dividing m chosen as in* Lemma 2.3.

Proof. This is a direct consequence of Lemma 2.3 and Proposition 3.1. □

Corollary 3.4. *For a rational prime p and the Jacobian J_{p^2}, we have*

$$\dim_{\mathbb{F}_5} \mathrm{Sel}_\phi(J_{p^2}/\mathbb{Q}) = 2, \qquad \text{if } p \equiv 7, 8 \pmod{25}.$$

When $p \equiv 24 \pmod{25}$*, there is a generator* π_w *of* w *above* p *satisfies* $\pi_w = a + b\sqrt{5}$ *for* $a, b \in \frac{1}{2}\mathbb{Z}$*. Then,*

$$\dim_{\mathbb{F}_5} \mathrm{Sel}_\phi(J_{p^2}/\mathbb{Q}) = \begin{cases} 1 & b \not\equiv 0 \pmod{5}, \\ 3 & b \equiv 0 \pmod{5}. \end{cases}$$

Proof. In the proof, we denote J by J_{p^2}. We first consider the case of $p \equiv 7, 8 \pmod{25}$. We recall that $i_5 : K(S,5) \to K^\times/K^{\times 5}$, and $K(S,5)$ is generated by $\zeta_5, 1+\zeta_5, 2, 1-\zeta_5$ and a prime p, which is inert in $K/\mathbb{Q}$. Since

$$i_5(\zeta_5, 1+\zeta_5, 2, 1-\zeta_5, 7, 8) = (\beta\gamma\epsilon, \beta^2\gamma^4\delta^2\epsilon^4, \epsilon^3\eta, \alpha, 1, \epsilon^4\eta^3)$$

by Lemma 2.1, we have

$$F_5(J(K_5)/\phi J(K_5)) = \langle \delta, \epsilon, \eta \rangle, \qquad \mathrm{im}\, i_5 = \langle \beta\gamma\epsilon, \beta^2\gamma^4\delta^2\epsilon^4, \epsilon^3\eta, \alpha \rangle,$$

together with Proposition 3.1. A sort of linear algebra shows that

$$\mathrm{im}\, i_5 \cap F_5(J(K_5)/\phi J(K_5)) = \langle \epsilon^3\eta \rangle,$$

and

$$(i_5^{-1} \circ F_5)(J(K_5)/\phi J(K_5)) = \langle 2, p \rangle.$$

By Proposition 3.1, $F_\mathfrak{p}(J(K_\mathfrak{p})/\phi J(K_\mathfrak{p})) = \langle \alpha_\mathfrak{p} \rangle$ for a prime $\mathfrak{p}$ not above 5. Now, Proposition 3.3 gives

$$(i_2^{-1} \circ F_2)(J(K_2)/\phi J(K_2)) \supset \langle 2, p \rangle, \qquad (i_p^{-1} \circ F_p)(J(K_p)/\phi J(K_p)) \supset \langle 2, p \rangle,$$

which shows that $\dim_{\mathbb{F}_5} \mathrm{Sel}_\phi(J/\mathbb{Q}) = 2$.

When $p \equiv 24 \pmod{25}$, we choose the generators $\pi_w, \pi_{\overline{w}}$ above p by $a \pm b\sqrt{5}$ for $a, b \in \frac{1}{2}\mathbb{Z}$. We still have $F_5(J(K_5)/\phi J(K_5)) \cong \langle \delta, \epsilon, \eta \rangle$. By Lemma 2.1, the images under i_5 of the generators above $p \equiv 24$ are in $\langle \gamma\delta\epsilon \rangle$ and trivial when $b \equiv 0 \pmod{5}$. Hence,

$$\mathrm{im}\, i_5 \subset \langle \beta\gamma\epsilon, \beta^2\gamma^4\delta^2\epsilon^4, \epsilon^3\eta, \alpha, \gamma\delta\epsilon \rangle.$$

Since $(\beta\gamma\epsilon)^3(\beta^2\gamma^4\delta^2\epsilon^4)(\gamma\delta\epsilon)^3$ is trivial, the dimension of the space in the right hand side is 4. Hence, the similar argument gives

$$\mathrm{im}\, i_5 \cap F_5(J(K_5)/\phi J(K_5)) = \langle \epsilon^3\eta \rangle,$$

and

$$(i_5^{-1} \circ F_5)(J(K_5)/\phi J(K_5)) = \begin{cases} \langle 2 \rangle & \text{if } b \not\equiv 0 \pmod{5}, \\ \langle 2, \pi_w, \pi_{\overline{w}} \rangle & \text{if } b \equiv 0 \pmod{5}. \end{cases}$$

Together with Proposition 3.3, we know that the dimension of the Selmer group $\mathrm{Sel}_\phi(J_{p^2}/\mathbb{Q})$ is 1 or 3, and dimension 3 if and only if $b \equiv 0 \pmod{5}$. □

Proof of Theorem 1.1. By the Dirichlet theorem on arithmetic progressions for number fields, there are infinitely many primes in a ray class modulo an ideal. Let us denote two real embeddings by σ_1, σ_2. For a modulus $(50)\cdot\sigma_1\sigma_2$ and a ray class $(2+\sqrt{5})$, there are infinitely many prime elements π which are congruent modulo $(50)\cdot\sigma_1\sigma_2$ to one of $u_F^{2n}(2+\sqrt{5})$ where $u_F = (1+\sqrt{5})/2$.

Using an integral basis $\{1, u_F\}$ of $\mathcal{O}_F$, we may write

$$\pi = u_F^{2n}(2+\sqrt{5}) + 50z_1 + 50z_2u_F$$

for some $z_1, z_2 \in \mathbb{Z}$. Then, the norm of π is $-1 \pmod{25}$. Let a_n and b_n be integers satisfying

$$u_F^n = a_n + b_nu_F.$$

Then,

$$\begin{aligned}\pi &= u_F^{2n}(2+\sqrt{5} \pm 50z_1\,(a_{-2n} + b_{-2n}u_F) \pm 50z_2\,(a_{-2n+1} + b_{-2n+1}u_F)) \\ &= u_F^{2n}(2+\sqrt{5} \pm 25(z_1(2a_{-2n} + b_{-2n}) + z_2(2a_{-2n+1} + b_{-2n+1}) + \sqrt{5}(z_1b_{-2n} + z_2b_{-2n+1}))).\end{aligned}$$

For a rational prime $p \equiv 24 \pmod{25}$ divided by π, there is a generator of (π) satisfying the condition of Corollary 3.4 with $b \not\equiv 0 \pmod 5$. From the exact sequence

$$0 \longrightarrow \frac{J_{p^2}(\mathbb{Q})}{\phi J_{p^2}(\mathbb{Q})} \longrightarrow \mathrm{Sel}_\phi(J_{p^2}/\mathbb{Q}) \longrightarrow Ш(J_{p^2}/\mathbb{Q})[\phi] \longrightarrow 0$$

and $J_{p^2}(\mathbb{Q})_{\mathrm{tor}} \cong \mathbb{Z}/5\mathbb{Z}$ (see [9, p. 286] and [8, p. 80], or [1, Theorem 4.1]. Note that the latter contains a detailed proof), one can deduce that $J_{p^2}(\mathbb{Q}) \cong \mathbb{Z}/5\mathbb{Z}$.

Also, for a prime $p \equiv 7, 8 \pmod{25}$ we have

$$\mathbb{Z}/5\mathbb{Z} \le J_{p^2}(\mathbb{Q}) \le \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}, \qquad w(p^2) = -1$$

by Corollary 3.4 and Corollary 2.5. Under the parity conjecture, the algebraic rank is also an odd number when the root number is -1. This proves the second part of the theorem. □

We note that the machinery also works for the totally split primes, even though one need to compute everything directly.

Proposition 3.5. *The Mordell–Weil rank of $J_{101^2}/\mathbb{Q}$ is zero.*

Proof. We will show that $\dim_{\mathbb{F}_5} \mathrm{Sel}_\phi(J_{101^2}/\mathbb{Q}) = 1$. We note that Sagemath [4] runs most of computation in the proof. Let $\mathfrak{p}_j$ for $j = 1, 2, 3, 4$ be a prime ideal of K above $p = 101$, and let us choose generators π_j by

$$\zeta_5^3 + 3\zeta_5^2 - \zeta_5 + 1, \qquad 3\zeta_5^3 + 4\zeta_5^2 + 2\zeta_5 + 2, \qquad -4\zeta_5^3 - 2\zeta_5^2 - \zeta_5 - 2, \qquad -2\zeta_5^3 - \zeta_5^2 + 2\zeta_5.$$

We note that $\pi_1\pi_2\pi_3\pi_4 = 101$. Also,

$$K(S,5) = \langle 2, \zeta_5, 1+\zeta_5, 1-\zeta_5, \pi_1, \pi_2, \pi_3, \pi_4\rangle.$$

Now we want to compute the image of $i_1 := i_{\pi_1} : K(S,5) \to K_{\mathfrak{p}_1}^\times/K_{\mathfrak{p}_1}^{\times 5}$ of the above generators. In Section 2 we showed that $K_{\mathfrak{p}_1}^\times/K_{\mathfrak{p}_1}^{\times 5}$ is generated by two elements $\alpha_{\mathfrak{p}_1}, \beta_{\mathfrak{p}_1}$ which is $\pi_{\mathfrak{p}_1}$ and ζ_{25}, respectively. Let $\rho_1 : \mathcal{O}_{K,\mathfrak{p}_1} \to \mathcal{O}_{K,\mathfrak{p}_1}/\mathfrak{p}_1\mathcal{O}_{K,\mathfrak{p}_1} \cong \mathbb{F}_{101}$ be a projection map. Then,

$$\rho_1(2, \zeta_5, 1+\zeta_5, 1-\zeta_5, \pi_2, \pi_3, \pi_4) = (2, 95, 96, 7, 92, 89, 81).$$

We also denote ρ_1 as a composition of the previous map and the quotient $\mathbb{F}_{101}^\times \to \mathbb{F}_{101}^\times/\mathbb{F}_{101}^{\times 5}$. Then, we know that

$$\rho_1(2,\zeta_5,1+\zeta_5,1-\zeta_5,\pi_2,\pi_3,\pi_4) = (\overline{2},\overline{1},\overline{3},\overline{3},\overline{8},\overline{2},\overline{2}).$$

Note that $\overline{2}^3 = \overline{8}$ and $\overline{2}$ is a multiplicative inverse of $\overline{3}$. Since the elements above are not divided by π_1, we can describe the images of elements in $K(S,5)$ in $K_{\mathfrak{p}_1}^{\times}/K_{\mathfrak{p}_1}^{\times 5}$. Now

$$\begin{array}{ccc} & y+m & y-m \\ [(0,m)-\infty] & 2m & (2m)^{-1} \end{array}$$

Therefore, $F_{\mathfrak{p}_1}(J(K_{\mathfrak{p}_1})/\phi J(K_{\mathfrak{p}_1}))$ is generated by the product of $\alpha_{\mathfrak{p}_1}$ and the image of 2. Hence,

$$(i_1^{-1}\circ F_{\mathfrak{p}_1})(J(K_{\mathfrak{p}_1})/\phi J(K_{\mathfrak{p}_1})) = \langle 2\pi_1,\zeta_5,2(1+\zeta_5),2(1-\zeta_5),2^2\pi_2,2^4\pi_3,2^4\pi_4\rangle.$$

Similarly, we have

$$\rho_2(2,\zeta_5,1+\zeta_5,1-\zeta_5,\pi_1,\pi_3,\pi_4) = (\overline{2},\overline{1},\overline{3},\overline{8},\overline{2},\overline{8},\overline{2}),$$

so $F_{\mathfrak{p}_2}(J(K_{\mathfrak{p}_2})/\phi J(K_{\mathfrak{p}_2}))$ is generated by the product of $\alpha_{\mathfrak{p}_2}$ and the image of 2. Hence,

$$(i_2^{-1}\circ F_{\mathfrak{p}_2})(J(K_{\mathfrak{p}_2})/\phi J(K_{\mathfrak{p}_2})) = \langle 2\pi_2,\zeta_5,2(1+\zeta_5),2^2(1-\zeta_5),2^4\pi_1,2^2\pi_3,2^4\pi_4\rangle.$$

Also,

$$\rho_3(2,\zeta_5,1+\zeta_5,1-\zeta_5,\pi_1,\pi_2,\pi_4) = (\overline{2},\overline{1},\overline{3},\overline{3},\overline{2},\overline{2},\overline{8}),$$
$$\rho_4(2,\zeta_5,1+\zeta_5,1-\zeta_5,\pi_1,\pi_2,\pi_3) = (\overline{2},\overline{1},\overline{2},\overline{8},\overline{8},\overline{2},\overline{2})$$

and

$$(i_3^{-1}\circ F_{\mathfrak{p}_3})(J(K_{\mathfrak{p}_3})/\phi J(K_{\mathfrak{p}_3})) = \langle 2\pi_3,\zeta_5,2(1+\zeta_5),2(1-\zeta_5),2^4\pi_1,2^4\pi_2,2^2\pi_4\rangle,$$
$$(i_4^{-1}\circ F_{\mathfrak{p}_4})(J(K_{\mathfrak{p}_4})/\phi J(K_{\mathfrak{p}_4})) = \langle 2\pi_4,\zeta_5,2^4(1+\zeta_5),2^2(1-\zeta_5),2^2\pi_1,2^4\pi_2,2^4\pi_3\rangle.$$

We denote each vector space $(i_j^{-1}\circ F_{\mathfrak{p}_j})(J(K_{\mathfrak{p}_j})/\phi J(K_{\mathfrak{p}_j}))$ over $\mathbb{F}_5$ by V_j for $j=1,2,3,4$. One can check that

$$W := V_1\cap V_2\cap V_3\cap V_4 = \langle \zeta_5, 2\pi_1\pi_2\pi_3\pi_4, 2^2\pi_2\pi_4(1-\zeta_5), 2^4(1-\zeta_5)^2(1+\zeta_5)^4\pi_1\pi_3\pi_4^3\rangle.$$

We recall that our embedding of K into K_5 maps ζ_5 to $1-\pi_5$. Then, π_1,π_2,π_3,π_4 are also mapped to

$$\begin{array}{ll} \pi_1 & \mapsto -(1+3\pi_5+4\pi_5^2+\pi_5^3+\pi_5^4) \\ \pi_2 & \mapsto 1+\pi_5+3\pi_5^2+2\pi_5^3+3\pi_5^4+4\pi_5^5 \\ \pi_3 & \mapsto 1+2\pi_5+\pi_5^2+4\pi_5^3+2\pi_5^4 \\ \pi_4 & \mapsto -(1+4\pi_5+2\pi_5^2+3\pi_5^3+\pi_5^5) \end{array}$$

modulo $O(\pi_5^6)$. So $-\pi_1,\pi_2,\pi_3,-\pi_4$ correspond to the $U^{(1)}$-part. By a routine computation, we have

$$i_5(\pi_1,\pi_2,\pi_3,\pi_4) = (\beta^3\gamma\delta^2\epsilon^2\eta^3, \beta\gamma^3\delta^4\epsilon\eta^3, \beta^2\delta^4\epsilon^4\eta^2, \beta^4\gamma\epsilon^3\eta^2).$$

We already know that

$$i_5(2,\zeta_5,1+\zeta_5,1-\zeta_5) = (\epsilon^3\eta, \beta\gamma\epsilon, \beta^2\gamma^4\delta^2\epsilon^4, \alpha)$$

and $F_5(J_{m^2}(K_5)/\phi J_{m^2}(K_5)) = \langle\delta,\epsilon,\eta\rangle$ by Proposition 3.1. The images of our basis members of W in the quotient space $(K_5^\times/K_5^{\times 5})/F_5(J_{m^2}(K_5)/\phi J_{m^2}(K_5))$ are $\overline{\beta\gamma}, \overline{1}, \overline{\alpha\gamma^4}, \overline{\alpha^2}$, respectively. Therefore $\mathrm{Sel}_\phi(J_{101^2}/\mathbb{Q})$ is one dimensional vector space generated by $2\pi_1\pi_2\pi_3\pi_4$. □

We conclude this section with an example on general m which is not divided by a rational prime equivalent to one modulo five.

Example 3.6 ($m = p_1p_2$ where $(p_1, p_2) \equiv (3, 4) \pmod{25}$.). Let $p_1 \equiv 3$ and $p_2 \equiv 4 \pmod{25}$, and π_w and $\pi_{\overline{w}}$ be prime elements $a \pm b\sqrt{5}$ for $a, b \in \frac{1}{2}\mathbb{Z}$ of K lying over p_2. Then, by Remark 3.2 and Lemma 2.1,

$$F_5(J(K_5)/\phi(J(K_5))) = \langle \gamma\delta, \epsilon, \eta\rangle \text{ and } \operatorname{im} i_5 = \langle \beta\gamma\epsilon, \beta^2\gamma^4\delta^2\epsilon^4, \epsilon^3\eta, \alpha, \epsilon\eta^2, (\gamma\delta\epsilon)^b\rangle.$$

So the previous argument shows that

$$(i_5^{-1} \circ F_5)(J(K_5)/\phi J(K_5)) = \begin{cases} \langle 2, p_1\rangle & \text{if } b \not\equiv 0 \pmod{5}, \\ \langle 2, p_1, \pi_w, \pi_{\overline{w}}\rangle & \text{if } b \equiv 0 \pmod{5}. \end{cases}$$

For the other bad primes $\mathfrak{p}$ we have $(i_{\mathfrak{p}}^{-1} \circ F_{\mathfrak{p}})(J(K_{\mathfrak{p}})/\phi J(K_{\mathfrak{p}}))$ contains $\langle 2, p_1, \pi_w, \pi_{\overline{w}}\rangle$, by Proposition 3.3. Therefore, for such $m = p_1p_2$,

$$\dim_{\mathbb{F}_5} \mathrm{Sel}_\phi(J_m/\mathbb{Q}) = \begin{cases} 2 & \text{if } b \not\equiv 0 \pmod{5}, \\ 4 & \text{if } b \equiv 0 \pmod{5}. \end{cases}$$

4. Special values of L-functions

In this section we will find sufficient conditions on A such that $L(1, J_A)$ becomes nonzero. By [3, Theorem 4], there is a Hecke character η_A of K such that

$$L(s, J_A) = L(s, \eta_A).$$

Following [9, Section 2], we denote $F := \mathbb{Q}(\sqrt{5})$ and $\chi_A := \eta_A|\cdot|_{\mathbb{A}}^{1/2}$ with $\mathbb{A} := \mathbb{A}_F$ the ring of adèles so that

$$L(1, J_A) = L(1, \eta_A) = L\left(\frac{1}{2}, \chi_A\right).$$

From now on, we assume that the global root number of χ_A is 1. Based on the work of [10, 12], Stoll and Yang give the following:

Proposition 4.1 ([9, Proposition 3.1])**.** *With the notation in* [9], *we have*

$$L(1, \eta_A) = \frac{\pi^2}{50C_1C_2}\left|\sum_{x\in F}\prod_{v\nmid 2A}\phi_v(x)\prod_{v|2A} I_v(x)\right|^2$$

for some constant C_1 and C_2.

Here $\phi = \prod_v \phi_v \in S(\mathbb{A})$ is an appropriately chosen Schwartz–Bruhat function and

$$I_v(x) = \int_{G_v} \omega_{\alpha,\chi_A,v}(g)\phi_v(x)dg \tag{5}$$

as in [9, p. 277]. We will introduce more precise notations later. Stoll and Yang further give a concrete choice of ϕ_v for $v \nmid 5A$ and infinite v. It allows them to compute $L(1, \eta_1)$. In this paper, we choose ϕ_v for $v \mid 5A$ and consider when $I_v(x)$ is non-zero.

Since the global root number of χ_A is $+1$, there is a unique $\alpha \in F^\times$ up to norm from $K^\times$ such that

$$\prod_{\substack{w \text{ places of } K \\ w|v}} \epsilon\left(\frac{1}{2}, \chi_{A,w}, \frac{1}{2}\psi_{K_w}\right)\chi_{A,w}(\delta) = \epsilon_v(\alpha)$$

for all places v of F (cf. [9, p. 276]). Here $\delta := \zeta_5^{-2} - \zeta_5^2$, ψ is an additive character of $\mathbb{A}_F$ given by $\psi = \prod_v \psi_v$ for $\psi_v(x) = e^{-2\pi\sqrt{-1}\lambda_v(x)}$ where

$$\lambda_v : F_v \xrightarrow{\mathrm{Tr}_{F_v/\mathbb{Q}_p}} \mathbb{Q}_p \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \mathbb{Q}/\mathbb{Z}\ ,$$

and $\psi_K := \psi \circ \mathrm{Tr}_{K/F}$. Also, ϵ on the left hand side are the local root numbers as in [9, Proposition 2.2], and ϵ_v is the local part of the Hecke character belonging to K/F. We let rings act on additive characters defined on them by multiplication with arguments. For example,

$$\left(\frac{1}{2}\psi_{K_w}\right)(x) := \psi_{K_w}\left(\frac{1}{2}x\right).$$

Since we only concern the case where A is a square not divisible by 2, [9, Lemma 2.3] tells us that we may choose

$$\alpha \in \left(\prod_{2\neq p|A} p\right) \cdot N_{K/F}K^\times$$

where $N_{K/F}$ denotes the norm. Next, we need to choose an appropriate Schwartz–Bruhat function $\phi = \prod_v \phi_v \in S(\mathbb{A})$ as in [9, p. 277]. To be more precise, we introduce more notations in [9, Section 2]. We fix an embedding $K \hookrightarrow \mathbb{C}$ such that $\zeta_5 \mapsto \exp(2\pi\sqrt{-1}/5)$. We also fix a CM type $\Phi = \{\sigma_2, \sigma_4\}$ of K where $\sigma_r(\zeta_5) = \exp(2\pi r\sqrt{-1}/5)$. Then the following lemma tells us a possible choice of ϕ_v for almost all places v.

Lemma 4.2 ([9, Lemma 3.2])**.** *Denote* char(X) *the characteristic function of the set X. Then,*

$$\phi_v(x) = \begin{cases} \mathrm{char}(\mathcal{O}_{F,v})(x) & v \nmid 10A\infty, \alpha \in \mathcal{O}_{F,v}^\times, \\ |2\sigma_j(\alpha\delta^3)|^{1/4}\exp\left(-\pi|\sigma_j(\alpha\delta^3)|\sigma_j(x)^2\right) & v = \sigma_j \in \{\sigma_2, \sigma_4\}. \end{cases}$$

If we choose $\alpha \in F^\times$ as above such that $\alpha \in \mathbb{Z}_2^\times$, then [9, Corollary 5.8] tells us that we may choose

$$\phi_2 = \mathrm{char}\left(\frac{1}{2} + \mathcal{O}_{F,2}\right).$$

We note that $\phi_2 = I_2$ and I_2 is a constant function (See [9, §4]). At $v = \sqrt{5}$, [12, Proposition 1.2, Corollary 1.4] tell us that we may choose

$$\phi_{\sqrt{5}} = 5^{\frac{2n(\chi_{A,\lambda})-1}{4}}\xi_\lambda \cdot \mathrm{char}(\mathcal{O}_{F,\sqrt{5}}).$$

Here, by denoting $\Delta := \delta^2$,

(1) $\lambda := 1 - \zeta_5 \in K$ is a prime element lying over $\sqrt{5}$.
(2) $n(\chi_{A,\lambda})$ is the conductor exponent of $\chi_{A,\lambda}$ which is completely determined by $q_5(A) = (A^4 - 1)/5$ (see [9, Proposition 2.2 (5)]):

$$n(\chi_{A,\lambda}) = \begin{cases} 1 & \text{if} \quad 5 \mid q_5(A), \\ 2 & \text{if} \quad 5 \nmid q_5(A). \end{cases}$$

(3) With $G = \{\pm 1\} \times U_K^{(1)}$, write $g = x + y\delta \in G$ and set

$$\xi_\lambda(g) = \begin{cases} \chi_{A,\lambda}(\delta(g-1))(\Delta, -y)_F & \text{if} \quad g \in U_K^{(1)}, \\ \chi_{A,\lambda}(\delta(g-1))(\Delta, -2\alpha)_F \epsilon(\frac{1}{2}, \epsilon_{K_w/F_v}, \psi_{K_\lambda}) & \text{if} \quad g \in G \setminus U_K^{(1)}. \end{cases}$$

This comes from [12, Proposition 1.2 (1)].[1]

By Proposition 4.1 and Lemma 4.2, we obtain

$$(6) \qquad L(1, \eta_A) = \frac{\pi^2}{50 C_1 C_2} \cdot 5^{\frac{2n(\chi_{A,\lambda})-1}{2}} \cdot \left| \sum_{x \in X'_A} \xi_\lambda(x) \phi_{\sigma_2}(x) \phi_{\sigma_4}(x) \cdot \left(\prod_{v \mid A} I_v(x) \right) \right|^2$$

where

$$X'_A = F \cap \left(\bigcap_{v \nmid 2A\infty} \mathcal{O}_{F,v} \right) \cap \left(\frac{1}{2} + \mathcal{O}_{F,2} \right).$$

For $v \mid A$ and w a place of K dividing v, we always have $n(\chi_{A,w}) = 1$ by [7, Proposition 3.3]. First, we consider the case $v \mid A$ splits in K/F. In this case we apply [10, Section 2]. Under the identification

$$K_v \cong \frac{F[t]}{(t^2 - \Delta)} \otimes_F F_v \cong F_v \cdot \delta \oplus F_v \cdot (-\delta)$$

we have $\delta = (1, -1) \in F_v \oplus F_v$. Denote $\pi_{F_v} \in \mathcal{O}_{F,v}$ by a uniformizer and in this case $n_v = 1$. To get $\phi_v = \phi_{v,1}$, following the notation of [10, Theorem 2.15], we first compute

$$\begin{aligned} \rho\left(\operatorname{char}\left(1 + \pi_{F_v}\mathcal{O}_{F,v}\right)\right)(x) &:= |\alpha|_v^{\frac{1}{2}} \psi_v\left(\frac{\alpha x^2}{2}\right) \int_{F_v} \psi_v(\alpha x y) \psi_v\left(\frac{\alpha y^2}{4}\right) \operatorname{char}\left(1 + \pi_{F_v}\mathcal{O}_{F,v}\right)(y) dy \\ &= |\alpha|_v^{\frac{1}{2}} \psi_v\left(\frac{\alpha x^2}{2}\right) \int_{1 + \pi_{F_v}\mathcal{O}_{F,v}} \psi_v(\alpha x y) dy \\ &= |\alpha|_v^{\frac{1}{2}} \psi_v\left(\frac{\alpha x^2}{2}\right) \int_{\pi_{F_v}\mathcal{O}_{F,v}} \psi_v(\alpha x (y+1)) dy \\ &= |\alpha|_v^{\frac{1}{2}} \psi_v\left(\frac{\alpha x^2}{2} + \alpha x\right) \int_{\pi_{F_v}\mathcal{O}_{F,v}} \psi_v(\alpha x y) dy \\ &= |\alpha|_v^{\frac{1}{2}} \psi_v\left(\frac{\alpha x^2}{2} + \alpha x\right) \operatorname{meas}(\pi_{F_v}\mathcal{O}_{F,v}) \operatorname{char}\left(\pi_{F_v}^{-2}\mathcal{O}_{F,v}\right)(x). \end{aligned}$$

Hence we get

$$\phi_v = \operatorname{meas}(\mathcal{O}_{F,v})^{-\frac{1}{2}} \operatorname{meas}(\pi_{F_v}\mathcal{O}_{F,v}) q_v^{\frac{1}{2}} |\alpha|_v^{\frac{1}{2}} \psi_v\left(\frac{\alpha x^2}{2} + \alpha x\right) \operatorname{char}\left(\pi_{F_v}^{-2}\mathcal{O}_{F,v}\right)(x).$$

[1] It seems that there is a typo in [12, Proposition 1.2 (1)]. Compare the statement and its proof [12, pp. 354–355].

To apply [9, Proposition 3.1], we need to compute

$$
\begin{aligned}
I_v(x) &:= \int_{\mathcal{O}_{F,v}^{\times}} \omega_{\alpha,\chi_A,v}(g)\phi_v(x)dg \\
&= \int_{\mathcal{O}_{F,v}^{\times}} \chi_{A,v}(g)|g|_v^{\frac{1}{2}}\phi_v(xg)dg \\
&= \int_{\mathcal{O}_{F,v}^{\times}} \phi_v(xg)dg \\
&= \mathrm{meas}(\mathcal{O}_{F,v})^{-\frac{1}{2}}\mathrm{meas}(\pi_{F_v}\mathcal{O}_{F,v})q_v^{\frac{1}{2}}|\alpha|_v^{\frac{1}{2}} \int_{\mathcal{O}_{F,v}^{\times}} \psi_v\left(\frac{\alpha}{2}(xg)^2+\alpha(xg)\right)\mathrm{char}\left(\pi_{F_v}^{-2}\mathcal{O}_{F,v}\right)(xg)dg \\
&= \mathrm{meas}(\mathcal{O}_{F,v})^{-\frac{1}{2}}\mathrm{meas}(\pi_{F_v}\mathcal{O}_{F,v})q_v^{\frac{1}{2}}|\alpha|_v^{\frac{1}{2}}\mathrm{char}\left(\pi_{F_v}^{-2}\mathcal{O}_{F,v}\right)(x) \int_{\mathcal{O}_{F,v}^{\times}} \psi_v\left(\frac{\alpha}{2}(xg)^2+\alpha(xg)\right)dg.
\end{aligned}
$$

We note that the action of Weil representation ω is described in [10, Corollary 2.10]. Since there is a representative

$$\alpha \in \left(\prod_{2\neq p|A} p\right)\cdot N_{K/F}K^{\times},$$

we choose α such that $\psi_v\left(\frac{\alpha}{2}(xg)^2+\alpha(xg)\right) = 1$ for $g \in \mathcal{O}_{F,v}^{\times}$ and $x \in \pi_{F_v}^{-2}\mathcal{O}_{F,v}$ for all $v \mid A$ splitting in K/F. Then

$$I_v|_{\pi_{F_v}^{-2}\mathcal{O}_{F,v}} = \mathrm{meas}(\mathcal{O}_{F,v})^{-\frac{1}{2}}\mathrm{meas}(\pi_{F_v}\mathcal{O}_{F,v})q_v^{\frac{1}{2}}|\alpha|_v^{\frac{1}{2}}\int_{\mathcal{O}_{F,v}^{\times}} dg = \frac{\mathrm{meas}(\mathcal{O}_{F,v}^{\times})}{\mathrm{meas}(\mathcal{O}_{F,v})^{\frac{1}{2}}}\mathrm{meas}(\pi_{F_v}\mathcal{O}_{F,v})q_v^{\frac{1}{2}}|\alpha|_v^{\frac{1}{2}}$$

is a non-zero constant. Therefore, there is a non-zero constant $c_v(\alpha)$ such that

$$I_v(x) = c_v(\alpha)\mathrm{char}(\pi_{F_v}^{-2}\mathcal{O}_{F,v})(x), \tag{7}$$

when $v \mid A$ splits in K/F.

Finally, consider the case $v \mid A$ is inert in K. Following the notation of [12, p. 339], we have

$$n(\psi'_{K_v}) = n\left(\frac{\alpha\delta}{4}\psi_{K_v}\right) = n(\psi_{K_v}) - \mathrm{ord}_{F_v}(\alpha) = -\mathrm{ord}_{F_v}(\alpha).$$

We choose α so that $\mathrm{ord}_{F_v}(\alpha) = 1$ and $n(\psi'_{K_v}) = -1$. Since we have $n(\chi_{A,v}) = 1$ and $w \mid v$ is unramified, we are in the case of [12, Proposition 1.5] with $\eta = 1$ the trivial character. Then we may choose,

$$
\begin{aligned}
\phi_v(x) &= \mathrm{char}(\pi_{F_v}\mathcal{O}_{F,v})(\pi_{F_v}x) \\
&\quad + \frac{1}{2G(\psi''_{F_v})}\sum_{\substack{(S,T)\in\kappa_v^2 \\ S^2-T^2\equiv\Delta \bmod \pi_{F_v}}} \xi_v^{-1}\left(\frac{S+\delta}{T}\right)\left(\frac{T}{\kappa_v}\right)\psi''_{F_v}\left(\frac{\Delta\alpha}{2}S(\pi_{F_v}x)^2\right)\mathrm{char}(\mathcal{O}_{F,v})(\pi_{F_v}x)
\end{aligned}
$$

when $\xi_v(-1) = \left(\frac{-1}{\kappa_v}\right)$, or

$$\phi_v(x) := \mathrm{char}(1+\pi_{F_v}\mathcal{O}_{F,v})(\pi_{F_v}x) - \mathrm{char}(-1+\pi_{F_v}\mathcal{O}_{F,v})(\pi_{F_v}x) + \frac{1}{G(\psi''_{F_v})} \sum_{\substack{(S,T)\in\kappa_v^2 \\ S^2-T^2\equiv\Delta \bmod \pi_{F_v}}} \xi_v^{-1}\left(\frac{S+\delta}{T}\right)\left(\frac{T}{\kappa_v}\right)\psi''_{F_v}(S(\pi_{F_v}x)^2-2T\pi_{F_v}x+S)\mathrm{char}(\mathcal{O}_{F,v})(\pi_{F_v}x)$$

when $\xi_v(-1) = -\left(\frac{-1}{\kappa_v}\right)$ and $\xi_v^{-1} \neq \eta_0$, where $\kappa_v := \mathcal{O}_{F,v}/\pi_{F_v}$ is the residue field of F_v. Note that ψ''_{F_v} in [12, Proposition 1.5] has conductor $\pi_{F_v}\mathcal{O}_{F,v}$ (see the proof of [11, Proposition 3.4] for the detail) so we regard ψ''_{F_v} as a character of κ_v and $G(\psi''_{F_v})$ is the Gauss sum of ψ''_{F_v}. Together with (6), we obtain

Proposition 4.3. *Let A be a square integer such that the root number of η_A is $+1$. Then, there is a non-zero constant $c_v(\alpha)$ such that*

$$L(1,\eta_A) = \frac{\pi^2}{50C_1C_2}\cdot 5^{\frac{2n(\chi_{A,\lambda})-1}{2}} \cdot \prod_{\substack{v|A \\ v\text{ split}}} c_v(\alpha)\cdot \left|\sum_{x\in X_A}\xi_\lambda(x)\phi_{\sigma_2}(x)\phi_{\sigma_4}(x)\cdot \prod_{\substack{v|A \\ v\text{ inert}}} I_v(x)\right|^2 \tag{8}$$

where $I_v(x)$ is taken from (5) *and*

$$X_A = F\cap\left(\bigcap_{v\nmid 2A\infty}\mathcal{O}_{F,v}\right)\cap\left(\frac{1}{2}+\mathcal{O}_{F,2}\right)\cap\left(\bigcap_{\substack{v|A \\ v\text{ split}}}\pi_{F_v}^{-2}\mathcal{O}_{F,v}\right).$$

Proof of Theorem 1.2. When $5^2 \mid (A^4-1)$, we have $n(\chi_{A,\lambda}) = 1$ which implies that ξ_λ is trivial (See [12, Proposition 1.2, Corollary 1.4]). Since every prime divisor of A splits in K/F, we obtain that

$$L(1,\eta_A) = \frac{\pi^2}{50C_1C_2}\cdot 5^{\frac{1}{2}}\cdot \prod_{\substack{v|A \\ v\text{ split}}} c_v(\alpha)\cdot\left|\sum_{x\in X_A}\phi_{\sigma_2}(x)\phi_{\sigma_4}(x)\right|^2. \tag{9}$$

Recall that σ_2 and σ_4 have real values on F and X_A is a subset of F. Therefore,

$$\phi_{\sigma_2}(x)\phi_{\sigma_4}(x) = \sqrt{2}\alpha^{\frac{1}{2}}5^{\frac{3}{8}}\exp\left(-\pi\alpha\left(\left(2\sin\frac{2\pi}{5}\right)^3\sigma_2(x)^2+\left(2\sin\frac{4\pi}{5}\right)^3\sigma_4(x)^2\right)\right)$$

is positive and the last term of (9) does not vanish. Hence $L(1,\eta_A)$ is non-zero. □

Proof of Corollary 1.3. We note that $q_5(101^2)$ is divided by 5. Now the result follows from Proposition 3.5 and Theorem 1.2. □

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